Exponential Estimates in Adiabatic Quantum Evolution[1]

Alain Joye[2] **and Charles-Edouard Pfister**[3]

Abstract

We review recent results concerning the exponential behaviour of transition probabilities across a gap in the adiabatic limit of the time-dependent Schrödinger equation. They range from an exponential estimate in quite general situations to asymptotic Landau-Zener type formulae for finite dimensional systems, or systems reducible to this case.

1 Introduction

The notion of adiabatic evolution or adiabatic process is an important theoretical concept, which occurs at several places in Physics. In Quantum Mechanics, this process is usually described by the equation $i\hbar\frac{\partial}{\partial t'}\psi(t') = H(\varepsilon t')\psi(t')$, where ε is a small parameter, such that $1/\varepsilon$ gives the typical time-scale over which the Hamiltonian changes significantly. Setting $\hbar = 1$ and introducing a rescaled time, $t = \varepsilon t'$, we can rewrite the time-dependent Schrödinger equation for the evolution operator U as

$$i\varepsilon\frac{\partial}{\partial t}U(t,s) = H(t)U(t,s)\,, \quad U(s,s) = \mathbf{I}\,, \quad \forall t,s \in \mathbf{R}. \tag{1}$$

The adiabatic limit corresponds to the singular limit $\varepsilon \to 0$ of the equation (1). If the state of the system is an eigenfunction $\psi(t_0)$ for the eigenvalue $e(t_0)$ at $t = t_0$, then, in the adiabatic limit, the state of the system at time $t = t_1$ is an eigenfunction $\psi(t_1)$ for the eigenvalue $e(t_1)$, provided the energy-level $e(t)$ is isolated in the spectrum of the Hamiltonian $H(t)$ for all t in the time-interval $[t_0, t_1]$ [4]. When the system performs a cycle, $H(t_0) = H(t_1)$, and the energy-level $e(t)$ is non-degenerate, the eigenfunction $\psi(t_1)$ defined in the adiabatic limit differs from $\psi(t_0)$ by a phase, which can be decomposed into an ε-dependent dynamical phase and an ε-independent geometrical phase related to the spectral subspaces visited during the adiabatic evolution. This is the fundamental observation of Berry, which gave rise to extensive developments (see the collection of papers in [32]). In this note we review a complementary aspect, namely the estimation of the probability that the final state of the system is *not* an eigenstate $\psi(t_1)$ for $e(t_1)$; such a transition is called adiabatic or nonadiabatic transition in the physical literature. There are two kinds of results. On the one hand for small systems (or systems reducible to this case), typically two-level systems, one can derive explicit formulae for the probability of such a transition. A particular case is the Landau-Zener formula used when the two levels display an avoided crossing. For general systems, on the other hand, one can usually obtain upper estimates for the transition probability only. Those estimates are reviewed in the second part. The whole discussion is lead in a scattering setting under the assumption that the time-dependence of the Hamiltonian is analytic. More precisely

H1 *The self-adjoint family $\{H(t)\}_{t\in\mathbf{R}}$ is defined on a common dense domain D of a separable Hilbert space $\mathcal{H}$; it is uniformly bounded from below; for each $\phi \in D$, $H(t)\phi$ has an analytic extension in a fixed open strip $S \subset \mathbf{C}$ including the real axis. Moreover, there exist $H(\pm\infty)$, two self-adjoint operators defined on D, and constants $C, \alpha > 0$ such*

[1]Proceedings of XII$^{\text{th}}$ International Congress of Mathematical Physics, Brisbane, Australia, 1997.
[2]Institut Fourier, Université Grenoble-1, B.P. 74, F-38402 Saint-Martin d'Hères
[3]Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne

that for $t\gtrless 0$, $\sup_{s\,|\,t+is\in S}\|(H(t+is)-H(\pm\infty))\varphi\|\leq C\frac{\|\varphi\|+\|H(\pm\infty)\varphi\|}{(1+|t|)^{1+\alpha}}$.

H2 *The spectrum* $\sigma(t)$ *of* $H(t)$ *consists in two disjoint parts* $\sigma(t)=\sigma_1(t)\cup\sigma_2(t)$, *such that* $\inf_{t\in\mathbf{R}}$ *dist* $(\sigma_1(t),\sigma_2(t))\geq g>0$ *and* $\sigma_1(t)$ *is bounded.*

Denoting by $P_j(t)$ the spectral projector associated with $\sigma_j(t)$, $j=1,2$, the transition probability between the subspaces $P_1(t_0)\mathcal{H}$ et $P_2(t_1)\mathcal{H}$ is defined as $\mathcal{P}(t_1,t_0,\varepsilon)=\|P_2(t_1)U(t_1,t_0)P_1(t_0)\|^2$. The Adiabatic Theorem of Quantum Mechanics [4], [21], [28], [1] states that $\mathcal{P}(t_1,t_0,\varepsilon)=\mathcal{O}(\varepsilon^2)$ as soon as $H(t)$ is C^3. Under the assumption H1 (analyticity *and* existence of $H(\pm\infty)$) we have $\mathcal{P}(+\infty,-\infty,\varepsilon)=\mathcal{O}(\varepsilon^\infty)$ [25], [29], [22]. The difficult questions are to prove exponential estimates and asymptotic formulae for $\mathcal{P}(+\infty,-\infty,\varepsilon)$.

2 Asymptotic formulae

Assume the Hamiltonian $H(t)$ is an $n\times n$ hermitian matrix with non degenerate spectrum $\sigma(t)=\{e_1(t),\cdots,e_n(t)\}$. We determine up to a global phase factor a corresponding basis of normalized eigenvectors $\varphi_j(t)$, $j=1,\cdots,n$, by requiring that $\langle\varphi_j(t)|\varphi_j'(t)\rangle\equiv 0$, $t\in\mathbf{R}$. We can expand the solution $\psi(t)=U(t)\psi(0)$ of (1) as

$$\psi(t)=\sum_{j=1}^{n}c_j(t)\mathrm{e}^{-i\int_0^t e_j(t')dt'/\varepsilon}\varphi_j(t),\tag{2}$$

where the $c_j's$ are complex valued coefficients and the phase factors $\mathrm{e}^{-i\int_0^t e_j(t')dt'/\varepsilon}$ are introduced for convenience. It follows from H1 that the limits $\lim_{t\to\pm\infty}c_j(t)=c_j(\pm\infty)$ exist. Thus, choosing $c_j(-\infty)=\delta_{jk}$ as initial conditions, we get that $|c_j(\infty)|^2$ yields the transition probability from the eigenspace associated with $e_k(-\infty)$ to the eigenspace associated with $e_j(+\infty)$, which we denote by $\mathcal{P}_{jk}(\varepsilon)$. Born and Fock [4] showed that

$$c_j(t)=c_j(-\infty)+\mathcal{O}(\varepsilon)\tag{3}$$

uniformly in $t\in\mathbf{R}$, from which the Adiabatic Theorem $\mathcal{P}_{jk}(\varepsilon)=\mathcal{O}(\varepsilon^2)$ follows.

In case $n=2$, for real symmetric analytic Hamiltonians, the pioneering works [24], [34] and [5] gave arguments in favour of the Landau-Zener-Dykhne formula $\mathcal{P}_{12}(\varepsilon)\simeq\mathrm{e}^{-2\gamma/\varepsilon}$ with $\gamma>0$ explicited below. A convincing derivation of this formula based on the integration of (1) in the complex t-plane can be found in the important contribution [9]. However, for *complex* hermitian matrices, this formula misses a prefactor of geometrical nature, as we now show. The proof is based on the fact that the solution ψ of (1) is analytic throughout the strip S, since H is, whereas φ_j and e_j have multivalued extensions. More precisely, their only possible branching points are the complex crossing points $z_0\in S$, such that $e_1(z_0)=e_2(z_0)$. Generically, $e_2(z)-e_1(z)\simeq\sqrt{z-z_0}$. Let η_0 be a negatively oriented loop based at the origin, which encircles z_0. Denoting by $f(z|\eta_0)$ the analytic continuation along η_0 of a function $f(z)$ defined in a neighbourhood of the origin, we can write $e_1(z|\eta_0)=e_2(z)$. It follows that $\varphi_1(z|\eta_0)$ is proportional to $\varphi_2(z)$ and we define $\theta_{21}(\eta_0)\in\mathbf{C}$ by $\varphi_1(z|\eta_0)=\mathrm{e}^{-i\theta_{21}(\eta_0)}\varphi_2(z)$. Since $\psi(z|\eta_0)=\psi(z)$, we deduce from the foregoing and (2) the key identity

$$c_1(z|\eta_0)=\mathrm{e}^{i\theta_{21}(\eta_0)}\mathrm{e}^{i\int_{\eta_0}e_1(z')dz'/\varepsilon}c_2(z).\tag{4}$$

Let us assume, for simplicity, that there exists a unique generic crossing point z_0 in S with $\mathrm{Im}\,z_0>0$. By analyticity of ψ, we can integrate (1) from $-\infty$ to $+\infty$ with

$c_j(-\infty) = \delta_{1j}$ along the real axis or along any path $\beta \subset S$ which passes above z_0. Denoting by $\tilde{c_1}$ the analytic continuation of c_1 along such a path β, we have, due to (4), $c_2(+\infty) = \mathrm{e}^{-i\theta_{21}(\eta_0)}\mathrm{e}^{-i\int_{\eta_0} e_1(z')dz'/\varepsilon}\tilde{c_1}(+\infty)$. It thus remains to prove the estimate $\tilde{c_1}(+\infty) = 1 + \mathcal{O}(\varepsilon)$, which is the equivalent along β of the estimate (3). As is well known from complex WKB methods, such an estimate can be proven under a *global* dissipativity condition on the path β only. By definition, a path $\beta \subset S$ parametrized by $t \mapsto \beta(t)$ with $\lim_{t\pm\infty} \mathrm{Re}\, \beta(t) = \pm\infty$ is called *dissipative* (for the indices $\{1,2\}$) if $\mathrm{Im} \int_{\beta(t)}(e_1(z) - e_2(z))dz$ is non-decreasing in $t \in \mathbf{R}$, where $\int_{\beta(t)}$ means integration from $-\infty$ to $\beta(t)$ along $\beta(s), -\infty \leq s \leq t$. Sumerizing the above discussion we have

Theorem 1 [12] *Assume H1, H2 for $H(t)$ a 2×2 matrix and suppose there exists a unique generic crossing point z_0 in S with Im $z_0 > 0$. Then, provided there exists a dissipative path in S going from $-\infty$ to $+\infty$ above z_0, we have for $\varepsilon > 0$ small enough*

$$\mathcal{P}_{21}(\varepsilon) = e^{2\mathrm{Im}\, \theta_{21}(\eta_0)}e^{2\mathrm{Im}\, \int_{\eta_0} e_1(z)dz/\varepsilon}(1 + \mathcal{O}(\varepsilon)). \tag{5}$$

Remarks: The complex factor $\mathrm{e}^{-i\theta_{21}(\eta_0)}$ is actually obtained by analytic continuation along η_0 of the quantity yielding the geometric phase when considered on the real axis. When $H(t)$ is real symmetric, Im $\theta_{21}(\eta_0) = 0$ and we recover the Landau-Zener-Dykhne formula. However, in general, Im $\theta_{21}(\eta_0) \neq 0$, thus giving rise to a non-trivial prefactor. This prefactor was independently discovered by Berry [3].
The (non-trivial) questions of existence of dissipative paths and competition between several crossing points in S are analyzed in details in [12].
In case no dissipative path above z_0 exists, an exponential bound follows from integration along dissipative paths close to the real axis [12], [20]. See also Theorem 2.
We refer the reader to tables 1 and 2 in [16] and references therein for similar asymptotic formulae in more general 2-level systems as well as in non-generic situations.

When $H(t)$ is an $n \times n$ matrix with $n > 2$ the same strategy is applicable in principle. Let us see this for $n = 3$, to fix the ideas, in the following simple setting. Suppose there exist two distinct crossing points z_0 and z_1 only in S, with Im $z_j > 0$. Let z_0 be a generic crossing point for the analytic continuations of $e_1(t)$ and $e_2(t)$ around a loop η_0 encircling z_0 only, as above. Assume the analytic continuation of $e_3(t)$ in a neighborhood of η_0 is analytic at z_0, as is generically the case. By hypothesis, z_1 is a generic crossing point for the analytic continuations of $e_2(t)$ and $e_3(t)$ around a similar loop η_1 encircling z_1 only, whereas the analytic continuation of $e_1(t)$ in a neighbourhood of η_1 is analytic at z_1. We get, as above, that (4) holds and must be completed by

$$c_2(z|\eta_1) = \mathrm{e}^{i\theta_{32}(\eta_1)}\mathrm{e}^{i\int_{\eta_1} e_2(z')dz'/\varepsilon}c_3(z), \tag{6}$$

where $\theta_{32}(\eta_1)$ is defined similarly. Assume the composition of the two negatively oriented loops, based at the origin, η_0 followed by η_1 can be deformed into one negatively oriented loop, based at the origin, encircling both z_0 and z_1. Then, if $\tilde{c_1}$ denotes the analytic continuation of c_1 along a path going from $-\infty$ to $+\infty$ above z_0 and z_1, we get from (4) and (6) $c_3(+\infty) = \mathrm{e}^{-i\theta_{21}(\eta_0)}\mathrm{e}^{-i\int_{\eta_0} e_1(z')dz'/\varepsilon}\mathrm{e}^{-i\theta_{32}(\eta_1)}\mathrm{e}^{-i\int_{\eta_1} e_2(z')dz'/\varepsilon}\tilde{c_1}(+\infty)$. Again, it remains to prove $\tilde{c_1}(+\infty) = 1 + \mathcal{O}(\varepsilon)$ with initial conditions $c_j(-\infty) = \delta_{1j}$. Such an estimate can be proven in a dissipative domain, defined as follows. A domain $P \subset S$ is *dissipative* if it extends from $-\infty$ to $+\infty$ and if any point $z \in P$ can be reached from $-\infty$ by a dissipative path $\beta_2 \subset P$ for the indices $\{1,2\}$ *and* by a dissipative path $\beta_3 \subset P$ for the indices $\{1,3\}$. However, generally, dissipative domains going above z_0 and z_1 do not extend from $-\infty$ to $+\infty$ [6], [9]. The way out is to perform a perturbative analysis.

Assume $H(t,\delta)$ depends on a supplementary parameter $\delta \geq 0$, satisfies H1 for each fixed δ and is regular enough in (t,δ). Suppose that for $\delta = 0$, the (analytic) eigenvalues $e_j(t,\delta = 0)$, $t \in \mathbf{R}$, labelled in increasing order at $t = -\infty$, are non degenerate, except for two generic real crossings at $t_0 < t_1$ where $e_1(t_0,0) = e_2(t_0,0)$ and $e_1(t_1,0) = e_3(t_1,0)$. Assume these degeneracies are lifted as soon as $\delta > 0$, turning t_0 and t_1 into *avoided crossings*. As a consequence, for $\delta > 0$ small enough, there exist crossing points $z_0(\delta)$ and $z_1(\delta)$, with the properties discussed above and $\lim_{\delta\to 0} z_j(\delta) = t_j$. Furthermore, the existence of a suitable dissipative domain P can be proven perturbatively so that we get

Theorem 1' [11] *Assume $H(t,\delta)$ is a 3×3 matrix satisfying the hypotheses (loosely) stated in the previous paragraph. Then, for $\delta, \varepsilon > 0$ small enough,*

$$\mathcal{P}_{31}(\varepsilon) = e^{2\mathrm{Im}\ \theta_{21}(\eta_0,\delta)} e^{2\mathrm{Im}\ \int_{\eta_0} e_1(z',\delta)dz'/\varepsilon} e^{2\mathrm{Im}\ \theta_{32}(\eta_1,\delta)} e^{2\mathrm{Im}\ \int_{\eta_1} e_2(z',\delta)dz'/\varepsilon}(1+\mathcal{O}(\varepsilon)), \quad (7)$$

where $\mathcal{O}(\varepsilon)$ is uniform in δ and $\lim_{\delta\to 0} \mathrm{Im}\ \theta_{ij}(\eta_k,\delta) = \lim_{\delta\to 0} \mathrm{Im}\ \int_{\eta_k} e_j(z',\delta)dz' = 0$.

Remarks: The above result and generalizations in [11], [18] show that it is possible to get asymptotic formulae for *certain* transition probabilities in n-level systems displaying avoided crossings, despite the difficulties induced by the notion of dissipativity.

The same approach was used when two levels only display one avoided crossing [10]. It yields back Theorem 1 without need to check dissipativity conditions in the complex plane. Moreover, assuming the generic behaviour $e_2(t) - e_1(t) \overset{\delta\to 0}{\simeq} \sqrt{a^2t^2+\delta^2}$, we get the Landau-Zener formula: 2 Im $\int_{\eta_0} e_1(z',\delta)dz'/\varepsilon = -\frac{\pi\delta^2}{2a\varepsilon}(1+\mathcal{O}(\delta))$ and Im $\theta_{21}(\eta_0,\delta) = \mathcal{O}(\delta)$ [10]. Further refinements of the Landau-Zener formula can be found in [27].

3 Exponential estimates

Let us come back to general abstract time-dependent Hamiltonians. In such cases we can prove that the transition probability across the gap is exponentially small.

Theorem 2 [13] *Let $H(t)$ be a time-dependent Hamiltonian satisfying assumptions H1 and H2. Then, there exist $C, \gamma > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*

$$\mathcal{P}(+\infty,-\infty,\varepsilon) = \lim_{\substack{t_0\to-\infty\\ t_1\to+\infty}} \|P_2(t_1)U(t_1,t_0)P_1(t_0)\|^2 \leq Ce^{-2\gamma/\varepsilon} \quad (8)$$

The proof is made by integrating (1) in the complex plane along generalized dissipative paths suitable for operators (see [13], section 5), following [12]. A weaker result can be found in [19]. Similar estimates were known for finite dimensional systems of ODE's, see e.g. [7] and the literature quoted in [6]. Theorem 2 was then recovered by different methods, some of which allowing a better control on the exponential decay rate γ as a function of the gap g. See [26] for an approach using microlocal analysis, [30] and [15] for superadiabatic techniques, see below, and [33] for a pseudo-differential point of view. It follows from [15] and [26] that $\gamma \geq cg$, g large, for some constant c.

4 Superadiabatic renormalization and reduction theory

The root of this method is the work [8] and subsequent generalizations and adaptations [29], [31], [14] on iterative schemes. Since the work [21], the Adiabatic Theorem is often proven by showing that the *adiabatic evolution* $V(t,s)$ defined by the equation

$$i\varepsilon\frac{\partial}{\partial t}V(t,s)=(H(t)+i\varepsilon[P_1'(t),P_1(t)])U(t,s)\,,\quad V(s,s)=\mathbf{I}\,,\quad \forall t,s\in\mathbf{R} \tag{9}$$

satisfies the intertwining property $V(t,s)P_j(s)=P_j(t)V(t,s)$ and $\|U(t,s)-V(t,s)\|=\mathcal{O}(\varepsilon)$ [1]. Such an approach can be used in a wider context, see [23]. An iterative scheme consists in generating from $H(t)$, by standard perturbation in ε, a sequence of Hamiltonians $\{H_q(t,\varepsilon)\}_{q\in\mathbf{N}}$ with the following features:
i) The Hamiltonians $H_q(t,\varepsilon)$ share the same general properties as $H(t)$. In particular, the gap hypothesis is true for $H_q(t,\varepsilon)$ so that the spectral projectors $P_{j,q}(t,\varepsilon)$ are well defined and tend to $P_j(t)$ as $\varepsilon\to 0$.
ii) The adiabatic evolution $V_q(t,s)$, defined by (9) with $H_q(t,\varepsilon)$ and $P_{j,q}(t,\varepsilon)$ in place of $H(t)$ and $P_j(t)$, approximates $U(t,s)$ up to order $\mathcal{O}(\varepsilon^q)$ instead of $\mathcal{O}(\varepsilon)$.

It follows that the transition probability between spectral subspaces of $H_q(t,\varepsilon)$ is $\mathcal{O}(\varepsilon^{2q})$ instead of $\mathcal{O}(\varepsilon^2)$ for finite times. Berry [2] formally showed for two-level systems that one can push the estimates up to exponential order by truncating the scheme at optimal $q\in\mathbf{N}$. In the present setting, Nenciu proved such exponential estimates in [30] using closely related ideas. The result of Nenciu was recovered in [15] by optimal truncation of the iterative scheme proposed in [14].

Theorem 3 [30], [15] *Assume H1 and H2. Then there exist constants $C,\gamma,\varepsilon_0>0$ and a self adjoint operator $H_*(t,\varepsilon)$ defined on D such that for all $0<\varepsilon<\varepsilon_0$ and $t\in\mathbf{R}$, $\|H_*(t,\varepsilon)-H(t)\|\le C\varepsilon/(1+|t|)^{(1+\alpha)}$. Moreover, the adiabatic evolution $V_*(t,s)$ associated with $H_*(t,\varepsilon)$ satisfies $\sup_{t,s\in\mathbf{R}}\|V_*(t,s)-U(t,s)\|\le Ce^{-\gamma/\varepsilon}$ as well as $V_*(t,s)P_{j,*}(s,\varepsilon)=P_{j,*}(t,\varepsilon)V_*(t,s)$. Finally, $\gamma\ge Cg$, for g large.*

Remarks: Theorem 2 becomes a corollary of Theorem 3, since $P_{j,*}(t,\varepsilon)\overset{t\to\pm\infty}{\to}P_j(\pm\infty)$. The evolution $V_*(t,s)$ is called *superadiabatic evolution* due to the exponential estimate.

The main interest of this construction, however, is that it allows to set up a rigorous reduction theory. Assume $P_1(t)\mathcal{H}$ is finite dimensional, say two-dimensional, such that $\sigma_1(t)=\{e_1(t),e_2(t)\}$ where $e_j(t)$ satisfy the gap assumption. Then the transition probability $\mathcal{P}_{21}(\varepsilon)$ between eigenspaces associated with $e_1(-\infty)$ and $e_2(+\infty)$ can be computed modulo errors $\mathcal{O}(\mathrm{e}^{-\gamma/\varepsilon})$ by replacing the evolution $U(t,s)$ by the superadiabatic evolution $V_*(t,s)$ which describes the evolution *inside* the $P_{j,*}(t,\varepsilon)\mathcal{H}$. This leads to an *effective* two-dimensional problem, with a corresponding effective 2×2 Hamiltonian shown to be a perturbation of $H(t)P_1(t)$ and to have the same analyticity properties as $H(t)$ in S. Hence, provided the conditions on the analytic continuations of e_j stated in Theorem 1 are satisfied, we get with the same notations [15]

$$\mathcal{P}_{21}(\varepsilon)=\mathrm{e}^{2\mathrm{Im}\theta_{21}(\eta_0)}\mathrm{e}^{2\mathrm{Im}\int_{\eta_0}e_1(z)dz/\varepsilon}(1+\mathcal{O}(\varepsilon))+\mathcal{O}(\mathrm{e}^{-\gamma/\varepsilon}) \tag{10}$$

The condition $\gamma>|2\mathrm{Im}\ \int_{\eta_0}e_1(z)dz|$ is satisfied if g is large enough [15] or in the avoided crossing regime described in Theorem 1' [10], [11].

Another use of the superadiabatic renormalization consists in performing the analysis of section 2 for finite dimensional systems when the decomposition (2) is replaced by $\psi(t)=\sum_{j=1}^n c_{j,*}(t)\mathrm{e}^{-i\int_0^t e_{j,*}(t',\varepsilon)dt'/\varepsilon}\varphi_{j,*}(t,\varepsilon)$ where $e_{j,*}(t,\varepsilon)$ and $\varphi_{j,*}(t,\varepsilon)$ are the eigenvalues and eigenvectors of $H_*(t,\varepsilon)$. After showing that they have analytic continuations in

S with similar properties as those of e_j and φ_j, we get very accurate asymptotic formulae by making use of Theorem 3 "in the complex plane". Under the conditions of Theorem 1

$$\mathcal{P}_{21}(\varepsilon) = \mathrm{e}^{2\mathrm{Im}\theta_{21,*}(\varepsilon)}\mathrm{e}^{2\mathrm{Im}\int_{\eta_0} e_{1,*}(z,\varepsilon)dz/\varepsilon}(1+\mathcal{O}(\mathrm{e}^{-\gamma/\varepsilon}))), \quad \gamma > 0, \tag{11}$$

as shown in [17], [11]. The loop η_0 and $\theta_{21,*}(\varepsilon)$ are defined as in section 2.

References:

[1] J.E. Avron, R. Seiler, L.G. Yaffe: *Commun.Math.Phys* **110** (1987) 33-49.

[2] M.V. Berry: *Proc.Roy.Soc.Lond.A* **429** (1990) 61-72.

[3] M.V. Berry: *Proc.Roy.Soc.London A***430** (1990) 405-411.

[4] M. Born, V. Fock: *Zeit.f.Physik* **51** (1928) 165-180.

[5] A.M. Dykhne: *Sov.Phys.JETP.* **14** (1962) 941-943.

[6] M. Fedoriuk: "Méthodes Asymptotiques pour les Equations Différentielles Ordinaires Linéaires", Mir Moscou 1987.

[7] N. Fröman, P.O. Fröman, " JWKB Approximation, Contributions to the Theory", North Holland 1965.

[8] L.M. Garrido: *J.Math.Phys.* **5** (1964) 335-362.

[9] J.-T. Hwang, P. Pechukas: *J.Chem.Phys.***67** (1977) 4640-4653.

[10] A. Joye: *Asymp. Anal.* **9** (1994) 209-258.

[11] A. Joye: *SIAM J. Math. Anal.* **28** (1997) 669-703.

[12] A. Joye, H. Kunz, C.-E. Pfister: *Ann.Phys.* **208** (1991) 299-332.

[13] A. Joye, C.-E. Pfister: *Commun.Math.Phys.***140** (1991) 15-41.

[14] A. Joye, C.-E. Pfister: *J.Phys.A***24** (1991) 753-766.

[15] A. Joye, C.-E. Pfister: *J.Math.Phys.***34** (1993) 454-479.

[16] A. Joye, C.-E. Pfister in NATO ASI Series B: Physics **324** Plenum, New York, (1994) 139-148.

[17] A. Joye, C.-E. Pfister : *SIAM J. Math. Anal.* **26** (1995) 944-977.

[18] A. Joye and C.-E.Pfister : *CPT-Marseille Preprint* (1997).

[19] V. Jaksic, J. Segert: *Rev.Math.Phys.***4** (1992) 529-574.

[20] V. Jaksic, J. Segert: *J.Math.Phys.***34** (1993) 2807-2820.

[21] T. Kato: *J.Phys.Soc.Japan* **5** (1950) 435-439.

[22] M. Klein, R. Seiler: *Commun.Math.Phys.***128** (1990) 141-160.

[23] S.G. Krein, "Linear Differential Equations in Banach Space", AMS, Providence 1971.

[24] L.D. Landau: *Collected Papers of L.D.Landau*, Pergamon Press, Oxford London Edinburgh New York Paris Frankfurt 1965.

[25] A. Lenard: Adiabatic Invariance to all Orders, *Ann.Phys* **6** (1959) 261-276.

[26] A. Martinez: *J.Math.Phys.* **35** (1994) 3889-3915.

[27] P.A. Martin, G. Nenciu: *Rev.Math.Phys.* **7** (1995) 193-242.

[28] G. Nenciu: *J.Phys.A***13** (1980) L15-L18.

[29] G. Nenciu: *Commun.Math.Phys.***82** (1981) 121-135.

[30] G. Nenciu: *Commun.Math.Phys.***152** (1993) 479-512

[31] G. Nenciu, G. Rasche: *Helv.Phys.Acta***62** (1989) 372-388.

[32] A. Shapere, F. Wilczek: "Geometric Phases in Physics", World Scientific, Singapore, New Jersey, London, Hong Kong 1989.

[33] J. Sjöstrand: *C.R.Acad.Sci.Paris* t.317 Sér.I **22** (1993) 217-220.

[34] C. Zener: *Proc.Roy.Soc.London***137** (1932) 696-702.