Preprint math/9807141, 1998

Wavelets in Banach Spaces*

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July 24, 1998

Abstract

We describe a construction of wavelets (coherent states) in Banach spaces generated by "admissible" group representations. Our main targets are applications in pure mathematics while connections with quantum mechanics are mentioned. As an example we consider operator valued Segal-Bargmann type spaces and the Weyl functional calculus.

*Supported by grant INTAS 93–0322–Ext.

Keywords and phrases. Wavelets, coherent states, Banach spaces, group representations, covariant, contravariant (Wick) symbols, Heisenberg group, Segal-Bargmann spaces, Weyl functional calculus (quantization), second quantization, bosonic field.

1991 *Mathematical Subject Classification.* Primary: 43A85; Secondary: 32M99, 43A32, 46E10, 47A60, 47A67, 47C99, 81R30, 81S10.

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1 Introduction

> Questioning the nature of essence, are we not in danger of losing ourselves in the void of the commonplace which suffocates all thought?
>
> M. Heidegger [26].

Wavelets [11, 13, 21, 27] (or coherent states [1, 31, 35, 41, 42, 45, 53]) belong to a field of intensive research, which was carried from many different viewpoints. This interest is supported by important applications of the theory in quantum mechanics, signal processing, etc. The huge potential of wavelets in pure mathematics is still to be explored [15, 35, 36, 37, 39]. It is impossible to give a short overview of all aspect of the theory here. We will refer to the recent survey [1] which contains an extensive bibliography (137 entries).

In this paper we investigate a subject which is outside the mainstream of research: wavelets in Banach space[1]. The first paper in this direction known to the present author is the paper [6] of Bargmann. It extends results of the early paper [5] which describes wavelets in a Hilbert space associated with a representation of the Heisenberg group in the Segal-Bargmann space [59].

[1] In fact, our construction is applicable for an arbitrary locally convex topological vector space, see Remark 3.14.

BARGMANN considered in [6] wavelets which are tempered distributions. J. KLAUDER and J. MCKENNA [44] extended this idea to (nontempered) distributions by using a fiducial vector from $\mathcal{D}$ rather than from $\mathcal{S}$, but they did not publish their results (see however [45]). H.G. FEICHTINGER and K.H. GROECHENIG in a series of important papers started by [23, 24] described atomic decomposition in Banach spaces of functions connected with unitary irreducible representations of groups in Hilbert spaces. This approach describes most of the technique used in real function theory and allows to construct new tools in this area. Unfortunately the approach has not received an essential appreciation by function theorists which it deserves. Note that the Hilbert space technique is his principal tool in papers [6, 23, 24].

There is also another predecessor from an unexpected field: combinatorics. The theory of umbral calculus as it developed by G.-C. Rota and his coauthors [51, 58, 55, 56] contain all principal formulas for a wavelet transform and its inverse for particularly selected group $\mathbb{Z}$ (or even convoloid $\mathbb{N}$ [38]). The key idea of the present paper (separation a vacuum vector from a test functional in topological or Banach spaces) is explicitly used in these papers. Due to elementary nature of this approach its realizations could be probably found in many different areas of mathematics.

One could find a reason which wavelets in Banach space are underrepresented for: two main areas of wavelets application are primary interested in Hilbert spaces. In signal processing the finite energy is defined by the L_2 norm and in quantum mechanics states of a systems form a Hilbert space. However, motivated by pure mathematics one is interested in Banach or topological spaces also. One could think on an unified treatment of Hardy spaces H_p without distinguishing value $p = 2$ or Segal-Bargmann type spaces F_p (see Subsection 4.2). A need of wavelets in Banach spaces arises particularly in a study of principal series representation of semisimple Lie groups [39]. Such representations are not square integrable and therefore the standard theory of wavelets from group representations [11] is not directly applicable here. Another appeal arises from operator theory. To make an advance in functional calculus of operators we need wavelets defined in Banach algebras [36, 40]. Even from quantum mechanical point of view Banach spaces are not completely useless: non-unitary evolution in Banach spaces could describe decay and creation of particles [61].

In this paper we try to present wavelets in Banach spaces as close as possible to Hilbert space case but without an explicit use of the Hilbert space results (cf. [6, 23, 24]). An abstract scheme in Section 2 consolidates many results of [11, 23, 24, 45, 53] and we receive the standard wavelet theory if assume the space under consideration to be a Hilbert one. In Section 3 we apply wavelets for consideration of symbolic and functional calculi of operators. As a first application we will consider theory of co- and contravariant symbols of operators [7, 8, 10] as its direct consequence. We give practical examples of wavelets in Banach spaces in Section 4 by operator valued Segal-Bargmann type spaces. Such spaces could be interested for their connections with the Weyl functional calculus, spectral measures [2] and as a model in quantum mechanics for second quantization of bosonic fields [10, 59]. More examples particularly to wavelets in Banach algebras [40] will be given elsewhere.

We avoid a formulation of the idea in the bigger generality. Such formulations while be applicable in more cases require however more efforts to be applied in any particular situation. We select an alternative path: to present the mainstream of the theory and mention in remarks possible modifications for a generalized setting (see Subsection 3.3).

2 Coherent States for Banach Spaces

2.1 Abstract Nonsence

Let G be a group and H be its closed normal subgroup. Let $X = G/H$ be the corresponding homogeneous space with an invariant measure $d\mu$ and $s : X \to G$ be a Borel section in the principal bundle $G \to G/H$. Let π be a continuous representation of a group G by invertible isometry operators π_g, $g \in G$ in a (complex) Banach space B.

The following definition simulates ones from the Hilbert space case [1, § 3.1].

Definition 2.1 Let G, H, $X = G/H$, $s : X \to G$, $\pi : G \to \mathcal{L}(B)$ be as above. We say that $b_0 \in B$ is a *vacuum vector* if for all $h \in H$

$$\pi(h)b_0 = \chi(h)b_0, \qquad \chi(h) \in \mathbb{C}. \tag{2.1}$$

We will say that set of vectors $b_x = \pi(x)b_0$, $x \in X$ form a family of *coherent states* if there exists a continuous non-zero linear functional $l_0 \in B^*$ such that

1. $\|b_0\| = 1$, $\|l_0\| = 1$, $\langle b_0, l_0 \rangle \neq 0$;
2. $\pi(h)^* l_0 = \bar{\chi}(h) l_0$, where $\pi(h)^*$ is the adjoint operator to $\pi(h)$;
3. The following equality holds

$$\int_X \left\langle \pi(x^{-1})b_0, l_0 \right\rangle \left\langle \pi(x)b_0, l_0 \right\rangle \, d\mu(x) = \left\langle b_0, l_0 \right\rangle . \tag{2.2}$$

The functional l_0 is called the *test functional.* According to the strong tradition we call the set (G, H, π, B, b_0, l_0) *admissible* if it satisfies to the above conditions.

We note that mapping $h \to \chi(h)$ from (2.1) defines a character of the subgroup H. The following Lemma demonstrates that condition (2.2) could be relaxed.

Lemma 2.2 *For the existence of a vacuum vector b_0 and a test functional l_0 it is sufficient that there exists a vector b'_0 and continuous linear functional l'_0 satisfying to* (2.1) *and* 2.1.2 *correspondingly such that the constant*

$$c = \int_X \left\langle \pi(x^{-1})b'_0, l'_0 \right\rangle \left\langle \pi(x^{-1})b'_0, l'_0 \right\rangle \, d\mu(x) \tag{2.3}$$

is non-zero and finite.

PROOF. There exist a $x_0 \in X$ such that $\left\langle \pi(x_0^{-1})b'_0, l'_0 \right\rangle \neq 0$, otherwise one has $c = 0$. Let $b_0 = \pi(x^{-1})b'_0 \left\| \pi(x^{-1})b'_0 \right\|^{-1}$ and $l_0 = l'_0 \left\| l'_0 \right\|^{-1}$. For such b_0 and l_0 we have 2.1.1 already fulfilled. To obtain (2.2) we change the measure $d\mu(x)$. Let $c_0 = \langle b_0, l_0 \rangle \neq 0$ then $d\mu' = \left\| \pi(x^{-1})b'_0 \right\| \left\| l'_0 \right\| c_0 c^{-1} d\mu$ is the desired measure. □

REMARK 2.3 Conditions (2.2) and (2.3) are known for unitary representations in Hilbert spaces as *square integrability* (with respect to a subgroup H). Thus our definition describes an analog of square integrable representations for Banach spaces. Note that in Hilbert space case b_0 and l_0 are often the same function, thus condition 2.1.2 is exactly (2.1). In the particular but still important case of trivial $H = \{e\}$ (and thus $X = G$) all our results take simpler forms.

Convention 2.4 In that follow we will usually write $x \in X$ and x^{-1} instead of $s(x) \in G$ and $s(x)^{-1}$ correspondingly. The right meaning of "x" could be easily found from the context (whether an element of X or G is expected there).

The wavelet transform (similarly to the Hilbert space case) could be defined as a mapping from B to a space of bounded continuous functions over G via representational coefficients

$$v \mapsto \widehat{v}(g) = \left\langle \pi(g^{-1})v, l_0 \right\rangle = \left\langle v, \pi(g^{-1})^* l_0 \right\rangle .$$

Due to 2.1.2 such functions have simple transformation properties along orbits gH, i.e. $\widehat{v}(gh) = \bar{\chi}(h)\widehat{v}(g)$, $g \in G$, $h \in H$. Thus they are completely defined by their values indexed by points of $X = G/H$. Therefore we prefer to consider so called reduced wavelet transform.

Definition 2.5 The *reduced wavelet transform* $\mathcal{W}$ from a Banach space B to a space of function $F(X)$ on a homogeneous space $X = G/H$ defined by a representation π of G on B, a vacuum vector b_0 and a test functional l_0 is given by the formula

$$\mathcal{W} : B \rightarrow F(X) : v \mapsto \widehat{v}(x) = [\mathcal{W}v](x) = \left\langle \pi(x^{-1})v, l_0 \right\rangle = \langle v, \pi^*(x) l_0 \rangle . \quad (2.4)$$

There is a natural representation of G in $F(X)$. For any $g \in G$ there is a unique decomposition of the form $g = s(x)h$, $h \in H$, $x \in X$. We will define $r : G \rightarrow H : r(g) = h = (s^{-1}(g))^{-1} g$ from the previous equality and write a formal notation $x = s^{-1}(g)$. Then there is a geometric action of G on $X \rightarrow X$ defined as follows

$$g : x \mapsto g^{-1} \cdot x = s^{-1}(g^{-1}s(x)).$$

We define a representation $\lambda(g) : F(X) \rightarrow F(X)$ as follow

$$[\lambda(g)f](x) = \chi(r(g^{-1} \cdot x)) f(g^{-1} \cdot x). \quad (2.5)$$

We recall that $\chi(h)$ is a character of H defined in (2.1) by the vacuum vector b_0. For the case of trivial $H = \{e\}$ (2.5) becomes the left regular representation $\rho_l(g)$ of G.

Proposition 2.6 *The reduced wavelet transform $\mathcal{W}$ intertwines π and the representation λ (2.5) on $F(X)$:*

$$\mathcal{W}\pi(g) = \lambda(g)\mathcal{W}.$$

PROOF. We have:

$$\begin{aligned}
[\mathcal{W}(\pi(g)v)](x) &= \left\langle \pi(x^{-1})\pi(g)v, l_0 \right\rangle \\
&= \left\langle \pi((g^{-1}s(x))^{-1})v, l_0 \right\rangle \\
&= \left\langle \pi(r(g^{-1} \cdot x)^{-1})\pi(s(g^{-1} \cdot x)^{-1})v, l_0 \right\rangle \\
&= \left\langle \pi(s(g^{-1} \cdot x)^{-1})v, \pi^*(r(g^{-1} \cdot x)^{-1}) l_0 \right\rangle \\
&= \chi(r(g^{-1} \cdot x)^{-1})[\mathcal{W}v](g^{-1}x) \\
&= \lambda(g)[\mathcal{W}v](x).
\end{aligned}$$

□

Corollary 2.7 *The function space $F(X)$ is invariant under the representation λ of G.*

Definition 2.8 The *inverse wavelet transform* $\mathcal{M}$ from $F(X)$ to B is given by the formula:

$$\begin{aligned}\mathcal{M}: F(X) \rightarrow B : \widehat{v}(x) \mapsto \mathcal{M}[\widehat{v}(x)] &= \int_X \widehat{v}(x) b_x \, d\mu(x) \\ &= \int_X \widehat{v}(x)\pi(x)\, d\mu(x) b_0. \qquad (2.6)\end{aligned}$$

Proposition 2.9 *The inverse wavelet transform $\mathcal{M}$ intertwines the representation λ on $F(X)$ and π on B:*

$$\mathcal{M}\lambda(g) = \pi(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned}\mathcal{M}[\lambda(g)\widehat{v}(x)] &= \mathcal{M}[\chi(r(g^{-1}\cdot x))\widehat{v}(g^{-1}\cdot x)] \\ &= \int_G \chi(r(g^{-1}\cdot x))\widehat{v}(g^{-1}\cdot x) b_x \, d\mu(x) \\ &= \chi(r(g^{-1}\cdot x)) \int_G \widehat{v}(x') b_{g\cdot x'} \, d\mu(x') \\ &= \pi_g \int_G \widehat{v}(x') b_{x'} \, d\mu(x') \\ &= \pi_g \mathcal{M}[\widehat{v}(x')],\end{aligned}$$

where $x' = g^{-1}\cdot x$. □

Corollary 2.10 *The image $\mathcal{M}(F(X)) \subset B$ of subspace $F(X)$ under the inverse wavelet transform $\mathcal{M}$ is invariant under the representation π.*

The following proposition explain the usage of the name for $\mathcal{M}$.

Theorem 2.11 *The operator*

$$\mathcal{P} = \mathcal{M}\mathcal{W} : B \rightarrow B \qquad (2.7)$$

is a projection of B to its linear subspace for which b_0 is cyclic.

Particularly if π is an irreducible representation then the inverse wavelet transform $\mathcal{M}$ is a left inverse *operator on B for the wavelet transform $\mathcal{W}$:*

$$\mathcal{M}\mathcal{W} = I.$$

PROOF. It follows from Propositions 2.6 and 2.9 that operator $\mathcal{M}\mathcal{W} : B \rightarrow B$ intertwines π with itself. Then Corollaries 2.7 and 2.10 imply that the image $\mathcal{MW}$ is a π-invariant subspace of B containing b_0. Because $\mathcal{MW}b_0 = b_0$ we conclude that $\mathcal{MW}$ is a projection.

From irreducibility of π by Schur's Lemma [34, § 8.2] one concludes that $\mathcal{MW} = cI$ on B for a constant $c \in \mathbb{C}$. Particularly

$$\mathcal{MW}b_0 = \int_X \left\langle \pi(x^{-1})b_0, l_0 \right\rangle \pi(x)b_0 \, d\mu(x) = cb_0.$$

From the condition (2.2) it follows that $\langle cb_0, l_0\rangle = \langle \mathcal{MW}b_0, l_0\rangle = \langle b_0, l_0\rangle$ and therefore $c = 1$. □

We denote by $\mathcal{W}^* : F^*(X) \to B^*$ and $\mathcal{M}^* : B^* \to F^*(X)$ the adjoint (in the standard sense) operators to $\mathcal{W}$ and $\mathcal{M}$ respectively.

Corollary 2.12 *We have the following identity:*

$$\langle \mathcal{W}v, \mathcal{M}^*l\rangle_{F(X)} = \langle v, l\rangle_B , \qquad \forall v \in B, \quad l \in B^* \tag{2.8}$$

or equivalently

$$\int_X \left\langle \pi(x^{-1})v, l_0 \right\rangle \langle \pi(x)b_0, l\rangle \, d\mu(x) = \langle v, l\rangle . \tag{2.9}$$

Proof. We show the equality in the first form (2.9) (but will apply it often in the second one):

$$\langle \mathcal{W}v, \mathcal{M}^*l\rangle_{F(X)} = \langle \mathcal{MW}v, l\rangle_B = \langle v, l\rangle_B .$$

□

Corollary 2.13 *The space $F(X)$ has the reproducing formula*

$$\widehat{v}(y) = \int_X \widehat{v}(x)\, \widehat{b}_0(x^{-1} \cdot y) \, d\mu(x), \tag{2.10}$$

where $\widehat{b}_0(y) = [\mathcal{W}b_0](y)$ is the wavelet transform of the vacuum vector b_0.

Proof. Again we have a simple application of the previous formulas:

$$\begin{aligned}
\widehat{v}(y) &= \left\langle \pi(y^{-1})v, l_0 \right\rangle \\
&= \int_X \left\langle \pi(x^{-1})\pi(y^{-1})v, l_0 \right\rangle \langle \pi(x)b_0, l_0\rangle \, d\mu(x) \qquad (2.11)\\
&= \int_X \left\langle \pi(s(y \cdot x)^{-1})v, l_0 \right\rangle \langle \pi(x)b_0, l_0\rangle \, d\mu(x) \\
&= \int_X \widehat{v}(y \cdot x)\, \widehat{b}_0(x^{-1}) \, d\mu(x) \\
&= \int_X \widehat{v}(x)\, \widehat{b}_0(x^{-1}y) \, d\mu(x),
\end{aligned}$$

where transformation (2.11) is due to (2.9). □

REMARK 2.14 To possess a reproducing kernel—is a well-known property of spaces of analytic functions. The space $F(X)$ shares also another important property of analytic functions: it belongs to a kernel of a certain first order differential operator with Clifford coefficients (the Dirac operator) and a second order operator with scalar coefficients (the Laplace operator) [4, 37, 39, 46].

Let us now assume that there are two representations π' and π'' of the same group G in two different spaces B' and B'' such that two admissible sets $(G, H, \pi', B', b'_0, l'_0)$ and $(G, H, \pi'', B'', b''_0, l''_0)$ could be constructed for the same normal subgroup $H \subset G$.

Proposition 2.15 *In the above situation if $F'(X) \subset F''(X)$ then the composition $\mathcal{T} = \mathcal{M}''\mathcal{W}'$ of the wavelet transform $\mathcal{W}'$ for π' and inverse $\mathcal{M}''$ for π'' is an intertwining operator between π' and π'':*

$$\mathcal{T}\pi' = \pi''\mathcal{T}.$$

$\mathcal{T}$ is defined as follows

$$\mathcal{T} : b \mapsto \int_X \left\langle \pi'(x^{-1})b, l'_0 \right\rangle \pi''(x) b''_0 \, d\mu(x). \tag{2.12}$$

This transformation defines a B''-valued linear functional (a distribution for function spaces) on B'.

The Proposition has an obvious proof. This simple result is a base for an alternative approach to functional calculus of operators [36, 37] and will be used in Subsection 3.2. Note also that formulas (2.4) and (2.6) are particular cases of (2.12) because $\mathcal{W}$ and $\mathcal{M}$ intertwine π and λ.

2.2 Wavelets and a Positive Cone

The above results are true for wavelets in general. In applications a Banach space B is usually equipped with additional structures and wavelets are interplay with them. We consider an example of such interaction.

We recall [49], [32, Chap. X] the notion of positivity in Banach spaces. Let $C \subset B$ be a sharp cone, i.e. $x \in C$ implies that $\lambda x \in C$ and $-\lambda x \notin C$ for $\lambda > 0$. We call elements $x \in C$ *positive vectors*, we say also that $x \geq y$ iff $x - y$ is positive. There is the *dual cone* $C^* \in B^*$ defined by the condition

$$C^* = \{f \mid f \in B^*, \ \langle b, f \rangle \geq 0 \ \ \forall x \in C\}.$$

An operator $A : B \to B$ is called *positive* if $Ab \geq 0$ for all $b \geq 0$. If A is positive with respect to C then A^* is positive with respect to C^*.

Definition 2.16 We call a representation $\pi(g)$ *positive* if there exists a vector $b_0 \in C$ such that $\pi(x)b_0 \in C$ for all $x \in X$. A linear functional $f \in B^*$ is *positive* ($f > 0$) with respect to a vacuum vector b_0 if $\langle \pi(x)b_0, f\rangle \geq 0$ for all $x \in X$ and $\langle \pi(x)b_0, f\rangle$ is not identically 0.

Lemma 2.17 *For any positive representation $\pi(g)$ and vacuum vector b_0 there exists a positive test functional.*

PROOF. Obvious. □

We finish with an estimation of positive linear functionals.

Proposition 2.18 *Let $b \in B$ be a vector such that $b = \int_X \widehat{b}(x) b_x \, d\mu(x)$. Let*

1. *$D(b) = \{\langle \pi(x^{-1})b, l_0\rangle \mid x \in X\}$ be the set of value of reduced wavelets transform;*
2. *$\breve{D}(b)$ be a convex shell of the values of $\widehat{b}(x)$;*
3. *$\hat{D}(b) = \{\langle b, f\rangle \mid f \in C^*, \|f\| = 1, f \geq 0\}$.*

Then

$$D(b) \subset \hat{D}(b) \subset \breve{D}(b).$$

PROOF. The first inclusion is obvious. The second could be easily checked:

$$\langle b, f\rangle = \left\langle \int_X \widehat{b}(x) b_x \, d\mu(x), f \right\rangle = \int_X \widehat{b}(x) \, \langle b_x, f\rangle \, d\mu(x).$$

□

2.3 Singular Vacuum Vectors

In many important cases the above general scheme could not be carried out because the representation π of G is not square-integrable or even not square-integrable modulo a subgroup H. Thereafter the vacuum vector b_0 could not be selected within the original space B which the representation π acts on. The simplest mathematical example is the Fourier transform (see Example 4.1). In physics this is the well-known problem of *absence of vacuum state* in the constructive algebraic quantum field theory [60, 62, 63]. The absence of the vacuum within the linear space of system's states is another illustration to the old thesis *Natura abhorret vacuum*[2] or even more specifically *Natura abhorret vectorem vacui*[3].

We will present a modification of our construction which works in such a situation. For a singular vacuum vector the algebraic structure of group representations could not describe the situation alone and requires an essential assistance from analytical structures.

Definition 2.19 Let G, H, $X = G/H$, $s : X \rightarrow G$, $\pi : G \rightarrow \mathcal{L}(B)$ be as in Definition 2.1. We assume that there exist a topological linear space $\widehat{B} \supset B$ such that

1. B is dense in $\widehat{B}$ (in topology of $\widehat{B}$) and representation π could be uniquely extended to the continuous representation $\widehat{\pi}$ on $\widehat{B}$.
2. There exists $b_0 \in \widehat{B}$ be such that for all $h \in H$
$$\widehat{\pi}(h)b_0 = \chi(h)b_0, \qquad \chi(h) \in \mathbb{C}. \tag{2.13}$$
3. There exists a continuous non-zero linear functional $l_0 \in B^*$ such that $\pi(h)^* l_0 = \bar{\chi}(h) l_0$, where $\pi(h)^*$ is the adjoint operator to $\pi(h)$;
4. The composition $\mathcal{M}\mathcal{W} : B \rightarrow \widehat{B}$ of the wavelet transform (2.4) and the inverse wavelet transform (2.6) maps B to B.
5. For a vector $p_0 \in B$ the following equality holds
$$\left\langle \int_X \left\langle \pi(x^{-1})p_0, l_0 \right\rangle \pi(x) b_0 \, d\mu(x), l_0 \right\rangle = \left\langle p_0, l_0 \right\rangle, \tag{2.14}$$
where the integral converges in the weak topology of $\widehat{B}$.

As before we call the set of vectors $b_x = \pi(x)b_0$, $x \in X$ by *coherent states*; the vector b_0—a *vacuum vector*; the functional l_0 is called the *test functional* and finally p_0 is the *probe vector*.

This Definition is more complicated than Definition 2.1. The equation (2.14) is a substitution for (2.2) if the linear functional l_0 is not continuous in the topology of $\widehat{B}$. Example 4.1 shows that the Definition does not describe an empty set. The function theory in $\mathbb{R}^{1,1}$ constructed in [39] provides a more exotic example of a singular vacuum vector.

We shall show that 2.19.5 could be satisfied by an adjustment of other components.

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[2]Nature is horrified by (any) vacuum (Lat.).

[3]Nature is horrified by a carrier of nothingness (Lat.). This illustrates how far a humane beings deviated from Nature.

Lemma 2.20 *For the existence of a vacuum vector b_0, a test functional l_0, and a probe vector p_0 it is sufficient that there exists a vector b'_0 and continuous linear functional l'_0 satisfying to* 2.19.1–2.19.4 *and a vector $p'_0 \in B$ such that the constant*

$$c = \left\langle \int_X \left\langle \pi(x^{-1})p_0, l_0 \right\rangle \, \pi(x) b_0 \, d\mu(x), l_0 \right\rangle$$

is non-zero and finite.

The proof follows the path for Lemma 2.2. The following Proposition summarizes results which could be obtained in this case.

Proposition 2.21 *Let the wavelet transform $\mathcal{W}$* (2.4)*, its inverse $\mathcal{M}$* (2.6)*, the representation $\lambda(g)$* (2.5)*, and functional space $F(X)$ be adjusted accordingly to Definition* 2.19*. Then*

1. *$\mathcal{W}$ intertwines $\pi(g)$ and $\lambda(g)$ and the image of $F(X) = \mathcal{W}(B)$ is invariant under $\lambda(g)$.*

2. *$\mathcal{M}$ intertwines $\lambda(g)$ and $\widehat{\pi}(g)$ and the image of $\mathcal{M}(F(B)) = \mathcal{M}\mathcal{W}(B) \subset B$ is invariant under $\pi(g)$.*

3. *If $\mathcal{M}(F(B)) = B$ (particularly if $\pi(g)$ is irreducible) then $\mathcal{M}\mathcal{W} = I$ otherwise $\mathcal{M}\mathcal{W}$ is a projection $B \rightarrow \mathcal{M}(F(B))$. In both cases $\mathcal{M}\mathcal{W}$ is an operator defined by integral*

$$b \mapsto \int_X \left\langle \pi(x^{-1})b, l_0 \right\rangle \, \pi(x) b_0 \, d\mu(x), \tag{2.15}$$

4. *Space $F(X)$ has a reproducing formula*

$$\widehat{v}(y) = \left\langle \int_X \widehat{v}(x) \, \pi(x^{-1}y) b_0 \, dx, l_0 \right\rangle \tag{2.16}$$

which could be rewritten as a singular convolution

$$\widehat{v}(y) = \int_X \widehat{v}(x) \, \widehat{b}(x^{-1}y) \, dx$$

with a distribution $b(y) = \langle \pi(y^{-1}x) b_0, l_0 \rangle$ defined by (2.16).

The proof is algebraic and completely similar to Subsection 2.1.

3 Wavelets in Operator Algebras

We are going to apply the above abstract scheme to special spaces, which our main targets—wavelets on operator algebras. This gives a possibility to study operators by means of functions—symbols of operators.

3.1 Co- and Contravariant Symbols of Operators

We construct a realization of the wavelet transform as co- and contravariant symbols (also known as Wick and anti-Wick symbols) of operators. These symbols and their connections with wavelets in Hilbert spaces are known for a while [7, 8, 9, 10]. However their realization (described bellow) as wavelets in Banach algebras seems to be new.

Let $\pi(g)$ be a representation of a group G in a Banach space B by isometry operators. Then we could define two new representations for groups G and $G\times G$ correspondingly in the space $\mathcal{L}(B)$ of bounded linear operators $B\to B$:

$$\begin{aligned} \widehat{\pi} &: G\to \mathcal{L}(\mathcal{L}(B)) : A\mapsto \pi(g)^{-1}A\pi(g), && (3.1)\\ \widetilde{\pi} &: G\times G\to \mathcal{L}(\mathcal{L}(B)) : A\mapsto \pi(g_1)^{-1}A\pi(g_2), && (3.2)\end{aligned}$$

where $A\in\mathcal{L}(B)$. Note that $\widehat{\pi}(g)$ are algebra automorphisms of $\mathcal{L}(B)$ for all g. Representation $\widetilde{\pi}(g_1,g_2)$ is an algebra homomorphism from $\mathcal{L}(B)$ to the algebra $L_{(g_1,g_2)}(B)$ of linear operators on B equipped with a composition

$$A_1\circ A_2 = A_1\,\pi(g_1)^{-1}\pi(g_2)\,A_2$$

with the usual multiplication of operators in the right-hand side. The rôle of such algebra homomorphisms in a symbolical calculus of operators was explained in [29]. It is also obvious that $\widehat{\pi}(g)$ is the restriction of $\widetilde{\pi}(g_1,g_2)$ to the diagonal of $G\times G$.

Let there are selected a vacuum vector $b_0\in B$ and a test functional $l_0\in B^*$ for π. Then there are the canonically associated vacuum vector $P_0\in\mathcal{L}(B)$ and test functional in $f_0\in L^*(B)$ defined as follows:

$$\begin{aligned} P_0 &: B\to B : b\mapsto P_0 b = \langle b, l_0\rangle\, b_0; && (3.3)\\ f_0 &: \mathcal{L}(B)\to\mathbb{C} : A\mapsto \langle Ab_0, l_0\rangle\,. && (3.4)\end{aligned}$$

They define the following coherent states and transformations of the test functional

$$\begin{aligned} P_g &= \widehat{\pi}(g)P_0 = \langle\cdot, l_g\rangle\, b_g, & P_{(g_1,g_2)} &= \widetilde{\pi}(g_1,g_2)P_0 = \langle\cdot, l_{g_1}\rangle\, b_{g_2},\\ f_g &= \widehat{\pi}^*(g)f_0 = \langle\,\cdot\, b_g, l_g\rangle\,, & f_{(g_1,g_2)} &= \widehat{\pi}^*(g_1,g_2)f_0 = \langle\,\cdot\, b_{g_1}, l_{g_2}\rangle\end{aligned}$$

where as usually we denote $b_g=\pi(g)b_0$, $l_g=\pi^*(g)l_0$. All these formulas take simpler forms for Hilbert spaces if $l_0=b_0$.

Definition 3.1 The *covariant (pre-)symbol* $A(x)$ ($A(x_1, x_2)$) of an operator A acting on a Banach space B defined by $b_0 \in B$ and $l_0 \in B^*$ is its wavelet transform with respect to representation $\widehat{\pi}(g)$ (3.1) ($\widetilde{\pi}(g_1, g_2)$ (3.2)) and the functional f_0 (3.4), i.e. they are defined by the formulas

$$\begin{aligned} A(x) &= (\widehat{\pi}(x)A, f_0) = \left\langle \pi(x)^{-1} A \pi(x) b_0, l_0 \right\rangle = \left\langle A b_x, l_x \right\rangle, && (3.5)\\ A(x_1, x_2) &= (\widetilde{\pi}(x_1, x_2)A, f_0) = \left\langle \pi(x_1)^{-1} A \pi(x_2) b_0, l_0 \right\rangle = \left\langle A b_{x_2}, l_{x_1} \right\rangle. && (3.6)\end{aligned}$$

The *contravariant (pre-)symbol* of an operator A is a function $\breve{A}(x)$ (a function $\breve{A}(x_1, x_2)$ correspondingly) such that A is the inverse wavelet transform of $\breve{A}(x)$ (of $\breve{A}(x_1, x_2)$ correspondingly) with respect to $\widehat{\pi}(g)$ ($\widetilde{\pi}(g_1, g_2)$), i.e.

$$\begin{aligned} A &= \int_X \breve{A}(x) \widehat{\pi}(x) P_0 \, d\mu(x) = \int_X \breve{A}(x) P_x \, d\mu(x), && (3.7)\\ A &= \int_X \int_X \breve{A}(x_1, x_2) \widetilde{\pi}(x_1, x_2) P_0 \, d\mu(x_1) \, d\mu(x_2) \\ &= \int_X \int_X \breve{A}(x_1, x_2) P_{(x_1, x_2)} \, d\mu(x_1) \, d\mu(x_2), && (3.8)\end{aligned}$$

where the integral is defined in the weak sense.

Obviously the covariant symbol $\breve{A}(x)$ is the restriction of the covariant presymbol $\breve{A}(x_1, x_2)$ to the diagonal of $G \times G$.

Proposition 3.2 *Mapping $\sigma : A \mapsto \sigma_A(x_1, x_2)$ of operators to their covariant symbols is the algebra homomorphism from algebra of operators on B to algebra of integral operators on $F(G)$, i.e.*

$$\sigma_{A_1 A_2}(x_1, x_3) = \int_X \sigma_{A_1}(x_1, x_2) \sigma_{B_2}(x_2, x_3) \, d\mu(x_2). \qquad (3.9)$$

PROOF. One could easily see that:

$$\begin{aligned} &\int_X \sigma_{A_1}(x_1, x_2) \sigma_{A_2}(x_2, x_3) \, d\mu(x_2) \\ &\quad= \int_X \left\langle \pi(x_1) A_1 \pi(x_2^{-1}) b_0, l_0 \right\rangle \left\langle \pi(x_2) A_2 \pi(x_3^{-1}) b_0, l_0 \right\rangle \, d\mu(x_2) \\ &\quad= \int_X \left\langle \pi(x_2^{-1}) b_0, A_1^* \pi^*(x_1) l_0 \right\rangle \left\langle \pi(x_2) A_2 \pi(x_3^{-1}) b_0, l_0 \right\rangle \, d\mu(x_2) \\ &\quad= \left\langle A_2 \pi(x_3^{-1}) b_0, A_1^* \pi^*(x_1) l_0 \right\rangle && (3.10)\\ &\quad= \left\langle \pi(x_1) A_1 A_2 \pi(x_3^{-1}) b_0, l_0 \right\rangle \\ &\quad= \sigma_{A_1 A_2}(x_1, x_3),\end{aligned}$$

where transformation (3.10) is due to (2.9). □

The following proposition is obvious.

Proposition 3.3 *An operator A could be reconstructed from its covariant presymbol $A(g_1, g_2)$ by the formula*

$$Av = \int_G \int_G A(g_1, g_2)\widehat{v}(g_2)\, d\mu(g_2) b_{g_1}\, d\mu(g_1).$$

We have a particular interest in operators closely connected with the representation π_g.

Proposition 3.4 *Let an operator A on B is defined by the formula*

$$Av = \int_G a(g)\, \pi_g v\, d\mu(g)$$

for a function $a(g)$ on G. Then $A'\mathcal{W} = \mathcal{W}A$ where A' is a two-sided convolution on G defined by the formula

$$[A'\widehat{v}](h) = \int_G \int_G a(g_1)\widehat{b}_0(g_2)\widehat{v}(g_1^{-1}hg_2)\, d\mu(g_1)\, d\mu(g_2).$$

For operator algebras there are the standard notions of positivity: any operator of the form A^*A is positive; if algebra is realized as operators on a Hilbert space H then $b \in H$ defines a positive functional $f_b(A) = \langle Ab, b\rangle$. Thus the following proposition is a direct consequence of the Proposition 2.18.

Proposition 3.5 [7, Thm. 1] *Let A be an operator, let $D(A)$ be the set of values of the covariant symbol $A(x)$, let $\breve{D}(A)$ be a convex shell of the values of contravariant symbol $\breve{A}(x)$. Let $\hat{D}(A)$ be the set of values of the quadratic form $\langle Ab, b\rangle$ for all vectors $\|b\| = 1$. Then*

$$D(b) \subset \hat{D}(b) \subset \breve{D}(b).$$

Example 3.6 There are at least two very important realizations of symbolical calculus of operators. The theory of pseudodifferential operators (PDO) [18, 64, 68] is based on the Schrödinger representation of the Heisenberg group $\mathbb{H}^n$ (see Subsection 4.1) on the spaces of functions $L_p(\mathbb{R}^n)$ [29]. The Wick and anti-Wick symbolical calculi [7, 10] arise from the Segal-Bargmann representation [59, 5] (see Subsection 4.3) of the same group $\mathbb{H}^n$. Connections (intertwining operators) between these two representations were exploited in [29] to obtain fundamentals of the theory of PDO.

3.2 Functional Calculus and Group Representations

This Subsection illustrates a new approach to functional calculus of operators outlined in [36, 37]. The approach uses the intertwining property for two representations instead of an algebraic homomorphism.

Let $\mathfrak{B}$ be a Banach algebra and $\mathbf{T} \subset \mathfrak{B}$ be its subset of elements. Let G be a group, H be its normal subgroup and $X = G/H$—the corresponding homogeneous space. We assume that there is a representation τ depending from $\mathbf{T} \subset \mathfrak{B}$ defined on measurable functions from $L(X, \mathfrak{B})$ by the formula

$$\tau(g)f(x) = t(g,x)f(g^{-1} \cdot x), \qquad f(x) \in L(X, \mathfrak{B}), \tag{3.11}$$

where $t(g,x) : \mathfrak{B} \to \mathfrak{B}$ depends from $x \in X$ and $g \in G$. It is convenient to use a linear functional $l \in \mathfrak{B}'$ to make the situation more tractable by reducing it to the scalar case. Using l we could define a representation $\tau_l(x)$ on $F(X)$ by the following formula

$$\tau_l(x) : f_l(y) = \langle f(y), l \rangle \mapsto [\tau_l(x) f_l](y) = \langle \tau(x) f(y), l \rangle \,, \tag{3.12}$$

where $f(y) \in L(X, \mathfrak{B})$, $l \in \mathfrak{B}'$. We will understand convergence of all integrals involving τ in a weak sense, i.e. as convergence of all corresponding integrals with τ_l, $l \in \mathfrak{B}'$. We also say that τ is *irreducible* if all τ_l are irreducible.

REMARK 3.7 If $\mathfrak{B}$ is realized as an algebra of operators on a Banach space B then $l \in \mathfrak{B}'$ could be realized as an element of $B \otimes B'$. In this case the formula (3.12) looks like (3.5). The important difference is the following. In (3.5) the representation in the operator algebra $\mathfrak{B}$ arises from a representation in Banach space B and is the same for all elements of $\mathfrak{B}$. Representation τ_l in (3.12) is defined via the representation τ which depends in its turn from a set $\mathbf{T} \subset \mathfrak{B}$. Such representations are usually connected with some (non-linear) geometric actions of a group directly on operator algebra. Examples of these geometric actions are the representation of the Heisenberg group (4.10) leading to the Weyl functional calculus, fractional-linear transformations of operators leading [37] to Dunford-Riesz functional calculus and monogenic functional calculus [36]. Thus such representations contain important information on $\mathbf{T}$.

We also assume that there is a representation π of G in $F(\Omega)$ with a vacuum vector b_0, a test functional l_0 and the system of wavelets (coherent states) b_x, $x \in X$, which were main actors in the previous Section. Let $\pi^*(g) = \pi(g^{-1})^*$ be the adjoint representation of $\pi(g)$ in $F'(\Omega)$.

We need a preselected element $T_0(x) \in L(X, \mathfrak{B})$ which plays a rôle of a *vacum vector* for the representation τ, it is defined by the condition:

$$\int_X \widehat{b}_0(x')\, \tau(x') T_0(x)\, dx' = T_0(x), \tag{3.13}$$

where $\widehat{b}_0(x) = \langle \pi(x^{-1}) b_0, l_0 \rangle$ is the wavelet transform of the vacuum vector $b_0 \in F(\Omega)$ for π.

Lemma 3.8 *A vacuum vector for τ always exists and is given by the formula*

$$T_0(x) = \int_X \widehat{b}_0(x')\, \tau(x') T(x)\, dx', \tag{3.14}$$

where $T(x) \in L(X, \mathfrak{B})$ is an arbitrary element which the integral (3.14) *converges for. If τ is irreducible (i.e. all τ_l* (3.12) *are irreducible) in the linear span of $\tau(x')T(x)$, $x' \in X$ then $T_0(x)$ does not depend from a particular chose of $T(x)$.*

PROOF. First we could easily verify condition (3.13) for the T_0 defined by (3.14):

$$\begin{aligned}
\int_X \widehat{b}_0(x)\, \tau(x) T_0(x_1)\, dx &= \int_X \widehat{b}_0(x)\, \tau(x) \int_X \widehat{b}_0(x')\, \tau(x') T(x_1)\, dx'\, dx \\
&= \int_X \int_X \widehat{b}_0(x)\, \widehat{b}_0(x')\, \tau(x)\tau(x') T(x_1)\, dx'\, dx \\
&= \int_X \left(\int_X \widehat{b}_0(x) \widehat{b}_0(x^{-1}x'')\, dx \right) \tau(x'') T(x_1)\, dx'' \qquad (3.15) \\
&= \int_X \widehat{b}_0(x'')\, \tau(x'') T(x_1)\, dx'' \qquad (3.16) \\
&= T_0(x_1).
\end{aligned}$$

Here we use the change of variables $x'' = x \cdot x'$ in (3.15) and reproducing property (2.10) of $\widehat{b}_0(x)$ in (3.16).

To prove that for any admissible $T(x)$ we will receive the same $T_0(x)$ is enough to pass from the representation τ to representations τ_l (3.12) defined by $l \in \mathfrak{B}'$. Then we deal with scalar valued (not operator valued) functions and knew that one could use any admissible vector $T_l(x) = \langle T(x), l \rangle$ as a vacuum vector in the reconstruction formula (2.6). □

Now we could specify the Definition 1.1 from [36] as follows.

Definition 3.9 Let G, H, $X = G/H$, $\mathfrak{B}$, $\mathbf{T}$, τ, π, F, b_0, T_0 be as described above. One says that a continuous linear $\mathfrak{B}$-valued functional $\Phi_{\mathbf{T}}(\cdot, x) : F(\Omega) \to \mathfrak{B}$, parametrized by a point $x \in X$ and depending from $\mathbf{T} \subset \mathfrak{B}$:

$$\Phi_{\mathbf{T}}(\cdot, x) : f(y) \mapsto [\Phi_{\mathbf{T}} f](x) = \int_{\Omega} f(y) \Phi_{\mathbf{T}}(y, x)\, dy$$

is a *functional calculus* if

1. $\Phi_{\mathbf{T}}$ is an intertwining operator between $\pi(g)$ and $\tau(g)$, namely

$$[\Phi_{\mathbf{T}} \pi(g) f(y)](x) = \tau(g) [\Phi_{\mathbf{T}} f(y)](x), \tag{3.17}$$

   for all $g \in G$ and $f(y) \in F(\Omega)$

2. $\Phi_{\mathbf{T}}$ maps the vacuum vector $b_0(y)$ for the representation π to the vacuum vector $T_0(x)$ for the representation τ:

$$[\Phi_{\mathbf{T}} b_0(y)](x) = T_0(x). \tag{3.18}$$

$\mathfrak{B}$-valued distribution $\Phi_{\mathbf{T}}(y, x_0)$, $s(x_0) = e \in G$ associated with $\mathfrak{B}$-valued linear functional on $F(\Omega)$ is called a *spectral decomposition* of operators $\mathbf{T}$.

Representation τ in (3.17) is defined by (3.11):

$$\tau(g)[\Phi_{\mathbf{T}} f(y)](x) = t(g, x)[\Phi_{\mathbf{T}} f(y)](g^{-1} \cdot x).$$

We could state (3.17) equivalently as

$$[I_y \otimes \tau_x(g)] \Phi(y, x) = [\pi_y^*(g^{-1}) \otimes I_x] \Phi(y, x).$$

REMARK 3.10 The functional calculus $\Phi_{\mathbf{T}}(y, x)$ as defined here has the explicit covariant property with respect to variable x. Thus it could be restored by the representation τ from a single value, e.g. $\Phi_{\mathbf{T}}(y, s^{-1}(e))$, where e is the identity of G. We particularly will calculate only $[\Phi_{\mathbf{T}} f](s^{-1}(e))$ in Subsection 4.3 as the value of a functional calculus. This value is usually denoted by $f(\mathbf{T})$ and is exactly the functional calculus of operators in the traditional meaning.

In particular cases different characteristics of the spectral decomposition could give relevant information on the set of operators $\mathbf{T}$, e.g. the support $\mathrm{supp}_y \Phi_{\mathbf{T}}(y, x_0)$ of $\Phi_{\mathbf{T}}(y, x_0)$

$$f(y) = 0 \ \forall y \in \mathrm{supp}_y \Phi_{\mathbf{T}}(y, x_0) \quad \Rightarrow \quad [\Phi_{\mathbf{T}} f](x_0) = 0$$

is called *(joint) spectrum* of set $\mathbf{T} \subset \mathfrak{B}$. This definition of the spectrum is connected with the *Arveson-Connes spectral theory* [3, 17, 66] while there are several important differences mentioned in [36, Rem. 4.4].

In the paper [36] the approach was illustrated by a newly developed functional calculus for several non-commuting operators based on Möbius transformations of the unit ball in $\mathbb{R}^n$. It was shown in [37, § 7] that the classic Dunford-Riesz functional calculus is generated by a representation of $SL(2, \mathbb{R})$ within this procedure. However an abstract scheme of the approach was not presented yet. We give some its elements here.

From Proposition 2.15 we know a general form of an intertwining operator of two related representations of a group, which could be employed here. Let $l_0(y)$ be the distribution corresponding to a test functional l_0 for the representation π on $F(\Omega)$ such that we could write

$$\langle f(y), l_0 \rangle_{F(\Omega)} = \int_\Omega f(y) l_0(y)\, dy.$$

We denote also by $\pi^*(x) l_0(y)$, $x \in X$, $y \in \Omega$ distributions corresponding to linear functionals $\pi^*(x) l_0$, where $\pi^*(x)$ is the adjoint representation to π on the space $F(\Omega)$.

Proposition 3.11 (Spectral syntesis) *Under assumption of Proposition 2.15 the functional calculus exists and is unique. The spectral decomposition $\Phi_{\mathbf{T}}(y, x)$ as a distribution on Ω is given by the formula*

$$\Phi_{\mathbf{T}}(y, x) = \int_X \pi^*(x) l_0(y)\, \tau(x) T_0(x)\, dx. \tag{3.19}$$

The functional calculus $\Phi_{\mathbf{T}}(\cdot, x)$ as a mapping $F(\Omega) \to \mathfrak{B}$ is given correspondingly

$$\Phi_{\mathbf{T}}(\cdot, x) : f(y) \mapsto [\Phi_{\mathbf{T}} f(y)](x) = \int_X \int_\Omega f(y)\, \pi^*(x') l_0(y)\, \tau(x') T_0(x)\, dy\, dx'. \tag{3.20}$$

Proof. Obviously (3.19) and (3.20) are equivalent. Thus we will prove (3.20) only. For an arbitrary $f(y) \in F(\Omega)$ we could write

$$\begin{aligned}
[\Phi_{\mathbf{T}} f(y)](x) &= \left[\Phi_{\mathbf{T}} \int_X \widehat{f}(x') \pi(x') b_0(y)\, dx'\right](x) && (3.21)\\
&= \int_X \widehat{f}(x') [\Phi_{\mathbf{T}} \pi(x') b_0(y)](x)\, dx' && (3.22)\\
&= \int_X \widehat{f}(x') \tau(x') [\Phi_{\mathbf{T}} b_0(y)](x)\, dx' && (3.23)
\end{aligned}$$

$$= \int_X \widehat{f}(x')\, \tau(x')T_0(x)\, dx' \tag{3.24}$$

$$= \int_X \langle f(y), \pi^*(x')l_0\rangle\ \tau(x')T_0(x)\, dx' \tag{3.25}$$

We use in (3.21) that functions in $F(\Omega)$ are superpositions of coherent states, transformation (3.22) is made by linearity and continuity of $\Phi_{\mathbf{T}}$, step (3.23) is due to condition (3.17) and we finally apply (3.18) to receive (3.24). Thus it is proven that the functional calculus which is continuous, linear, and satisfies to (3.17) and (3.18) (if exists) is unique and given by (3.25). Now we should check that (3.25) really gives the right answer.

We will check first that (3.25) satisfies to (3.17):

$$\begin{aligned}
\tau(g)\Phi(y,x) &= \tau(x_1)\int_X \pi^*(x')l_0(y)\,\tau(x')T_0(x)\,dx' \\
&= \int_X \pi^*(x')l_0(y)\,\tau(g\cdot x')T_0(x)\,dx' \\
&= \int_X \pi^*(g^{-1}\cdot x'')l_0(y)\,\tau(x'')T_0(x)\,dx'' \qquad (3.26)\\
&= \pi^*(g^{-1})\int_X \pi^*(x'')l_0(y)\,\tau(x'')T_0(x)\,dx'' \\
&= \pi^*(g^{-1})\Phi(y,x),
\end{aligned}$$

where we made substitution $x'' = g\cdot x'$ in (3.26). Finally (3.18) directly follows from the condition (3.13). $\square$

Let there exists $L_0(x)\in L'(X,\mathfrak{B})$—a test functional for a vacuum vector $T_0(x)$ and representation τ, i.e.

$$\langle T_0, L_0\rangle_{L(X,\mathfrak{B})} = \int_X \left\langle \tau(x^{-1}T_0), L_0\right\rangle_{L(X,\mathfrak{B})} \langle \tau(x)T_0, L_0\rangle_{L(X,\mathfrak{B})}\, dx,$$

where

$$\langle T_0, L_0\rangle_{L(X,\mathfrak{B})} = \int_X \langle T_0(x), L_0(x)\rangle_{\mathfrak{B}}\, dx$$

and $\langle T_0(x), L_0(x)\rangle_{\mathfrak{B}}$ is the pairing between $\mathfrak{B}$ and $\mathfrak{B}'$.

Proposition 3.12 (Spectral analysis) *If a $\mathfrak{B}$-valued function $F(x)$ from $L(X,\mathfrak{B})$ belongs to the closer of the linear span of $\tau(x')T_0(x)$, $x'\in X$ then*

$$F(x) = [\Phi_{\mathbf{T}} f(y)](x),$$

where

$$f(y) = \int_X \left\langle \tau(x^{-1}F), L_0\right\rangle_{L(X,\mathfrak{B})} \pi(x)b_0(y)\, dx. \tag{3.27}$$

PROOF. The formula (3.27) is just another realization of intertwining operator (2.12) from Proposition 2.15. □

Let $K : F(\Omega_2) \to F(\Omega_2)$ be an intertwining mapping between two representations π_1 and π_2 of groups G_1 and G_2 in spaces $F(\Omega_1)$ and $F(\Omega_1)$ respectively. Let $K(z, y)$, $z \in \Omega_1$, $y \in \Omega_2$ be the Schwartz kernel of K.

Theorem 3.13 (Mapping of spectral decompositions) *Let*

$$\begin{aligned} f_1(z) = [Kf_2](z) &= \int_{\Omega_2} f_2(y) K(z, y)\, dy \\ \Phi_{\mathbf{T}_1}(z, x) &= \int_{\Omega_2} K(z, y) \Phi_{\mathbf{T}_2}(y, x)\, dy, \end{aligned}$$

where a functional calculus $\Phi_{\mathbf{T}_2}$ is defined by representations π_2 and τ. Then $\Phi_{\mathbf{T}_2}(z, x)$ is a functional calculus for π_1 and τ and we have an identity:

$$[\Phi_{\mathbf{T}_1} f_1(z)](x) = [\Phi_{\mathbf{T}_2} f_2(y)](x). \tag{3.28}$$

PROOF. The intertwining property for $\Phi_{\mathbf{T}_2}(z, x)$ follows from transitivity. The identity (3.28) is a simple application of the Fubini theorem:

$$\begin{aligned} [\Phi_{\mathbf{S}} g(z)](x) &= \int_{\Omega_1} g(z) \Phi_{\mathbf{S}}(z, x)\, dz \\ &= \int_{\Omega_1} g(z) \int_{\Omega_2} K(z, y) \Phi_{\mathbf{T}}(y, x)\, dy\, dz \\ &= \int_{\Omega_2} \int_{\Omega_1} g(z) K(z, y)\, dz\, \Phi_{\mathbf{T}}(y, x)\, dy \\ &= \int_{\Omega_2} g(f(y)) \Phi_{\mathbf{T}}(y, x)\, dy \\ &= [\Phi_{\mathbf{T}} g(f(y))](x). \end{aligned}$$

□

This Theorem could be turned in the spectral mapping theorem under suitable conditions [36, Thm. 3.19].

3.3 Remarks on Future Generalizations

We would like to make some final remarks and mention possible generalizations.

REMARK 3.14 Our consideration could be applied even for a more general setting: coherent states in an arbitrary locally convex topological vector space. The transition to topological case is straightforward: anywhere one should substitute norm estimations by topological conditions in the standard way.

REMARK 3.15 Irreducibility of π guarantees that a vacuum vector and a test functional exist, even any vector could be taken as a vacuum and coherent states will be dense in whole B. But irreducibility is not necessary for the theory in fact.

Example 3.16 One could take a normal base $\{b_j\}$, $j \in \mathbb{Z}$ in B indexed by integers (both positive and negative) and take a biortogonal system of linear functionals $\{l_j\}$: $\langle b_j, l_k\rangle = \delta_{jk}$. Then the group $\mathbb{Z}$ of integers has a representation σ on B, which is defined on the base as follows $\sigma(k)b_j = b_{j+k}$, $k \in \mathbb{Z}$. The adjoint representation is $\sigma^*(k)l_j = l_{j-k}$, $k \in \mathbb{Z}$. Then b_0 and l_0 could be taken as a vacuum vector and test functional correspondingly and b_0 is cyclic for B. All results of this Subsection 2 are true then. However the representation σ is far from irreducible one. One could find a proof that cyclic vectors exist for a wide family of reducible representations in [19].

REMARK 3.17 Many of our results (e.g., (3.9)) are based only on the formula (2.2) and are not related to a group structure within the set $X = G/H$, which indexed coherent states. Thus one could give a general definition of coherent states based only on (2.2) interpreted as a *resolution of unity*. For Hilbert space such general definitions are used in [7, 10, 41, 43].

However other results like (2.10) use the group structure and future incites could be given by the harmonic analysis of particular groups involved into subject. For this reasons we stay with our Definition 2.1, which is not being of the extreme generality could give more information in particular cases.

REMARK 3.18 We often meet in applications non-linear representations. But such representations could be linearized in the standard way. Let a non-linear representation π of G be defined in a linear space B such that $\pi(g) : B \to B$ are uniformly bounded for all $g \in G$. In such a case for any G-homogeneous space X with an invariant measure $d\mu(x)$ all linear integrals of the form

$$f(b) = \int_X f(x)\pi(x)b\,d\mu(x), \qquad f \in L_1(X),\ b \in B, \|b\| = 1 \tag{3.29}$$

converge in B. They form an invariant linear subspace B_X of B with respect to an associated linear representation $\widetilde{\pi}$ of G (or even of the convolution algebra $L_1(G)$) defined as follows:

$$\widetilde{\pi}(g) : B_X \rightarrow B_X : f(b) \mapsto \widetilde{\pi}(g)f(b) = \int_X f(g \cdot x)\pi(x)b\, d\mu(x), \qquad (3.30)$$

where $f \in L_1(G)$, $b \in B$. To make integrals (3.29) convergent for locally convex topological spaces one may restrict symbols $f(x)$ to smooth functions with compact support. Then many of previous results could be translated to this setting.

REMARK 3.19 There is a powerful way to generalize the above construction. One could define the wavelet transform $\mathcal{W} : b \rightarrow \langle \pi(x^{-1})b, l_0 \rangle$ not via a scalar $\mathbb{C}$-valued product with a linear functional l_0 but as $L_0(\pi(x^{-1})b)$ for a linear map $L_0 : B \rightarrow \mathfrak{B}$ where $\mathfrak{B}$ is the space of bounded linear operators on B. Then the wavelet transform $\widehat{b}(x)$ is $\mathfrak{B}$-valued function on X. The inverse wavelet transform then could be again defined by the formula (2.6). For $\mathfrak{B}$-valued wavelet transform $\chi(h)$ should not be a $\mathbb{C}$-valued character of H. Instead of it one could consider arbitrary representation of H in B such that $L_0(\pi(h)b) = \chi(h)L_0(b)$ for all $b \in B$, $h \in H$. As a simple example of a wavelet transform one could consider $\mathbb{R}^{1,1}$-valued function theory constructed in [39].

REMARK 3.20 Another generalization is possible within a framework of multiresolution wavelet theory [13], which is very useful for certain reducible representation [15]. Namely, one considers not a single vacuum vector b_0 but a family of such vectors $b_{0,j}$, $1 \leq j \leq n$ and associated set of test functionals $l_{0,j}$. Coherent states $b_{x,j}$ are indexed now by $x \in X$ and $1 \leq j \leq n$. We have a family of wavelet transforms $\mathcal{W}_j$:

$$\widehat{b}(j,x) = [\mathcal{W}_j b](x) = \left\langle \pi(x^{-1})b, l_{0,j} \right\rangle, \qquad x \in X,\ 1 \leq j \leq n$$

and single reconstruction formula

$$\mathcal{M}\widehat{b}(j,x) = \int_X \sum_{j=1}^{n} \widehat{b}(j,x)\pi(x)b_{0,j}\, dx.$$

It is interesting to note that this reconstruction formula could be found already in papers of M.G. KREIN [47].

4 Examples

We are going to demonstrate that the above construction is not only algebraically attractive but also belongs to the heart of analysis. More examples could be found in [15, 37, 39].

4.1 The Heisenberg Group and Schrödinger Representation

We will consider a realization of the previous results in a particular cases of the Fourier transform and Segal-Bargmann [5, 59] type spaces $F_p(\mathbb{C}^n)$. They arise from representations of the Heisenberg group $\mathbb{H}^n$ [25, 28, 69] on $L_p(\mathbb{R}^n)$.

The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ spanned by $\{T, P_j, Q_j\}$, $n = 1, \ldots, n$ is defined by the commutation relations:

$$[P_i, Q_j] = T\delta_{ij}. \tag{4.1}$$

They are known from quantum mechanics as the canonical commutation relations of coordinates and momentum operators. An element $g \in \mathbb{H}^n$ could be represented as $g = (t, z)$ with $t \in \mathbb{R}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and the group law is given by

$$g * g' = (t, z) * (t', z') = (t + t' + \frac{1}{2}\sum_{j=1}^{n} \Im(\bar{z}_j z'_j), z + z'), \tag{4.2}$$

where $\Im z$ denotes the imaginary part of a complex number z. The Heisenberg group is (non-commutative) nilpotent step 2 Lie group.

We take a representation of $\mathbb{H}^n$ in $L_p(\mathbb{R}^n)$, $1 < p < \infty$ by operators of shift and multiplication [69, § 1.1]:

$$g = (t, z) : f(y) \to [\sigma_{(t,z)} f](y) = e^{i(2t - \sqrt{2}vy + uv)} f(y - \sqrt{2}u), \qquad z = u + iv, \tag{4.3}$$

i.e., this is the Schrödinger type representation with parameter $\hbar = 1$. These operators are isometries in $L_p(\mathbb{R}^n)$ and the adjoint representation $\pi^*_{(t,z)} = \pi_{(-t,-z)}$ in $L_q(\mathbb{R}^n)$, $p^{-1} + q^{-1} = 1$ is given by a formula similar to (4.3).

4.2 Wavelet Transforms for $\mathbb{H}^n$ in Function Spaces

Example 4.1 We start from the subgroup $H = \mathbb{R}^{n+1} = \{(t, z) \mid \Im(z) = 0\}$. Then $X = G/H = \mathbb{R}^n$ and an invariant measure coincides with the Lebesgue measure. Mappings $s : \mathbb{R}^n \to \mathbb{H}^n$ and $r : \mathbb{H}^n \to H$ are defined by the identities $s(x) = (0, ix)$, $s^{-1}(t, z) = \Im z$, $r(t, u + iv) = (t, u)$. The composition law $s^{-1}((t, z) \cdot s(x)) = x + u$ reduces to Euclidean shifts on $\mathbb{R}^n$. We also find $s^{-1}((s(x_1))^{-1} \cdot s(x_2)) = x_2 - x_1$ and $r((s(x_1))^{-1} \cdot s(x_2)) = 0$.

We consider the representation $\sigma(g)$ of $\mathbb{H}^n$ in the space of smooth rapidly decreasing functions $B = \mathcal{S}(\mathbb{R}^n)$. As a character of $H = \mathbb{R}^{n+1}$ we take the $\chi(t, u) = e^{2it}$. The corresponding test functional l_0 satisfying to 2.19.3 is the integration $l_0(f) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(y)\,dy$. Thus the wavelet transform is as follows

$$\widehat{f}(x) = \int_{\mathbb{R}^n} \sigma(s(x)^{-1})f(y)\,dy = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\sqrt{2}xy}f(y)\,dy \tag{4.4}$$

and is nothing else but the Fourier transform[4].

Now we arrive to the absence of a vacuum vector in B, indeed there is no a $f(x) \in \mathcal{S}(\mathbb{R}^n)$ such that

$$[\sigma(t,u)f](y) = \chi(t,u)f(y) \iff e^{i2t}f(y-\sqrt{2}u) = e^{i2t}f(y).$$

There is a way out accordingly to Subsection 2.3. We take $B' = L_\infty(\mathbb{R}^n) \supset B$ and the vacuum vector $b_0(y) \equiv (2\pi)^{-n/2} \in B'$. Then coherent states are $b_x(y) = (2\pi)^{-n/2}e^{-i\sqrt{2}xy}$ and the inverse wavelet transform is defined by the inverse Fourier transform

$$f(y) = \int_{\mathbb{R}^n} \widehat{f}(y)b_x(y)\,dx = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(y)e^{-i\sqrt{2}xy}\,dx.$$

The condition 2.19.4 $\mathcal{MW} : B \to B$ follows from the composition of two facts $\mathcal{W} : B \to B$ and almost identical to it $\mathcal{M} : B \to B$, which are proved in standard analysis textbooks (see for example [33, § IV.2.3]). To check scaling (2.14) according to the tradition in analysis [28] we take a probe vector $p_0 = e^{-y^2/2} \in B$. Due to well known formula $\int_{-\infty}^{+\infty} e^{-y^2/2}dy = (2\pi)^{1/2}$ of real analysis we have

$$\begin{aligned}
\left\langle \int_X \langle \widehat{p}_0(x), l_0\rangle\, b_x\,dx, l_0 \right\rangle &= (2\pi)^{-n}\int\int\int e^{i\sqrt{2}xy}e^{-y^2/2}\,dy e^{-i\sqrt{2}xw}\,dx\,dw \\
&= (2\pi)^{n/2}\int_{\mathbb{R}^n} e^{-y^2/2}dy \\
&= \langle p_0, l_0\rangle.
\end{aligned}$$

Thus our scaling is correct. $\mathcal{W}$ and $\mathcal{M}$ intertwine the left regular representation — multiplication by $e^{i\sqrt{2}yv}$ with operators

$$\begin{aligned}
[\lambda(g)f](x) &= \chi(r(g^{-1}\cdot x))f(g^{-1}\cdot x) \\
&= e^{i\sqrt{2}\cdot 0}f(x-\sqrt{2}u) = f(x-\sqrt{2}u),
\end{aligned}$$

i.e. with Euclidean shifts. From the identity $\langle \mathcal{W}v, \mathcal{M}^*l\rangle_{F(X)} = \langle v, l\rangle_B$ (2.8) follows the Plancherel's identity:

$$\int_{\mathbb{R}^n} \widehat{v}(y)\widehat{l}(y)\,dy = \int_{\mathbb{R}^n} v(x)l(x)\,dx.$$

[4]The *inverse* Fourier transform in fact. In our case the signs selection is opposite to the standard one, but we will neglect this difference.

These are basic and important properties of the Fourier transform.

The Schrödinger representation is irreducible on $\mathcal{S}(\mathbb{R}^n)$ thus $\mathcal{M} = \mathcal{W}^{-1}$. Thereafter integral formulas (2.15) and (2.16) representing operators $\mathcal{M}\mathcal{W} = \mathcal{W}\mathcal{M} = 1$ correspondingly give an integral resolution for a convolution with the Dirac delta $\delta(x)$. We have integral resolution for the Dirac delta

$$\delta(x-y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi(x-y)}\, d\xi.$$

All described results on the Fourier transform are a part of any graduate curriculum. What is a reason for a reinvention of a bicycle here? First, the same path works with minor modifications for a function theory in $\mathbb{R}^{1,1}$ described in [39]. Second we will use this interpretation of the Fourier transform in Example 4.5 for a demonstration how the Weyl functional calculus fits in the scheme outlined in [36, 37] and Subsection 3.2.

Remark 4.2 Of course, the Heisenberg group is not the only possible source for the Fourier transform. We could consider the "$ax+b$" group [69, Chap. 7] of the affine transformations of Euclidean space $\mathbb{R}^n$. The normal subgroup $H = \mathbb{R}$ of dilations generates the homogeneous space $X = \mathbb{R}^n$ on which shifts act simply transitively. The Fourier transform deduced from this setting will naturally exhibit scaling properties. We could alternatively consider a group $\mathbb{M}^n$ of Möbius transformation [14, Chap. 2] in $\mathbb{R}^{n+1}$ which map upper half plane to itself. Then there is an induced action of $\mathbb{M}^n$ on $\mathbb{R}^n$—the boundary of upper half plane. $\mathbb{M}^n$ generated by composition of the affine transformations and the Kelvin inverse [14, Chap. 2]. If we take the normal subgroup H generated by dilations and the Kelvin inverse then the quotient space X will again coincide with $\mathbb{R}^n$ and we immediately arrive to the above case. On the other hand the Fourier transform derived in such a way could be easily connected with the plane wave decomposition [65] in Clifford analysis [12, 22].

Example 4.3 As a subgroup H we select now the center of $\mathbb{H}^n$ consisting of elements $(t, 0)$. Of course $\Omega = G/H$ isomorphic to $\mathbb{C}^n$ and mapping $s : \mathbb{C}^n \to G$ simply is defined as $s(z) = (0, z)$. The Haar measure on $\mathbb{H}^n$ coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [69, § 1.1] thus the invariant measure on Ω also coincides with the Lebesgue measure on $\mathbb{C}^n$. Note also that composition law $s^{-1}(g \cdot s(z))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s^{-1}((s(z_1))^{-1} \cdot s(z_2)) = z_2 - z_1$ and $r((s(z_1))^{-1} \cdot s(z_2)) = \frac{1}{2}\Im \bar{z}_1 z_2$.

As a "vacuum vector" we will select the original *vacuum vector* of quantum mechanics—the Gauss function $f_0(x) = e^{-x^2/2}$ which belongs to all $L_p(\mathbb{R}^n)$. Its transformations are defined as follow:

$$f_g(x) = [\pi_{(t,z)} f_0](x) \quad = \quad e^{i(2t-\sqrt{2}vx+uv)}\, e^{-(x-\sqrt{2}u)^2/2}$$

$$\begin{aligned} &= e^{2it-(u^2+v^2)/2} e^{-((u-iv)^2+x^2)/2+\sqrt{2}(u-iv)x} \\ &= e^{2it-z\bar{z}/2} e^{-(\bar{z}^2+x^2)/2+\sqrt{2}\bar{z}x}. \end{aligned}$$

Particularly $[\pi_{(t,0)} f_0](x) = e^{-2it} f_0(x)$, i.e., it really is a vacuum vector in the sense of our definition with respect to H. For the same reasons we could take $l_0(x) = e^{-x^2/2} \in L_q(\mathbb{R}^n)$, $p^{-1} + q^{-1} = 1$ as the test functional.

It could be shown that $[\pi_{(0,z)} f_0](x)$ belongs to $L_q(\mathbb{R}^n) \otimes L_p(\mathbb{C}^n)$ for all $p > 1$ and $q > 1$, $p^{-1} + q^{-1} = 1$. Thus transformation (2.4) with the kernel $[\pi_{(0,z)} f_0](x)$ is an embedding $L_p(\mathbb{R}^n) \to L_p(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned} \widehat{f}(z) &= \left\langle f, \pi_{s(z)} f_0 \right\rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} f(x)\, e^{-z\bar{z}/2}\, e^{-(z^2+x^2)/2+\sqrt{2}zx}\, dx \\ &= e^{-z\bar{z}/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(x)\, e^{-(z^2+x^2)/2+\sqrt{2}zx}\, dx. \end{aligned} \tag{4.5}$$

Then $\widehat{f}(g)$ belongs to $L_p(\mathbb{C}^n, dg)$ or its preferably to say that function $\breve{f}(z) = e^{z\bar{z}/2}\widehat{f}(t_0, z)$ belongs to space $L_p(\mathbb{C}^n, e^{-|z|^2}dg)$ because $\breve{f}(z)$ is analytic in z. Such functions for $p = 2$ form the *Segal-Bargmann space* $F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ of functions [5, 59], which are analytic by z and square-integrable with respect the Gaussian measure $e^{-|z|^2}dz$. For this reason we call the image of the transformation (4.5) by *Segal-Bargmann type space* $F_p(\mathbb{C}^n, e^{-|z|^2}dg)$. Analyticity of $\breve{f}(z)$ is equivalent to condition $(\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2} z_j I)\widehat{f}(z) = 0$.

The integral in (4.5) is the well-known Segal-Bargmann transform [5, 59]. Inverse to it is given by a realization of (2.6):

$$\begin{aligned} f(x) &= \int_{\mathbb{C}^n} \widehat{f}(z) f_{s(z)}(x)\, dz \\ &= \int_{\mathbb{C}^n} \widehat{f}(u,v) e^{iv(u-\sqrt{2}x)}\, e^{-(x-\sqrt{2}u)^2/2}\, du\, dv \qquad (4.6) \\ &= \int_{\mathbb{C}^n} \breve{f}(z) e^{-(\bar{z}^2+x^2)/2+\sqrt{2}\bar{z}x}\, e^{-|z|^2}\, dz. \end{aligned}$$

The corresponding operator $\mathcal{P}$ (2.7) is an identity operator $L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ and (2.7) gives an integral presentation of the Dirac delta.

Integral transformations (4.5) and (4.6) intertwines the Schrödinger representation (4.3) with the following realization of representation (2.5):

$$\begin{aligned} \lambda(t,z) f(w) &= \widehat{f}_0(z^{-1} \cdot w) \bar{\chi}(t + r(z^{-1} \cdot w)) && (4.7) \\ &= \widehat{f}_0(w - z) e^{it + i\Im(\bar{z}w)} && (4.8) \end{aligned}$$

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|z|^2}dg) \to F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ is of a separate interest and is a principal ingredient in Berezin quantization [10, 16]. We could easy find its kernel from (2.10). Indeed, $\widehat{f}_0(z) = e^{-|z|^2}$, then the kernel is

$$\begin{aligned} K(z,w) &= \widehat{f}_0(z^{-1}\cdot w)\bar{\chi}(r(z^{-1}\cdot w)) \\ &= \widehat{f}_0(w-z)e^{i\Im(\bar{z}w)} \\ &= \exp\left(\frac{1}{2}(-|w-z|^2 + w\bar{z} - z\bar{w})\right) \\ &= \exp\left(\frac{1}{2}(-|z|^2 - |w|^2) + w\bar{z}\right). \end{aligned}$$

To receive the reproducing kernel for functions $\check{f}(z) = e^{|z|^2}\widehat{f}(z)$ in the Segal-Bargmann space we should multiply $K(z,w)$ by $e^{(-|z|^2+|w|^2)/2}$ which gives the standard reproducing kernel $= \exp(-|z|^2 + w\bar{z})$ [5, (1.10)].

4.3 Operator Valued Representations of $\mathbb{H}^n$

We proceed now with our main targets: wavelets in operator algebras. We shell show that well-known and new functional calculi are realizations of the scheme from Subsection 3.2.

Convention 4.4 [2] Let B be a Banach space. We will say that an operator $A : B \to B$ is *unitary* if A is invertible and $\|Ab\| = \|b\|$ for all $b \in B$. An operator $A : B \to B$ is called *self-adjoint* if the operator $\exp(iA)$ is unitary. In the Hilbert space case this convention coincides with the standard definition.

Let $T_1, \ldots, T_n$ be an n-tuples of selfadjoint linear operators on a Banach space B. We put for our convenience $T_0 = I$—the identical operator. It follows from the Trotter-Daletskii[5] formula [54, Thm. VIII.31] that any linear combination $\sum_{j=0}^{n} a_j T_j$ is again a selfadjoint operator. We will consider a set of unitary operators

$$T(a_0, a_1, \ldots, a_n) = \exp\left(i\sum_{j=0}^{n} a_j T_j\right) \tag{4.9}$$

parametrized by vectors $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$. Particularly $T(0, 0, \ldots, 0) = I$. A family of their transformations $\omega(t, z)$, $t \in \mathbb{R}$, $z \in \mathbb{C}^n$ is defined by the

[5]The formula is usually attributed to Trotter alone. It is widely unknown that the result appeared in [20] also.

rule

$$\omega(t,z)T(a_0,a_1,\ldots,a_n) = T\left(a_0 + t + \sum_{j=1}^{n}(u_j v_j - \sqrt{2}a_j u_j),\right.$$
$$\left. a_1 - \sqrt{2}v_1, \ldots, a_n - \sqrt{2}v_n\right), \qquad (4.10)$$

where $z_j = u_j + iv_j$. A direct calculation shows that $\omega(t', z')\omega(t'', z'') = \omega(t' + t'' + \frac{1}{2}\Im(\bar{z}'z''), z' + z'')$—this is a non-linear geometric representation of the Heisenberg group $\mathbb{H}^n$. We could observe that

$$T(a_0,a_1,\ldots,a_n) = \omega(a_0,a)T(0,0,\ldots,0) = \omega(a_0,a)T_0 = \omega(a_0,a)I,$$

where $a = (ia_1, \ldots, ia_n)$. Obviously all transformations $\omega(t,z)$ are isometries if the norm of elements $T(a_0,a_1,\ldots,a_n)$ is defined as their operator norm.

The representation ω (4.10) is not linear and we would like to use the procedure outlined in Remark 3.18. We construct the linear space of operator valued functions $L(\mathbb{R}^n, \mathfrak{B})$ for a $\mathbb{H}^n$-homogeneous space X as follows

$$[\mathcal{T}f](t) = \int_X f(x)\,\omega(s(x))\,dx\,T(t), \qquad t \in \mathbb{R}^n,\ s(x) \in \mathbb{H}^n. \qquad (4.11)$$

We also extend the representation ω to $L(X, \mathfrak{B})$ in accordance witn (3.30).

We will go on with coherent states defined by such a representation. In the notations of Subsection 3.2 operators $T_1, \ldots, T_n$ form a set $\mathbf{T}$ defining the representation $\tau = \omega$ in (4.10) with $T_0(x) = I$ being a vacuum vector.

Example 4.5 We are ready to demonstrate that the Weyl functional calculus is an application of Definition 3.9 and Example 4.1 as was announced in [36, Remark 4.3]. Consider again the subgroup $H = \mathbb{R}^{n+1} = \{(t,z) \mid \Im(z) = 0\}$ and a realization of scheme from Subsection 2.3 for this subgroup. Then the first paragraph of Example 4.1 is applicable here.

We could easily see that $\omega(t, u_1, \ldots, u_n)I = e^{it+i\sum_1^n u_j} I$ and we select the identity operator I times $(2\pi)^{-n/2}$ as the vacuum vector $T_0(x)$ of the representation ω. Thereafter the transformation $\mathcal{T} : S(\mathbb{R}^n) \to L(\mathbb{R}^n, \mathfrak{B})$ (4.11) is exactly the inverse wavelet transform for the representation ω. This transformation is defined at least for all $f \in S(\mathbb{R}^n)$. The space $S(\mathbb{R}^n)$ is the image of the wavelet (Fourier) transform (4.4). Thus as outlined in Proposition 3.11 we could construct an intertwining operator $\mathcal{F}$ between σ (4.3) and ω (4.10) from the formula (3.20) as follow (see Remark 3.10):

$$\begin{aligned}[\Phi_{\mathbf{T}}f](0) &= \mathcal{M}_\omega \mathcal{W}_\sigma f = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(x)\,\omega(0,x_1,\ldots,x_n)\,I\,dx \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(x)\,e^{i\sum_1^n x_j T_j}\,dx. \qquad (4.12)\end{aligned}$$

This formula is exactly the integral formula for the Weyl functional calculus [2, 52, 67]. As an example one could define a function

$$e^{-\sum_1^n T_j^2/2} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} e^{i\sum_1^n x_j T_j}\, dx. \tag{4.13}$$

As we have seen (3.19) one could formally write the integral kernel from (4.12) as convolution of the integral kernels for $\mathcal{W}_\sigma$ and $\mathcal{M}_\omega$:

$$\Phi(y,0) = \int_{\mathbb{R}^n} e^{-i\sum_1^n y_j x_j} e^{i\sum_1^n x_j T_j}\, dx. \tag{4.14}$$

This expression looks very formal, but it is possible to give it a precise mathematical meaning as an operator valued distribution. Such an approach was explored by ANDERSON in [2]. The support of this distribution was defined as the *Weyl joint spectrum* for n-tuple of non-commuting operators $T_1, \ldots, T_n$ and studied in [2].

REMARK 4.6 As was mentioned in Remark 4.2 one could construct the Fourier transform from representations of $ax+b$ group or the group $\mathbb{M}^n$ of Möbius transformations of the upper half plane. Analogously one could deduce the Weyl functional calculus as an intertwining operator between two representation of this group. The Cauchy kernel $G(x)$ [12, § 9] in Clifford analysis is the kernel of the Cauchy integral transform

$$f(y) = \int_{\partial\Omega} G_y(x)\vec{n}(x) f(x)\, d\sigma(x), \qquad y \in \Omega,$$

where $\vec{n}(x)$ is the outer unit vector orthogonal to $\partial\Omega$ and $d\sigma(x)$ is the surface element at a point x. The Cauchy integral formula (as any wavelet transform) intertwines two representations acting on Ω and $\partial\Omega$ of $\mathbb{M}^n$ similarly to the case of complex analysis [37, § 6]. Thus we could apply here Theorem 3.13 on a mapping of spectral distributions. The formula (3.28) take the form

$$\Phi_W f = \int_{\partial\Omega} \Phi_W(G)(x)\vec{n}(x) f(x)\, d\sigma(x),$$

where Φ_W stands for the Weyl functional calculus. This gives another interpretation for the main result of the paper [30, Thm. 5.4].

There is no a reason to limit ourself only to the case of subgroup $H = \mathbb{R}^{n+1} = \{(t,z) \mid \Im(z) = 0\}$. Thus we proceed with the next example.

Example 4.7 In an analogy with Example 4.3 let us consider now the wavelet theory associated to the subgroup $H = \mathbb{R}^1 = \{(t,0)\}$ and the representation ω. The first paragraph of Example 4.3 depends only on $G = \mathbb{H}^n$ and $H = \mathbb{R}^1$ and thus is applicable in our case.

It is easy to see from formula (4.10) that any operator valued function $[\mathcal{T}f](x)$ (4.11) is an eigen vector for $\omega(h)$, $h \in H$. To be concise with function models we select as a vacuum vector the operator $\exp(-\sum_1^n T_j^2)$ (4.13). Then the condition (3.13) immediately follows from (4.13) Thus we could define a functional calculus $\Phi : F_p(\mathbb{C}^n) \to L(\mathbb{C}^n, \mathfrak{B})$ by the formula (see Remark 3.10):

$$\begin{aligned}
[\Phi_{\mathbf{T}} f](0) &= \int_{\mathbb{C}^n} f(z)\omega(0,z)\exp(-\sum_1^n T_j^2/2)\,dz \\
&= \int_{\mathbb{C}^n} f(z)\omega(0,z)(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} e^{i\sum_1^n x_j T_j}\,dx\,dz \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} f(z) \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2}\omega(0,z) e^{i\sum_1^n x_j T_j}\,dx\,dz \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{3n}} \exp\sum_{j=1}^n \left(-\frac{x_j^2}{2} + i(v_j - \sqrt{2}x_j)u_j + i(x_j - \sqrt{2}v_j)T_j\right) \\
&\qquad \times f(z)\,dx\,du\,dv. \qquad (4.15)
\end{aligned}$$

The last formula could be rewritten for mutually commuting operators T_j as follows:

$$\begin{aligned}
[\Phi_{\mathbf{T}} f](0) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \exp\sum_{j=1}^n \left(-\frac{x_j^2}{2} + i x_j (T_j - \sqrt{2}u_j)\right)\,dx \\
&\qquad \times \exp\sum_{j=1}^n i(v_j u_j - \sqrt{2} v_j T_j)\, f(z)\,du\,dv \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} \exp\sum_{j=1}^n \left(i v_j (u_j - \sqrt{2}T_j) - \frac{(T_j - \sqrt{2}u_j)^2}{2}\right) f(z)\,dz
\end{aligned}$$

where the exponent of operator is defined in the standard sense, e.g. via the Weyl functional calculus (4.12) or the Taylor expansion. The last formula is similar to (4.6). This is very natural for commuting operators as well as that for non-commuting operators fromula (4.15) is more complicated.

The spectral distribution

$$\Phi_{\mathbf{T}}(z,0) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\sum_{j=1}^n \left(-\frac{x_j^2}{2} + i(v_j - \sqrt{2}x_j)u_j + i(x_j - \sqrt{2}v_j)T_j\right)\,dx$$

derived from (4.15) contains at least as much information on operators T_1, ..., T_n as the Weyl distribution (4.14) and deserves a careful separate investigation. We will just mention in conclusion that the Segal-Bargmann space is an example of the Fock space—space of second quantization for bosonic fields. Thus the functional calculus based on the Segal-Bargmann model sketched here seems to be an appropriate model for quantized bosonic fields.

Acknowledgments

The author was supported by the grant INTAS 93–0322–Ext. I am grateful to the Prof. H. Feichtinger, Prof. J. Klauder, Dr. D.S. Kalyuzhny, and Prof. I. Segal for their valuable comments on the subject of the paper. The history of the Trotter-Daletskii formula mentioned in Footnote 5 was told me by Dr. V. Kushnirevitch. I am in debt to Dr. J. Cnops, Dr. M.V. Kuzmin, M.Z. Neyman, and O.P. Pilipenko who shared with me their philological knowledge.

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