Semiclassical Asymptotics of the Residues of the Scattering Matrix for Shape Resonances

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July 6, 1998

Abstract

The aim of this study is to give complete semiclassical asymptotics of the residues $\mathrm{Res}[S(\lambda,\omega,\omega'),\rho]$ at some pole ρ of the distributional kernel of the scattering matrix $S(\lambda)$ corresponding to a semiclassical two-body Schrödinger operator $P = -h^2\Delta + V$, and considered as a meromorphic operator-valued function with respect to the energy λ. We do it in the case where the pole ρ considered is a shape resonance of P. This is a continuation of [Be] where an extra geometrical condition was assumed (namely the absence of caustics near the energy level $\mathrm{Re}\rho$). Here we drop this assumption by using an FBI transform which permits to work in the complexified phase space. Then we show that some semiclassical WKB expansions are global, and this allows us to find out estimates for the residue of the type $\mathcal{O}(h^N e^{-2S_0/h})$ where S_0 is the Agmon width of the potential barrier, and N may be arbitrarily large depending on an explicit geometrical location of the incoming and outgoing waves ω and ω' one consider. Full asymptotic expansions are obtained under some additional generic geometric assumption on the potential V.

1 Introduction

As it is well known, one of the main purposes of scattering experiments is to detect hidden objects inside an unknown region of space. From a quantum mechanical point of view, these hidden objects can be considered e.g. as bounded regions where the particles have a great probability of presence. Now, using the semiclassical approximation, these regions may consist in classically allowed regions, surrounded by a classically forbidden one. The simplest case of such a model is given by the "potential well in an island" as described in [HeSj], which gives rise to so-called shape resonances to which a rather large literature has been devoted (see e.g. [BCD], [Ge-Ma1], [HeSj], [HiSi], [Na1], [Na2], ...). In particular, in the case where the well is a non degenerate minimum of the potential, very precise semiclassical constructions can be done, which permit to foresee rather well some quantitative experimental results. This is the case we consider in this paper : let

$$P = -h^2\Delta + V(x)$$

be the semiclassical two-body Schrödinger operator on $\mathbf{R}^n$, where the (smooth) potential V satisfies, for some fixed positive energy λ_0 :

$$\{V > \lambda_0\} = \ddot{O}\backslash\{x_0\} \text{ where } \ddot{O} \text{ is bounded and connected, and } x_0 \text{ is a point of } \ddot{O} \text{ ;}$$
$$V''(x_0) \text{ is positive definite.}$$

Then, under additional assumptions of analyticity and decay at infinity on V, it is well known that for h small enough P admits resonances near λ_0 ("shape resonances"), which can be viewed either as complex eigenvalues of P acting on some weighted L^2-type space (see [HeSj]) or as well as poles of the scattering matrix $S(\lambda)$ associated with P (see [GeMa2]). If one denotes $S(\lambda,\omega,\omega')$ the distributional kernel of $S(\lambda)$, this also means that they are poles of $S(\lambda,\omega,\omega')$ for some value of $(\omega,\omega')\in \mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$, and the main question we are interested in is to know if the quantitative knowledge of $S(\lambda,\omega,\omega')$ for λ real permits to detect (in a semiclassical sense to be specified) the presence of the well x_0 inside the island $\ddot{O}$.

Mathematically, this question leads to estimate the residues of $S(\lambda,\omega,\omega')$ at the shape resonances, and see in which case it is or not neglectible when compared with the imaginary part of the resonances. Actually, such a problem has already been studied by S.Nakamura in [Na2], and by the first author in [Be] :

In [Na2], by using Agmon inequalities it has been proved (in an even more general situation) that for any $\omega,\omega'\in\mathbf{S}^{n-1}$ and for any $\varepsilon>0$ the residue at such a shape resonance ρ satisfies:

$$\text{Residue}[S(\lambda,\omega,\omega');\rho] = \mathcal{O}(e^{-(2S_0-\varepsilon)/h})$$

where S_0 denotes the Agmon distance (that is the pseudodistance associated to the degenerated metric $(V(x)-\lambda_0)_+dx^2$) between x_0 and $\partial\ddot{O}$.

In [Be], using the WKB constructions of [HeSj] and assuming that no caustics appear in the complementary of $\ddot{O}$, it has been proved that this residue satisfies :

$$\text{If } \ (\omega,\omega')\notin L\times(-L), \text{ then}$$

$\text{Residue}[\mathrm{S}(\lambda,\omega,\omega');\rho] = \mathcal{O}(h^\infty e^{-2S_0/h})$;

If $(\omega,\omega')\in L\times(-L)$, then $\text{Residue}[S(\lambda,\omega,\omega');\rho] = \mathcal{O}(h^{N(n,\rho)}e^{-2S_0/h})$;where $L\subset\mathbf{S}^{n-1}$ is the set of the limit directions (as the time tends to $+\infty$) of all the rays starting at a point of $\partial\ddot{O}$ which can be joined to x_0 by a smooth minimal geodesic with respect to the metric $(V(x)-\lambda_0)_+dx^2$ (the so called *points of type 1* in the terminology of [HeSj], which will be used here too). Moreover, $N(n,\rho)$ is a real number depending only on the dimension and on ρ.

Here, we plan to generalize and specify the latter result to cases where caustics may appear in $\mathbf{R}^n\backslash\ddot{O}$, that is dropping out the corresponding assumption made in [Be]. The main tool for such an extension will consist in the use of a global Fourier-Bros-Iagolnitzer (in short FBI) transform T, namely by showing that if u is a resonant state of P, then Tu admits anyway a global WKB expansion near the bicharacteristics of P starting at $(x_1,0)$ where x_1 is any point of type 1. Such a property is not very surprising, since it is well known in analytic microlocal analysis that one of the effects of the FBI transform is precisely to 'suppress' the real caustics : see [Sj1]. However, and as far as we know, until now this phenomenon has been scarsely exploited to get *global* WKB constructions in the semiclassical context.

Once we have obtained these WKB expansions, the proof follows essentially the arguments of [Be], with the difference that now one has to work in phase space instead of configuration space. In particular, the set L has to be defined in a slightly different way : now it consists in the limit directions of the projections on the position space, of all the bicharacteristics starting at some $(x_1, 0) \in \mathbf{R}^{2n}$ with x_1 of type 1. Moreover, an additional study is done in order to obtain a full asymptotic expansion of the residue in some cases.

The structure of the paper is the following :

In section 2, we specify our assumptions, introduce some geometrical objects, and state our main result. Section 3 is devoted to a representation formula for the residues of the scattering matrix, which is based on the constructions of Isozaki and Kitada [IsKi1] but also using a global FBI transform. In section 4 a global phase function is constructed on phase space, which permits in section 5 to get global WKB expansions. Finally the proof is completed in section 6, by extending and specifying the arguments of [Be].

Acknowledgements: Part of this work was done while the first author was preparing a PhD thesis at the University of Paris-Nord (Villetaneuse, France). Moreover, the first complete version of the paper has been achieved while the second author was visiting Tokyo University in May 1998, invited there by Shu Nakamura to whom he wishes to express his gratitude.

2 Assumptions and Results

Let $P = -h^2\Delta + V$ be a Schrödinger operator on $L^2(\mathbf{R}^n)$ where $h \in]0,1]$ is a semiclassical small parameter . Here V denotes the operator of multiplication by a fuction $V(x)$ satisfying the following assumptions :

Assumption H. 1 *The potential $V \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is analytic and admits a holomorphic extension in*

$$D = \{z \in \mathbf{C}^n : |\Im m z| \leq \frac{1}{C_0}\langle \Re e z\rangle\} \tag{2.1}$$

and satisfies

$$|V(z)| \leq C_0 \langle z\rangle^{-\tau} \textit{ in } D,$$

where C_0 and τ are positive constants. (Here we have denoted $\langle z\rangle = (1+|z|^2)^{1/2}$.)

Assumption H. 2 *There exist $\lambda_0 > 0$, a connected bounded open set $\ddot{O} \subset \mathbf{R}^n$ and $x_0 \in \ddot{O}$, such that:*

$$V(x_0) = \lambda_0,\ \ V''(x_0) > 0\ ,\ V > \lambda_0 \textit{ on } \ddot{O}\backslash\{x_0\},\ V < \lambda_0 \textit{ on } \mathbf{R}^n\backslash\overline{\ddot{O}} \tag{2.2}$$

and λ_0 is a non trapping energy level outside $\ddot{O}$, that is :

$$\{(x,\xi) \in p^{-1}(\lambda_0)\ ;\ x \in \ddot{O}^c,\ |\exp sH_p(x,\xi)| \not\longrightarrow \infty \textit{ when} |s| \to \infty\} = \emptyset \tag{2.3}$$

where $H_p = \sum_{i=1}^n (\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j})$ *is the Hamiltonian field of $p(x,\xi) = \xi^2 + V(x)$.*

Under the above hypothesis, it is well known (see e.g. [BCD], [HeSj], [HiSi]) that P admits resonances near λ_0, and that for any interval of the form $I = [\lambda_0, \lambda_0 + Ch)$ with $C > 0$ arbitrarily large, the resonances ρ_1, ρ_2, ..., of P in $I - [0,\delta]$ ($\delta > 0$ small enough) admit, as $h \to 0$, asymptotic expansions of the type :

$$\rho_j \sim \lambda_0 + \sum_{k=2}^{\infty} \rho_{j,k} h^{k/2}$$

where $\rho_{1,2} = e_1$, $\rho_{2,2} = e_2$, ... are the eigenvalues of the harmonic oscillator $H_0 = -\Delta + \frac{1}{2}\langle V''(x_0)x, x\rangle$.

In the particular case where e_j is simple (e.g. if $e_j = e_1$ is the ground energy level of H_0) then, for some $\delta > 0$ and for all h small enough, P admits a *unique* resonance inside $[\lambda_0 + (e_j - \delta)h, \lambda_0 + (e_j + \delta)h] + i[-\delta, \delta]$. Moreover, it has been proved in [HeSj] that under the same assumption (i.e. the simplicity of $\rho_{j,2}$), the resonant state ψ_j of P associated to this unique resonance ρ_j, admits near x_0 a WKB expansion of the type :

$$\psi_j(x,h) = h^{-\frac{n}{4} - m_j} a_j(x,h) e^{-f(x)/h} \tag{2.4}$$

where $a_j(x,h) \sim \sum_{k \geq 0} a_{j,k}(x) h^{k/2}$ is the realization of an analytic symbol in the sense of [Sj1], $f(x) = d_V(x_0, x)$ denotes the distance from x to x_0 relatively to the degenerate metric $(V(x) - \lambda_0)_+ dx^2$ (the so called Agmon distance), and $h^{-\frac{n}{4} - m_j}$ is just a normalization factor which is made in such a way that one has :

$$\| h^{-\frac{n}{4} - m_j} a_j e^{-f/h} \|_{L^2(\Omega)} = 1 + \mathcal{O}(h^\infty)$$

for any small enough fixed neighborhood Ω of x_0. Actually the expansion (2.4) can be continued along any d_V-geodesic starting at x_0, until any point of type 1 of the boundary $\partial \ddot{O}$ is reached (let us recall that this points are defined as the elements of $\overline{B_V(x_0, S_0)} \cap \partial \ddot{O}$, where $S_0 = d_V(x_0, \partial \ddot{O})$ and $B_V(x_0, S_0) = \{x \in \mathbf{R}^n \ ; \ d_V(x_0, x) < S_0\}$). At such a point both a_j and f develop singularities, but going inside the complex domain by avoiding the caustic set :

$$\mathcal{C} = \text{Closure}\left(\{x \in \ddot{O} \ ; \ d_V(x_0, x) = S_0 + d_V(x, \partial \ddot{O})\} \right),$$

one can prove that the WKB behaviour of ψ_j can be extended through $\partial \ddot{O}$, and gives rise outside $\ddot{O}$ (close enough, however, to a point of type 1 of $\partial \ddot{O}$) to the following expansion :

$$\psi_j(x,h) = h^{-\frac{n}{4} - m_j} \widetilde{a}_j(x,h) e^{-\widetilde{f}(x)/h} \tag{2.5}$$

where $\widetilde{f}$ is the analytic continuation of f along a path around $\mathcal{C}$ such that $\Im m \widetilde{f} \leq 0$ outside $\ddot{O}$, and $\widetilde{a}_j$ is a realization of the analytic symbol obtained by continuing analytically the $a_{j,k}$'s along the same complex path as for $\widetilde{f}$. Moreover, $\Re e \widetilde{f} \geq S_0$ with equality exactly on the integral curves of $\nabla_x \Im m \widetilde{f}$ starting at a point of type 1 of the boundary of $\ddot{O}$. Adding the following assumption of non degeneracy :

Assumption H. 3 *The set $\mathcal{C} \cap \partial \ddot{O} = \overline{B_V(x_0, S_0)} \cap \partial \ddot{O}$ is a submanifold Γ of $\partial \ddot{O}$, and $\mathcal{C}$ has a contact of order exactly 2 with $\partial \ddot{O}$ along Γ,* then it can be also shown that the transversal Hessian of $\Re e\tilde{f}$ on $\bigcup_{s>0 \text{ small}} \exp s\nabla_x \Im m\tilde{f}(\Gamma)$ is positive definite. Finally, outside neighborhoods of the points of type 1, it is shown in Prop 9.11 of [HeSj] that ψ_j is neglectible in the precise following sense : there exists $\varepsilon > 0$ such that

$$\psi_j(x) = \mathcal{O}(e^{-(S_0+\varepsilon)/h}) \tag{2.6}$$

uniformly with respect to h small enough, and locally uniformly with respect to x in an open neighborhood of the set $\overline{\ddot{O}} \setminus \overline{B_V(x_0, S_0)}$.

Before stating our result, let us write more precisely the definition of the set L already invoked in the introduction :

$$L = \bigcap_{T>0} \overline{\left\{ \frac{\pi_x \exp sH_p(x,0)}{|\pi_x \exp sH_p(x,0)|} \ ; \ s \geq T \ , \ x \in \partial\ddot{O} \text{ is of type } 1 \right\}} \quad (\subset \mathbf{S}^{n-1}). \tag{2.7}$$

Then the result is :

Theorem 2.1 *Let $P = -h^2\Delta + V$ with V satisfying assumptions H.1, H.2 and H.3, let e_j be a simple eigenvalue of the harmonic oscillator $H_0 = -\Delta + \langle V''(x_0)x, x\rangle/2$, and let ρ_j be the unique resonance of P in $[\lambda_0 + (e_j - \delta)h, \lambda_0 + (e_j + \delta)h] + i[-\delta, \delta]$ ($\delta > 0$ small enough). Then, the residue at ρ_j of the distribution kernel $S(\lambda; \omega, \omega')$ of the scattering matrix $S(\lambda)$ associated to P, satisfies :*

1. *If $(\omega, \omega') \notin L \times (-L)$, then :*

$$\text{Res}\ (S(\lambda; \omega, \omega'), \rho_j) = \mathcal{O}(h^\infty e^{-2S_0/h}) \tag{2.8}$$

uniformly with respect to h small enough, where S_0 is the Agmon distance between the well x_0 and the sea $\mathbf{R}^n \backslash \ddot{O}$.

2. *If $(\omega, \omega') \in L \times (-L)$, then :*

$$\text{Res}\ (S(\lambda; \omega, \omega'), \rho_j) = \mathcal{O}(h^{d+2-\frac{n}{2}-2m_j} e^{-2S_0/h}) \tag{2.9}$$

uniformly with respect to h small enough, where $m_j \in \mathbf{R}$ is the number appearing in (2.4), and d is the codimension of Γ in $\partial\ddot{O}$.

3. *If $(\omega, \omega') \in L \times (-L)$ and $d = n-1$, then the residues admit a full asymptotic expansion of the type :*

$$\text{Res}\ (S(\lambda; \omega, \omega'), \rho_j) = h^{\frac{n}{2}+1-2m_j} e^{-2S_0/h} \left[\sum_{\substack{x \in \Gamma_\omega \\ x' \in \Gamma_{-\omega'}}} e^{i(C_{x'} - C_x)/h} \sum_{k=0}^{\infty} \alpha_{j,k}(x, x') h^{k/2} + \mathcal{O}(h^\infty) \right] \tag{2.10}$$

where one has denoted, for $\omega \in L$:

$$\Gamma_\omega = \{x \in \Gamma \ ; \ \frac{\pi_x \exp sH_p(x,0)}{|\pi_x \exp sH_p(x,0)|} \to \omega \quad (s \to +\infty)\}$$

and where, denoting $(y(s),\eta(s)) = \exp sH_p(x,0)$, C_x is defined as :

$$C_x = \int_0^T y(s).\nabla V(y(s))ds$$

where $T > 0$ is some fixed large enough constant, that can be taken to $+\infty$ in the short-range case $\tau > 1$ (so that in this case C_x can be interpreted formally as the classical action between $(x,0)$ and infinity). Moreover, in the case $j = 1$ the complex numbers $\alpha_{1,0}(x,x')$ never vanish.

Remark- For $j = 1$ formula (2.10) permits to get a semiclassical equivalent of the residue in rather generic situations (e.g. if Γ_ω and $\Gamma_{-\omega'}$ are reduced to single points). However, cancellations could happen in particular situations where a greater number of points of type 1 are contained in Γ_ω or $\Gamma_{-\omega'}$.

In the followings it will be more convenient for us to use the notation (ω_1,ω_2) instead of the traditional one (ω,ω') for the variables of $\mathbf{S}^{n-1}$ appearing in the distribution kernel of $S(\lambda)$.

3 A Representation Formula

Using the constructions of Isozaki and Kitada [IsKi1], it has been shown in [Be] (prop 3.1) that under the assumptions of theorem 2.1, the residue of $S(\lambda,\omega_1,\omega_2)$ corresponding to the pole ρ_j can be represented by the following formula :

$$\mathrm{Res}(S(\lambda,\omega_1,\omega_2),\rho_j) = \frac{1}{2}\rho_j^{(n-2)/2}\langle U_\mu\psi_j, U_{\bar\mu}e^{i\varphi_1/h}t_1\rangle_{L^2(\mathbf{R}^n)}\langle U_\mu e^{i\varphi_2/h}t_2, U_{\bar\mu}\bar\psi_j\rangle_{L^2(\mathbf{R}^n)}$$

where ψ_j is the resonant state associated to ρ_j, U_μ is a complex distorsion operator of the type :

$$(U_\mu\varphi)(x) = (J_\mu(x))^{1/2}\varphi((1+\mu\chi_0(|x|))x) \tag{3.1}$$

with

$$\chi_0 \in C^\infty(\mathbf{R}_+),\ \ \chi_0 = \begin{cases} 0 & \text{if } |x| \le 3R \\ 1 & \text{if } |x| \ge 4R, \end{cases}$$

(here $J_\mu(x)$ is the determinant of the Jacobian of the application $x \mapsto (1+\mu\chi_0(|x|))x$ and μ is a small enough complex number with $\Im m\mu > 0$), and where the functions $\varphi_k = \varphi_k(x,\eta)\Big|_{\eta=\sqrt{\rho_j}\omega_k}$ and $t_k = t_k(x,\eta;h)\Big|_{\eta=\sqrt{\rho_j}\omega_k}$ $(k=1,2)$ satisfy to the following properties :

1. There exist $d > 0$, $R > 0$ and $-1 < \sigma_1^- < \sigma_1^+ < 0 < \sigma_2^- < \sigma_2^+ < 1$ such that φ_k is C^∞ and real in the following domain :

$$\Gamma(R,d,\sigma_k^-,\sigma_k^+) = \{(x,\eta) \in \mathbf{R}^{2n}\ ;\ |x| \ge R\ ,\ |\eta| \ge d\ ,\ \cos(x,\eta) \in [-1,\sigma_k^-] \cup [\sigma_k^+,1]\} \tag{3.2}$$

   and satisfies there the eikonal equation :

$$|\nabla_x\varphi_k|^2 + V(x) = \eta^2$$

   and the estimates :

$$|D_x^\alpha D_\eta^\beta(\varphi_k(x,\eta) - x\eta)| = \mathcal{O}(\langle x\rangle^{1-\tau-|\alpha|}) \tag{3.3}$$

   for all $\alpha,\beta \in \mathbf{N}^n$.

2. The function t_k is C^∞ and supported in $\Gamma(R,d,\sigma_k^-,\sigma_k^+)$, admits there an asymptotic expansion of the type
$$t_k(x,\eta;h) \sim h\sum_{l\geq 0} t_{k,l}(x,\eta)h^l \tag{3.4}$$
and is exponentially small in $\Gamma(2R,d,\sigma_k^- - \delta,\sigma_k^+ + \delta)$ for some $\delta > 0$, in the sense that there exists some $\varepsilon_k > 0$ such that :
$$t_k(x,\eta;h) \;=\; \mathcal{O}(e^{-\varepsilon_k\langle x\rangle\langle\eta\rangle/h}) \tag{3.5}$$
uniformly for $(x,\eta)\in\Gamma(2R,d,\sigma_k^- - \delta,\sigma_k^+ + \delta)$ and $h > 0$ small enough.

3. Moreover, the functions φ_k and t_k can be extended analytically with respect to $r = |x|$ and $\sigma = |\xi|$ for r in $\{\Re e r > 3R, |\Im m r| < \varepsilon\langle\Re e r\rangle\}$ ($\varepsilon > 0$ sufficiently small), and σ in a complex neighborhood of $\sqrt{\lambda_0}$ (so that $U_{\overline{\mu}}(e^{i\varphi_1/h}t_1)$ and $U_\mu(e^{i\varphi_2/h}t_2)$ have a well defined meaning, and actually belong to $L^2(\mathbf{R}^n)$). Moreover, their extensions continue to satisfy estimates analog to (3.3) and (3.5).

Since our purpose here is to suppress the problems due to possible caustics of the Hamiltonian flow outside $\ddot{O}$, we shall instead use another representation of the residue of $S(\lambda,\omega_1,\omega_2)$, which uses the following FBI transformation (cf.[Sj1],[Sj2]) :
$$T_\nu u(x,h) = \int e^{-\nu(x-y)^2/2h}u(y,h)dy \tag{3.6}$$
where ν is a positive parameter which will be choosen large enough later on.
It is standard to see that for any $C > 0$, T_ν sends $H^{loc}_{C|\Im m x|}$ into $H^{loc}_{\Phi_0}$ where $\Phi_0(x) = \nu(\Im m x)^2/2$ and H^{loc}_f denotes the Sjöstrand space associated to the weight f (see [Sj2] for more details).
Furthermore T_ν is an isometry from $L^2(\mathbf{R}^n,dy)$ into $L^2(\mathbf{C}^n,\left(\frac{\pi h}{\nu}\right)^{\frac{3n}{2}}e^{-2\Phi_0(x)/h}i^n dx\wedge d\overline{x})$, and if we define also the transformation $\widetilde{T}_\nu$ by :
$$\begin{aligned}\widetilde{T}_\nu u(\alpha) &= 2^{-\frac{n}{2}}\left(\frac{\nu}{\pi h}\right)^{\frac{3n}{4}}\int e^{i\nu(\alpha_x-y)\alpha_\xi/h-\nu(\alpha_x-y)^2/2h}u(y)dy &(3.7)\\ &= 2^{-\frac{n}{2}}\left(\frac{\nu}{\pi h}\right)^{\frac{3n}{4}}e^{-\nu\alpha_\xi^2/2h}(T_\nu u)(\alpha_x - i\alpha_\xi), &(3.8)\end{aligned}$$
(where $\alpha = (\alpha_x,\alpha_\xi)\in\mathbf{C}^{2n}$), then we get :
$$||\widetilde{T}_\nu u||_{L^2(\mathbf{R}^{2n})} = ||u||_{L^2(\mathbf{R}^n)}.$$

Now, for $\delta > 0$ denote $\mathcal{A}_\delta$ the set of entire functions φ on $\mathbf{C}^n$ which satisfy :
$$\exists\,\varepsilon > 0 \text{ such that } \lim_{|z|\to\infty,|\arg z|<\delta} e^{\varepsilon z^2}\varphi(z) = 0$$
Then, using the fact that $\widetilde{T}_\nu$ is an isometry from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$, and making a change of contour of integration, it is easy to see that for all $u,v\in\mathcal{A}_\delta$ and for all $t > 0$ small enough, one has :

$$\langle u, v\rangle_{L^2(\mathbf{R}^n)} \quad = \quad \langle \widetilde{T}_\nu u, \widetilde{T}_\nu v\rangle_{L^2(\mathbf{R}^{2n})} = \int_{\alpha_x \in (1+it)}$$

$\mathbf{R}^{n}_{\alpha_\xi \in (1+it)^{-1}\mathbf{R}^n \widetilde{T}_\nu u(\alpha)\overline{\widetilde{T}_\nu v(\overline{\alpha})} d\alpha}$

In the followings we denote $\Lambda_t = (1+it)\mathbf{R}^n \times (1+it)^{-1}\mathbf{R}^n$, and for $f = f(x,\xi)$ defined on Λ_t we set :

$$\|f\|_{L^2(\Lambda_t)} = \left(\int_{\mathbf{R}^{2n}} |f((1+it)x, (1+it)^{-1}\xi)|^2 dx d\xi\right)^{1/2}.$$

We also use the generic notation $\overline{f}(z) := \overline{f(\overline{z})}$ (whenever the right-hand side is meaningful), and we start by giving some preliminary results.

Lemma 3.1 *If ψ denotes a resonant state of P associated to a resonance close enough to λ_0, then for all $t > 0$ sufficiently small one has :*

$$\widetilde{T}_\nu \psi \in L^2(\Lambda_t),$$
$$\widetilde{T}_\nu \overline{\psi} \in L^2(\Lambda_{-t}),$$

and

$$\|\widetilde{T}_\nu \psi\|_{L^2(\Lambda_t)} + \|\widetilde{T}_\nu \overline{\psi}\|_{L^2(\Lambda_{-t})} = \mathcal{O}(\|\psi((1+it)y)\|_{L^2(\mathbf{R}^n_y)})$$

uniformly for $h > 0$ sufficiently small. Moreover, this estimate remains valid if ψ is replaced by any element of $\mathcal{A}_\delta$ (and $|t| < \delta$).

Proof

It is well known (see e.g. [BaSk] for the short-range case, and [HeSj] in the general case) that for any $\varepsilon > 0$, one has $e^{-\varepsilon|x|/h}\psi \in L^2(\mathbf{R}^n)$ for h sufficiently small, so that $\widetilde{T}_\nu\psi$ is a well defined smooth function. Moreover, by standard arguments of non-characteristic deformations (see e.g. [HeMa]), one sees that ψ can be extended holomorphically in a complex sector around $\mathbf{R}^n$, and that for $t > 0$ sufficiently small, $\psi((1+it)x) \in L^2(\mathbf{R}^n)$ (which obviously also implies that $\overline{\psi}((1-it)x) \in L^2(\mathbf{R}^n)$). Then the result can be easily obtained by making, in the expression of $\widetilde{T}_\nu\psi$, the change of contour of integration : $y \mapsto (1+it)y$ and, in the expression of $\widetilde{T}_\nu\overline{\psi}$ the change of contour of integration : $y \mapsto (1-it)y$. $\square$

Now, for $k \in \mathbf{N}$, denote

$$u_k(x) = \left(\frac{\pi}{k}\right)^{n/2} \int e^{-k(x-y)^2}\psi(y)dy. \tag{3.9}$$

Then $u_k \in \mathcal{A}_\delta$, and by a change of contour of integration as before, one can see that $u_k \to \psi$ in $L^2((1+it)\mathbf{R}^n)$ as $k \to +\infty$, and therefore, using the previous lemma :

$$\begin{aligned} &\widetilde{T}_\nu u_k \to \widetilde{T}_\nu \psi \text{ in } L^2(\Lambda_t) \\ &\widetilde{T}_\nu \overline{u_k} \to \widetilde{T}_\nu \overline{\psi} \text{ in } L^2(\Lambda_{-t}). \end{aligned} \tag{3.10}$$

Using instead the change of contour : $y \mapsto y + \mu\chi_0(|y|)y$, one also obtains for any complex number μ small enough such that $\Im m\mu > 0$:

$$\begin{aligned} &U_\mu u_k \to U_\mu \psi \text{ in } L^2(\mathbf{R}^n) \\ &U_{\overline{\mu}} \overline{u_k} \to U_{\overline{\mu}} \overline{\psi} \text{ in } L^2(\mathbf{R}^n). \end{aligned} \tag{3.11}$$

As a consequence, since for all $u, v \in \mathcal{A}_\delta$ one has :

$$\langle U_\mu u, U_{\overline{\mu}} v\rangle_{L^2(\mathbf{R}^n)} = \langle u, v\rangle_{L^2(\mathbf{R}^n)} = \langle \widetilde{T}_\nu u, \widetilde{T}_\nu v\rangle_{L^2(\mathbf{R}^{2n})} = \int_{\Lambda_t} \widetilde{T}_\nu u(\alpha)\overline{\widetilde{T}_\nu v(\overline{\alpha})}d\alpha$$

we deduce from (3.10) and (3.11) that for all $u \in \mathcal{A}_\delta$:

$$\langle U_\mu \psi, U_{\overline{\mu}} u\rangle_{L^2(\mathbf{R}^n)} = \int_{\Lambda_t} \widetilde{T}_\nu \psi(\alpha)\overline{\widetilde{T}_\nu u(\overline{\alpha})}d\alpha \tag{3.12}$$

and

$$\langle U_\mu u, U_{\overline{\mu}} \overline{\psi}\rangle_{L^2(\mathbf{R}^n)} = \int_{\Lambda_t} \widetilde{T}_\nu u(\alpha)\overline{\widetilde{T}_\nu \overline{\psi}(\overline{\alpha})}d\alpha. \tag{3.13}$$

Now, for $C > 0$ large enough, denote $f_C(\alpha_\xi) = (|\alpha_\xi| - C)_+$. Then we have the estimate:

Lemma 3.2 *There exist $\delta_0 > 0$ and $C > 0$ such that for all $t > 0$ sufficiently small:*

$$\|e^{2\delta_0 f_C/h}\widetilde{T}_\nu \psi\|_{L^2(\Lambda_t)} + \|e^{2\delta_0 f_C/h}\widetilde{T}_\nu \overline{\psi}\|_{L^2(\Lambda_{-t})} = \mathcal{O}(\|\psi((1+it)y)\|_{L^2(\mathbf{R}^n_y)})$$

uniformly for h sufficiently small.

Proof
We observe that for $|\xi|$ sufficiently large, the symbol of P satisfies

$$|p(x,\xi)| \geq \frac{1}{C'}|\xi|^2$$

for some positive constant C'. Actually, if $\delta_0 > 0$ is small enough, the same kind of estimate holds for the symbol of $e^{\delta_0\langle\xi\rangle/h}Pe^{-\delta_0\langle\xi\rangle/h}$, and even if (x,ξ) stands on $\Lambda_{\pm t}$ instead of $\mathbf{R}^{2n}$. Then the result is a consequence of standard a priori estimates for elliptic analytic pseudodifferential operators : see e.g. [Sj1] for more details, or [HeSj] and [Ma3]-[Ma4] for versions more adapted to the spaces $\Lambda_{\pm t}$. □

Lemma 3.3 *There exists $C_1 > 0$ such that for all $t > 0$ sufficiently small, one has :*

$$\|e^{-\delta_0 f_C/h}\widetilde{T}_\nu(t_2 e^{i\varphi_2/h})\|_{L^2(\Lambda_t)} = \mathcal{O}\left(e^{C_1 t/h}\|U_{it}(t_2e^{i\varphi_2/h})\|_{L^2(\mathbf{R}^n)}\right)$$

where U_{it} is the distorsion introduced in (3.1).

Proof
We set $g = t_2 e^{i\varphi_2/h}$ and we use the change of contour $(y, y') \mapsto (\omega_t(y), \omega_t(y'))$, with $\omega_t(y) = y + it\chi_0(|y|)y$. Then, denoting $\tilde{\alpha}_x = (1+it)\alpha_x$ and $\tilde{\alpha}_\xi = (1+it)^{-1}\alpha_\xi$, we have for $(\alpha_x, \alpha_\xi) \in \mathbf{R}^{2n}$:

$$(\widetilde{T}_\nu g)((1+it)\alpha_x, (1+it)^{-1}\alpha_\xi)$$

$$= \int e^{i\nu(\tilde{\alpha}_x - y)\tilde{\alpha}_\xi/h - \nu(\tilde{\alpha}_x - y)^2/2h} f(y)dy = \int e^{i\nu(\tilde{\alpha}_x - \omega_t(y))\tilde{\alpha}_\xi/h - \nu(\tilde{\alpha}_x - \omega_t(y))^2/2h}(U_{it}g)(y)d\omega_t(y)$$

$$= I_1(\tilde{\alpha}_x, \tilde{\alpha}_\xi) + I_2(\tilde{\alpha}_x, \tilde{\alpha}_\xi)$$

where I_1 (resp. I_2) are obtained by introducing the function $\chi(y)$ (resp. $1-\chi(y)$) in the integrand, with $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi(y) = 1$ for $|y| \leq 4R$, $\mathrm{Supp}\chi \subset \{|y| \leq 5R\}$.

Since $\omega_t(y) = (1+it)y$ in $\mathrm{Supp}(1-\chi)$, we get easily that:

$$\|I_2\|_{L^2(\Lambda_t)} = \mathcal{O}(\|U_{it}g\|).$$

On the other hand, since the scalar product $\tilde{\alpha}_x \cdot \tilde{\alpha}_\xi$ remains real, and $|y| \le 5R$ on $\mathrm{Supp}\chi$, we obtain for t sufficiently small:

$$|I_1(\tilde{\alpha}_x, \tilde{\alpha}_\xi)| = \mathcal{O}(e^{[6\nu Rt|\alpha_\xi| - \nu(\alpha_x - y)^2/4]h})|U_{it}g(y)|$$

from which the result follows. $\square$

As a consequence of lemma 3.3, and since $U_{it}(\mathcal{A})$ is dense in $L^2(\mathbf{R}^n)$, have the following corollary:

Corollary 3.1 *There exists a sequence $\{v_k\}$ of elements of $\mathcal{A}$ such that $e^{-\delta_0 f/h}\widetilde{T}_\nu v_k \to e^{-\delta_0 f/h}\widetilde{T}_\nu(t_2 e^{i\varphi_2/h})$ in $L^2(\Lambda_t)$ as $k \to +\infty$.*

Indeed, choosing a sequence v_k similar to the one of (3.9) (but with $t_2 e^{i\varphi_2/h}$ instead of ψ), we also have $U_\mu v_k \to U_\mu(t_2 e^{i\varphi_2/h})$ and therefore, formula (3.13), lemma 3.2 and lemma 3.3 permit to get at the limit :

$$\langle U_\mu t_2 e^{i\varphi_2/h}, U_{\overline{\mu}}\overline{\psi}\rangle_{L^2(\mathbf{R}^n)} = \int_{\Lambda_t} \widetilde{T}_\nu(t_2 e^{i\varphi_2/h})(\alpha)\overline{\widetilde{T}_\nu\overline{\psi}(\overline{\alpha})}d\alpha. \quad (3.14)$$

In a similar way, one also has :

$$\langle U_\mu \psi, U_{\overline{\mu}} t_1 e^{i\varphi_1/h}\rangle_{L^2(\mathbf{R}^n)} = \int_{\Lambda_t} \widetilde{T}_\nu\psi(\alpha)\overline{\widetilde{T}_\nu(t_1 e^{i\varphi_1/h})(\overline{\alpha})}d\alpha. \quad (3.15)$$

Finally, denoting

$$\mathcal{T}_{t,\nu}(u,v) = \int_{\Lambda_t} \widetilde{T}_\nu u(\alpha)\overline{\widetilde{T}_\nu v(\overline{\alpha})}d\alpha$$

and using the first representation formula given in this section, we have proved :

Proposition 3.1 *Under the assumptions of theorem 2.1, one has for all $\omega_1, \omega_2 \in S^{n-1}$:*

$$\mathrm{Res}(S(\lambda, h, \omega_1, \omega_2), \rho_j) = \frac{1}{2}\rho_j{}^{(n-2)/2}\mathcal{T}_{t,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j})\mathcal{T}_{t,\nu}(\psi_j, e^{i\varphi_1/h}t_1)$$

Before giving a first estimate on $\mathcal{T}_{t,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j})$ and $\mathcal{T}_{t,\nu}(\psi_j, e^{i\varphi_1/h}t_1)$, let us observe that, using lemma 3.2 and the fact that $\widetilde{T}_\nu u(\alpha)$ and $\overline{\widetilde{T}_\nu u(\overline{\alpha})}$ are holomorphic functions of α for any $u \in \mathcal{S}'(\mathbf{R}^n)$, one can make in the definition of $\mathcal{T}_{t,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j})$ and $\mathcal{T}_{t,\nu}(\psi_j, e^{i\varphi_1/h}t_1)$ a change of contour of integration which transform Λ_t into the contour :

$$\widetilde{\Lambda}_t \;:\; \mathbf{R}^{2n} \ni (\alpha_x, \alpha_\xi) \mapsto$$

$((1+it\chi_1(\alpha_x))\alpha_x\ ,\ (1+it\chi_1(\alpha_x))^{-1}\alpha_\xi)$ (3.16) where $\chi_1 \in C^\infty(\mathbf{R}^n\ ;\ [0,1])$, $\mathrm{Supp}\chi_1 \subset \{|\alpha_x| \ge 5R\}$, $\chi_1 = 1$ in $\{|\alpha_x| \ge 6R\}$.

Now we give a first estimate which will permit us to neglect, in the previous expression of the residue, the parts of $\mathcal{T}_{t,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j})$ and $\mathcal{T}_{t,\nu}(\psi_j, e^{i\varphi_1/h}t_1)$ corresponding to large values of $|\alpha|$.

Let us denote :

$$\mathcal{T}_{R,\nu}^{-}(u,v) = \int_{\substack{\alpha\in\widetilde{\Lambda}_t \\ |\alpha|\leq 7R}} \widetilde{T}_\nu u(\alpha)\overline{\widetilde{T}_\nu v(\overline{\alpha})}d\alpha \tag{3.17}$$

and

$$\mathcal{T}_{R,\nu}^{+}(u,v) = \int_{\substack{\alpha\in\widetilde{\Lambda}_t \\ |\alpha|\geq 7R}} \widetilde{T}_\nu u(\alpha)\overline{\widetilde{T}_\nu v(\overline{\alpha})}d\alpha \tag{3.18}$$

Then we have :

Lemma 3.4 *There exists $\delta_1 > 0$ such that for all R sufficiently large and for all $t > 0$ small enough, one has :*

$$|\mathcal{T}_{R,\nu}^{+}(e^{i\varphi_2/h}t_2, \overline{\psi}_j)| + |\mathcal{T}_{R,\nu}^{+}(\psi_j, e^{i\varphi_1/h}t_1)| = \mathcal{O}(e^{-(S_0+\delta_1 tR)/h}) \tag{3.19}$$

uniformly with respect to h small enough.

Proof.

By construction, if $|y| \geq 4R$ we have for some $\delta_2 > 0$ independent of t and R :

$$|e^{i\varphi_2/h}t_2(\omega_t(y))| + |\overline{e^{i\varphi_1/h}t_1}(\omega_t(y))| = \mathcal{O}(e^{-\delta_2 t\langle y\rangle/h}) = \mathcal{O}(e^{-\delta_2 t(\langle y\rangle+\langle 4R\rangle)/2h})$$

and if $|y| \leq 4R$ and $|\alpha_x| \geq 6R$, then $\mathrm{Re}(\alpha_x - y)^2 \geq R^2/2$. As a consequence, cutting the integral in two parts one gets easily for α in $\widetilde{\Lambda}_t \cap \{|\alpha_x| \geq 6R\}$:

$$\left|\widetilde{T}_\nu(e^{i\varphi_2/h}t_2)(\alpha)\right| + \left|\overline{\widetilde{T}_\nu(e^{i\varphi_1/h}t_1)}(\alpha)\right| = \mathcal{O}(e^{-\delta_2 t(R+|\alpha_x|)/h}). \tag{3.20}$$

On another hand, one can prove by Agmon-type inequalities (see [HeSj]) that for any $\varepsilon > 0$, ψ_j is locally uniformly $\mathcal{O}(e^{-(S_0-\varepsilon)/h})$ in a complex neighborhood of $\ddot{O}^c$. Since the operator $P((1+it)x, (1+it)^{-1}hD_x) - \lambda_0$ is elliptic for $|x|$ large enough, it follows from microlocal a priori exponential weighted estimates (see e.g. [Ma4]) that, for $\delta_0 > 0$ and f_C as in lemma 3.2 :

$$\|e^{2\delta_0 f_C/h}\widetilde{T}_\nu\psi_j\|_{L^2(\Lambda_t\cap\{|\alpha_x|\geq 6R\})} + \|e^{2\delta_0 f_C/h}\widetilde{T}_\nu\overline{\psi}_j\|_{L^2(\Lambda_{-t}\cap\{|\alpha_x|\geq 6R\})} = \mathcal{O}(e^{-(S_0-\varepsilon)/h}) \tag{3.21}$$

for any $\varepsilon > 0$. (Here we have also used the fact that $\|\psi_j((1+it)y)\|_{L^2(\mathbf{R}_y^n)} = \mathcal{O}(1)$ uniformly with respect to h.)

Then, the result follows from (3.20), (3.21) and the fact that if $|\alpha_x| \leq 6R$ and $|\alpha| \geq 7R$, then $|\alpha_\xi| \geq R/2$. □

Now, thank to proposition 3.1 and lemma 3.4, we see that it remains to estimate the two quatities $\mathcal{T}_R^-(\psi, e^{i\varphi_1/h}t_1)$ and $\mathcal{T}_R^-(e^{i\varphi_2/h}t_2, \overline{\psi})$. For this purpose, we show in the next two sections that $\widetilde{T}_\nu\psi_j$ admits a WKB-type expansion near the points which give the main contributions to these quantities.

4 Global Phase Functions

We start by recalling some basic notions attached to the transformation T_ν (we refer to [Sj2] for more details). Let K_ν be the canonical transformation associated to T_ν, defined by : $K_\nu(x,\xi) = (x - \frac{i}{\nu}\xi, \xi)$. We also denote $\Lambda_{\Phi_0} = \{(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)), x \in \mathbf{C}^n\} = \{(x,\xi) \in \mathbf{C}^{2n}; \xi = -\nu \Im m x\} = K_\nu(\mathbf{R}^{2n})$. If $y_0 \in \partial \ddot{O}$ is a point of type 1 and $\mathcal{U}_1$ is a small open complex neighborhood of y_0, then the set :

$$\Lambda = \{(y, i\widetilde{f}'(y))), y \in \mathbf{C}^n, \Re e y \in \ddot{O}^c,\ y \in \mathcal{U}_1\} \tag{4.1}$$

is a $\mathbf{C}$-Lagrangian manifold with respect to the complex symplectic 2-form $\sigma = dy \wedge d\eta$ on $\mathbf{C}^{2n}$, included in $\{(y,\eta) \in \mathbf{C}^{2n}\ ;\ \eta^2 + V(y) - \lambda_0 = 0\}$. Applying the transformation K_ν we obtain :

$$K_\nu(\Lambda) = \{(x,\xi) \in \mathbf{C}^{2n}\ ;\ \xi = i\widetilde{f}'(y),\ x = y - \frac{i}{\nu}\xi,\ \Re e y \in \ddot{O}^c,\ y \in \mathcal{U}_1\} \tag{4.2}$$

which is a $\mathbf{C}$-Lagrangian manifold too, since K_ν is a symplectic complex transformation.

We denote by $\widetilde{\Lambda}$ the extention of Λ by the flow of H_p :

$$\widetilde{\Lambda} = \{\exp sH_p(\lambda)\ ;\ \lambda \in \Lambda\ ,\ s \in \mathbf{R}\}$$

and we set $\widetilde{p} = p \circ K_\nu^{-1}$. Let us also set $\theta = K_\nu \circ \mathbf{c} \circ K_\nu^{-1}$ where $\mathbf{c}$ is the complex conjugation : $\mathbf{c}(x,\xi) = (\overline{x}, \overline{\xi})$. A straighforward computation gives

$$\theta(x,\xi) = (\overline{x} - \frac{2i}{\nu}\overline{\xi}, \overline{\xi})$$

and since $\det K_\nu = 1$, we see that :

$$H_{\widetilde{p}} = K_\nu \circ H_p \circ K_\nu^{-1}.$$

This implies : $\exp sH_{\widetilde{p}} = K_\nu \circ \exp sH_p \circ K_\nu^{-1}$, and hence $d(\exp sH_{\widetilde{p}}) = K_\nu \circ d(\exp sH_p) \circ K_\nu^{-1}$. Furthermore, since p is real on the real we have $H_p \circ \mathbf{c} = \mathbf{c} \circ H_p$, and thus, for all $\rho \in \mathbf{C}^{2n}$, $d(\exp sH_p)_\rho \circ \mathbf{c} = \mathbf{c} \circ d(\exp sH_p)_{\overline{\rho}}$. We deduce from this that for all $\rho \in \mathbf{C}^{2n}$:

$$d(\exp sH_{\widetilde{p}})_\rho \circ \theta = \theta \circ d(\exp sH_{\widetilde{p}})_{\theta(\rho)}. \tag{4.3}$$

On another hand, consider the quadratic form Q defined for $u \in \mathbf{C}^{2n}$ by :

$$Q(u) = \frac{1}{2i}\sigma(u, \theta(u)).$$

Another straightforward computation gives $Q(x,\xi) = -\frac{1}{\nu}|\xi|^2 + \Im m(\xi\overline{x})$, and thus in particular :

$$Q\ |_{x=0} = -\frac{1}{\nu}|\xi|^2. \tag{4.4}$$

Now, let $y_0 = \pi_x \exp s_0 H_p(y_0, 0)$ with $s_0 > 0$ small enough (so that y_0 is a point of type 1 in $\ddot{O}^C$ close enough to y_0), and denote $\rho_0 = K_\nu(y_0, i\widetilde{f}'(y_0)) \in K_\nu(\Lambda)$. Then the tangent space of $K_\nu(\Lambda)$ at ρ_0 is :

$$\mathbf{T}_{\rho_0}K_\nu(\Lambda) = \left\{(y + \frac{1}{\nu}\widetilde{f}''(y_0)y\ ,\ i\widetilde{f}''(y_0)y)\ ;\ y \in \mathbf{C}^{2n}\right\}$$

and therefore, if $u = (x,\xi) = (y + \frac{1}{\nu}\widetilde{f}''(y_0)y\ ,\ i\widetilde{f}''(y_0)y) \in \mathbf{T}_{\rho_0}K_\nu(\Lambda)$, one finds :

$$Q(u) = -\frac{1}{\nu}|\widetilde{f}''(y_0).y|^2 + \Im m\,\langle i\widetilde{f}''(y_0).y, \overline{y} + \frac{1}{\nu}\overline{\widetilde{f}''(y_0).\overline{y}}\rangle = \langle f_1''(y_0).y_0, y_0\rangle + \langle f_1''(y_0).y_2, y_2\rangle$$

where $y_0 = \Re e y$, $y_2 = \Im m y$ and $f_1 = \Re e \widetilde{f}$. In particular, since $f_1''(y_0) \geq 0$, one gets :

$$Q\mid_{\mathbf{T}_{\rho_0}K_\nu(\Lambda)} \geq 0 \tag{4.5}$$

and thus, using (4.4) we see that $K_\nu(\Lambda)$ is necessarily transversal to the direction $\{x=0\}$ at ρ_0. Moreover, (4.3) implies that Q is invariant under the flow of $H_{\widetilde{p}}$ at any point ρ such that $\rho = \theta(\rho)$, and thus at any $\rho = K_\nu(\exp sH_p(y_0,0))$ since p is real on the real. As a consequence, using also the fact that $K_\nu(\widetilde{\Lambda})$ is invariant under $\exp sH_{\widetilde{p}}$, we deduce from (4.5) that for any $\rho \in K_\nu(\{\exp sH_p(y_0,0)\ ;\ s\in\mathbf{R}\})$, one has :

$$Q\mid_{\mathbf{T}_\rho K_\nu(\widetilde{\Lambda})} \geq 0.$$

Therefore, we get by (4.4) that $K_\nu(\widetilde{\Lambda})$ is everywhere transversal to the direction $\{x=0\}$, and thus by standard arguments on Lagragian manifolds, $K_\nu(\widetilde{\Lambda})$ is described by a global phase function, in the sense that there exists Φ holomorphic on $\pi_x\left(K_\nu(\widetilde{\Lambda})\right)$ such that :

$$K_\nu(\widetilde{\Lambda}) = \{(x,\xi)\in\mathbf{C}^{2n}\ ;\ \xi = \Phi'(x)\ ,\ x\in\pi_x\left(K_\nu(\widetilde{\Lambda})\right)\}$$

that is, for some complex neighborhood $\mathcal{V}$ of $\{\pi_x(\exp sH_p(y_0,0))\ ;\ t\in\mathbf{R}\}$:

$$K_\nu(\widetilde{\Lambda}) = \{(x,\xi)\in\mathbf{C}^{2n}\ ;\ \xi = \Phi'(x)\ ,\ x + \frac{i}{\nu}\Phi'(x)\in\mathcal{V}\} =: \Lambda_{\Im m\,\Phi} \tag{4.6}$$

(here the notation is justified by the fact that $\Phi'(x) = \frac{2}{i}\frac{\partial \Im m\,\Phi}{\partial x}$).

Notice that by construction, and possibly by adding a constant to Φ, if x is such that $x + \frac{i}{\nu}\Phi'(x)$ is close to some $\pi_x(\exp sH_p(y_0,0))$ with $s>0$ small enough, then $\Phi(x)$ can be written in the following way (see also ([He-Sj3] formula 4.40) :

$$\Phi(x) = c.v_y(i\nu\frac{(x-y)^2}{2} + i\widetilde{f}(y) - iS_0). \tag{4.7}$$

where $c.v_y$ denotes the critical value with respect to the y variables, and where the corresponding critical point $y_c(x) = x + \frac{i}{\nu}\Phi'(x)$ satisfies :

$$\left(x, i\widetilde{f}'(y_c(x))\right) = K_\nu\left(y_c(x), i\widetilde{f}'(y_c(x))\right). \tag{4.8}$$

Moreover, with this choice of normalization we claim that :

$$\begin{cases} -\Im m\Phi(x) \leq \Phi_0(x), & \text{with equality exactly on} \\ \Gamma_1 := \pi_x\left(K_\nu\left(\widetilde{\Lambda}\cap\bigcup_{s\in\mathbf{R}}\exp sH_p(\Gamma\times\{0\})\right)\right) \end{cases} \tag{4.9}$$

where Γ is the set introduced in assumption H.3. Indeed, using (4.7), (4.8) and the fact that $y_c(x)$ is critical, one finds for x close to some $\pi_x(K_\nu(\exp sH_p(y_0,0)))$ with $s>0$ small enough and $y_0\in\Gamma$:

$$\Phi''(x)=i\nu(\mathbf{I}-y_c'(x))=i\nu\left(\mathbf{I}-\left(\mathbf{I}+\frac{1}{\nu}\widetilde{f}''(y_c(x))\right)^{-1}\right)$$

and thus, since Φ is holomorphic :

$$\frac{\partial^2\Im m\,\Phi}{\partial\Re e x^2}=-\frac{\partial^2\Im m\,\Phi}{\partial\Im m x^2}=\Im m\;\Phi''(x)=\nu\mathbf{I}-\nu(1+A)^{-1/2}(1+B^2)^{-1}(1+A)^{-1/2} \qquad (4.10)$$

where $A:=\nu^{-1}\Re e\,\widetilde{f}''(y_c(x))$ and $B:=\nu^{-1}(1+A)^{-1/2}\Im m\,\widetilde{f}''(y_c(x))(1+A)^{-1/2}$. Moreover, one also has :

$$\frac{\partial^2\Im m\,\Phi}{\partial\Re e x\partial\Im m x}=\Re e\;\Phi''(x)=-\nu(1+A)^{-1/2}B(1+B^2)^{-1}(1+A)^{-1/2} \qquad (4.11)$$

Using the fact that $y_c(x)$ is real when $x\in\pi_x\left(K_\nu\left(\exp sH_p(\Gamma\times\{0\})\right)\right)$, $s>0$ small enough (which is a consequence of (4.8)), one can deduce from (4.10) and the fact that $\Re e\,\widetilde{f}''(y_c(x))$ has rank d (and is non negative) :

$$\text{Rank Hess }\left(\Phi_0+\Im m\,\Phi\right)(x)=n+d$$

at any point $x=\pi_x(K_\nu(\exp sH_p(y_0,0)))$ with $s>0$ small enough and $y_0\in\Gamma$. But since the (real) codimension of Γ_1 in $\mathbf{C}^n$ is precisely $n+d$, and by construction $\Phi_0+\Im m\,\Phi=\text{const.}=0$ on Γ_1, this means that the transversal Hessian of $\Phi_0+\Im m\,\Phi$ is non degenerate at such points x. Moreover one can see on (4.10)-(4.11) that Hess $\left(\Phi_0+\Im m\,\Phi\right)(x)$ is non negative, and therefore the previous transversal Hessian is necessarily positive definite. Since both $K_\nu(\widetilde{\Lambda})$ and Λ_{Φ_0} are invariant by the flow of $H_{\widetilde{p}}$, and their tangent spaces are described by Hess $\Im m\,\Phi$ and Hess Φ_0 respectively, this positivity propagates along the curves $K_\nu(\exp sH_p(y_0,0))\in K_\nu(\widetilde{\Lambda})\cap\Lambda_{\Phi_0}$ $(s\in\mathbf{R})$ for any $y_0\in\Gamma$, and (4.9) follows. □

5 Global WKB expansions

Now we study the WKB-type expansions of $T_\nu\psi_j$. Denote

$$\widetilde{\Lambda}_+=\{\exp sH_p(\lambda)\;;\;\lambda\in\Lambda\;,\;s\geq 0\}.$$

We have :

Proposition 5.1 *Under the assumptions of theorem 2.1, let ψ_j be the resonnant state associated to the resonance ρ_j. Then, if $\nu>0$ is fixed large enough, $T_\nu\psi_j$ admits in $\pi_x(K_\nu(\widetilde{\Lambda}_+))$ the following WKB representation :*

$$T_\nu\psi_j=h^{\frac{n}{4}-m_j}b_je^{i\Phi/h-S_0/h} \qquad (5.1)$$

where $b_j=b_j(x,h)\sim\sum_{k\geq 0}b_{j,k}(x)h^{k/2}$ is a semiclassical analytic symbol defined for x in $\pi_x(K_\nu(\tilde{\Lambda}))$, and where the semiclassical expansion is locally uniform with respect to x.

Proof
We know that near $\partial \ddot{O}$ and out of $\ddot{O}$, ψ_j admits an asymptotic expansion in a neighborhood of the bicharacteristic curves starting at a point of type 1. Let γ be such a bicharacteristic curve starting at $y_0 \in \Gamma$, and let $y_1 = \gamma(s_1)$ with $s_1 > 0$ small enough. Let also ω be a complex sufficiently small neighborhood of y_1 so that $\psi_j(y)$ admits on ω the following WKB expansion (see (2.5)) :

$$\psi_j(y) = h^{-\frac{n}{4}-m_j}\tilde{a}_j(y,h)e^{-\widetilde{f}(y)/h} \tag{5.2}$$

where we recall that $\tilde{a}_j$ is a semiclassical analytic symbol. Let X_ν be a ν-dependant open subset of $\pi_x(K_\nu(\widetilde{\Lambda}_+))$ such that the set $\widetilde{X}_\nu := \{y_c(x)\ ;\ x \in X_\nu\}$ (where $y_c(x) := x + \frac{i}{\nu}\Phi'(x)$) is a relatively compact subset of ω. Then, for all $x \in X_\nu$ and $y \in \omega^c$ one has :

$$\Re e\left((x-y)^2\right) = \left[\Re e\left(y_c(x) + \frac{1}{\nu}\widetilde{f}'(y_c(x))\right) - y\right]^2 - (\Im m x)^2 \ \geq\ -(\Im m x)^2 + \delta \tag{5.3}$$

where $\delta > 0$ can be taken independant of ν if X_ν is well choosen (since $\Re e\ \widetilde{f}'(y_0) = 0$). Then, we take $\nu > 0$ sufficiently large so that

$$\delta\nu > 4S_0. \tag{5.4}$$

and from now on we keep these choices in such a way that the following analysis remains valid near all the points of type 1 of $\partial \ddot{O}$.

$1^{\mathbf{st}}$ **step** : Let us compute $T_\nu\psi_j(x)$ for $x \in X_\nu$.
In view of (5.2) (and denoting $\omega_{\mathbf{R}} = \omega \cap \mathbf{R}^n$), one has :

$$T_\nu\psi_j(x) = h^{-\frac{n}{4}-m_j}e^{-S_0/h}\int_{\omega_{\mathbf{R}}} e^{iG(x,y)/h}\tilde{a}_j(y,h)\,dy + \int_{\omega_{\mathbf{R}}^c} e^{-\nu(x-y)^2/2h}\psi_j(y)\,dy \tag{5.5}$$

with

$$G(x,y) = i\nu(x-y)^2/2 + i\widetilde{f}(y) - iS_0. \tag{5.6}$$

Now, for $y \in \omega^c$ and $x \in X_\nu$, one has by (5.3) and (5.4) : $e^{-\nu(x-y)^2/2h} = \mathcal{O}(e^{\Phi_0(x)-\nu\delta/h}) = \mathcal{O}(e^{(\Phi_0(x)-4S_0)/h})$, and thus, using standard a priori estimates on ψ_j (or alternatively making a change of contour of integration), one gets :

$$\int_{\omega_{\mathbf{R}}^c} e^{-\nu(x-y)^2/2h}\psi_j(y)\,dy = \mathcal{O}(e^{(\Phi_0-3S_0)/h}) \tag{5.7}$$

uniformly for x in X_ν and h small enough. In particular, by (4.9), and possibly by shrinking X_ν again, this gives :

$$\int_{\omega_{\mathbf{R}}^c} e^{-\nu(x-y)^2/2h}\psi_j(y)\,dy = \mathcal{O}(e^{(-\Im m\,\Phi-2S_0)/h}). \tag{5.8}$$

To compute the integral on ω, we try to apply the complex stationnary-phase theorem. Indeed, the analysis made at the end of the previous section proves that the function $y \mapsto G(x,y)$ admits a unique (complex) critical point $y_c(x)$ in ω, which is real when $x \in X_\nu \cap \Gamma_1$. Moreover the critical value is exactly $\Phi(x)$, and one has :

$$(\mathrm{Hess}_{yy}\ \Im m G)\,(x, y_c(x)) = 2\nu\mathbf{I} + \Re e\left[\widetilde{f}''(y_c(x))\right]$$

which is therefore positive definite if ω is small enough (or alternatively if ν is taken large enough). Then the complex stationnary-phase theorem can be applied (see e.g. [Sj1]), and gives the existence of a semiclassical analytic symbol $\widetilde{b}_j(x,h)$ of order 0 such that :

$$\int_{\omega_{\mathbf{R}}} e^{iG(x,y)/h}\widetilde{a}_j(y,h)\,dy = h^{\frac{n}{2}}\widetilde{b}_j(x,h)e^{i\Phi(x)/h}. \tag{5.9}$$

Using (5.5), this gives for $x \in X_\nu$:

$$T_\nu\psi_j(x) = h^{\frac{n}{4}-m_j}b_j(x,h)e^{i\Phi(x)-S_0/h} \tag{5.10}$$

with $b_j(x,h) = \widetilde{b}_j(x,h) + e^{(S_0-i\Phi(x))/h}\int_{\omega_{\mathbf{R}}^c} e^{-\nu(x-y)^2/2h}\psi_j(y)\,dy$ which, by (5.8), is again an analytic symbol of order 0 (actually with the same semiclassical expansion as $\widetilde{b}_j$).

$2^{\mathbf{nd}}$ **step** : Now we propagate the previous construction.
It is well known (see e.g. [Sj1], [Sj2] or [Ma4]) that $T_\nu P = \tilde{P}T_\nu$, where $\tilde{P}$ is a pseudo-differential operator with principal symbol $\widetilde{p}$. Moreover, by construction the phase $\Phi(x)$ satisfies on $\pi_x(K_\nu(\widetilde{\Lambda}))$ the eikonal equation :

$$\widetilde{p}(x,\Phi'(x)) = \lambda_0$$

(this is a consequence of the fact that $p\Big|_{\widetilde{\Lambda}} = \lambda_0$). Therefore, one can formally extend the symbol $b(x,h)$ on $\pi_x(K_\nu(\widetilde{\Lambda}))$ by solving iteratively the transport equations coming out from the following equation :

$$(\tilde{P}-\rho_j)(b_je^{i\Phi(x)/h}) = 0. \tag{5.11}$$

In fact, using the same technic as in [Ma1], we see that the previous equation can be written in the following formal way (that is in the sense of formal symbols) :

$$\sum_{\alpha\in\mathbf{N}^n}\frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha\widetilde{p}_\nu(x,\Phi'(x),h)$$

$\partial_y^\alpha(b_j(y,h)e^{i\mathcal{H}(x,y)/h})\,|_{x=y}= \rho_jb_j(x,h)(5.12)$where $\widetilde{p}_\nu$ is the full symbol of $\tilde{P}$ (for the standard quantization), and

$$\mathcal{H}(x,y) = \Phi(y)-\Phi(x)-(y-x)\Phi'(x) = \mathcal{O}(|x-y|^2).$$

We obtain in a classical way the following system of transport equations:

$$\left\{\begin{array}{rcl} 2\nabla_x\Phi\nabla_xb_{j,0}+(\Delta_x\Phi-ie_j)b_{j,0} &=& 0\\ 2\nabla_x\Phi\nabla_xb_{j,k}+(\Delta_x\Phi-ie_j)b_{j,k} &=& F_k(b_{j,0},b_{j,1},\dots,b_{j,k-1})\end{array}\right. \tag{5.13}$$

where $F_k(b_{j,0},b_{j,1},\dots,b_{j,k-1})$ are analytic functions and $b_j(x,h)\sim\sum b_{j,k}(x)h^{k/2}$. The formal resolution of this system (with Cauchy data given by the values of $b_j(x,h)$ already known) is standard, and after resummation we obtain for the solution a classical analytic symbol in the sense of Sjöstrand [Sj1], which extends $b_j(x,h)$ on $\pi_x(K_\nu(\widetilde{\Lambda}))$.

Finally, setting

$$u := b_j - h^{m_j-\frac{n}{4}}e^{S_0/h-\Phi_0/h}T_\nu\psi_j$$

one can see that u admits in $\pi_x(K_\nu(\widetilde{\Lambda}_+))$ a local a priori estimates of the type : $u = \mathcal{O}(e^{\varepsilon/h})$ for any $\varepsilon > 0$. (Such estimates, however, fail to be true at $x = y_0$.) Since, by (5.10) and (4.9), u is exponentially small in X_ν, using a standard argument of propagation, one gets as in [HeSj], section 9 (see also [Sj1] and [Ma2]), that $u(x,h)$ is exponentially small on $\pi_x(K_\nu(\widetilde{\Lambda}_+))$ as $h \to 0$, locally uniformly with respect to x. In particular, the function $b_j - u$ is an analytic symbol (with the same semiclassical expansion as b_j) which satisfies to the conclusions of proposition 5.1. $\square$

Remark 5.1 *The previous construction, applied to any point $y_0 \in \partial\ddot{O}$ of type 1, shows that $T_\nu\psi_j$ admits a WKB expansion in a whole complex neighborhood of the set :*

$$\widetilde{\Gamma}_+ := \pi_x(K_\nu(\bigcup_{s>0} \exp sH_p(\Gamma \times \{0\})) = \pi_x(\bigcup_{s>0} \exp sH_{\widetilde{p}}(\Gamma \times \{0\})). \tag{5.14}$$

Then, noticing that :

$$(T_\nu\overline{\psi}_j)(x) = \overline{T_\nu\psi_j(\overline{x})}$$

we also get that $T_\nu\overline{\psi}_j$ admits a WKB expansion in a complex neighborhood of

$$\widetilde{\Gamma}_- := \mathbf{c}(\widetilde{\Gamma}_+)$$

given by :

$$T_\nu\overline{\psi}_j(x,h) = h^{\frac{n}{4}-m_j}\overline{b_j}(x,h)e^{-i\overline{\Phi}(x)/h - S_0/h} \tag{5.15}$$

where, for any holomorphic function f, we still denote $\overline{f}(x) := \overline{f(\overline{x})}$.

Now, out of $\widetilde{\Gamma}_+$, we have the following result :

Proposition 5.2 *For any compact subset K of $\mathbf{C}^n \cap \{\Re e x \in \ddot{O}^c\}$ such that $K \cap \widetilde{\Gamma}_+ = \emptyset$, there exists $\varepsilon_K > 0$ such that :*

$$T_\nu\psi_j(x,h) = \mathcal{O}(e^{(\Phi_0(x) - S_0 - \varepsilon_K)/h}) \tag{5.16}$$

uniformly for $x \in K$ and $h > 0$ small enough.

Proof
Denote :

$$A = \{(y,\xi) \in \mathbf{R}^{2n} \ ; \ y \in \ddot{O}^c \ , \ (y,\xi) \notin \bigcup_{s>0} \exp sH_p(\Gamma \times \{0\})\}.$$

Then obviously $K_\nu(A) \cup \widetilde{\Gamma}_+ = \mathbf{C}^n \cap \{\Re e x \in \ddot{O}^c\}$, and A is invariant under the flow of H_p.

By assumption H.2, if $(y,\xi) \in A \cap p^{-1}(\lambda_0)$, then $|\exp sH_p(y,\xi)| \to \infty$ as $s \to \pm\infty$. In this case, the same argument used in [HeSj],§9 shows that $v := e^{(S_0 - \Phi_0)/h}T_\nu\psi_j$ is exponentially small near $\pi_x K_\nu(\exp sH_p(y,\xi))$ for s sufficiently negative : this is essentially due to the fact that the weight function $G(y,\xi)$ used to define the resonances of P in [HeSj] has the property of tending to $-\infty$ along any curve $\exp sH_p(y,\xi)$ with $(y,\xi) \in A \cap p^{-1}(\lambda_0)$, as $s \to -\infty$. (But this can also be seen directly on the expression of $T_\nu\psi_j$ by using the fact that $\psi_j((1+it)x) \in L^2(\mathbf{R}^n)$.) Moreover, since one also has good local a priori estimates along such curves, of the type : $v = \mathcal{O}_\varepsilon(e^{\varepsilon/h})$ for any $\varepsilon > 0$, except at those points where $\Re e\ x \in \partial\ddot{O}$ is of type 1, one can use again a standard propagation result (see e.g. [Ma2]) to deduce that v is exponentially small near any point of the part of these curves included in $\mathbf{C}^n\backslash\{\Re e\, x \in \partial \ddot{O}$ is of type 1$\}$ and containing $\pi_x K_\nu(\exp sH_p(y,\xi))$ for $s << -1$.

Now, if $(y,\xi) \in A\backslash p^{-1}(\lambda_0)$, then $P-\lambda_0$ is elliptic near (y,ξ), and then it is also standard to show by elliptic a priori estimates that v is exponentially small near $\pi_x K_\nu(y,\xi)$. □

Remark 5.2 *Since*

$$\widetilde{T}_\nu u(\alpha_x,\alpha_\xi) = 2^{-\frac{n}{2}}\nu^{\frac{3n}{4}}\pi h^{-\frac{3n}{4}} e^{-\nu\alpha_\xi^2/h} T_\nu u(\alpha_x - i\alpha_\xi),$$

we see that propositions 5.1 and 5.2 permit in particular to get precise estimates for $\widetilde{T}_\nu\psi_j$ *and* $\widetilde{T}_\nu\overline{\psi}_j$ *on the set* $\{\alpha = (\alpha_x,\alpha_\xi) \in \mathbf{C}^{2n}\ ;\ \Re e\alpha_x + \Im m\alpha_\xi \in \ddot{O}^C\}$.

6 Completion of the proof

We first prove :

Proposition 6.1 *Let L be as in (2.7). For $R>0$ fixed large enough, one has :*

1. *If $\omega_2 \notin -L$, then*
$$\mathcal{T}_{R,\nu}^-(e^{i\varphi_2/h}t_2,\overline{\psi_j}) = \mathcal{O}(h^\infty e^{-S_0/h})\ ; \tag{6.1}$$
2. *If $\omega_1 \notin L$, then*
$$\mathcal{T}_{R,\nu}^-(\psi_j, e^{i\varphi_1/h}t_1) = \mathcal{O}(h^\infty e^{-S_0/h})\ ; \tag{6.2}$$

uniformly with respect to $h>0$ small enough.

Proof
By definition, one has for some constant $c_{n,\nu}>0$ (and writing $\alpha=(\alpha_x,\alpha_\xi)$) :

$$\mathcal{T}_{R,\nu}^-(e^{i\varphi_2/h}t_2,\overline{\psi_j}) = c_{n,\nu}h^{-\frac{3n}{2}}\int_{\substack{\alpha\in\widetilde{\Lambda}_t\\ |\alpha|\leq 7R}} e^{-\nu\alpha_\xi^2/h}T_\nu(e^{i\varphi_2/h}t_2)(\alpha_x - i\alpha_\xi)T_\nu\psi_j(\alpha_x+i\alpha_\xi)\, d\alpha$$

and since $t_2(y)$ is supported in $\{|y|\geq R\}$, it is easy to see that, choosing R sufficiently large, the previous integral concentrates on $\{|\Re e\alpha_x|\geq R/2\}$, in the sense that :

$$\mathcal{T}_{R,\nu}^-(e^{i\varphi_2/h}t_2,\overline{\psi_j}) = c_{n,\nu}h^{-\frac{3n}{2}}\int_{\widetilde{\Lambda}_{t,R}} e^{-\nu\alpha_\xi^2/h}T_\nu(e^{i\varphi_2/h}t_2)(\alpha_x - i\alpha_\xi)T_\nu\psi_j(\alpha_x+i\alpha_\xi)\, d\alpha + \mathcal{O}(e^{-2S_0/h}) \tag{6.3}$$

where

$$\widetilde{\Lambda}_{t,R} = \widetilde{\Lambda}_t \cap \{|\alpha|\leq 7R\ ;\ |\Re e\alpha_x|\geq R/2\}.$$

In particular $\widetilde{\Lambda}_{t,R}$ is compact, and if in turns t is taken sufficiently small, then $\widetilde{\Lambda}_{t,R} \subset \{\Re e\alpha_x + \Im m\alpha_\xi \in \ddot{O}^C\}$.

Now, introducing in (6.3) a partition of unity, the study reduces to integrals of the type :

$$I_\theta = \int_{\widetilde{\Lambda}_{t,R}} e^{-\nu\alpha_\xi^2/h}T_\nu(e^{i\varphi_2/h}t_2)(\alpha_x - i\alpha_\xi)T_\nu\psi_j(\alpha_x+i\alpha_\xi)\theta(\alpha)\, d\alpha \tag{6.4}$$

where $\theta\in C_0^\infty(\widetilde{\Lambda}_{t,R})$ is supported in a small neighborhood $\mathcal{V}_\theta$ of some point $\alpha^\theta = (\alpha_x^\theta,\alpha_\xi^\theta)\in \widetilde{\Lambda}_{t,R}$.

Now we distinguish several cases :

Case 1 - If $\Re e\alpha_x^\theta + i\Re e\alpha_\xi^\theta \notin \widetilde{\Gamma}^+$: Then, using proposition 5.2 it is easy to see that if t has been choosen small enough, one has $I_\theta = \mathcal{O}(e^{-(S_0+\varepsilon)/h})$ for some $\varepsilon > 0$.

Case 2 - If $\Re e\alpha_x^\theta + i\Re e\alpha_\xi^\theta \in \widetilde{\Gamma}^+$: Then near α^θ one can use the WKB expansion of $T\psi_j(\alpha_x + i\alpha_\xi)$ given by proposition 5.1, which gives :

$$I_\theta = h^{\frac{n}{4}-m_j} e^{-S_0/h} \int_{\widetilde{\Lambda}_{t,R}} e^{i\Psi(y,\alpha)/h} t_2(y) b_j(\alpha_x + i\alpha_\xi)\theta(\alpha)\, dy\, d\alpha$$

where

$$\Psi(y,\alpha) = i\nu\alpha_\xi^2 + i\nu(\alpha_x - i\alpha_\xi - y)^2/2 + \varphi_2(y) + \Phi(\alpha_x + i\alpha_\xi).$$

(Here we have omited the dependence of t_2 and φ_2 on $\sqrt{\rho_j}\omega_2$.)

Subcase 2.a - If $|\Re e\alpha_x^\theta| \geq 5R$: Then we make in I_θ the following change of contour of integration with respect to y :

$$\mathbf{R}^n \ni y \mapsto \omega_t(y)$$

where $\omega_t(y) = y + it\chi_0(|y|)y$, with χ_0 as in (3.1). Along this contour, we have :

$$\begin{aligned}\Im m\ \Psi(y,\alpha) = \nu[(\Re e\alpha_\xi)^2 - (\Im m\alpha_\xi)^2] + \nu[\Re e(\alpha_x - i\alpha_\xi - y)]^2/2 - \nu[\Im m(\alpha_x - i\alpha_\xi - y)]^2/2\\ +\Im m\varphi_2(y) + \Im m\Phi(\alpha_x + i\alpha_\xi)\end{aligned}$$

and therefore, using (4.9) and (3.3) (plus the fact that φ_2 is real on the real and $\Im m\rho_j$ is exponentially small) :

$$\begin{aligned}\Im m\ \Psi(y,\alpha) \geq \nu[(\Re e\alpha_\xi)^2 - (\Im m\alpha_\xi)^2] + \nu[\Re e(\alpha_x - i\alpha_\xi - y)]^2/2 - \nu[\Im m(\alpha_x - i\alpha_\xi - y)]^2/2\\ +\Im m(\sqrt{\rho_j}y.\omega_2) + \mathcal{O}(|\Im my|\langle y\rangle^{-\tau} + e^{-\varepsilon/h}\langle y\rangle) - \nu[\Im m(\alpha_x + i\alpha_\xi)]^2/2\end{aligned}$$

which gives (using that $\Im m\alpha_x$, $\Im m\alpha_\xi$ and $\Im my$ are all $\mathcal{O}(tR)$ and $\rho_j = \lambda_0 + \mathcal{O}(h)$) :

$$\begin{aligned}\Im m\ \Psi(y,\alpha) \geq \nu[\Re e(\alpha_x - i\alpha_\xi - y)]^2/2 - \nu\Im my.\Re e\alpha_\xi + (\sqrt{\lambda_0} + \mathcal{O}(h))\Im my.\omega_2\\ +\mathcal{O}(t^2R^2 + |\Im my|\langle y\rangle^{-\tau} + e^{-\varepsilon/h}\langle y\rangle).\end{aligned} \tag{6.5}$$

Now, if $|\Re ey| \leq 4R$, since $|\Re e\alpha_x|$ is closed to $|\Re e\alpha_x^\theta|$ which is $\geq 5R$, we get easily :

$$\Im m\ \Psi(y,\alpha) \geq \nu R^2/8 + \mathcal{O}(tR^2) \geq \nu R^2/10$$

if t is small enough, so that the corresponding part of the integral is exponentially small as $h \to 0_+$.

If instead $|\Re ey| \geq 4R$, then $\chi_0(\Re ey) = 1$, and thus $\Im my = t\Re ey$ on the contour. In this case, (6.5) can be transformed into :

$$\begin{aligned}\Im m\ \Psi(y,\alpha) \geq \nu[\Re e(\alpha_x - i\alpha_\xi - y + t\alpha_\xi)]^2/2 - \nu t\Re e\alpha_x.\Re e\alpha_\xi + (\sqrt{\lambda_0} + \mathcal{O}(h))t\Re ey.\omega_2\\ +\mathcal{O}(t^2R^2 + t\langle y\rangle^{1-\tau} + e^{-\varepsilon/h}\langle y\rangle).\end{aligned} \tag{6.6}$$

Then we observe two things : The first one is, that if $\Re ey.\omega_2 \leq \sigma_2^- - \delta$, then by (3.5) $t_2(y)$ is exponentially small there, and thus the same is true for the corresponding part of the integral. The second one is that

$$\Re e\alpha_x.\Re e\alpha_\xi < 0 \tag{6.7}$$

for α in a neighborhood of α^θ. Indeed, since $\Re e\alpha_x^\theta + i\Re e\alpha_\xi^\theta \in \tilde{\Gamma}^+$, we have $\Re e\alpha_x^\theta + i\Re e\alpha_\xi^\theta = \Pi_x K_\nu(\exp sH_p(y_0,0))$ for some $s > 0$ and $y_0 \in \partial\ddot{O}^c$ of type 1. In particular, denoting $(y(s),\xi(s)) = \exp sH_p(y_0,0)$, we get :

$$\Re e\alpha_x^\theta = y(s) \quad ; \quad \Re e\alpha_\xi^\theta = -\frac{1}{\nu}\xi(s)$$

and therefore

$$\Re e\alpha_x^\theta.\Re e\alpha_\xi^\theta = -\frac{1}{\nu}y(s).\xi(s).$$

Since s must be large enough so that $|\Re e\alpha_x^\theta| \geq 5R$, and $y(s) \sim 2s\xi(s)$, $|\xi(s)|^2 \to \lambda_0$ as $s \to +\infty$, (6.7) follows. In particular, since also $\sigma_2^- - \delta > 0$, we see on (6.6) that at the points y where t_2 is not exponentially small, one has (with some positive constant $\delta_3 > 0$ independent of R) :

$$\Im m\ \Psi(y,\alpha) \geq 2\delta_3 tR + \mathcal{O}(t^2R^2 + tR^{1-\tau} + e^{-\varepsilon/h}R) \geq \delta_3 tR$$

if t is small enough with respect to R^{-1}. As a consequence, even these points give an exponentially small contribution to the integral.

Subcase 2.b - If $|\Re e\alpha_x^\theta| < 5R$: Then one can assume that $\chi_1(\Re e\alpha_x) = 0$ on the support of θ, so that α and y are both real, and we get there :

$$\Im m\ \Psi(y,\alpha) \geq \nu(\alpha_x - y)^2/2 + \mathcal{O}(e^{-\varepsilon/h}\langle y\rangle) \geq \mathcal{O}(e^{-\varepsilon/h}R). \tag{6.8}$$

Then we try to apply the non stationary phase theorem to estimate the integral. Looking for a real critical point of Ψ, we find the equations :

$$\begin{cases} -\nu\alpha_\xi - i\nu(\alpha_x - y) + \nabla_y(\varphi_2(y,\sqrt{\rho_j}\omega_2)) = 0 \\ \Phi'(\alpha_x + i\alpha_\xi) + i\nu(\alpha_x - y) + \nu\alpha_\xi = 0 \end{cases} \tag{6.9}$$

and thus

$$\begin{cases} y = \alpha_x + \mathcal{O}(e^{-\varepsilon/h}) \\ \nu\alpha_\xi = \sqrt{\rho_j}\omega_2 + \mathcal{O}(R^{-\tau} + e^{-\varepsilon/h}) = -\Phi'(\alpha_x + i\alpha_\xi) + \mathcal{O}(R^{-\tau} + e^{-\varepsilon/h}). \end{cases} \tag{6.10}$$

But, by the same argument as at the end of the subcase 2.a, we see that α_ξ^θ behaves like $-\dfrac{1}{s}y(s)$ with s large enough, so that $\alpha_\xi/|\alpha_\xi|$ remains arbitrarily closed to the set $-L$ when R is large. As a consequence, if $\omega_2 \notin -L$, we see on (6.10) that Ψ admits no real critical point. Since we also have (6.8), the estimate (6.1) follows from the non stationary phase theorem. The second estimate (6.2) can be proved in the same way. □

Now we treat the case where $\omega_2 \in -L$ and $\omega_1 \in L$.

Proposition 6.2 *For $R > 0$ large enough we have:*

1. *If $\omega_2 \in -L$*

$$\mathcal{T}_{R,\nu}^-(e^{i\varphi_2/h}t_2, \overline{\psi_j}) = \mathcal{O}(h^{\frac{d}{2}-\frac{n}{4}+1-m_j}e^{-S_0/h}). \tag{6.11}$$

If moreover $d = n-1$, then $\mathcal{T}^-_{R,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j})$ admits an asymptotic expansion of the form :

$$\mathcal{T}^-_{R,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j}) = h^{\frac{n}{4}+\frac{1}{2}-m_j} e^{-S_0/h} \left[\sum_{x\in\Gamma_{-\omega_2}} e^{iC_x/h} \sum_{k=0}^{\infty} \alpha^{(2)}_{j,k}(x) h^{k/2} + \mathcal{O}(h^\infty) \right] \quad (6.12)$$

where the $\alpha^{(2)}_{j,k}(x)$'s are complex constants, and $\alpha^{(2)}_{1,0}(x)$ never vanishes.

2. *If $\omega_1 \in L$*

$$\mathcal{T}^-_{R,\nu}(\psi_j, e^{i\varphi_1/h}t_1) = \mathcal{O}(h^{\frac{d}{2}-\frac{n}{4}+1-m_j} e^{-S_0/h}). \quad (6.13)$$

If moreover $d = n-1$, then $\mathcal{T}^-_{R,\nu}(\psi_j, e^{i\varphi_1/h}t_1)$ admits an asymptotic expansion of the form :

$$\mathcal{T}^-_{R,\nu}(\psi_j, e^{i\varphi_1/h}t_1) = h^{\frac{n}{4}+\frac{1}{2}-m_j} e^{-S_0/h} \left[\sum_{x\in\Gamma_{\omega_1}} e^{-iC_x/h} \sum_{k=0}^{\infty} \alpha^{(1)}_{j,k}(x) h^{k/2} + \mathcal{O}(h^\infty) \right] \quad (6.14)$$

where the $\alpha^{(1)}_{j,k}(x)$'s are complex constants, and $\alpha^{(1)}_{1,0}(x)$ never vanishes.

Proof
As in the proof of proposition 6.1, we use a partition and reduce our study to expressions of the type I_θ given in (6.4). Then the study of the cases 1 and 2-a are the same as before, and do not give significant contributions to the quantity $\mathcal{T}^-_{R,\nu}(e^{i\varphi_2/h}t_2, \overline{\psi_j})$.

Now, in the subcase 2-b, we slightly modify the phase by replacing $\varphi_2(y, \sqrt{\rho_j}\omega_2)$ by $\widetilde{\varphi}_2(y) = \varphi_2((y, \sqrt{\lambda_0}\omega_2)$. Doing this we make an error $\mathcal{O}(h)$ in the phase of the integral, which equivals to multiply the symbol by some positive function. Then, since $\widetilde{\varphi}_2(y)$ is real on the real, the equations of the possible real critical point become :

$$y = \alpha_x \quad (6.15)$$

$$\nu\alpha_\xi = \nabla\widetilde{\varphi}_2(y) \quad (6.16)$$

$$\nu\alpha_\xi = -\Phi'(\alpha_x + i\alpha_\xi). \quad (6.17)$$

Let $(\alpha_x, \alpha_\xi) \in \mathrm{Supp}\theta$ be such that $\alpha_x + i\alpha_\xi \in \widetilde{\Gamma}^+$. Then, as we have already seen, such a point is of the type $(y(s), -\xi(s)/\nu)$, where $(y(s), \xi(s)) = \exp sH_p(y_0, 0)$, $y_0 \in \partial\ddot{0}$ is of type 1, and s is sufficiently large so that $|y(s)| \geq R/2$. Moreover, by definition $K_\nu(y(s), \xi(s)) \in K_\nu(\widetilde{\Lambda})$ and thus $\xi(s) = \Phi'(x(s) - i\xi(s)/\nu)$, which means exactly that (6.17) is satisfied. In particular, (6.15) and (6.17) are always satisfied by the point $(\alpha_x, \alpha_x, \alpha_\xi)$, and it remains to study (6.16). Analysing the
Hamilton-Jacobi equations, it is easy to see that $\xi(s)$ always admits
a limit $\xi_\infty \in \sqrt{\lambda_0}L$ when $s \to +\infty$, and we distinguish 2 cases :

Case A - If $\xi_\infty \neq -\sqrt{\lambda_0}\omega_2$. Then, as before we see that there is no real critical point if R is taken large enough.

Case B - If $\xi_\infty = -\sqrt{\lambda_0}\omega_2$. Then we claim that $(\alpha_x, \alpha_x, \alpha_\xi)$ satisfies (6.16). First, we have to refer to the construction of φ_2 made in [IsKi1] (see also [GeMa2] section II), which gives in particular that for any $(y, \eta) \in \mathbf{R}^{2n}$ large enough such that $y.\eta \leq \sigma_2^-|y|.|\eta|$, one has :

$$\pi_\xi \exp tH_p(y, \nabla_y\varphi_2(y, \eta)) \to \eta \quad (t \to -\infty). \quad (6.18)$$

Since $y(s).\xi(s)/|y(s)|.|\xi(s)| \to 1$ as $s \to +\infty$, we see that for s large enough we can apply (6.18) to $y = y(s)$ and $\eta = \sqrt{\lambda_0}\omega_2$, and therefore obtain :

$$\pi_\xi \exp tH_p(y(s), \nabla\tilde{\varphi}_2(y(s))) \to \sqrt{\lambda_0}\omega_2 \quad (t \to -\infty). \tag{6.19}$$

On another hand, since $p(x,\xi)$ is even with respect to ξ and $\xi(0) = 0$, we have $(y(s), -\xi(s)) = (y(-s), \xi(-s))$ and therefore :

$$\pi_\xi \exp tH_p(y(s), -\xi(s)) = \pi_\xi \exp(t-s)H_p(y_0, 0) = \xi(t-s) = -\xi(s-t) \to \sqrt{\lambda_0}\omega_2 \quad (t \to -\infty). \tag{6.20}$$

It also comes out from [IsKi1] and [GeMa2] that for any $y \in \mathbf{R}^n$ large enough, the application

$$\eta \mapsto \lim_{t\to-\infty} \pi_\xi \exp tH_p(y, \eta)$$

is a global diffeomorphism from $\{\eta \; ; \; y.\eta \leq \sigma_2^- |y|.|\eta|\}$ to its image. As a consequence, we can deduce from (6.19) and (6.20) that :

$$\nabla\tilde{\varphi}_2(y(s)) = -\xi(s) \tag{6.21}$$

for all s large enough, which gives exactly (6.16).

The conclusion of this study is that if $(\alpha_x^\theta, -\nu\alpha_\xi^\theta)$ belongs to a nul bicharacteristic $\gamma = \gamma(s)$ of $p - \lambda_0$ such that $\pi_\xi\gamma(s)$ tends to $-\sqrt{\lambda_0}\omega_2$ as s tends to $+\infty$, then there is at least a whole curve of real critical points of the phase inside $\mathbf{R}^n \times \text{Supp }\theta$. Moreover, at these points the value of the phase is :

$$\tilde{\Psi}(y(s), y(s), -\frac{1}{\nu}\xi(s)) = \tilde{\varphi}_2(y(s)) + \Re e\; \Phi(y(s) - \frac{i}{\nu}\xi(s))$$

for some s large enough (where $\tilde{\Psi}$ is the new phase obtained by substituing $\tilde{\varphi}_2$ to φ_2 in the expression of Ψ). Then we observe that

$$\frac{d}{ds}\Re e\; \Phi(y(s) - \frac{i}{\nu}\xi(s)) = \Re e\; \left[\Phi'(y(s) - \frac{i}{\nu}\xi(s)).(2\xi(s) - i\nabla V(y(s)))\right] = 2|\xi(s)|^2 \tag{6.22}$$

while, by (6.21),

$$\frac{d}{ds}\tilde{\varphi}_2(y(s)) = -2|\xi(s)|^2.$$

As a consequence, the critical value $C(y_0) = \tilde{\Psi}(y(s), y(s), -\frac{1}{\nu}\xi(s))$ ($\in \mathbf{R}$) does not depend on s (for s large enough), but only on the starting point $y(0) = y_0 \in \partial\ddot{O}$ of type 1. Moreover, using (4.10)-(4.11) one can see that

$$\text{Rank Hess } \Im m\; \tilde{\psi}(y(s), y(s), -\frac{1}{\nu}\xi(s)) = 2n + d$$

so that we can use the stationnary phase theorem (see [MeSj]) at least in $2n + d$ variables, which gives :

$$I_\theta = h^{\frac{n}{4} - m_j + n + \frac{d}{2}} e^{-S_0/h} \int_{\mathbf{R}_s \times \mathbf{R}_z^{n-1-d}} e^{i\psi_1(s,z)/h} c(s, z; h) ds dz \tag{6.23}$$

where ψ_1 is a new phase function with non negative imaginary part, which satisfies

$$\psi_1(s, 0) = C(y_0) \tag{6.24}$$

and $c(s,z;h) \sim h \sum_{k=0}^{\infty} c_k(s,z)h^{k/2}$ is a (C^∞) semiclassical symbol with compact support included in some set of the form $[s_0, s_1] \times K$ where K is a small neighborhood of 0 in $\mathbf{R}^{n-1-d}$. Moreover, c_0 satisfies :

$$c_0(s,0) = t_{2,0}(y(s))b_{j,0}(y(s) - \frac{i}{\nu}\xi(s))\theta(y(s), -\frac{1}{\nu}\xi(s)) \tag{6.25}$$

for all $s \in [s_0, s_1]$. In the general case, the estimate (6.11) follows directly from (6.23) and (6.3). In the particular case where $d = n-1$, the integral in (6.23) reduces to an integral over $s \in \mathbf{R}$, and using (6.24) one gets :

$$I_\theta = h^{\frac{7n}{4} - m_j + \frac{1}{2}} e^{(iC(y_0) - S_0)/h} \left[\sum_{k=0}^{\infty} \widetilde{c}_k h^{k/2} + \mathcal{O}(h^\infty) \right] \tag{6.26}$$

with $\widetilde{c}_k = \int c_k(s)ds$.

Now, looking more carefully at the construction of φ_2 made in [IsKi1] (see also [GeMa2]), we see that :

$$\widetilde{\varphi}_2(y(s)) = -y(s)\xi(s) + \int_s^T y(t)\nabla V(y(t))dt$$

for some $T > 0$ large enough, while by (6.22) (and making an integration by parts) :

$$\Re e\ \Phi(y(s) - \frac{i}{\nu}\xi(s)) = \Re e\ \Phi(y_0) + 2\int_0^s |\xi(t)|^2 dt = y(s)\xi(s) + \int_0^s y(t)\nabla V(y(t))dt$$

where we have also used the fact that $\Re e\ \Phi(y_0) = \Im m\ \widetilde{f}(y_0) = 0$.

As a consequence, $C(y_0) = \int_0^T y(t)\nabla V(y(t))dt$, and (6.12) follows from (6.26) and (6.3).

When $j = 1$, the fact that $\alpha_{1,0}^{(2)}$ does not vanish is a consequence of the ellipticity of $b_{1,0}$ along $\{y(s) - \frac{i}{\nu}\xi(s)\ ;\ s \in \mathbf{R}\}$ and the fact that for x close to some $y(s)$, $t_{2,0}(x)$ is of the form (see [IsKi1] or [GeMa2]) :

$$t_{2,0}(x) = e^{-i\widetilde{\varphi}_2(x)/h}(P - \lambda_0)\left((1 - \chi(x))e^{i\widetilde{\varphi}_2(x)/h}\right) + \mathcal{O}(h)$$

where χ is a cut-off function which equals 1 in $\{|x| \leq R\}$.

Similar arguments permit to get also (6.13) and (6.14). □

Now, theorem 2.1 is an easy consequence of proposition 6.2, 6.1, lemma 3.4, and proposition 3.1. □

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