KAM Theorem and Quantum Field Theory

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Abstract

We give a new proof of the KAM theorem for analytic Hamiltonians. The proof is inspired by a quantum field theory formulation of the problem and is based on a renormalization group argument treating the small denominators inductively scale by scale. The crucial cancellations of resonances are shown to follow from the Ward identities expressing the translation invariance of the corresponding field theory.

1 Introduction

Consider the Hamiltonian

$$H(I,\phi) \,=\, \omega\cdot I+\tfrac{1}{2}I\cdot\mu I+\lambda\, U(\phi,I) \tag{1.1}$$

with $\phi\in\mathbf{R}^d/(2\pi\mathbf{Z}^d)\equiv\mathbf{T}^d$, $I\in\mathbf{R}^d$, $\omega\in\mathbf{R}^d$ with the components ω_i independent over $\mathbf{Z}^d$ and μ a real symmetric $d\times d$ matrix. It generates the Hamiltonian flow given by the equations of motion

$$\dot{\phi}=\omega+\mu I+\lambda\,\partial_I U\,,\quad \dot{I}=-\lambda\partial_\phi U. \tag{1.2}$$

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For the parameter $\lambda = 0$ and the initial condition $(\phi_0, 0)$, the flow $(\phi_0 + \omega t, 0)$ is quasiperiodic and spans a d-dimensional torus in $\mathbf{T}^d \times \mathbf{R}^d$. KAM-theorem deals with the question under what conditions such quasiperiodic solutions persist as the parameter λ is turned on.

Consider a quasiperiodic solution in the form

$$(\phi(t), I(t)) = (\phi_0 + \omega t + \Theta(\phi_0 + \omega t),\ J(\phi_0 + \omega t)).$$

Eqs. (1.2) require that $Z = (\Theta, J) : \mathbf{T}^d \to \mathbf{R}^d \times \mathbf{R}^d$ satisfies the relation

$$\mathcal{D}Z(\phi) = -\lambda\, \partial U(\phi + \Theta(\phi),\ J(\phi)), \tag{1.3}$$

where $\partial = (\partial_\phi, \partial_I)$ and

$$\mathcal{D} = \begin{pmatrix} 0 & \omega \cdot \partial_\phi \\ -\omega \cdot \partial_\phi & \mu \end{pmatrix}. \tag{1.4}$$

Note that if Z is a solution of Eq. (1.3) then so is Z_β for $\beta \in \mathbf{R}^d$ and

$$Z_\beta(\phi) \;=\; Z(\phi - \beta) - (\beta, 0). \tag{1.5}$$

Eq. (1.3) is a fixed point problem for the function Z of a difficult type: the straightforward linearization $\mathcal{D} + \lambda \partial\partial U(\phi, 0)$ is not invertible for any interesting U (see e.g. [8]). Also, one can expect to have a solution only for sufficiently irrational $\omega \in \mathbf{R}^d$, e.g. satisfying a Diophantine condition

$$|\omega \cdot q| \;>\; a|q|^{-\nu} \quad \text{for} \quad q \in \mathbf{Z}^d,\ q \neq 0 \tag{1.6}$$

with some q-independent $a, \nu > 0$. There have been traditionally two approaches to the problem:

1. The KAM approach. (1.3) is solved by a Newton method that constructs a sequence of symplectic changes of coordinates defined on shrinking domains that, in the limit, transform the problem to the $\lambda = 0$ case [1, 2, 17, 18].

2. Perturbation theory. For U analytic (see below) one can attempt to solve (1.3) by iteration. This leads to a power series in λ, the Lindstedt series: $Z = \sum_n Z_n \lambda^n$. Each Z_n is given as a sum of several terms (see Sect. 9), some of which are very large, proportional to $(n!)^a$ with $a > 0$, due to piling up of “small denominators” $(\omega{\cdot}q)$ from the momentum space representation of operator $\mathcal{D}^{-1}$. However, the KAM method also yields the analyticity of Z in λ [19]. Thus the Lindstedt series must converge. To see this directly turned out to be rather hard and was finally done by Eliasson [8] who, by regrouping terms, was able to produce an absolutely convergent series that gives the quasiperiodic solution. Subsequently Eliasson’s work was simplified and extended by Gallavotti [9, 10, 11], by Chierchia and Falcolini [6, 7] and by Bonetto, Gentile, Mastropietro [12, 13, 14, 15, 3, 4].

In the present paper we shall develop a new iterative scheme to solve Eq. (1.3). It is based on a direct application of the renormalization group (RG) idea of quantum field theory (QFT) to the problem. The idea is to split the operator $\mathcal{D}$ (or rather its inverse, see Sect. 2) into a small denominator and large denominator part, where small and large are defined with respect to a scale of order unity. The next step is to solve the large denominator problem which results in a new effective equation of the type (1.3) for the small denominator part, with a new right hand side. The procedure is iterated, with the scale separating small and large at the n^{th} step equal to η^n for some fixed $\eta < 1$. As a result we get a sequence of effective problems that converge to a trivial one as $n \to \infty$. A generic step is solved by a simple application of the Banach Fixed Point Theorem in a big space of functionals of Z representing the right hand side of Eq. (1.3) in the n^{th} iteration step.

Our iteration can be viewed as an iterative resummation of the Lindstedt series, as will be discussed in Sect. 9. This iterative approach trivializes the rather formidable combinatorics of the small denominators. The functional formulation in terms of effective problems removes also the mystery behind the subtle cancellations in the Lindstedt series: they turn out to be an easy consequence of a symmetry in the problem as formulated in terms of the so called Ward identities of QFT. The QFT analogy of the problem (1.3) has been forcefully emphasized by Gallavotti *et al.* [11, 12]. The proof of Eliasson's theorem by these authors was based on a separation into scales of the graphical expressions entering the Lindstedt series and was a direct inspiration for the present work.

An important part of the standard RG theory is an approximate scale invariance of the problem that is exhibited and exploited by the RG method. The KAM problem also is expected to have this aspect: as the coupling λ is increased the solution with a given ω eventually ceases to exist. For suitable "scale invariant" ω (e.g. in $d = 2$ for $\omega = (1, \gamma)$ with γ a "noble" irrational) the solution at the critical λ is expected to exhibit a power law decay of Fourier coefficients and periodic orbits converging to it have peculiar "universal" scaling properties [16, 21]. We hope that the present approach will shed some light on these problems in the future.

While the main goal of this paper is to develop a new method, we use it to reprove the following (classical) result:

Theorem 1. *Let U be real analytic in ϕ and analytic in I in a neighborhood of $I = 0$. Assume that ω satisfies condition* (1.6). *Then Eq.* (1.3) *has a solution which is analytic in λ and real analytic in ϕ provided that either*

(a) (the non-isochronous case) *μ is an invertible matrix and $|\lambda|$ is small enough (in a μ-dependent way).*

(b) (the isochronous case) $\mu = 0$, $\int_{\mathbf{T}^d} \partial_I U(\phi, 0)\, d\phi = 0$, *the $d \times d$ matrix with elements $\int_{\mathbf{T}^d} \partial_{I_k} \partial_{I_l} U(\phi, 0)\, d\phi$, $k, l = 1, \ldots, d$, is invertible and $|\lambda|$ is small enough.*

The above solutions are unique up to translations (1.5).

Remark. Actually, we show that the solution is an analytic function not only of λ, but of the potential U, when the latter belongs to a small ball in a Banach space of analytic functions (see Sect. 3 for the introduction of such spaces). This allows us to consider more general Hamiltonians of the form

$$H(I, \phi) \;=\; H_0(I) + U(\phi, I).$$

with H_0 and U analytic and U small. Indeed, we may expand H_0 around I_0 s.t. $\partial_I H_0(I_0) = \omega$, with ω satisfying condition (1.6):

$$H_0(I) = H(I_0) + \omega \cdot (I - I_0) + \tfrac{1}{2}\,(I - I_0) \cdot \mu(I - I_0) + \tilde{H}_0(I)$$

and define $\tilde{U} = U + \tilde{H}_0$ so as to include in it all the terms of order higher than two in the expansion of H_0. Replacing $I - I_0$ by I, we may apply Theorem 1 provided that $\tilde{U}$ satisfies the corresponding hypotheses. Also, more general cases where μ is a degenerate matrix can be treated.

The organization of the paper is as follows. In Sect. 2 we explain the RG formalism. In Sect. 3, we introduce spaces of analytic functions on Banach spaces; such spaces will be used to solve our RG equations. In Sect. 4, we state the main inductive estimates which are proved in Sect. 6 after an interlude on the Ward identities in Sect. 5. Theorem 1 is proved then in Sect. 7. Sect. 8 explains the connection of our formalism to QFT for those familiar with the latter. We should emphasize that the QFT is solely a source of intuition, the simple RG formalism of Sect. 2 is independent of it. Finally, in Sect. 9, the connection with the Lindstedt series is explained.

2 Renormalization group scheme

In this section we explain the iterative RG scheme without spelling out the technical assumptions that are needed to carry it out. We refer the reader to Sect. 9 for a graphical representation of the main quantities introduced here.

We shall work with Fourier transforms, denoting by lower case letters the Fourier transforms of functions of ϕ, the latter being denoted by capital letters:

$$F(\phi) = \sum_{q \in \mathbf{Z}^d} \mathrm{e}^{-iq\cdot\phi} f(q), \qquad \text{where} \qquad f(q) = \int_{\mathbf{T}^d} \mathrm{e}^{iq\cdot\phi} F(\phi)\, d\phi$$

with $d\phi$ standing for the normalized Lebesgue measure on $\mathbf{T}^d$.

Note first that we may use the translations (1.5) to limit our search for the solution of Eq. (1.3) to the subspace of Θ with zero average, i.e. with $\theta(0) = 0$ in the Fourier language. It will be convenient to separate the constant mode of J explicitly by writing $Z = X + (0, \zeta)$ where X has zero average. Let us define

$$W_0(\phi; X, \zeta) = \lambda\, \partial U((\phi, \zeta) + X(\phi)). \tag{2.1}$$

Denote by G_0 the operator $-\mathcal{D}^{-1}$ acting on $\mathbf{R}^{2d}$-valued functions on $\mathbf{T}^d$ with zero average. In terms of the Fourier transforms,

$$(G_0 x)(q) = \begin{pmatrix} \mu(\omega\cdot q)^{-2} & i(\omega\cdot q)^{-1} \\ -i(\omega\cdot q)^{-1} & 0 \end{pmatrix} x(q). \tag{2.2}$$

for $q \neq 0$ and $(G_0 x)(0) = 0$. Writing Eq. (1.3) separately for the averages (i.e. $q = 0$) and the rest, we may rewrite it as the fixed point equations

$$X \;=\; G_0 P W_0(X, \zeta), \tag{2.3}$$

$$(0, \mu\zeta) \;=\; -\int_{\mathbf{T}^d} W_0(\phi; X, \zeta)\, d\phi, \tag{2.4}$$

where P projects out the constants: $PF = F - \int_{\mathbf{T}^d} F(\phi)d\phi$. Our strategy is to solve Eq. (2.3) by an inductive RG method for given ζ. This turns out to be possible quite generally without any nondegeneracy assumptions on U. The latter enter only in the solution of Eq. (2.4). Below, we shall treat W_0 given by Eq. (2.1) as a map on a space of $\mathbf{R}^{2d}$-valued functions X on $\mathbf{T}^d$ with arbitrary averages[3]. The vector ζ will be treated as a parameter and we shall often suppress it in the notation for W_0.

For the inductive construction of the solution of Eq. (2.3), we shall decompose

$$G_0 \;=\; G_1 + \Gamma_0, \tag{2.5}$$

where Γ_0 will effectively involve only the Fourier components with $|\omega \cdot q|$ larger than $\mathcal{O}(1)$ and G_1 the ones with $|\omega \cdot q|$ smaller than that (see Sect. 4). In particular, we shall have $\Gamma_0 = \Gamma_0 P$. Upon writing $X = Y + \tilde{Y}$, Eq. (2.3) becomes

$$Y + \tilde{Y} \;=\; (G_1 + \Gamma_0) P W_0(Y + \tilde{Y}). \tag{2.6}$$

Suppose that $\tilde{Y} = \tilde{Y}_0$ where $\tilde{Y}_0$ solves for fixed Y the "large denominator" equation:

$$\tilde{Y}_0 \;=\; \Gamma_0 W_0(Y + \tilde{Y}_0). \tag{2.7}$$

Then Eq. (2.6) reduces to the relation

$$Y \;=\; G_1 P W_1(Y) \tag{2.8}$$

if we define $W_1(Y) = W_0(Y + \tilde{Y}_0)$. We have thus reduced the orginal problem (2.3) to the one from which the largest denominators were eliminated, at the cost of solving the easy large denominator problem (2.7) and of replacing the map W_0 by W_1.

Note that, with these definitions, $\tilde{Y}_0 = \Gamma_0 W_1(Y)$ and thus W_1 satisfies the fixed point equation

$$W_1(Y) \;=\; W_0(Y + \Gamma_0 W_1(Y)). \tag{2.9}$$

Conversely, this equation, which we shall solve for W_1 by the Banach Fixed Point Theorem in a suitable space, implies that $\tilde{Y}_0 = \Gamma_0 W_1(Y)$ satisfies Eq. (2.7) and thus that

$$X \;=\; Y + \Gamma_0 W_1(Y) \;\equiv\; F_1(Y) \tag{2.10}$$

is a solution of Eq. (2.3) if and only if Y solves Eq. (2.8).

After $n-1$ inductive steps, the solution of Eq. (2.3) will be given as

$$X \;=\; F_{n-1}(Y), \tag{2.11}$$

where Y solves the equation

$$Y \;=\; G_{n-1} P W_{n-1}(Y) \tag{2.12}$$

[3]That the solution X of Eq. (2.3) has zero average follows from the form of the equation.

and G_{n-1} contains only the denominators $|\omega \cdot q| \le \mathcal{O}(\eta^n)$ where η is a positive number smaller than 1 fixed once for all. The next inductive step consists of decomposing $G_{n-1} = G_n + \Gamma_{n-1}$ where Γ_{n-1} involves $|\omega \cdot q|$ of order η^n and G_n the ones smaller than that. We define $W_n(Y)$ as the solution of the fixed point equation

$$W_n(Y) \;=\; W_{n-1}(Y + \Gamma_{n-1} W_n(Y)) \tag{2.13}$$

and set

$$F_n(Y) \;=\; F_{n-1}(Y + \Gamma_{n-1} W_n(Y)) \tag{2.14}$$

(which is consistent with relation (2.10) if we take $F_0(Y) = Y$). Then replacing Y in Eqs. (2.11) and (2.12) by $Y + \Gamma_{n-1} W_n(Y)$, we infer that $X = F_n(Y)$ if $Y = G_n P W_n(Y)$ completing the next inductive step. Note also the cumulative formulas that follow easily by induction:

$$W_n(Y) \;=\; W_0(Y + \Gamma_{<n} W_n(Y))\,, \tag{2.15}$$

$$F_n(Y) \;=\; Y + \Gamma_{<n} W_n(Y)\,, \tag{2.16}$$

where $\Gamma_{<n} = \sum\limits_{k=0}^{n-1} \Gamma_k$ contains all the denominators larger than $\mathcal{O}(\eta^n)$.

We shall control the transformations (2.13) and (2.14) in suitable norms. The point of the inductive procedure is that $PW_n(Y)$ becomes effectively linear in Y for large n, see Remark 1 after Proposition 3 below, so that $Y = 0$ is a better and better approximation to a solution of the equation $Y = G_n P W_n(Y)$. In fact, as follows from the cumulative relations (2.15) and (2.16), $X_n \equiv F_n(0) = \Gamma_{<n} W_n(0)$ solves the approximate problem:

$$X_n \;=\; \Gamma_{<n} W_0(X_n), \tag{2.17}$$

obtained from Eq. (2.3) by replacing G_0 by $\Gamma_{<n}$ (since $\Gamma_{<n} P = \Gamma_{<n}$). We shall construct the solution X of Eq. (2.3) as the limit of the approximate solutions

$$X \;=\; \lim_{n \to \infty} X_n\,. \tag{2.18}$$

3 Spaces

Let us rewrite the definition (2.1) in terms of the Fourier transforms:

$$w_0(q;y) \;=\; \lambda \int_{\mathbf{T}^d} \mathrm{e}^{iq\cdot\phi}\, \partial U((\phi,\zeta) + Y(\phi))\, d\phi,$$

where we recall that y refers to the Fourier transform of Y. Let us explain here how to view w_0 as an analytic functional on a suitable Banach space. The analyticity of U implies the following. There exist $\rho > 0$, $\alpha > 0$ and $b < \infty$ such that the coefficients $U_{m+1}(\phi,\zeta)$, belonging to the space of m-linear maps $\mathcal{L}(\mathbf{C}^{2d}, \ldots, \mathbf{C}^{2d}; \mathbf{C}^{2d})$, of the Taylor expansion

$$\partial U((\phi,\zeta) + Y) = \sum_{m=0}^{\infty} \tfrac{1}{m!} U_{m+1}(\phi,\zeta)(Y, \ldots, Y)$$

are analytic in $|\zeta|<\rho$ and their Fourier transforms satisfy the bounds

$$\sum_q \mathrm{e}^{\alpha|q|}\, \|u_{m+1}(q,\zeta)\|_{\mathcal{L}(\mathbf{C}^{2d},\ldots,\mathbf{C}^{2d};\mathbf{C}^{2d})} \leq\; b\, m!\, \rho^{-m}. \tag{3.1}$$

For later convenience, we shall use in $\mathbf{C}^{2d}\cong\mathbf{C}^d\times\mathbf{C}^d$ the norm $|\cdot|_0$ defined by

$$|(z_1,z_2)|_0 \;\equiv\; |z_1|+|z_2| \tag{3.2}$$

and the induced norms on the spaces of linear maps. Inserting the Fourier series for Y we end up with the expansion

$$\begin{aligned} w_0(q;y) \;&=\; \sum_{m=0}^{\infty}\sum_{\mathbf{q}} \tfrac{1}{m!}\, u_{m+1}(q-\textstyle\sum q_i,\zeta)\,(y(q_1),\ldots,y(q_m)) \\ &\equiv\; \sum_{m=0}^{\infty}\sum_{\mathbf{q}} w_0^{(m)}(q,q_1,\ldots,q_m;\zeta)\,(y(q_1),\ldots,y(q_m)), \end{aligned} \tag{3.3}$$

where $\mathbf{q}=(q_1,\ldots,q_m)\in\mathbf{Z}^{md}$. This formula suggests to consider w_0 as an analytic function of y, where y belongs to a suitable Banach space h. We take

$$h=\{\, y=(y(q)) \mid y(q)\in\mathbf{C}^{2d},\;\; \|y\|\equiv\sum_q |y(q)|_0<\infty\,\}.$$

Let $B(r_0)$ be the open ball of radius r_0 in h centered at the origin and let $H^\infty(B(r_0),h)$ denote the Banach space of analytic functions [5] $w : B(r_0)\to h$, equipped with the supremum norm, which we shall denote by $|||w|||$. The bound (3.1) implies that $w_0\in H^\infty(B(r_0),h)$ for r_0 small enough, but before stating this, it is convenient to encode the decay property of the kernels $w_0^{(m)}$ inherited from the estimate (3.1) as a property of the functional w_0.

For that let τ_β denote the translation by $\beta\in\mathbf{R}^d$, $(\tau_\beta Y)(\phi)=Y(\phi-\beta)$. On h, τ_β is realized by $(\tau_\beta y)(q)=e^{i\beta\cdot q}y(q)$. It induces a map $w\mapsto w_\beta$ from $H^\infty(B(r_0),h)$ to itself if we set

$$w_\beta(y)=\tau_\beta(w(\tau_{-\beta}y)).$$

On the kernels $w^{(m)}$, this is given by

$$w_\beta^{(m)}(q;q_1,\ldots,q_m) \;=\; \mathrm{e}^{i\beta\cdot(q-\sum q_j)}\, w^{(m)}(q;q_1,\ldots,q_m). \tag{3.4}$$

and makes sense also for $\beta\in\mathbf{C}^d$. We have

$$|||w_{0\beta}||| \;\leq\; \sum_{m=0}^{\infty}\sup_{q_1,\ldots,q_m}\sum_q \mathrm{e}^{-\mathrm{Im}\beta\cdot(q-\sum q_j)}\,|w_0^{(m)}(q;q_1,\ldots,q_m;\zeta)|\, r_0^m$$

Combining this with the bound (3.1) we can summarize the above discussion by

Proposition 1. *There exists $r_0>0$, $\alpha>0$ and $D<\infty$, such that $w_{0\beta}\in H^\infty(B(r_0),h)$ and it extends to an analytic function of β in the region $|\mathrm{Im}\beta|<\alpha$ with values in $H^\infty(B(r_0),h)$ satisfying the bound*

$$|||w_{0\beta}||| \;\leq\; D|\lambda|. \tag{3.5}$$

Moreover, $w_{0\beta}$ is analytic in ζ for $|\zeta| < r_0$.

Remarks. 1. The analyticity of $w_{0\beta}$ in a strip in $\mathbf{C}^d$ centered on $\mathbf{R}^d$ followed from the exponential decay of the kernels $w_0^{(m)}(q; q_1, \ldots, q_m)$ as functions of $q - \sum q_j$. In order to show that the solution of Eq. (2.3) is real analytic in ϕ, we shall need to inductively establish such a decay for the kernels $w_n^{(m)}$ of the Fourier transforms w_n of effective functionals W_n. This will follow once we establish the analyticity of $w_{n\beta}$ in β with uniform bounds.

2. From now on, analyticity in ζ will always be understood to hold for $|\zeta| < r_0$.

We finish this section by collecting, for convenience, some standard properties of bounded analytic functions defined on open balls in Banach spaces (that are identical to those of analytic functions on finite dimensional spaces, see [5]). Let h, h', h'' be Banach spaces, $B(r) \subset h$, $B(r') \subset h'$ and $w_i \in H^\infty(B(r), h')$, $w \in H^\infty(B(r'), h'')$. Then

Composition property. If $|||w_i||| < r'$ then $w \circ w_i \in H^\infty(B(r), h'')$ and

$$|||w \circ w_i||| \;\le\; |||w|||. \tag{3.6}$$

Inequalities. First of all, one deduces from the Cauchy estimate that for $r_1 < r'$,

$$\sup_{\|x\|<r_1} \|Dw(x)\|_{\mathcal{L}(h';h'')} \le (r' - r_1)^{-1} |||w|||, \tag{3.7}$$

where $\mathcal{L}(h';h'')$ denotes the space of bounded linear operators from h' to h''. Taking $r_1 = \frac{1}{2} r'$, we infer that if $|||w_i||| \le \frac{1}{2} r'$ then

$$|||w \circ w_1 - w \circ w_2||| \;\le\; \tfrac{2}{r'}\, |||w|||\; |||w_1 - w_2|||. \tag{3.8}$$

Moreover, if $\delta_k w(x) = w(x) - \sum\limits_{\ell=0}^{k-1} \frac{1}{\ell!} D^\ell w(0)(x)$, then

$$\sup_{\|x\|\le\gamma r'} \|\delta_k w(x)\| \;\le\; \tfrac{\gamma^k}{1-\gamma} |||w||| \tag{3.9}$$

for $0 \le \gamma < 1$.

4 Inductive bounds

In this section, we first define the operators Γ_n used in the RG transformations. Then, we state an easy result, namely that the fixed point equations (2.13) and (2.15) may be solved for any n, if we choose λ small enough in an n-dependent way. Then, we define the spaces in which our RG equations (2.13) are eventually solved inductively for $|\lambda|$ small uniformly in n and state precisely our inductive assumptions.

In the Fourier variables, the fixed point equations (2.13) and (2.15) may be written in the form

$$\begin{aligned} w_{n\beta}(y) \;&=\; w_{(n-1)\beta}(y + \Gamma_{n-1} w_{n\beta}(y)), \qquad (4.1)\\ w_{n\beta}(y) \;&=\; w_{0\beta}(y + \Gamma_{<n} w_{n\beta}(y)), \qquad (4.2)\end{aligned}$$

where we have introduced β assuming that the operators Γ_n are diagonal in Fourier space and hence commute with τ_β. Similarly, the equations (2.14) and (2.16) translate in the Fourier space to the relations

$$
\begin{aligned}
f_{n\beta}(y) &= f_{(n-1)\beta}(y+\Gamma_{n-1}w_{n\beta}(y)), &(4.3)\\
f_{n\beta}(y) &= y+\Gamma_{<n}w_{n\beta}(y). &(4.4)
\end{aligned}
$$

To define the operators Γ_n, we shall use an analytic partition of unity (χ_n) dividing the positive line into scales. Let

$$\chi_0(\kappa)=1-\mathrm{e}^{-\kappa^6}, \qquad \chi_n(\kappa)=\mathrm{e}^{-(\eta^{-n+1}\kappa)^6}-\mathrm{e}^{-(\eta^{-n}\kappa)^6} \quad \text{for } n\geq 1. \tag{4.5}$$

Clearly,

$$\sum_{n=0}^{\infty}\chi_n(\kappa)=1, \qquad \chi_n(\kappa)=\chi_1(\eta^{-n+1}\kappa) \quad \text{for } n\geq 1. \tag{4.6}$$

Note that $\kappa^{-6}\chi_n(\kappa)$ are entire functions of κ and that, for $|\mathrm{Im}\kappa|<B$ and $\ell=0,\ldots,6$,

$$
\begin{aligned}
|\kappa^{-\ell}\chi_0(\kappa)| &\leq C, &(4.7)\\
|\kappa^{-\ell}\chi_1(\kappa)| &\leq C\,\mathrm{e}^{-\frac{1}{2}|\kappa|^6}, &(4.8)
\end{aligned}
$$

for some B-dependent constant C. Define

$$\Gamma_n(q,q') = \chi_n(\omega\cdot q)\,G_0(q,q') = \gamma_n(\omega\cdot q)\,\delta_{q,q'}, \tag{4.9}$$

where the matrix γ_n is of the block form:

$$\gamma_n(\kappa) = \chi_n(\kappa)\kappa^{-2}\begin{pmatrix}\mu & i\kappa\\ -i\kappa & 0\end{pmatrix} \tag{4.10}$$

and we denote the kernel of an operator a in $\mathcal{L}(h;h)$ by $a(q,q')\in\mathrm{End}(\mathbf{C}^{2d})$. We shall also need below more general operators $\Gamma_n(\kappa)$ with shifted kernels,

$$\Gamma_n(\kappa)(q,q') = \gamma_n(\omega\cdot q+\kappa)\,\delta_{q,q'}. \tag{4.11}$$

It follows easily from the bounds (4.7) and (4.8) that for $n\geq 1$,

$$\|\Gamma_{n-1}(\kappa)\|_{\mathcal{L}(h;h)},\ \|\Gamma_{<n}(\kappa)\|_{\mathcal{L}(h;h)} \leq C\eta^{-2n} \qquad \text{if} \qquad |\kappa|<\eta^n B \tag{4.12}$$

with a new constant C (say, twice bigger).

Our goal is to show that w_n and f_n exist as analytic functionals provided that $|\lambda|$ is taken small in an n-independent way. For later purposes it will be useful to prove this first for $|\lambda|$ small in an n-dependent way. Although this is very easy to carry out, it illustrates an important part of the main analytic argument in the general step.

Proposition 2. *For any sufficiently small $r>0$, $|\lambda|<\lambda_n(r)$ and $|\mathrm{Im}\beta|<\alpha$, the equations* (4.2) *have a unique solution $w_{n\beta}\in H^\infty(B(r^n),h)$ with*

$$|||w_{n\beta}||| \leq D|\lambda|, \tag{4.13}$$

where D is as in Proposition 1. The maps $f_{n\beta}$ defined by Eqs. (4.4) *belong to* $H^\infty(B(r^n),h)$. *They satisfy the bounds* $|||f_{n\beta}||| \leq 2r^n$. *Moreover,* $w_{n\beta}$ *and* $f_{n\beta}$ *are analytic in* λ, β *and* ζ *and they satisfy the recursive relations* (4.1) *and* (4.3)*, respectively.*

Postponing the proof to the end of the section, we shall state the bounds for w_n that will be inductively established for $|\lambda|$ small in an n-independent way. Due to the smallness of $|\omega \cdot q|$ in the $n^{\rm th}$ scale, γ_n will have very different effects in the variables θ and j in $y = (\theta, j)$. It will be therefore convenient to choose n-dependent norms for $n \geq 1$. Let us first do it for $\mathbf{C}^{2d}$ by defining

$$|(z_1, z_2)|_{\pm n} \;\equiv\; |z_1| + \tfrac{1}{\eta^{\pm n}}|z_2|. \tag{4.14}$$

We shall use the notation $|\cdot|_{n;m}$ for the matrix norms induced by viewing a $2d \times 2d$ matrices as maps from $\mathbf{C}^{2d}$ with the norm $|\cdot|_n$ to $\mathbf{C}^{2d}$ with the norm $|\cdot|_m$. Next we set

$$\|y\|_n = \sum_q |y(q)|_n \, {\rm e}^{\eta^{-n}|\omega\cdot q|}. \tag{4.15}$$

The weight ${\rm e}^{\eta^{-n}|\omega\cdot q|}$ will facilitate dealing with non-dangerous large denominators $|\omega \cdot q|$. For w, it turns out to be useful to introduce the norms

$$\|w\|_{-n} = \sum_q |w(q)|_{-n} \, {\rm e}^{-\eta^{-n}|\omega\cdot q|}. \tag{4.16}$$

Let $h_{\pm n}$ denote the corresponding Banach spaces. Note the natural embeddings for $n \geq 2$

$$h_n \;\longrightarrow\; h_{n-1} \;\longrightarrow\; h\,, \qquad\qquad h \;\longrightarrow\; h_{-n+1} \;\longrightarrow\; h_{-n} \tag{4.17}$$

with the norms bounded by 1:

$$\|\cdot\| \;\leq\; \|\cdot\|_{n-1} \;\leq\; \|\cdot\|_n\,, \qquad\qquad \|\cdot\|_{-n} \;\leq\; \|\cdot\|_{-n+1} \;\leq\; \|\cdot\|\,. \tag{4.18}$$

For $n \geq 2$ (but not for $n = 1$), the operator Γ_{n-1} or, more generally, operators $\Gamma_{n-1}(\kappa)$ may be considered as mapping h_{-n} into h_n. Indeed, it follows easily with the use of bound (4.8) that

$$\|\Gamma_{n-1}(\kappa)\|_{-n;n} \;\leq\; C\,\eta^{-2n} \qquad \text{if} \qquad |\kappa| < \eta^{n-1}B \tag{4.19}$$

with a new (n-independent) constant C.

To simplify notations, we shall denote by B_n the open ball in h_n of radius r^n and by $\mathcal{A}_n$ the space $H^\infty(B_n, h_{-n})$ of analytic functions on B_n with the supremum norm denoted by $|||\cdot|||$. Finally, for a linear operator $M : h_n \to h_m$ we use the abbreviated notation $\|\cdot\|_{n;m}$ for the norm in $\mathcal{L}(h_n, h_m)$. Due to the embeddings (4.17), we may regard the maps $w_{n\beta}$, whose existence for sufficiently small $|\lambda|$ is claimed in Proposition 2, as belonging to $\mathcal{A}_n$. Note that both sides of relation (4.1) are well defined for such maps due to the bound (4.19) and that their equality is implied by the results of Proposition 2. The next proposition states that, viewed as $\mathcal{A}_n$-valued functions of λ, $w_{n\beta}$'s may be analytically extended to an n-independent disc $|\lambda| < \lambda_0$ (provided we restrict somewhat the strip of β). It also lists the properties of such extensions.

Proposition 3. (a) *There exist positive constants r and λ_0 with $r < \eta^4$ such that, for $|\lambda| < \lambda_0$ and $|\mathrm{Im}\beta| < \alpha_n$, where*

$$\alpha_1 = \alpha, \quad \alpha_n = (1 - n^{-2})\alpha_{n-1}, \quad n \geq 2, \tag{4.20}$$

there exist solutions $w_{n\beta} \equiv w$ of Eqs. (4.1) belonging to $\mathcal{A}_n$, analytic in λ, β and ζ and coinciding with the solutions $w_{n\beta}$ of Proposition 2 for $|\lambda| < \lambda_n(r)$.

(b) *Writing*

$$w(y) = w(0) + Dw(0)y + \delta_2 w(y), \tag{4.21}$$

we have

$$\|Pw(0)\|_{-n} \;\leq\; \epsilon\, r^{2n}, \tag{4.22}$$

$$|||\delta_2 w||| \;\leq\; \epsilon\, r^{\frac{3}{2}n}, \tag{4.23}$$

where $\epsilon \to 0$ as $\lambda \to 0$.

(c)

$$\|Dw(y)\|_{n;-n} \;\leq\; \epsilon\, \eta^{2n}. \tag{4.24}$$

Remarks. 1. If we rescale the maps w_n by introducing $\tilde{w}_n(y) = \eta^{-2n} r^{-n} w_n(r^n y)$ then it follows from the above statements that $\tilde{w}_{n\beta} \equiv \tilde{w}$ are analytic maps from a unit ball in h_n to h_{-n} and $\tilde{w}(y) = \tilde{w}(0) + D\tilde{w}(0)y + \delta_2\tilde{w}(y)$ with

$$\|P\tilde{w}(0)\|_{-n} \leq \epsilon\, \eta^{-2n} r^n, \qquad |||\delta_2\tilde{w}||| \leq \epsilon\, \eta^{-2n} r^{\frac{1}{2}n}, \qquad \|D\tilde{w}(0)\|_{n,-n} \leq \epsilon.$$

Hence with growing n, $P\tilde{w}$ becomes an approximately linear map.

2. Let us explain the idea of the proof of Proposition 3. Consider the linearization of Eq. (4.1):

$$w_n = w_{n-1} + Dw_{n-1}\Gamma_{n-1}w_n \;+\; \ldots \tag{4.25}$$

In order to solve the above equation one has to invert the operator $1 - Dw_{n-1}\Gamma_{n-1}$:

$$w_n = (1 - Dw_{n-1}\Gamma_{n-1})^{-1} w_{n-1} \;+\; \ldots$$

However, operator Γ_{n-1} is of order η^{-2n} as a map from h_{-n} to h_n (recall the bounds (4.19)) and we need to show that Dw_{n-1} is effectively of order η^{2n} as a map from h_n to h_{-n}, which is, essentially, what Eq. (4.24) says with n shifted to $n-1$. Altogether, $Dw_{n-1}\Gamma_{n-1}$ remains of order ϵ as a map from h_{-n} to h_{-n} (this motivates also our choice of the norms) and $\|(1 - Dw_{n-1}\Gamma_{n-1})\|_{n;-n} \leq 1 + \mathcal{O}(\epsilon)$. In the proof of the estimate (4.24), we shall need the Ward identities discussed in Sect. 5. This is the only subtle part of our argument. Indeed, once the bound (4.24) is shown, the rest of the proof of Proposition 3 reduces to the standard Banach Fixed Point Theorem combined with the Diophantine property of ω.

The latter is used in the following way (which is similar to the way it enters the standard KAM proof): upon iteration, we consider smaller and smaller $|\omega \cdot q|$'s, of order η^n. This means $|q|$ is of order $\eta^{n/\tau}$, by the Diophantine condition (1.6). On the other hand, the introduction of the parameter β in (3.4) allows to preserve the exponential decay of the kernels $w_0^{(m)}(q; q_1, \ldots, q_m)$ in the size of $|q - \sum q_j|$. By shrinking at each step the analyticity region in β we show that the leading contribution to w_n's given by $w_n(0)$ (see Eq. (4.21)) contracts for $q \neq 0$. Actually, $\|Pw_n(0)\|_{-n}$ decays super-exponentially in n, see the estimate (6.14) below, which explains why we can choose r as small as we want.

Finally, the bound (4.23) is easy to understand. By definition, $\delta_2 w_n$ and its first derivative vanish for $y = 0$, and the norm $|||\delta_2 w_n|||$ is defined by taking the supremum over balls of radius r^n, hence one expects $|||\delta_2 w_n|||$ to be of order $(1 + \mathcal{O}(\epsilon))^n r^{2n}$ by the Cauchy estimate (3.9) (the weaker bound (4.23) is sufficient and is a convenient way to control the non-linear corrections to the iteration). Recall that, eventually, we construct our solution as a limit of $X_n = F_n(0)$, for which we need to control $w_n(y)$ only for $y = 0$, see Eq. (2.16). Thus we can let the radius r^n of the ball where our estimates hold tend to zero.

3. Combining all the bounds, we get

$$|||w_n - (1 - P)w_n(0)||| \;\leq\; C\,\epsilon\,\eta^{2n}\,. \tag{4.26}$$

The zero mode part $(1 - P)w_n(0)$ of $w_n(0)$ will be controlled later, see Eqs. (5.3) and the second of Eqs. (7.4) below from which it follows that it is of the form $(0, \xi_n)$ where $\xi_n = \mathcal{O}(\lambda)$ converges in $\mathbf{R}^d$ when $n \to \infty$. Note that, since w_n is multiplied by $\Gamma_{n-1} = \Gamma_{n-1}P$ in the argument of w_{n-1} in Eq. (4.1), the constant mode $(1 - P)w_n(0)$ may be decoupled from the iteration and we do not need to control it in order to prove Proposition 3.

4. We choose the constants as follows. $\eta < 1$ has been fixed first. B which enters the estimates (4.12) and (4.19) is chosen then large enough (see Eq. (6.35) below). Given those, r and then λ_0 are chosen small enough. It should be emphasized that all quantities that are bounded by an n-dependent power of r are easy to estimate and that these estimates do not involve the Ward identities, the latter entering only in bounds with η^{2n}. Finally, we denote by C a generic constant, independent of ϵ and n, which may vary from place to place.

Let us end this section with the easy

Proof of Proposition 2. Consider the fixed point equation (4.2) and write it as $w = \mathcal{F}(w)$ for $w = w_{n\beta}$ and

$$\mathcal{F}(w)(y) = w_{0\beta}(y + \Gamma_{<n} w(y)).$$

Let $\mathcal{B}_n$ denote the closed ball composed of $w \in H^\infty(B(r^n), h)$ with $|||w||| \leq D\lambda_n$ (where D is as in Proposition 1). Choose λ_n so that $C\eta^{-2n} D\lambda_n \leq r^n$ with C as in the bounds (4.12). It follows from the latter that for $w \in \mathcal{B}_n$ and $y \in B(r^n) \subset h$,

$$\|y + \Gamma_{<n} w(y)\| \leq r^n + C\eta^{-2n} D\lambda_n \leq 2r^n \leq \tfrac{1}{2} r_0 \tag{4.27}$$

for r sufficiently small. Thus $\mathcal{F}(w)$ is defined in $B(r^n)$ and, by Proposition 1, $\|\mathcal{F}w(y)\| \leq D|\lambda| \leq D\lambda_n$. Hence $\mathcal{F} : \mathcal{B}_n \to \mathcal{B}_n$. For $w_i \in \mathcal{B}_n$, $i = 1, 2$, use the property (3.8) to conclude that

$$|||\mathcal{F}(w_1)-\mathcal{F}(w_2)||| \le \frac{2}{r_0} C\eta^{-2n} D\lambda_n |||w_1-w_2||| \le \frac{2r^n}{r_0}|||w_1-w_2||| \le \frac{1}{2}|||w_1-w_2|||$$

for r as in the estimate (4.27), i.e. that $\mathcal{F}$ is a contraction. It follows that Eq. (2.15) possesses a unique solution $w_{n\beta}$ in $\mathcal{B}_n$ satisfying the bound (4.13) which, besides, is analytic in λ, β and ζ. Consider now for $n \ge 2$ the map $\mathcal{F}'$:

$$\mathcal{F}'(w)(y) = w_{0\beta}(y + \Gamma_{n-1}w_{n\beta}(y) + \Gamma_{<n-1}w(y)). \tag{4.28}$$

Again, $\mathcal{F}'$ is a contraction in $\mathcal{B}_n$ since, for $\|y\| \le r^n$, we have $\|y + \Gamma_{n-1}w_{n\beta}(y) + \Gamma_{<n-1}w(y)\| \le 3r^n \le \frac{1}{2}r_0$ for r sufficiently small. But Eqs. (4.2) imply that $w_{n\beta}$ and $w_{(n-1)\beta}\circ(1+\Gamma_{n-1}w_{n\beta})$, both in $\mathcal{B}_n$, are its fixed points and, consequently, they have to coincide. Hence the recursions (4.1) follow. By virtue of the estimate (4.27), $\|y+\Gamma_{<n}w(y)\| \le 2r^n$ for $y \in B(r^n)$. By definition (4.4), this gives the claimed bound on $|||f_{n\beta}|||$. The recursion (4.3) follows easily from Eq. (4.1). □

5 Ward identities and cancellation of resonances

The goal of this section is to prove the properties of the maps W_n which will be essential in the proof of part (c) of Proposition 3, see Lemma 2 below, and in the proof Theorem 1, see Lemma 1. These properties, which are proven by a simple integration by parts, result from the symmetries of W_0 and will be encoded in the identities which, in the QFT formulation of the problem explained in Sect. 8, can be interpreted as the *Ward identities* corresponding to the translation symmetry. Their function in the proof is to guarantee a partial cancellation of the repeated resonances that plague the Lindstedt series, see Sect. 9.

Indeed, as we shall see, the subtle part of the proof of the estimate (4.24) reduces to a bound on the diagonal elements of the kernel $Dw_n(0)(q,q)$ of the derivative Dw_n evaluated at zero. Our strategy will be to show that this kernel is actually a function of $\omega\cdot q$ only and is of the form $\left(\begin{smallmatrix}\mathcal{O}((\omega\cdot q)^2) & \mathcal{O}(\omega\cdot q)\\ \mathcal{O}(\omega\cdot q) & \mathcal{O}(1)\end{smallmatrix}\right)$ for $\omega\cdot q$ small. This, combined with our choice of the norms (4.14) and (4.16), will then be used to imply the estimate (4.24). To show such a behavior of $Dw_n(0)(q,q)$, we shall compute certain of its derivatives at $\omega\cdot q = 0$ and show that they vanish. This is the role of the Ward identities proven here. Since q is in $\mathbf{Z}^d$, to make sense of such derivatives we need to introduce a smooth interpolation of $Dw_n(q,q)$ viewed as a function of $\omega\cdot q$. This is the role of the functions π_n defined after Lemma 1 below.

For simplicity, we shall first state and prove the Ward identities for the maps W_n constructed in Proposition 2 for $|\lambda| < \lambda_n$ (by analyticity in λ, they will also hold for W_n's which will be constructed in Proposition 3 for $|\lambda| < \lambda_0$). The basic identity reads (recall that $Y^i = \Theta^i$ for $i \le d$)

$$\int_{\mathbf{T}^d} W_n^i(\phi;Y)\,d\phi \;=\; \int_{\mathbf{T}^d} Y^l(\phi)\;\partial_{\phi^i}W_n^l(\phi;Y)\;d\phi \quad \text{for} \;\; i\le d \tag{5.1}$$

or, in the Fourier language,

$$w_n^i(0;y) \;=\; i\sum_q y^l(q)\,q^i\,w_n^l(-q;y)\,, \tag{5.2}$$

where on the right hand sides the summations over the repeated index l from 1 to $2d$ are understood. Let us check first the $n=0$ case, see Eq. (2.1),

$$\begin{aligned}\int (\partial_i U)((\phi,\zeta)+Y(\phi))\,d\phi \;=\; &\int \partial_i[U((\phi,\zeta)+Y(\phi))]\;d\phi\\ &-\int (\partial_l U)((\phi,\zeta)+Y(\phi))\;\partial_i Y^l(\phi)\;d\phi.\end{aligned}$$

The first term on the right hand side vanishes and the second one yields the claim by integration by parts. For $n\geq 1$, using the relation (2.15), we obtain

$$\begin{aligned}\int W_n^i(\phi;Y)\,d\phi \;&=\; \int W_0^i(\phi;Y+\Gamma_{<n}W_n(Y))\,d\phi\\ &=\; \int (Y^l+(\Gamma_{<n}W_n)^l(Y))(\phi)\;\partial_{\phi^i}W_0^l(\phi;Y+\Gamma_{<n}W_n(Y))\,d\phi\\ &=\; \int Y^l(\phi)\,\partial_{\phi^i}W_n^l(\phi;Y)\,d\phi\;+\;\int (\Gamma_{<n}W_n(Y))^l(\phi)\;\partial_{\phi^i}W_n^l(\phi;Y)\,d\phi.\end{aligned}$$

The last integral can be written as $\int \Gamma_{<n}^{ll'}(\phi-\phi')\,W_n^{l'}(\phi',Y)\,\partial_{\phi^i}W_n^l(\phi;Y)\,d\phi\,d\phi'$ and two integrations by parts and the symmetry of $\Gamma_{<n}$ show that it is equal to its opposite, hence that it vanishes.

To derive the consequences of the Ward identity (5.2) used later, evaluate it first at $y=0$:

Lemma 1. $$w_n^i(0;y)|_{y=0} \;=\; 0 \qquad \text{for all}\quad n \quad \text{and} \quad i\leq d. \tag{5.3}$$

The next identities involve Dw_n. Let us first introduce smooth interpolations of the diagonal parts of the kernels of the derivatives $Dw_{n\beta}$ of the maps $w_{n\beta}$ constructed in Proposition 2 for $|\lambda|<\lambda_n$. These derivatives are given by the formula

$$Dw_{n\beta}(y) = [1-Dw_{0\beta}(y_n)\,\Gamma_{<n}]^{-1}\,Dw_{0\beta}(y_n) \tag{5.4}$$

with $y_n\equiv y+\Gamma_{<n}w_{n\beta}(y)$. This is obtained from the y-derivative of Eq. (4.2). We shall now proceed to show that the kernel $Dw_{n\beta}(q,q';y)$ on the diagonal $q=q'$ depends on q only through $\omega\cdot q$. For this purpose, let us introduce the continuous automorphism $t_p:\mathcal{L}(h;h)\to\mathcal{L}(h;h)$, for $p\in\mathbf{Z}^d$, shifting both arguments of the kernel of an operator a by p:

$$(t_p a)(q,q') = a(q+p,q'+p). \tag{5.5}$$

For $n=0$, $t_pDw_0=Dw_{0\beta}$, as follows from the explicit form (3.3) of the Taylor coefficients of w_0. Applying t_p to Eq. (5.4), we obtain the relation

$$t_pDw_{n\beta}(y) = [1-Dw_{0\beta}(y_n)\,t_p\Gamma_{<n}]^{-1}\,Dw_{0\beta}(y_n). \tag{5.6}$$

Note that $t_p\Gamma_{<n} = \Gamma_{<n}(\omega\cdot p)$, see the definition (4.11). It follows that $t_p Dw_{n\beta}$ depends on p only through the scalar product $\omega\cdot p$. Denote $Dw_{0\beta}(y) = \pi_{0\beta}(y)$ and define for $n\geq 1$ and $|\kappa| < \eta^n B$,

$$\pi_{n\beta}(\kappa; y) \;=\; [1 - \;\pi_{0\beta}(y_n)\,\Gamma_{<n}(\kappa)]^{-1}\pi_{0\beta}(y_n). \tag{5.7}$$

Since, by the inequalities (4.27), $\|y_n\| \leq \frac{1}{2}r_0$ for $y\in B(r^n)\subset h$, Proposition 1 and the Cauchy estimate (3.7) imply that $\|\pi_{0\beta}(y_n)\|_{\mathcal{L}(h;h)} \leq \frac{2}{r_0}D|\lambda|$. It follows then easily that $\pi_{n\beta}(\kappa; y)$ is analytic for $|\lambda| < \lambda_n(r)$, $|\mathrm{Im}\beta| < \alpha$, $|\kappa| < \eta^n B$ and $y\in B(r^n)\subset h$ with the norm arbitrarily small if $\lambda\to 0$, e.g.

$$\|\pi_{n\beta}(\kappa; y)\|_{\mathcal{L}(h;h)} \leq\; |\lambda|^{1/2}. \tag{5.8}$$

Comparing Eqs. (5.6) and (5.7), we infer that

$$t_p Dw_{n\beta}(y) = \pi_{n\beta}(\omega\cdot p; y) \quad \text{ and } \quad t_p\pi_{n\beta}(\kappa) = \pi_{n\beta}(\kappa + \omega\cdot p), \tag{5.9}$$

whenever defined, i.e. that $\pi_{n\beta}(\kappa)$ is a smooth interpolation of $t_p Dw_{n\beta}$. Note an explicit expression, which we shall need later, for the κ-derivative of $\pi_{n\beta}(\kappa)$ obtained by differentiating Eq. (5.7):

$$\partial_\kappa\pi_{n\beta}(\kappa; y) \;=\; \pi_{n\beta}(\kappa; y)\,\partial_\kappa\Gamma_{<n}(\kappa)\,\pi_{n\beta}(\kappa; y). \tag{5.10}$$

As is easy to check, the maps π_n satisfy for $\|y\| < r^n$ and $|\kappa| < \eta^n B$ the recursion relation

$$\pi_{n\beta}(\kappa; y) \;=\; [1 - \;\pi_{(n-1)\beta}(\kappa;\tilde{y})\,\Gamma_{n-1}(\kappa)]^{-1}\,\pi_{(n-1)\beta}(\kappa;\tilde{y}), \tag{5.11}$$

where we have denoted $\tilde{y}\equiv y + \Gamma_{n-1}w_{n\beta}(y)$.

The second consequence of the Ward identity is

Lemma 2.

$$\pi^{ij}_n(0,0;\kappa;y)|_{\substack{\kappa=0\\ y=0}} \;=\; 0 \quad \text{for } \; i \text{ or } j\leq d \tag{5.12}$$

$$\partial^\ell_\kappa\pi^{ij}_n(0,0;\kappa;y)|_{\substack{\kappa=0\\ y=0}} \;=\; 0 \quad \text{for } \; i \text{ and } j\leq d \quad \text{and } \ell < 2. \tag{5.13}$$

Remark. Eqs. (5.12), (5.13) and (5.8) imply that $\pi_n(0,0;\kappa;y)|_{y=0} = \left(\begin{smallmatrix}\mathcal{O}(\kappa^2) & \mathcal{O}(\kappa)\\ \mathcal{O}(\kappa) & \mathcal{O}(1)\end{smallmatrix}\right)$, the fact which will be used in an essential way in the next section.

Proof. First, evaluating the derivative of Eq. (5.2) w.r.t. y at $y=0$, we obtain:

$$Dw^{il}_n(0,q;0) \;=\; iq^i\,w^l_n(-q;0) \qquad \text{for } \; i\leq d. \tag{5.14}$$

The kernels of Dw_n possess the symmetry property

$$Dw^{ll'}_n(q,q';y) \;=\; Dw^{l'l}_n(-q',-q;y) \tag{5.15}$$

which follows by Eq. (5.4) from the similar property of $Dw^{ll'}_0(q,q';y)$. The latter comes from the fact that Dw_0 is the symmetric second derivative of the functional $\lambda\int U((\phi,\zeta)+$

$Y(\phi))d\phi$ and Eq. (5.15) encodes, at least formally, the analogous property of W_n, see Sect. 8. More generally,

$$\pi_n^{ll'}(q,q';\kappa;y) \;=\; \pi_n^{l'l}(-q',-q;-\kappa;y), \tag{5.16}$$

where we denote the kernels of the maps $\pi_n(\kappa;y)$ by $\pi_n(q,q';\kappa;y)$. Setting $q=0$ in Eq. (5.14) and using the symmetry (5.15), we obtain the first claim

$$Dw_n^{ij}(0,0;y)|_{y=0} \;=\; \pi_n^{ij}(0,0;\kappa;y)|_{\substack{\kappa=0\\y=0}} \;=\; 0 \quad \text{for } \; i \text{ or } j \le d. \tag{5.17}$$

For the second claim, we use the relation (5.10) for $\partial_\kappa \pi_n$ which, written in terms of the kernels $\pi_n(q,q';\kappa;y)$, yields:

$$\partial_\kappa \pi_n(q,q';\kappa;y)|_{\substack{\kappa=0\\y=0}} \;=\; \sum_{q''} Dw_n(q,q'';0)\,\partial_\kappa\gamma_{<n}(\omega\cdot q'')\,Dw_n(q'',q';0).$$

In particular, for $q=q'=0$,

$$\partial_\kappa \pi_n^{ij}(0,0;\kappa;y)|_{\substack{\kappa=0\\y=0}} \;=\; \sum_{q} Dw_n^{il}(0,q;0)\,\partial_\kappa\gamma_{<n}^{ll'}(\omega\cdot q)\,Dw_n^{jl'}(0,-q;0),$$

where we have also used the symmetry (5.15). Finally, for $i,j\le d$, substituting the Ward identity (5.14), we obtain the relation

$$\partial_\kappa \pi_n^{ij}(0,0;\kappa;y)|_{\substack{\kappa=0\\y=0}} \;=\; \sum_{q} q^i q^j\; w_n^l(-q;0)\,\partial_\kappa\gamma_{<n}(\omega\cdot q)^{ll'}\,w_n^{l'}(q;0). \tag{5.18}$$

The right hand side of the last expression is symmetric in indices i,j but, by the symmetry (5.16), the left hand side is antisymmetric, hence zero. We can also see this more directly since, as follows from Eq. (4.10), $\partial_\kappa\gamma_{<n}(\kappa) = \left(\begin{smallmatrix} a(\kappa) & b(\kappa)\\ -b(\kappa) & 0\end{smallmatrix}\right)$ with a odd and b even. Thus the expression summed on the right hand side of Eq. (5.18) is odd in q and the sum vanishes. $\Box$

6 Proof of Proposition 3

The proof contains two different parts. The inductive proofs of parts (a) and (b) are straightforward applications of the Banach Fixed Point Theorem and the contraction in n follows easily by combining analyticity with the Diophantine condition. The proof of part (c) may also be divided into two parts. First, one controls $Dw_n(y)-Dw_n(0)$ and the off-diagonal elements of $Dw_n(0)(q,q')$. This is also straightforward and similar to the proofs in parts (a) and (b). For $Dw_n(0)(q,q)$, we use Lemma 2 above.

For $n\le n_0$ with fixed n_0, the bounds (4.22) and (4.23) as well as (4.24) follow immediately from the estimate (4.13) and the inequalities $\|\cdot\|\le\|\cdot\|_n$, $\|\cdot\|_{-n}\le\|\cdot\|$ by taking λ small enough. For $n>n_0$ with n_0 large enough, we shall proceed inductively. It will be convenient to modify slightly the simplified notations of the text of Proposition 3 and so, below, w will stand for $w_{(n-1)\beta}$ and w' for $w_{n\beta}$. Finally, Γ will stand for Γ_{n-1}.

Proof of (a). Consider the recursive equation (4.1) for w' and use the decomposition (4.21) to rewrite it as

$$w'(y) = w(0) + Dw(0)(y + \Gamma w'(y)) + \delta_2 w(y + \Gamma w'(y))$$

from which we deduce that

$$w'(y) = Hw(0) + HDw(0)y + u(y), \tag{6.1}$$

where

$$u(y) \equiv H\delta_2 w(y + \Gamma w'(y)) = H\delta_2 w(\Gamma Hw(0) + \tilde{H}y + \Gamma u(y)) \tag{6.2}$$

with $H = (1 - Dw(0)\Gamma)^{-1}$ and $\tilde{H} = 1 + \Gamma HDw(0) = (1 - \Gamma Dw(0))^{-1}$.

In the inductive step, first we assume that w satisfies the bounds (4.22), (4.23) and (4.24) with n replaced by $n-1$. The bounds (4.24), (4.19) and (4.18) imply then that the operators H and $\tilde{H}$ are well defined with

$$\|H\|_{-n+1;-n+1}\ ,\ \|\tilde{H}\|_{n-1;n-1}\ \leq\ 1 + C\epsilon\ \leq\ 2, \tag{6.3}$$

for ϵ (i.e. $|\lambda|$) small enough.

We solve Eq. (6.2) using the Banach Fixed Point Theorem. Given its solution u, the existence of w' satisfying Eq. (6.1) follows. To solve Eq. (6.2), we consider the map $\mathcal{G}$ defined by

$$\mathcal{G}(u)(y) = H\delta_2 w(\tilde{y}) \qquad \text{with} \qquad \tilde{y} = \Gamma Hw(0) + \tilde{H}y + \Gamma u(y). \tag{6.4}$$

We claim that $\mathcal{G}$ is a contraction in the ball

$$\mathcal{B} = \{u \in H^\infty(B_\delta, h_{-n+1}) \mid |||u||| \leq 2\epsilon r^{\frac{3}{2}(n-1)}\}, \tag{6.5}$$

where $B_\delta \subset h_{n-1}$ is the open ball of radius $r^{n-\delta}$ for $0 \leq \delta < 1$ and $r < r(\delta)$. Indeed, for $\|y\|_{n-1} \leq r^{n-\delta}$, the inequalities (4.19), (6.3) and (4.22) combined with the identity $\Gamma Hw(0) = \Gamma HPw(0)$ ($HP = H$ follows from $\Gamma P = \Gamma$ and the definition of H) and the inequalities (4.18) imply that for $u \in \mathcal{B}$,

$$\|\tilde{y}\|_{n-1} \leq 2C\eta^{-2n}\epsilon r^{2(n-1)} + 2r^{n-\delta} + 2C\eta^{-2n}\epsilon r^{\frac{3}{2}(n-1)} \leq \tfrac{1}{2}r^{n-1} \tag{6.6}$$

if r is small enough. Thus $\tilde{y} \in B_{n-1}$, the domain of definition of w and hence of $\delta_2 w$. It follows that $\mathcal{G}(u)$ for u in the ball (6.5) may be considered as an analytic map of $y \in B_\delta$ with values in h_{-n+1}. Moreover

$$\|\mathcal{G}(u)(y)\|_{-n+1}\ \leq\ 2|||\delta_2 w|||\ \leq\ 2\epsilon r^{\frac{3}{2}(n-1)}, \tag{6.7}$$

where we used the bounds (4.23), (4.18) and (6.3). Hence $\mathcal{G} : \mathcal{B} \to \mathcal{B}$.

To prove that $\mathcal{G}$ is a contraction, we use the estimate (3.8) for $\tilde{y}_i(y) = \Gamma H w(0) + \tilde{H} y + \Gamma u_i(y)$ and $u_i \in \mathcal{B}$, $i = 1, 2$. By inequality (6.6), $\|\tilde{y}_i\|_{-n+1} \le \frac{1}{2} r^{n-1}$ and so the bounds (3.8), (4.23), (4.18) and (6.3) imply that

$$
\begin{aligned}
|||\mathcal{G}(u_1) - \mathcal{G}(u_2)||| &= \sup_{y \in B_\delta} \|H \delta_2 w(\tilde{y}_1) - H \delta_2 w(\tilde{y}_2)\|_{-n+1} \\
&\le 4 r^{-n+1} |||\delta_2 w||| \sup_{y \in B_\delta} \|\tilde{y}_1 - \tilde{y}_2\|_{n-1} \le 4 \epsilon\, r^{\frac{1}{2}(n-1)} \sup_{y \in B_\delta} \|\tilde{y}_1 - \tilde{y}_2\|_{n-1} \\
&\le 4 \epsilon\, r^{\frac{1}{2}(n-1)} C \eta^{-2n}\, |||u_1 - u_2||| \le \tfrac{1}{2} |||u_1 - u_2|||
\end{aligned}
$$

for r and ϵ small proving the contractive property of $\mathcal{G}$ on $\mathcal{B}$. Hence the existence of the fixed point $u \in \mathcal{B}$ of $\mathcal{G}$ solving the equation (6.2) and thus of $w' : B_\delta \to h_{-n+1}$ given by Eq. (6.1) follows. Using the natural embeddings (4.17), we may consider B_n as a subset of B_δ, and w' may be regarded as an element of the space $\mathcal{A}_n$ of analytic functions from $B_n \subset h_n$ to h_{-n}. Note also that, since $\tilde{y} = y + \Gamma w'(y)$ (see the argument of $\delta_2 w$ in Eq. (6.2)), the inequality (6.6) may be rewritten as

$$
\|y + \Gamma w'(y)\|_{n-1} \le \tfrac{1}{2} r^{n-1} \quad \text{for} \quad y \in B_\delta \tag{6.8}
$$

which implies that $y + \Gamma w'(y) \in B_{n-1}$ for such y.

It is easy to see inductively that for $|\lambda|$ small enough, the maps $w_{n\beta}$ constructed in Proposition 2 give rise via the decomposition (6.1) to u's which solve the fixed point equation (6.2) and belong to $\mathcal{B}$. It follows that w' constructed above coincide for small $|\lambda|$, and hence for all λ in the common domain of definition, with $w_{n\beta}$ of Proposition 2.

Proof of (b). Using the decomposition (6.1), we may write

$$
w'(y) = w'(0) + D w'(0) y + \delta_2 w'(y), \tag{6.9}
$$

with

$$
w'(0) = H w(0) + u(0), \qquad D w'(0) = H D w(0) + D u(0) \tag{6.10}
$$

and $\delta_2 w' = \delta_2 u$. Let us first iterate the bound (4.22). Note that

$$
P w'(0) = P H P w(0) + P u(0)
$$

since $H = HP$. As $u \in \mathcal{B}$, see the definition (6.5), we have,

$$
\|P u(0)\|_{-n+1} \le \|u(0)\|_{-n+1} \le 2 \epsilon r^{\frac{3}{2}(n-1)}, \tag{6.11}
$$

which, with the use of the estimates (6.3) and (4.22) to bound $\|H P w(0)\|_{-n+1}$, implies that

$$
\|P w'(0)\|_{-n} \le \|P w'(0)\|_{-n+1} \le 2 \epsilon\, r^{2(n-1)} + 2 \epsilon\, r^{\frac{3}{2}(n-1)}. \tag{6.12}
$$

This seems weaker than what is needed to iterate the estimate (4.22) but, as we shall see, one actually needs much less. The crucial point[4] is that the bound (6.12)

[4] This is the first of the two places where we gain by working with $w_{n\beta}$ for complex β.

holds for $|\mathrm{Im}\beta| < \alpha_{n-1}$, while we have to establish the estimate (4.22) for w' only for $|\mathrm{Im}\beta| < \alpha_n = (1-n^{-2})\alpha_{n-1}$. For such β, we infer, using the estimate (6.12), that, for $q \neq 0$,

$$|w'(q;0)|_{-n+1}\, e^{(\alpha_{n-1}-\alpha_n)|q|}\ \mathrm{e}^{-\eta^{-n+1}|\omega\cdot q|}\ \leq\ \|Pw'(0)\|_{-n+1}\ \leq\ \epsilon \tag{6.13}$$

if in the middle term we take w' corresponding to the value of β shifted to $\beta' = \beta - i\frac{(\alpha_{n-1}-\alpha_n)}{|q|}q$. It follows that

$$\|Pw'(q;0)\|_{-n}\ \leq\ \epsilon \sum_{q\neq 0} \mathrm{e}^{-n^{-2}\alpha_{n-1}|q|}\ \mathrm{e}^{-(1-\eta)\eta^{-n}|\omega\cdot q|}. \tag{6.14}$$

Since $\alpha_{n-1} \geq \prod\limits_{n=2}^{\infty}(1-n^{-2})\,\alpha > 0$, the sum on the right hand side is clearly bounded by Cn^2. However, we may extract from the sum factors that are super-exponentially small in n. Indeed, for $|\omega\cdot q| \leq \eta^{\frac{1}{2}n}$, we may extract from the first exponential under the sum a factor $\mathrm{e}^{-\mathcal{O}(\eta^{-\frac{n}{2\nu}}n^{-2})}$ due to the Diophantine condition (1.6). On the other hand, for $|\omega\cdot q| > \eta^{\frac{1}{2}n}$, we may extract a factor $\mathrm{e}^{-\mathcal{O}(\eta^{-\frac{1}{2}n})}$ from the second exponential. Hence the inductive bound (4.22) follows for $n \geq n_0$, and n_0 large enough.

Let us now iterate the relation (4.23) for $\delta_2 w'$ equal to $\delta_2 u$ (see Eq. (6.1)). Recall that $\|u(y)\|_{-n+1} \leq 2\epsilon r^{\frac{3}{2}(n-1)}$ for $\|y\|_{n-1} < r^{n-\delta}$, see the definition (6.5). The estimate (3.9) with $k=2$ and $\gamma = r^\delta$ and the bounds (4.18) imply then that, for $\|y\|_n < r^n$,

$$\|\delta_2 w'(y)\|_{-n}\ \leq\ \tfrac{2r^{2\delta-\frac{3}{2}}}{1-r^\delta}\,\epsilon\, r^{\frac{3}{2}n}. \tag{6.15}$$

Taking $\delta > \frac{3}{4}$ and $r < r(\delta)$, we infer that $\|\delta_2 w'(y)\|_{-n}\ \leq\ \epsilon\, r^{\frac{3}{2}n}$. This completes the inductive proof of (b).

Proof of (c). We shall use the maps $\pi_{n\beta}$ introduced in Sect. 5, and related to Dw_n by Eqs. (5.9). Recall that $\pi_{n\beta}$ was constructed as a map from $B(r^n) \subset h$ to $\mathcal{L}(h,h)$, for $|\lambda| < \lambda_n$. With the use of the embeddings (4.17), they may be viewed as maps from $B_n \subset h_n$ to $\mathcal{L}(h_n;h_{-n})$. As we shall see, they can be extended to $|\lambda| < \lambda_0$. Let us split $\pi_{n\beta}(\kappa;0)$ into the diagonal and the off-diagonal parts:

$$\pi_{n\beta}(\kappa;0)\ =\ \sigma_n(\kappa) + \rho_n(\kappa), \tag{6.16}$$

where $\sigma_n(q,q';\kappa) = \pi_{n\beta}(q,q;\kappa;0)\,\delta_{q,q'}$. Let us denote by D_n the disc $\{\kappa \in \mathbf{C} \mid |\kappa| < \eta^n B\}$.

Lemma 3. *The maps $\pi \equiv \pi_{n\beta} : D_n \times B_n \to \mathcal{L}(h_n;h_{-n})$ extend analytically to $|\lambda| < \lambda_0$. Their extensions satisfy the relations*

$$t_p Dw(y)\ =\ \pi(\omega\cdot p;\, y) \qquad \textit{and} \qquad t_p\pi(\kappa;y)\ =\ \pi(\kappa+\omega\cdot p;\, y), \tag{6.17}$$

whenever defined, and depend analytically on κ, y, β and ζ. They obey the bounds:

$$\|\delta_1\pi(\kappa;y)\|_{n;-n}\ \leq\ \epsilon\, r^{\frac{1}{2}n}, \tag{6.18}$$

$$\|\sigma(\kappa)\|_{n;-n}\ \leq\ \tfrac{1}{2}\epsilon\,\eta^{2n}, \tag{6.19}$$

$$\|\rho(\kappa)\|_{n;-n}\ \leq\ \epsilon\, r^{\frac{1}{2}n}, \tag{6.20}$$

where $\delta_1\pi(\kappa;y) \equiv \pi(\kappa;y) - \pi(\kappa;0)$.

Obviously, the bound (4.24) follows by combining Eqs. (6.17) for $p=0$, (6.18), (6.19) and (6.20). In particular, we may use estimate (4.24) as an inductive hypothesis in the proof of Lemma 3. Hence, we are left with

Proof of Lemma 3. We use the same notation as in the proof of Proposition 3. Let us show that we may use recursively the relations (5.11) to construct for $|\lambda| < \lambda_0$ the maps $\pi_{n\beta}$ satisfying the identities (6.17). First, differentiating Eq. (4.1), we obtain the relation

$$Dw'(y) \;=\; [1 - Dw(\tilde{y})\Gamma]^{-1} Dw(\tilde{y}), \tag{6.21}$$

where $\tilde{y} = y + \Gamma w'(y)$. The right hand side is well defined for $y \in B_\delta \subset h_{n-1}$, since, by inequality (6.8), $\tilde{y} \in B_{n-1}$ for such y's. The bound (4.19) and the inductive hypothesis (4.24) imply that $\|Dw(\tilde{y})\Gamma\|_{-n+1;-n+1} \leq C\epsilon$. Denoting $\pi \equiv \pi_{(n-1)\beta}$ we then define $\pi' \equiv \pi_{n\beta}$ by relation (5.11), i.e. by

$$\pi'(\kappa;y) \;=\; [1 - \;\pi(\kappa;\tilde{y})\,\Gamma(\kappa)]^{-1}\pi(\kappa;\tilde{y}). \tag{6.22}$$

The relations (6.17) for π' may be inferred by applying the automorphism (5.5) to Eqs. (6.21) and (6.22). The inductive hypotheses imply that, for $\kappa \in D_{n-1}$ and $y \in B_\delta$, $\pi'(\kappa;y)$ is defined in $\mathcal{L}(h_{n-1};h_{-n+1})$ and is an analytic function of its arguments. It follows by induction that it coincides for $|\lambda| < \lambda_n$ with the map $\pi_{n\beta}$ constructed before, see Eq. (5.7), also satisfying the recursion (5.11). Note that

$$\pi'(\kappa;0) \;=\; [1 - \;\pi(\kappa;\tilde{y}_0)\,\Gamma(\kappa)]^{-1}\pi(\kappa;\tilde{y}_0), \tag{6.23}$$

where $\tilde{y}_0 = \Gamma w'(0)$. The bounds

$$\|\pi'(\kappa;y)\|_{n-1;-n+1} \quad \leq \; \epsilon\,\eta^{2(n-1)}, \tag{6.24}$$

$$\|\delta_1\pi'(\kappa;y)\|_{n-1;-n+1} \;\leq\; 3\,\epsilon\, r^{\frac{1}{2}(n-1)} \tag{6.25}$$

follow easily: the stronger factor $r^{\frac{1}{2}(n-1)}$ instead of the weaker one $\eta^{2(n-1)}$ in the estimate of $\delta_1\pi'$ is supplied by the inductive bound (6.18) for the difference $\pi(\kappa;\tilde{y}) - \pi(\kappa;\tilde{y}_0) = \delta_1\pi(\kappa;\tilde{y}) - \delta_1\pi(\kappa;\tilde{y}_0)$. The inequality (6.25) is enough to iterate the bound (6.18). Indeed, the estimate (3.9) with $k=1$ and $\gamma = r^\delta$ permits to extract the additional factor $\frac{r^\delta}{1-r^\delta}$ and to obtain the improved bound (6.18) (if $\delta > \frac{1}{2}$ and $r < r(\delta)$) for $\|\delta_1\pi'(\kappa;y)\|_{n;-n}$ for y restricted to $B_n \subset h_n$.

Let us turn to the inductive proof of the bound (6.20) for $\|\rho'(\kappa)\|_{n;-n}$. We gain a very small factor from the restriction of the analyticity strip in β, as before in the control of $Pw'(0)$ in the proof of (b), see the inequalities (6.13). First, by the definition (4.16) of the norms, we have the following estimate of the kernel $\rho'(q,q';\kappa)$:

$$|\rho'(q,q';\kappa)|_{n-1;-n+1}\, \mathrm{e}^{(\alpha_{n-1}-\alpha_n)|q-q'|}\, \mathrm{e}^{-\eta^{-n+1}(|\omega\cdot q|+|\omega\cdot q'|)} \leq \|\pi'(\kappa;0)\|_{n-1;-n+1} \leq \epsilon,$$

where in the middle term we have shifted β to $\beta' = \beta - i\frac{(\alpha_{n-1}-\alpha_n)}{|q-q'|}(q-q')$. The last inequality follows from the bound (6.24). Moreover, for an operator with kernel $a(q,q')$,

$\|a\|_{n;-n} \leq \sup_{q'} \sum_q |a(q,q')|_{n;-n} \, \mathrm{e}^{-\eta^{-n}(|\omega\cdot q|+|\omega\cdot q'|)}$. This implies the estimate

$$\|\rho'(\kappa)\|_{n;-n} \leq \epsilon \sup_{q'} \sum_{q\neq q'} \mathrm{e}^{-n^{-2}\alpha_{n-1}|q-q'|} \, \mathrm{e}^{-(1-\eta)\eta^{-n}(|\omega\cdot q|+|\omega\cdot q'|)}$$

The expression on the right hand side may be bounded by a factor super-exponentially small in n, which is much more than is needed. Indeed, we may extract from it an extra factor $\mathrm{e}^{-\mathcal{O}(\eta^{-\frac{n}{2\nu}}n^{-2})}$ for $|\omega\cdot q|$ and $|\omega\cdot q'| \leq \eta^{\frac{1}{2}n}$ and hence $0 \neq |\omega\cdot(q-q')| \leq 2\eta^{\frac{1}{2}n}$ using the Diophantine condition (1.6) and an extra factor $\mathrm{e}^{-\mathcal{O}(\eta^{-\frac{1}{2}n})}$ if $|\omega\cdot q| > \eta^{\frac{1}{2}n}$ or $|\omega\cdot q'| > \eta^{\frac{1}{2}n}$. Hence the bound (6.20) for the off-diagonal operator $\rho'(\kappa)$.

We are left with the proof of the estimate (6.19) for the diagonal operator $\sigma'(\kappa)$. Let us define

$$s(z) \;=\; u^{n-1}\sigma(0,0;B\eta^{n-1}z)\,u^{n-1}, \tag{6.26}$$

where $u = \left(\begin{smallmatrix}1 & 0\\ 0 & \eta\end{smallmatrix}\right)$ is a block matrix. Similarly, we introduce the matrix $s'(z)$ related to σ'. With the use of symmetry (5.16), we write $s'(z) = \eta^{2(n+1)}\left(\begin{smallmatrix}\wp_0'(z) & \wp_1'(z)\\ \wp_1'(-z) & \wp_2'(z)\end{smallmatrix}\right)$. We shall prove that, for $|z|<1$,

$$|\wp_i'(z) - p_i'\,z^{2-i}| \;\leq\; A|z|^{3-i} \tag{6.27}$$

with $|p_i'| \leq (1-\frac{1}{n})\frac{\epsilon}{32}$ and $A \leq \frac{\epsilon}{32}$, assuming inductively similar bounds for $s(z)$. Note that such inductive assumptions, together with the identity $|M|_{n-1;-n+1} = |u^{n-1}Mu^{n-1}|_{0;0}$ for the matrix norms imply, in particular, the estimate

$$|\sigma(0,0,\kappa)|_{n-1;-n+1} \leq \tfrac{1}{8}\epsilon\,\eta^{2n} \tag{6.28}$$

for $\kappa\in D_{n-1}$. The bound (6.27) will follow from Lemma 2 expressing the cancellations of resonances. The leading Taylor coefficients p_i of $\wp_i(z)$ are *marginal* in the RG terminology and the higher ones are *irrelevant*. The presence of lower order *relevant* Taylor coefficients would spoil the iterative bounds. They are, however, forbidden by the Ward identities. Let us pass to the details.

Let us first prove the estimate (6.19) for σ' assuming the bound (6.28). We shall split

$$\sigma'(\kappa) \;=\; \sigma_0'(\kappa) + \sigma_1'(\kappa) \qquad \text{with} \qquad \sigma_0'(\kappa) \;=\; [1-\,\sigma(\kappa)\Gamma(\kappa)]^{-1}\sigma(\kappa), \tag{6.29}$$

compare with Eq. (6.23). Note that $\sigma(\kappa)\Gamma(\kappa)$ is an operator diagonal in Fourier space and hence, so is $\sigma_0'(\kappa)$. Since, by the inductive hypotheses (6.20) and (6.18),

$$\|\sigma(\kappa)-\pi(\kappa;\tilde{y}_0)\|_{n-1;-n+1} \;=\; \|\rho(\kappa)+\delta_1\pi(\kappa;\tilde{y}_0)\|_{n-1;-n+1} \;\leq\; 2\epsilon\,r^{\frac{1}{2}(n-1)}, \tag{6.30}$$

it follows that

$$\|\sigma_1'(\kappa)\|_{n;-n} \;\leq\; \tfrac{1}{4}\epsilon\,\eta^{2n}, \tag{6.31}$$

for r small enough. We pass to the estimation of $\sigma'_0(\kappa)$. Note that the bound (4.19) together with the inequalities (4.18) and the inductive hypothesis (6.19) imply that $\|\Gamma(\kappa)\sigma(\kappa)\|_{-n;-n} \leq \frac{1}{2}C\epsilon$ so that

$$\|\sigma'_0(\kappa)\|_{n;-n} \leq 2\|\sigma(\kappa)\|_{n;-n}. \tag{6.32}$$

For operators a diagonal in Fourier transform, $\|a\|_{n;-n} = \sup_q |a(q,q)|_{n;-n}\, \mathrm{e}^{-2\eta^{-n}|\omega\cdot q|}$. Hence it follows from the bound (6.32) that

$$\|\sigma'_0(\kappa)\|_{n;-n} \leq \sup_q\ 2\,|\sigma(q,q;\kappa)|_{n;-n}\, \mathrm{e}^{-2\eta^{-n}|\omega\cdot q|}. \tag{6.33}$$

For q with $|\omega\cdot q| < (1-\eta)\eta^{n-1}B$, we use for $\kappa \in D_n$ the equality $\sigma(q,q;\kappa) = \sigma(0,0;\tilde\kappa)$ with $\tilde\kappa = \kappa + \omega\cdot q$ which follows from the second identity (6.17) (observe that for such q's and for $\kappa\in D_n$, $\tilde\kappa \in D_{n-1}$). By virtue of the inequality (6.28),

$$|\sigma(0,0;\tilde\kappa)|_{n;-n} \leq |\sigma(0,0;\tilde\kappa)|_{n-1;-n+1} \leq \tfrac{1}{8}\,\epsilon\,\eta^{2n}. \tag{6.34}$$

Hence, for q with $|\omega\cdot q| < (1-\eta)\eta^{n-1}B$, we may bound the expression on the right hand side in the estimate (6.33) by $\frac{1}{4}\epsilon\eta^{2n}$. For $|\omega\cdot q| \geq (1-\eta)\eta^{n-1}B$, we instead extract an extra factor estimating

$$\begin{aligned} 2\,|\sigma(q,q;\kappa)|_{n;-n}\, \mathrm{e}^{-2\eta^{-n}|\omega\cdot q|} &\leq 2\,\|\sigma(\kappa)\|_{n-1;-n+1}\, \mathrm{e}^{-2\eta^{-1}(1-\eta)^2 B} \\ &\leq \mathrm{e}^{-2\eta^{-1}(1-\eta)^2 B}\,\epsilon\eta^{2(n-1)} \leq \tfrac{1}{4}\epsilon\eta^{2n} \end{aligned} \tag{6.35}$$

for B sufficiently large (this is the only place where B large is needed). Putting these estimates together with the inequality (6.31) for σ'_1, we infer the bound (6.19) for $\sigma'(\kappa)$.

We still have to iterate the crucial estimates (6.27) which is the only place in the proof of Proposition 3 where we use the Ward identities. Writing Eqs. (6.29) in terms of s, see the definition (6.26), we obtain

$$s'(z) = [1 - (\mathcal{L}s)(z)\tilde\gamma(z)]^{-1}(\mathcal{L}s)(z) + s'_1(z)$$

with the "linearized RG map" $\mathcal{L}$,

$$(\mathcal{L}s)(z) = u\,s(\eta z)\,u,$$

and $\tilde\gamma(z) = u^{-n}\gamma_{n-1}(B\eta^n z)u^{-n} = \eta^{-2n}(Bz)^{-2}\chi_1(B\eta^2 z)\left(\begin{smallmatrix}\mu & iBz\\ -iBz & 0\end{smallmatrix}\right)$, see Eqs. (4.6) and (4.10). The estimate (6.30) implies that the remainder s'_1 satisfies the bound $|s'_1(z)|_{0,0} \leq C\epsilon\, r^{\frac{1}{2}n}$. Combining the definition (4.5) (which implies that $\chi_1(z)$ is of order $|z|^6$ for small z) and the inductive bound for s, we infer that $|(\mathcal{L}s)(z)\tilde\gamma(z)|_{0;0} \leq C\epsilon|z|^4$. Thus altogether

$$|s'(z) - (\mathcal{L}s)(z)|_{0;0} \leq C\epsilon^2\eta^{2n}|z|^4 + C\epsilon\, r^{\frac{1}{2}n}.$$

The map $\mathcal{L}$ preserves p_i and contracts the constant A to ηA. The Ward identity, Lemma 2, implies that $\partial^j\wp'_i(0) = 0$ for $j < 2-i$. Since $|p'_i - p_i| \leq C\epsilon\,\eta^{-2n}r^{\frac{1}{2}n} \leq \frac{\epsilon}{32n^2}$ and $\eta A + C\epsilon^2 + C\epsilon\eta^{-2n}r^{\frac{1}{2}n} \leq A$ for r and ϵ small, we infer that s' satisfies the bound (6.27). This finishes the proof of Lemma 3 and of Proposition 3. □

7 Proof of Theorem 1

We shall first show that $X_n \equiv F_n(0) = \Gamma_{<n} W_n(0)$ converges to a real analytic function X with zero average as $n \to \infty$ and that X solves Eq. (2.3).

Recall that in Proposition 2, we have constructed for $|\lambda| < \lambda_n$ the analytic maps $f_{n\beta}$ from $B(r^n) \subset h$ into h, satisfying the relations (4.4) and (4.3) and the bound $|||f_{n\beta}||| \leq 2r^n$. They may be also viewed as analytic maps from $B(r^n) \subset h_n$ to h. As such, they may be analytically extended to $|\lambda| < \lambda_0$ for $n \geq n_0$ by iterated use of Eq. (4.3) if we recall the bound (6.8). The extensions are clearly uniformly bounded (e.g. by $2r^{n_0}$). Let us prove the convergence in h of $x_{n\beta} \equiv f_{n\beta}(0)$ obtained this way. The recursion (4.3) implies that

$$x_{n\beta} \;=\; x_{(n-1)\beta} + \delta_1 f_{(n-1)\beta}(\Gamma_{n-1} w_{n\beta}(0)). \tag{7.1}$$

Using the estimate (3.9) for $k = 1$, we deduce from the inequalities (4.19) and (4.22) that the $\|\cdot\|$ norm of the 2^{nd} term on the right hand side of Eq. (7.1) is bounded by $C\epsilon\,\eta^{-2n} r^{2n}$. Hence the convergence

$$x_{n\beta} \;\to\; x_\beta \qquad \text{in} \quad h \tag{7.2}$$

together with the bound $\|x_\beta\| \leq C\epsilon$ uniform in the strip $|\mathrm{Im}\beta| < \alpha' = \alpha \prod_{n=2}^{\infty}(1 - n^{-2})$. The latter implies the pointwise estimate

$$|x(q)| \;\leq\; C\epsilon\, \mathrm{e}^{-\alpha' |q|} \tag{7.3}$$

and, consequently, the real analyticity of the Fourier transform X of x.

For $|\lambda| < \lambda_n$, Eqs. (4.4) and (4.2) imply that

$$x_n \equiv f_n(0) = \Gamma_{<n} w_0(x_n) \qquad \text{and} \qquad w_0(x_n) \;=\; w_n(0). \tag{7.4}$$

In particular, it follows from the first of these equations that $x_n(q)|_{q=0} = 0$ and from the second one and Lemma 1 (i.e. the Ward identities) that $w_0^i(q; x_n)|_{q=0} = 0$ for $i \leq d$. By analyticity in λ, these relations have to hold for $|\lambda| \leq \lambda_0$. Since w_0 is analytic, we can take the $n \to \infty$ limit of Eqs. (7.4), and infer that

$$x(0) = 0, \quad x(q) = G_0 w_0(q; x) \;\text{ for }\; q \neq 0, \;\text{ and }\; w_0(0; x) \;=\; (0, \xi). \tag{7.5}$$

The first 2 of these equations are the Fourier transformed version of Eq. (2.3). The solution x is an analytic function of λ for $|\lambda| < \lambda_0$ and it vanishes for $\lambda = 0$. Recall that ζ is a parameter in w_0, and thus x is also analytic in ζ for $|\zeta| < r_0$.

We still have to solve Eq. (2.4) for ζ. In view of the 3^{rd} of Eqs. (7.5) and Eq. (2.1) it reduces to the equalities

$$\mu\zeta + \lambda \int_{\mathbf{T}^d} \partial_I U((\phi,\zeta) + X(\phi))\, d\phi \;=\; 0 \quad \text{or} \quad \int_{\mathbf{T}^d} \partial_I U((\phi,\zeta) + X(\phi))\, d\phi \;=\; 0 \tag{7.6}$$

in the, respectively, non-isochronous and isochronous cases. We shall solve the above equations for ζ by the Implicit Function Theorem. Note that $\lambda = 0$ and $\zeta = 0$ satisfies the non-isochronous equation and that the ζ-derivative of its left hand side at these points is μ, which is assumed to be invertible. Similarly, $\lambda = 0$ and $\zeta = 0$ solves the isochronous equation (since X vanishes for $\lambda = 0$ and we have assumed that $\int \partial_I U(\phi, 0)\, d\phi = 0$) and the ζ-derivative of its left hand side at these points is $\int \partial_I^2 U(\phi, 0)\, d\phi$, which is assumed to be invertible. The existence of the local solution $\zeta(\lambda)$ analytic for $|\lambda| < \lambda_0$ with λ_0 small enough and vanishing at $\lambda = 0$ follows in the both cases. The resulting solution $Z = X + (0, \zeta)$ of Eq. (1.3) depends analytically on λ for $|\lambda| < \lambda_0$ and vanishes for $\lambda = 0$. Its uniqueness up to translations (1.5) follows from the fact that the equations (2.3) and (2.4) completely determine the coefficients of the Taylor expansion of their solution in powers of λ (i.e. the Lindstedt series discussed in Sect. 9). This ends the proof of Theorem 1. $\square$

8 Field theory interpretation

Although arising in a problem of classical mechanics of d degrees of freedom, Eq. (1.3) has a field theory interpretation, see [11, 12]. Let us consider the (action) functional $S(Z)$ of maps $Z = (\Theta, J) : \mathbf{T}^d \to \mathbf{R}^d \times \mathbf{R}^d$,

$$S(Z) \;=\; \int\limits_{\mathbf{T}^d} \Big[\tfrac{1}{2} J(\phi) \cdot \mu J(\phi) - J(\phi) \cdot (\omega \cdot \partial_\phi)\Theta(\phi) + \lambda U(\phi + \Theta(\phi), J(\phi)) \Big]\, d\phi. \tag{8.1}$$

Note the translation symmetry

$$S(\tau_\beta Z - (\beta, 0)) \;=\; S(Z), \tag{8.2}$$

where, as before, $(\tau_\beta Z)(\phi) = Z(\phi - \beta)$ for $\beta \in \mathbf{R}^d$. As is easily seen, our basic equation (1.3) coincides with the equation $\delta S(Z) = 0$ for the extrema of functional S. The map Z with $\int \Theta(\phi)\, d\phi = 0$ minimizing S may, in turn, be interpreted as the limit $\hbar \to 0$ of the formal functional integral

$$\Big\langle Z(\phi) \Big\rangle_\hbar \;=\; \frac{\int Z(\phi)\; \mathrm{e}^{\frac{i}{\hbar} S(Z)} DZ}{\int \mathrm{e}^{\frac{i}{\hbar} S(Z)}\; DZ,} \tag{8.3}$$

where DZ denotes the formal Lebesgue measure on $L^2_0(\mathbf{T}^d, \mathbf{R}^d)/\mathbf{Z}^d \times L^2(\mathbf{T}^d, \mathbf{R}^d)$ where L^2_0 is composed of maps with zero average. Indeed, in this limit, the integral should be localized at the minimum of the functional (8.1).

While Eq. (8.3) has a purely formal meaning, we may gain intuition from it by some further manipulations. First, write $Z = (0, \zeta) + Y$ with Y of zero average. Then

$$\Big\langle Z(\phi) \Big\rangle_\hbar = \frac{\int [(0, \zeta) + Y(\phi)]\; \mathrm{e}^{\frac{i}{\hbar} V_0(Y, \zeta)}\; d\mu_0(Y) d\mu(\zeta)}{\int \mathrm{e}^{\frac{i}{\hbar} V_0(Y, \zeta)}\; d\mu_0(Y) d\mu(\zeta)}, \tag{8.4}$$

where

$$V_0(Y, \zeta) = \lambda \int\limits_{\mathbf{T}^d} U((\phi, \zeta) + Y(\phi)) d\phi \tag{8.5}$$

and we have used the quadratic part of the action functional to define the oscillatory "measures"

$$d\mu(\zeta) \;=(\det \tfrac{\mu}{2\pi i\hbar})^{\frac{1}{2}}\, \mathrm{e}^{\frac{i}{2\hbar}\,\zeta\cdot\mu\zeta}\; d\zeta \tag{8.6}$$

and $d\mu_0$. The latter is formally given by

$$d\mu_0(Y) \;=\; \mathrm{e}^{-\frac{i}{2\hbar}\,(Y,G_0^{-1}Y)}\; DY \;\Big/\; \int \mathrm{e}^{-\frac{i}{2\hbar}\,(Y,G_0^{-1}Y)}\; DY\,, \tag{8.7}$$

i.e. it is the "Gaussian measure" with mean zero and covariance $i\hbar G_0$ on the space of maps Y with zero average.

In quantum field theory, the idea of the renormalization group (RG) is to calculate functional integrals inductively. Let us explain a concrete realization of this idea. Let

$$G_0 = G_1 + \Gamma_0$$

be a decomposition of the operator G_0 into a sum of two operators. We may then write the measure $d\mu_0$ as a product measure according to the formula:

$$\int F(Y)\; d\mu_0(Y) \;=\; \int F(Y+\tilde{Y})\; d\mu_1(Y)\; d\nu_0(\tilde{Y}), \tag{8.8}$$

where $d\mu_1$ is the measure given by Eq. (8.7) with G_0 replaced by G_1 and $d\nu_0$ is the Gaussian measure with mean zero and covariance $i\hbar\Gamma_0$, both on the space of maps with zero average. Using this identity, we may rewrite Eq. (8.4) as

$$\big\langle Z(\phi)\big\rangle_\hbar = \frac{\int\,[(0,\zeta)+F_1(Y,\zeta;\phi)]\;\mathrm{e}^{\frac{i}{\hbar}\,V_1(Y,\zeta)}\; d\mu_1(Y)\,d\mu(\zeta)}{\int \mathrm{e}^{\frac{i}{\hbar}\,V_1(Y,\zeta)}\; d\mu_1(Y)\,d\mu(\zeta)} \tag{8.9}$$

if we set

$$\mathrm{e}^{\frac{i}{\hbar}\,V_1(Y,\zeta)} \;=\; \int \mathrm{e}^{\frac{i}{\hbar}\,V_0(Y+\tilde{Y},\zeta)}\; d\nu_0(\tilde{Y})\,, \tag{8.10}$$

$$F_1(Y,\zeta;\phi) \;=\; \frac{\int\,[Y(\phi)+\tilde{Y}(\phi)]\;\mathrm{e}^{\frac{i}{\hbar}\,V_0(Y+\tilde{Y},\zeta)}\; d\nu_0(\tilde{Y})}{\int \mathrm{e}^{\frac{i}{\hbar}\,V_0(Y+\tilde{Y},\zeta)}\; d\nu_0(\tilde{Y})}. \tag{8.11}$$

This is the first step of the iterative procedure. After the n subsequent decompositions,

$$G_0 \;=\; G_n \,+\, \sum_{k=0}^{n-1}\Gamma_k\,,$$

one arrives at the expression

$$\big\langle Z(\phi)\big\rangle_\hbar = \frac{\int\,[(0,\zeta)+F_n(Y,\zeta;\phi)]\;\mathrm{e}^{\frac{i}{\hbar}\,V_n(Y,\zeta)}\; d\mu_n(Y)\,d\mu(\zeta)}{\int \mathrm{e}^{\frac{i}{\hbar}\,V_n(Y,\zeta)}\; d\mu_n(Y)\,d\mu(\zeta)} \tag{8.12}$$

with

$$
\mathrm{e}^{\frac{i}{\hbar}V_n(Y,\zeta)} \;=\; \int \mathrm{e}^{\frac{i}{\hbar}V_{n-1}(Y+\tilde{Y},\zeta)}\; d\nu_{n-1}(\tilde{Y})\,, \tag{8.13}
$$

$$
F_n(Y,\zeta;\phi) \;=\; \frac{\int F_{n-1}(Y+\tilde{Y},\zeta;\phi)\; \mathrm{e}^{\frac{i}{\hbar}V_{n-1}(Y+\tilde{Y},\zeta)}\; d\nu_{n-1}(\tilde{Y})}{\int \mathrm{e}^{\frac{i}{\hbar}V_{n-1}(Y+\tilde{Y},\zeta)}\; d\nu_{n-1}(\tilde{Y})}\,. \tag{8.14}
$$

Such a procedure may lead to the calculation of the expectation $\langle Z(\phi)\rangle_{\hbar}$ if we have a good control of the asymptotic behavior of the effective interactions V_n and of the effective insertions F_n. Note that the translation symmetry (8.2) goes through the RG iteration (8.13):

$$
V_n(\tau_\beta Y - (\beta,0)) \;=\; V_n(Y) \tag{8.15}
$$

if we consider in the definitions (8.10) and (8.13) fields Y with arbitrary averages and if the covariances Γ_n commute with the translations τ_β. The latter property has been guaranteed by the explicit construction of the operators Γ_n (see Eq. (4.9)).

The inductive RG scheme described in Sect. 2 may be obtained as the formal $\hbar\to 0$ limit of the above iterative calculation of the functional integral (8.3). After the first step, we obtain

$$
V_1(Y) \;=\; -\tfrac{1}{2}\,(\tilde{Y}_0,\Gamma_0^{-1}\tilde{Y}_0) \;+\; V_0(Y+\tilde{Y}_0)\,, \qquad F_1(Y) \;=\; Y+\tilde{Y}_0, \tag{8.16}
$$

where $\tilde{Y}_0$ minimizes the right hand side of the first equation. Denoting

$$
W_n(\phi;Y) \;=\; \frac{\delta V_n(Y)}{\delta Y(\phi)}\,, \tag{8.17}
$$

we may rewrite Eqs. (8.16) in the differential form:

$$
W_1(Y) \;=\; W_0(Y+\tilde{Y}_0)\,, \qquad \tilde{Y}_0 \;=\; \Gamma_0 W_0(Y+\tilde{Y}_0) \tag{8.18}
$$

or, more conveniently, as the fixed point equation

$$
W_1(Y) \;=\; W_0(Y+\Gamma_0 W_1(Y)) \tag{8.19}
$$

whose solution determines F_1:

$$
F_1(Y) \;=\; Y+\tilde{Y}_0 = \; Y+\Gamma_0 W_1(Y)\,. \tag{8.20}
$$

These are Eqs. (2.9) and (2.10), respectively. Similarly, after n inductive steps, W_n and F_n are determined by relations (2.13) and (2.14), the $\hbar\to 0$ versions of Eqs. (8.13) and (8.14), respectively. The cumulative expressions (2.15) and (2.16) may be obtained as those for W_1 and F_1 by replacing Γ_0 by $\Gamma_{<n} = \sum\limits_{k<n}\Gamma_k$, i.e. by performing n RG steps at once.

The Ward identity (5.1) of Sect. 5, which assured the partial cancellation of the repeated resonances or, in the field theory language, the absence of the terms proportional to $\int |\Theta|^2$, $\int \Theta\cdot(\omega\cdot\partial)\Theta$, $\int J\cdot\Theta$ (and to $\int\Theta$) in the effective interactions V_n, is the infinitesimal version of the translation symmetry (8.15). It is obtained from the latter by the differentiation w.r.t. β^i at $\beta=0$, which yields

$$-\int \frac{\delta V_n(Y)}{\delta Y^i(\phi)}\, d\phi \;-\; \sum_l \int \frac{\delta V_n(Y)}{\delta Y^l(\phi)}\, \partial_{\phi^i} Y^l(\phi)\, d\phi \;=\; 0.$$

The integration by parts in the second term and the definition (8.17) give then the identity (5.1).

9 Renormalization Group and Lindstedt series.

In this section, we sketch the connection of our approach to the Lindstedt series and to the resummation of the latter by Eliasson. The Lindstedt series has a graphical representation and our RG method amounts to resumming at each step a particular subset of graphs. Let us first introduce the graphical representation. It will be convenient to use a common symbol Q for the momentum variable q and the vector index i, $Q=(q,i)\in\mathbf{Z}^d\times\{1,\ldots,2d\}\equiv\mathcal{J}$ and to write $x(Q)\equiv x^i(q)$ and $w_0^{(m)}(Q_0\ldots,Q_m)$ for the matrix elements of the kernels $w_0^{(m)}(q_0,q_1\ldots,q_m;\zeta)$ introduced in Eq. (3.3). Using the relations (2.3) and (3.3), we obtain

$$x(Q) \;=\; \sum_{m=0}^{\infty}\sum_{\mathbf{Q}}{}^{*} G_0(Q,Q_0)\; w_0^{(m)}(\mathbf{Q}) \prod_{\ell=1}^{m} x(Q_\ell), \tag{9.1}$$

where $G_0(Q,Q_0)\equiv G_0(q,q_0)^{ii_0}$ and $\mathbf{Q}=(Q_0,\ldots Q_m)$. The sum $\sum^*$ means that we sum over $q_i\neq 0$ (this is due to the projector P in Eq. (2.3)). Since $w_0^{(m)}$ is proportional to λ, this formula yields a power series solution in powers of λ of $x(Q)$ obtained by regarding the equality (9.1) as a fixed point equation solved iteratively, starting with $x=0$.

The resulting series can be conveniently expressed as a sum over tree graphs whose weights are as follows. Let $\mathcal{T}^k_{m+1}$ denote the set of connected tree graphs T on $m+1+k$ vertices $v\in\{0,\ldots,m+k\}\equiv V(T)$ such that the first $m+1$ ones, the "external vertices", have coordination number one. We shall assume that $m+1, k\geq 1$ and will call the first of the external vertices the root of T. The lines ℓ of T are pairs $\ell=(v,v')$, $v,v'\in V(T)$ which we order assuming that the unique path going from the root to v' goes through v. The set of lines is denoted by $\mathcal{L}(T)$, lines ℓ containing an external vertex are called external, $\ell\in\mathcal{L}_E(T)$, and the remaining ones internal, $\ell\in\mathcal{L}_I(T)$. Given a function $G:\mathcal{J}^2\to\mathbf{C}$ and a collection $(w^{(m)})_{m\geq 0}\equiv w$ of functions $w^{(m)}:\mathcal{J}^{m+1}\to\mathbf{C}$ symmetric in the last m-variables, we define the "amplitude" $A(T,G,w,\mathbf{Q})$ of T. For this purpose, we assign variables $Q_i\in\mathcal{J}$ to the external vertices $0\leq i\leq m$ of T and variables $P_{\ell v}\in\mathcal{J}$ to each internal line ℓ and a vertex v contained in it. We write $\mathbf{P}_\ell=(P_{\ell v},P_{\ell v'})$ for $\ell=(v,v')$. For $m<v\leq m+k$, we set $R_v=P_{\ell v}$ if there exists an internal line (v',v) or $R_v=Q_0$ otherwise (i.e. when $(0,v)$ is an external line), and $\mathbf{P}_v=\{P_{\ell v}\,|\,\ell\in\mathcal{L}_I(T),\ \ell=(v,v')\}$, $\mathbf{Q}_v=\{Q_i\,|\,\ell=(v,i)\in\mathcal{L}_E(T)\}$. We define:

$$A(T,G,w,\mathbf{Q}) \;=\; \tfrac{1}{m!\,k!}\sum_{\mathbf{P}}{}^{*}\prod_{\ell\in\mathcal{L}_I(T)} G(\mathbf{P}_\ell)\prod_{v=m+1}^{m+k} m_v!\, w^{(m_v)}(R_v,\mathbf{P}_v,\mathbf{Q}_v), \tag{9.2}$$

where $\mathbf{Q} = (Q_0, \ldots, Q_m)$, the sum $\sum\limits_{\mathbf{P}}^*$ runs over all $p_{\ell v} \neq 0$, and $m_v + 1$ is the coordination number of vertex v.

With this notation, the iterative solution of Eq. (9.1) is given by

$$x(Q) = \sum_{Q_0}^* \sum_{T \in \mathcal{T}_1} G_0(Q, Q_0)\, A(T, G_0, w_0, Q_0), \tag{9.3}$$

where the sum is over all tree graphs with one external line, $\mathcal{T}_1 = \cup_k \mathcal{T}_1^k$. Comparing with Eq. (9.1), we also infer that

$$w_0(Q_0; x) = \sum_{T \in \mathcal{T}_1} A(T, G_0, w_0, Q_0) \tag{9.4}$$

Remark. A formal way to derive the identity (9.3) is to start from the field theory formula (8.4), expand it in Feynman diagrams, and let $\hbar \to 0$. In that limit, only tree diagrams remain (each line has a power of $\hbar$, while each vertex carries a factor $\hbar^{-1}$; so, all graphs except tree graphs are multiplied by some positive power of $\hbar$), and we obtain Eq. (9.3).

As is well known since Poincaré [20], the series in Eq. (9.3) does not absolutely converge for nontrivial potentials U in the Hamiltonian (1.1), i.e. $\sum_{T \in \mathcal{T}_1} |A(T, G_0, w_0, Q_0)| = \infty$ (see e.g. [6] for a simple proof). This is due to the presence of repeated resonances, namely of long sequences of lines (connected by vertices with coordination number two) many of which have the same small denominators $\omega \cdot p_\ell$. However, many trees contribute to the same order in λ and, as shown by Eliasson [8], cancellations occur when one regroups terms in a suitable way.

Let us now explain the renormalization group in the graph language. The key idea is a combinatorial identity which performs a partial resummation in Eq. (9.3):

Proposition 4. *Let $G = G' + \Gamma$. Then,*

$$\sum_{T \in \mathcal{T}_{m+1}} A(T, G, w, \mathbf{Q}) = \sum_{T' \in \mathcal{T}_{m+1}} A(T', G', w', \mathbf{Q}) \tag{9.5}$$

with

$$w'^{(m)}(\mathbf{P}) = \sum_{T \in \mathcal{T}_{m+1}} A(T, \Gamma, w, \mathbf{P}) \tag{9.6}$$

The proof of the identity (9.5) is quite simple. Insert $G = G' + \Gamma$ in the expression of A on the left hand side of (9.5) and decompose the tree T into a family of connected subtrees $\{T_\alpha\}$ containing only Γ-lines and joined together by G'-lines. Let T' denote the tree which is obtained from T by contracting each tree T_α to a point. Now, let us fix T' and sum over the $\{T_\alpha\}'s$; this leads to the identity (9.5) with the "renormalized" vertices w' given by Eq. (9.6).

We may apply the identity (9.6), starting with $w = w_0$, $G = G_0 = G_1 + \Gamma_0$, $w' = w_1$, and then inductively to $w = w_{n-1}$, $G = G_{n-1} = G_n + \Gamma_{n-1}$, $w' = w_n$. It is easy to check that the $w_n's$ so defined coincide with those constructed through Eq. (2.13). Indeed, by the definition (2.9),

$$W_1(Y) = W_0(Y + \Gamma_0 W_1(Y)).$$

Writing both sides in Fourier transform and expanding in a Taylor series, we obtain:

$$\begin{aligned} w_1(Q; y) &= \sum_{m=0}^{\infty} \sum_{\mathbf{P}} w_1^{(m)}(Q, \mathbf{P}) \prod_{\ell=1}^{m} y(P_\ell) \\ &= \sum_{m=0}^{\infty} \sum_{\mathbf{P}} w_0^{(m)}(Q, \mathbf{P}) \prod_{\ell=1}^{m} (y(P_\ell) + \Gamma_0 w_1(y)(P_\ell)) \end{aligned} \tag{9.7}$$

Now expand the product over ℓ in the right hand side replacing $w_1(Q; y)$ in $\Gamma_0 w_1(y)(P_\ell) = \sum_Q \Gamma_0(P_\ell, Q)\, w_1(Q; y)$ by the right hand side of Eq. (9.7). This leads, upon iteration, to a sum over trees with a number of external lines to which a factor $y(P_\ell)$ is attached. The trees have an amplitude $A(T, \Gamma_0, w_0, \mathbf{Q})$, $\mathbf{Q} = (Q, \mathbf{P})$ and we can rewrite the right hand side of Eq. (9.7), using the definition (9.6), as

$$\sum_{m=0}^{\infty} \sum_{\mathbf{P}} w_0'^{(m)}(Q, \mathbf{P}) \prod_{\ell=1}^{m} y(P_\ell).$$

Comparison with the left hand side of Eq. (9.7) shows that $w_1 = w_0'$ defined here coincides with w_1 defined in Sect. 2. The same arguments apply inductively to w_n.

Note that the transformation (9.6) is rather easy to control (in the sense that, if w is small, w' is also small in a suitable norm) and could be used to give an alternative proof of Proposition 3. Indeed, and this is the main difference between our approach and the one of Eliasson, each transformation (9.6) involves small denominators (in Γ_n) on *only one scale*. To see intuitively how to use this fact, consider first all the trees having only vertices with coordination number different from 2. It is easy to see that these trees have a number of vertices with coordination number one which is proportional to their total number of lines. Thus, one can control for those trees the small denominators on the lines, all of the same size, by the exponential decay in $|q|$ of $w_n^{(0)}(Q)$ and the Diophantine condition (1.6). The next observation is that we may reduce ourselves to those trees by resumming the contributions coming from the vertices with $m = 1$ (i.e. coordination number 2). This is where the problem of repeated resonances appears in this formalism. We obtain a series of the form:

$$\sum_{k=0}^{\infty} (\Gamma_n w_n^{(1)})^k\, \Gamma_n = (1 - \Gamma_n w_n^{(1)})^{-1} \Gamma_n = \tilde{H} \Gamma_n$$

which has to be controlled using the Ward identities, as in the proof of Proposition 3. Here we see again that this is the subtle point of the proof.

Finally, let us translate in the tree language some of the formulas introduced in Sect. 2. Applying the resummation (9.5), (9.6) to Eq. (9.3), we obtain:

$$x(Q) = \sum_{Q_0}{}^{*} \sum_{T \in \mathcal{T}_1} G_0(Q, Q_0)\, A(T, G_1, w_1, Q_0) \tag{9.8}$$

and, inductively,

$$x(Q) = \sum_{Q_0}{}^{*} \sum_{T\in\mathcal{T}_1} G_0(Q,Q_0)\, A(T,G_n,w_n,Q_0) \tag{9.9}$$

which, upon the substitution $G_0 = G_n + \Gamma_{<n}$, may be rewritten as

$$\begin{aligned} x(Q) &= \sum_{Q_0}{}^{*} \sum_{T\in\mathcal{T}_1} G_n(Q,Q_0)\, A(T,G_n,w_n,Q_0) \\ &\quad + \sum_{Q_0}{}^{*} \sum_{T\in\mathcal{T}_1} \Gamma_{<n}(Q,Q_0)\, A(T,G_n,w_n,Q_0). \end{aligned} \tag{9.10}$$

This corresponds to the decomposition

$$X = F_n(Y) = Y + \Gamma_{<n} W_n(Y),$$

see Eq. (2.16), with Y satisfying $Y = G_n P W_n(Y)$. Indeed, the solution of the latter equation may be written, in the same way as the solution (9.3) of Eq. (2.3), as a sum over the tree graphs. This gives the first term of on the right hand side of Eq. (9.10). Moreover (compare with Eq. (9.4)),

$$w_n(Q_0; y) = \sum_{T\in\mathcal{T}_1} A(T,G_n,w_n,Q_0).$$

This gives the second term on the right hand side of Eq. (9.10).

Note also that X_n, defined by the equality $X_n = \Gamma_{<n} W_0(X_n)$, see Eq. (2.17), satisfies the equality

$$x_n(Q) = \sum_{Q_0}{}^{*} \sum_{T\in\mathcal{T}_1} \Gamma_{<n}(Q,Q_0)\, A(T,\Gamma_{<n},w_0,Q_0), \tag{9.11}$$

which may be derived from the equation $X_n = \Gamma_{<n} W_0(X_n)$ in the same way as we derived the identity (9.3). Applying the relations (9.5), (9.6) inductively, we we infer that

$$x_n(Q) = \sum_{Q_0}{}^{*} \Gamma_{<n}(Q,Q_0)\, w_n^{(0)}(Q_0) \tag{9.12}$$

which is equal to the term in the second sum of Eq. (9.10) corresponding to the tree with only one vertex. It is also easy to see why the other terms in Eq. (9.10) are small and therefore why $x_n \to x$, as shown in the proof of Theorem 1: in all the other terms, the $w_n^{(0)}(p)$'s attached to vertices with coordination number one are multiplied by some G_n, which forces the corresponding p to be large (using the support properties of G_n, the Diophantine property (1.6) and the fact that the sums here run over $p \neq 0$). Then the exponential decay of $w_n^{(0)}(p)$ implies that the contribution of those trees is small.

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