A New Look at the Multidimensional Inverse Scattering Problem[1]

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Abstract

As a prototype of an evolution equation we consider the Schrödinger equation

$$i(d/dt)\Psi(t) = H\,\Psi(t), \quad H = H_0 + V(\mathbf{x})$$

for the Hilbert space valued function $\Psi(\cdot) : \mathbb{R} \to \mathcal{H} = L^2(\mathbb{R}^\nu)$ which describes the state of the system at time t in space dimension $\nu \geq 2$. The kinetic energy operator H_0 may be $H_0 = -(1/2m)\,\Delta$ (nonrelativistic quantum mechanics for a particle of mass m), $H_0 = \sqrt{-\Delta + m^2}$ (relativistic kinematics, Klein-Gordon equation), the Dirac operator, or ..., while the potential $V(\mathbf{x}) \to 0$ suitably as $|\mathbf{x}| \to \infty$.

We present a *geometrical* approach to the inverse scattering problem. For given scattering operator S we show uniqueness of the potential, we give explicit limits of the high-energy behavior of the scattering operator, and we give reconstruction formulas for the potential.

Our mathematical proofs closely follow physical intuition. A key observation is that at high energies translation of wave packets dominates over spreading during the interaction time. Extensions of the method cover e.g. Schrödinger operators with magnetic fields, multiparticle systems, and wave equations.

1 Introduction, the Schrödinger Equation

The **Schrödinger equation** is a linear evolution equation for a function of time $t \in \mathbb{R}$ with values in a state space (phase space) $\mathcal{H}$ which is a Hilbert space:

$$\Psi(\cdot) : \mathbb{R} \to \mathcal{H}.$$

The initial value problem reads

$$i\frac{d}{dt}\,\Psi(t) = H\,\Psi(t), \quad \Psi(0) = \Psi, \tag{1.1}$$

with a linear operator H acting on $\mathcal{H}$. (We use units where Planck's constant $\hbar = 1$.) This type of equation includes as special cases nonrelativistic and relativistic quantum mechanics, the Dirac equation, the linear wave equation (with the usual method to transform a second order equation into a first order system), and other evolution

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equations. Splitting off the factor i is just for convenience because in important applications H is symmetric. Since the operator H is typically unbounded, care is needed to ensure solvability of the equation. In the models mentioned above the operator H is self-adjoint on a suitably chosen domain $\mathcal{D}(H)$. Then Stone's theorem (or the spectral theorem and functional calculus, see e.g. [16]) ensure that the exponential $\exp\{-itH\}$ is a well defined unitary operator for all $t \in \mathbb{R}$ and that

$$\Psi(t) = e^{-itH}\Psi \tag{1.2}$$

is the unique global solution of the initial value problem (1.1).

In the following we will describe our geometrical approach to the inverse problem for the Schrödinger equation as an equation which describes the motion of particles according to the laws of quantum mechanics. We will exploit physical intuition to help us solve the mathematical problems. The methods and results carry over to other evolution equations as well.

We will discuss the differences of nonrelativistic and relativistic kinematics as far as they are relevant here. The time scales for interaction and for spreading of wave functions differ at high energies. This implies the simplicity of the leading behavior of the scattering operator because only the translational part of the time evolution matters as long as the interaction is strong. We obtain explicit formulas for the high energy scattering operator which can be used to reconstruct the potential uniquely. In the present paper we want to explain *why* the statements are true and how physical intuition and mathematical proofs are closely analogous. While we give all major steps of the proofs for two typical examples we refer to the papers for some more technical estimates and further examples.

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2 Particles in Quantum Mechanics

We describe the state of a quantum mechanical particle in ν-dimensional space by a normalized vector $\Psi \in \mathcal{H}$. (Due to the superposition principle we may restrict ourselves to pure states and choose a normalized vector instead of the equivalence class of vectors differing by a global phase factor.) The vector can conveniently be represented in various ways (similar to basis changes in linear algebra), e.g. by a square integrable function $\psi(\cdot) \in L^2(\mathbb{R}^\nu, dx)$ with volume measure dx. Instead of a function depending on the *position* or *configuration space variable* $\mathbf{x} \in \mathbb{R}^\nu$ one can use its Fourier transform

$$\hat{\psi}(\cdot) \in L^2(\mathbb{R}^\nu, dp), \quad \hat{\psi}(\mathbf{p}) := (2\pi)^{-\nu/2} \int dx\, e^{-i\mathbf{px}}\, \psi(\mathbf{x}) \tag{2.1}$$

depending on the *momentum variable* $\mathbf{p} \in \mathbb{R}^\nu$ with normalization

$$||\Psi||^2 = \int dx\, |\psi(\mathbf{x})|^2 = \int dp\, |\hat{\psi}(\mathbf{p})|^2 = 1.$$

We use for the abstract state vector a capital letter Ψ, for its representation as a function of position $\psi(\mathbf{x})$, or its momentum space wave function $\hat{\psi}(\mathbf{p})$, respectively, and write

$$\begin{array}{lclcl} \mathcal{H} & \longleftrightarrow & L^2(\mathbb{R}^\nu,\, dx) & \longleftrightarrow & L^2(\mathbb{R}^\nu,\, dp) \\ \Psi & \longleftrightarrow & \psi(\mathbf{x}) & \longleftrightarrow & \hat{\psi}(\mathbf{p}) \end{array} \tag{2.2}$$

to indicate the switching between representations.

For a given state Ψ the probability measures $\mu_{\mathbf{x}}$ on configuration space and $\mu_{\mathbf{p}}$ on momentum space, respectively,

$$\mu_{\mathbf{x}}(A) = \int_A dx\, |\psi(\mathbf{x})|^2 \quad \text{and} \quad \mu_{\mathbf{p}}(B) = \int_B dp\, |\hat{\psi}(\mathbf{p})|^2 \tag{2.3}$$

describe the probabilities to find the particle in the (Lebesgue measurable) subset $A \subset \mathbb{R}^\nu$ of configuration space or $B \subset \mathbb{R}^\nu$ of momentum space. One may visualize such a state as a cloud of very many particles where $\mu_{\mathbf{x}}(A)$ describes the fraction of them which have their position in A and, similarly, $\mu_{\mathbf{p}}(B)$ is the fraction with momentum in B. Such a state is also called a *wave packet*. Classical point particles with a δ-like distribution in position and/or momentum space are impossible in quantum mechanics. The state space does not contain such idealized objects, in agreement with observations.

We extend the triple of representations of state vectors to the linear operators acting on them.

Abstract operator acting on $\mathcal{H}$	$\longleftrightarrow$	Action on configuration space wave functions	$\longleftrightarrow$	Action on momentum space wave functions

The Fourier transformation (2.1) interchanges differentiation and multiplication of a function with its argument. Thus we obtain for the position and momentum operators, respectively,

$$\mathbf{x} \longleftrightarrow \mathbf{x} \longleftrightarrow i\nabla_{\mathbf{p}}\,, \tag{2.4}$$

$$\mathbf{p} \longleftrightarrow -i\nabla_{\mathbf{x}} \longleftrightarrow \mathbf{p}. \tag{2.5}$$

In our notation sometimes we do not distinguish between the abstract operator on $\mathcal{H}$ and its representaion as a multiplication operator on the corresponding space on which it is "diagonal".

If the forces acting on the particle are described as the negative gradient of a potential function $V(\mathbf{x})$ (conservative mechanical system) then the generator H of the time evolution, the *Hamiltonian* or *Schrödinger operator*, is the energy operator

$$H = H_0 + V(\mathbf{x}) \tag{2.6}$$

which is a sum of the kinetic energy operator H_0 – responsible for the kinematics – and the real valued potential energy which determines the dynamics. The potential should decrease suitably as $|\mathbf{x}| \to \infty$, see Section 4 for precise conditions on V.

3 Kinematics

The kinetic energy operator or *free Hamiltonian* H_0 usually is a function $H_0(\mathbf{p})$ of the momentum of the particle. We will study two typical cases, nonrelativistic (NR) and relativistic (Rel) kinematics. In the first case

$$\text{NR:} \quad H_0(\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2. \tag{3.1}$$

It acts as a multiplication operator on $\hat{\phi}$ and as a differential operator on ϕ:

$$H_0\,\Phi \quad\longleftrightarrow\quad (H_0\,\phi)(\mathbf{x}) = -\frac{1}{2m}(\Delta\phi)(\mathbf{x}) \quad\longleftrightarrow\quad H_0(\mathbf{p})\,\hat{\phi}(\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2\,\hat{\phi}(\mathbf{p}).$$

Generally, the *velocity operator* is the change of position in time:

$$\mathbf{v}(\mathbf{p}) = \frac{d}{dt}\,e^{itH_0}\,\mathbf{x}\,e^{-itH_0}\Big|_{t=0} = i\,[H_0,\,\mathbf{x}] = \nabla_{\mathbf{p}}\,H_0(\mathbf{p})\,, \tag{3.2}$$

a function of the momentum operator. In the nonrelativistic case it is

$$\text{NR:}\quad \mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}}{m}\,. \tag{3.3}$$

Note that the velocity is unbounded, the speed tends to infinity if the kinetic energy or the momentum does so.

Let us now turn to the relativistic case corresponding e.g. to the Klein-Gordon equation.

$$\text{Rel:}\quad H_0(\mathbf{p}) = \sqrt{\mathbf{p}^2c^2 + m^2c^4} = \sqrt{\mathbf{p}^2 + m^2} \tag{3.4}$$

if we use units of measurement such that the speed of light $c = 1$. Here the velocity operator is

$$\text{Rel:}\quad \mathbf{v}(\mathbf{p}) = \nabla_{\mathbf{p}}\,H_0(\mathbf{p}) = c\,\frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2c^2 + m^2c^4}} = \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}}\,. \tag{3.5}$$

In this case the speed is bounded by $c = 1$ (the speed of light).

The **free time evolution** operator is a simple multiplication operator in momentum space [and a complicated oscillating convolution operator in configuration space]

$$e^{-itH_0}\Phi \quad\longleftrightarrow\quad (e^{-itH_0}\phi)(\mathbf{x}) \quad\longleftrightarrow\quad e^{-itH_0(\mathbf{p})}\hat{\phi}(\mathbf{p}). \tag{3.6}$$

While for short times the free classical and quantum time evolutions differ considerably they behave similarly for large times. Asymptotically, the distribution in configuration space of a quantum state is in good approximation the same as that of the corresponding cloud of free classical particles, of the "classical wave packet". For later applications we study a particular family of states $\Phi_{\bar{\mathbf{p}}}$ with compact momentum support around a very large average momentum $\bar{\mathbf{p}} \in \mathbb{R}^\nu$. The operator $\exp(i\bar{\mathbf{p}}\mathbf{x})$, a unitary function of the position operator $\mathbf{x}$, shifts a state in momentum space by $\bar{\mathbf{p}}$:

$$\Phi_{\mathbf{0}} \quad\longleftrightarrow\quad \phi_{\mathbf{0}}(\cdot) \quad\longleftrightarrow\quad \hat{\phi}_{\mathbf{0}}(\cdot) \in C_0^\infty(\mathbb{R}^\nu) \tag{3.7}$$

$$\Phi_{\bar{\mathbf{p}}} = e^{i\bar{\mathbf{p}}\mathbf{x}}\Phi_{\mathbf{0}} \longleftrightarrow \phi_{\bar{\mathbf{p}}}(\mathbf{x}) = e^{i\bar{\mathbf{p}}\mathbf{x}}\phi_{\mathbf{0}}(\mathbf{x}) \longleftrightarrow \hat{\phi}_{\bar{\mathbf{p}}}(\mathbf{p}) = \hat{\phi}_{\mathbf{0}}(\mathbf{p} - \bar{\mathbf{p}}). \tag{3.8}$$

Since $\phi_{\mathbf{0}}(\cdot) \in \mathcal{S}(\mathbb{R}^\nu)$, the Schwartz space of rapidly decreasing functions, these states are well localized in configuration space, too, uniformly in $\bar{\mathbf{p}}$. They have average velocities around $\mathbf{v}(\bar{\mathbf{p}}) \in \mathbb{R}^\nu$, where

$$\mathbf{v}(\bar{\mathbf{p}}) = \nabla H_0(\bar{\mathbf{p}}) =: v(\bar{\mathbf{p}})\,\boldsymbol{\omega} = \begin{cases} \bar{\mathbf{p}}/m & \text{NR}\,, \\ \bar{\mathbf{p}}/\sqrt{\bar{\mathbf{p}}^2 + m^2} & \text{Rel}\,. \end{cases} \tag{3.9}$$

In the given examples the unit direction vector $\boldsymbol{\omega} = \mathbf{v}(\bar{\mathbf{p}})/|\mathbf{v}(\bar{\mathbf{p}})| = \bar{\mathbf{p}}/|\bar{\mathbf{p}}|$ of the velocity $\mathbf{v}(\bar{\mathbf{p}})$ has the same direction as $\bar{\mathbf{p}}$ and the speed $v(\bar{\mathbf{p}}) = |\mathbf{v}(\bar{\mathbf{p}})|$ depends on $|\bar{\mathbf{p}}|$ only.

Consider in the following only "large" $\bar{\mathbf{p}}$ such that the minimal velocity in the support of $\hat{\phi}_{\bar{\mathbf{p}}}$ is at least $2/3^{\text{rd}}$ of the average velocity:

$$\inf \left\{ \frac{v(\mathbf{p})}{v(\bar{\mathbf{p}})} \;\middle|\; \mathbf{p} \in \operatorname{supp} \hat{\phi}_{\bar{\mathbf{p}}} \right\} \geq \frac{2}{3} \,. \tag{3.10}$$

In our examples e.g. $|\bar{\mathbf{p}}| \geq 3 \sup\{ |\mathbf{p}| \mid \mathbf{p} \in \operatorname{supp} \hat{\phi}_{\mathbf{0}}\}$ will do. As we are mainly interested in the high-energy behavior $|\bar{\mathbf{p}}| \to \infty$ this restriction (3.10) is harmless, it excludes components in the state with particles of low or zero velocities which would require special treatment below.

In our context we have to control the localization in configuration space of freely evolving wave packets. This depends mainly on the support of the state in velocity (momentum) space. Therefore, we have chosen compactly supported momentum space wave functions. Then in configuration space the states cannot have compact support as well but rapid falloff is sufficient there. A special case of such **propagation properties** (or non–propagation) of quantum wave packets for long times [5], [17], is

$$\int\limits_{|\mathbf{x}|<t\,v(\bar{\mathbf{p}})/2} dx \; |(e^{-itH_0}\phi_{\bar{\mathbf{p}}})(\mathbf{x})|^2 < \frac{\text{const}(\Phi_{\mathbf{0}},\, n)}{(1 + |t\, v(\bar{\mathbf{p}})|)^n} \tag{3.11}$$

for any $n \in \mathbb{N}$ *uniformly* for large $\bar{\mathbf{p}}$. A classical free particle which starts at time 0 from the origin and has momentum $\mathbf{p} \in \operatorname{supp} \hat{\phi}_{\bar{\mathbf{p}}}$ will be localized at time t in the region

$$\mathbf{x}(t) \in \left\{ \mathbf{x} = t\,\mathbf{v}(\mathbf{p}) \;\middle|\; \mathbf{p} \in \operatorname{supp} \hat{\phi}_{\bar{\mathbf{p}}} \right\} \subset \{\, \mathbf{x} \mid |\mathbf{x} - t\,\mathbf{v}(\bar{\mathbf{p}})| < v(\bar{\mathbf{p}})/3 \} \,. \tag{3.12}$$

The "classically forbidden" region $|\mathbf{x}| < t\,v(\bar{\mathbf{p}})/2$ is separated from the "allowed region" by at least $t\,v(\bar{\mathbf{p}})/6$. The state mainly propagates within the classically allowed region which moves away from the origin with a positive minimal speed. The "quantum tails" of the wave packet in the classically forbidden region do not vanish, nevertheless, they decay very fast in time, both in the future and past. This is physically and mathematically in close analogy to rays versus waves in optics. While the shadow behind an obstacle is not totally black due to diffraction it is, nevertheless, quite dark away from the region which can be reached by straight rays (the role of the increasing separation $t\,v(\bar{\mathbf{p}})/6$). We will need below only the estimate of non–propagation (3.11). It is proved with a stationary phase estimate of an integral with rapidly oscillating integrand.

Kinematics has a strong influence on the high-energy behavior of the scattering strength. Wave packets will be influenced by the potential only as long as they are mainly localized in the region where the potential is strong. Due to the propagation property this is essentially a finite time interval, the interaction time, proportional to $v(\bar{\mathbf{p}})^{-1}$. Thus the strength of the interaction (which is proportional to the duration of the interaction) will vanish as $|\bar{\mathbf{p}}| \to \infty$ in the nonrelativistic case and it will have a finite limit in the relativistic case. This is a crucial difference of the two cases.

4 Dynamics

Let us now turn to the interacting (perturbed) time evolution

$$e^{-itH}\Psi, \quad H = H_0 + V(\mathbf{x}). \tag{4.1}$$

The free Hamiltonian H_0 is self-adjoint on its domain

$$\mathcal{D}(H_0) = \{\Psi \in \mathcal{H} \mid \hat{\phi}(\bar{\mathbf{p}}) \text{ and } H_0(\mathbf{p})\,\hat{\phi}(\bar{\mathbf{p}}) \in L^2(\mathbb{R}^\nu,\, dp)\}.$$

The configuration space wave functions lie in the Sobolev spaces $W^{2,2}(\mathbb{R}^\nu)$ in the nonrelativistic case and in $W^{1,2}(\mathbb{R}^\nu)$ in the relativistic case, respectively. First we will consider *short-range* potentials $V(\mathbf{x}) = V^s(\mathbf{x})$ which are roughly those which decrease at least like $|\mathbf{x}|^{-(1+\varepsilon)}$, $\varepsilon > 0$, as $|\mathbf{x}| \to \infty$. More precisely, the set of short-range potentials is

$$\mathcal{V}^s = \left\{ V^s \;\middle|\; \int_0^\infty \sup_{|\mathbf{x}|\geq R} |V^s(\mathbf{x})| \; dR < \infty \right\}. \tag{4.2}$$

For simplicity of presentation we will restrict ourselves in this paper to bounded potentials. The sum is self-adjoint on the domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. Singular potentials can easily be included using standard techniques [16]. This covers the physically relevant local singularities like $1/|\mathbf{x}|$ for Coulomb or Yukawa potentials. One simply has to regularize the singularities by a free resolvent or by a function $f(\mathbf{p})$ of the momentum operator which has compact support and is the identity on $\Phi_\mathbf{0}$. In addition, with suitable adjustments we will treat long-range potentials V^ℓ which decrease towards infinity e.g. as slowly as the Coulomb potential, see Section 10. Then

$$V(\mathbf{x}) = V^s(\mathbf{x}) + V^\ell(\mathbf{x}), \quad V^s \in \mathcal{V}^s. \tag{4.3}$$

In the present context a short-range potential behaves similarly to a compactly supported one. Depending on the required accuracy it is essentially concentrated in a ball of some radius R around the origin.

The influence on the particle by the force $-\nabla V(\mathbf{x})$ is relevant only as long as the particle is essentially localized in the interaction region, i.e. where the potential is strong. Correspondingly, the state space can be split into two orthogonal components. The bound states remain forever under the influence of the potential, they constitute the *pure point* spectral subspace which is spanned by the eigenvectors of H. These states remain localized uniformly in time. Orthogonal to these are the scattering states which form the *continuous* spectral subspace $\mathcal{H}^{\text{cont}}(H) = \{\text{eigenvectors of } H\}^\perp$, they leave the interaction region for large times (see e.g. Theorem XI.115 in [17]). The latter are studied in scattering theory.

5 Scattering

As a general reference for mathematical scattering theory see e.g. [17] or [19] for the Dirac equation. It is well known that for short-range potentials ($H = H_0 + V^s$) the asymptotic motion of scattering states is an essentially free motion: For any scattering state $\Psi \in \mathcal{H}^{\text{cont}}(H)$ there exist free asymptotic configurations $\Phi^\pm \in \mathcal{H}$ such that

$$\left\| e^{-it\,[H_0+V^s]}\,\Psi - e^{-itH_0}\,\Phi^\pm \right\| \to 0 \quad \text{as } t \to \pm\infty. \tag{5.1}$$

This is usually called asymptotic completeness of the wave operators. Similarly, for any *incoming* configuration Φ^- or *outgoing* Φ^+ there is a corresponding state $\Psi \in \mathcal{H}^{\mathrm{cont}}(H)$ such that (5.1) holds (existence of wave operators). These statements can be proved for any dimension ν of configuration space using propagation estimates like (3.11). The proof of (7.2) below is similar.

A convenient tool to describe scattering is the scattering operator S which maps an incoming configuration Φ^- to the corresponding outgoing configuration Φ^+ of the same state Ψ. For given Φ^- let

$$\Psi = \lim_{t_- \to -\infty} e^{it_-[H_0+V^s]}\, e^{-it_-\,H_0}\,\Phi^- \qquad \text{and}$$

$$\Phi^+ = \lim_{t_+ \to \infty} e^{it_+ H_0}\, e^{-it_+[H_0+V^s]}\,\Psi.$$

Then

$$S(t_+, t_-) := e^{it_+ H_0}\, e^{-it_+[H_0+V^s]}\, e^{it_-[H_0+V^s]}\, e^{-it_-\,H_0}\,, \tag{5.2}$$

$$S := \operatorname*{s-lim}_{\substack{t_+ \to \infty \\ t_- \to -\infty}} S(t_+, t_-), \quad \text{satisfies} \quad S\,\Phi^- = \Phi^+. \tag{5.3}$$

For microscopic particles for which quantum mechanics is an adequate description one cannot really observe more details of the scattering process than those encoded in the scattering operator. We denote the mapping

$$\mathcal{V}^s \to L(\mathcal{H}), \quad V^s \mapsto S = S(V^s) \tag{5.4}$$

as the *scattering map* from short-range potentials to bounded (unitary) scattering operators on the Hilbert space of asymptotic configurations.

The **direct** problem of scattering theory is to determine for a given potential V the scattering operator while the **inverse** problem is to determine the potential(s) if the scattering operator or part of it is known.

6 Uniqueness of the Potential

We denote by $F(H_0 \geq E)$ the multiplication operator in momentum space with the characteristic function of the set $\{\mathbf{p} \in \mathbb{R}^\nu \mid H_0(\mathbf{p}) \geq E\}$, i.e. the spectral projection of the kinetic energy operator to energies above E. The main results about uniqueness are of the following form. They are a corollary of the asymptotic behavior of the scattering operator shown below.

Theorem 6.1 *The scattering map $S\colon \mathcal{V}^s \to L(\mathcal{H})$ is injective. Actually, the high-energy part of the scattering operator alone: $S\,F(H_0 \geq E)$, E arbitrarily large, determines the short-range potential uniquely.*

In the nonrelativistic case similar results go back to Faddeev (1956) and Berezanski (1958), the strongest results were by Saito (1984), all using time-independent methods. Our geometrical time-dependent proof covers a wider class of potentials and, more importantly, it is simpler. It has been extended to show analogous results for long-range potentials, magnetic fields, multiparticle systems, for the Dirac equation, and other systems, see Sections 9, 10, and 11.

7 Time Scales for Interaction and Spreading

For high energy states as constructed in (3.8) scattering theory becomes simple because two time scales, a short interaction time $T_I(\bar{\mathbf{p}})$ and a longer kinematical time of spreading $T_{Sp}(\bar{\mathbf{p}})$ satisfy $T_I(\bar{\mathbf{p}})/T_{Sp}(\bar{\mathbf{p}}) \to 0$ as $|\bar{\mathbf{p}}| \to \infty$. For a potential which is essentially supported in a ball of radius R the **interaction time** is of the order $T_I(\bar{\mathbf{p}}) = R/v(\bar{\mathbf{p}})$. Due to propagation like (3.11) we have

$$S(t_+, t_-)\, \Phi^-_{\bar{\mathbf{p}}} \approx S\, \Phi^-_{\bar{\mathbf{p}}} \quad \text{if both} \quad \pm\, t_\pm \gg T_I(\bar{\mathbf{p}}). \tag{7.1}$$

More precisely, for $\Phi_{\bar{\mathbf{p}}}$, $\Phi'_{\bar{\mathbf{p}}}$ as in (3.8) and any $\varepsilon > 0$ there is a radius $\rho(\varepsilon)$ such that uniformly for large $|\bar{\mathbf{p}}|$ (which satisfy (3.10))

$$|(\Phi'_{\bar{\mathbf{p}}},\, [\, S\, - S(t_+, t_-)\,]\, \Phi_{\bar{\mathbf{p}}})| < \frac{\varepsilon}{v(\bar{\mathbf{p}})} \quad \text{if} \quad \pm\, t_\pm > \rho(\varepsilon)/v(\bar{\mathbf{p}}). \tag{7.2}$$

Intuitively, $\rho(\varepsilon)$ measures the radius of the interaction region and the extension in configuration space of the states up to effects of size ε. The bound (7.2) is physically intuitive and it is a crucial estimate which will be used in Section 8 to justify the interchanging of limits.

To prove (7.2) one has to bound a term

$$\begin{aligned} &\left\| \lim_{t\to\infty} e^{it[H_0+V^s]}\, e^{-it\,H_0}\, \Phi'_{\bar{\mathbf{p}}} - e^{it_+[H_0+V^s]}\, e^{-it_+\,H_0}\, \Phi'_{\bar{\mathbf{p}}} \right\| \\ &= \left\| \int_{t_+}^{\infty} dt\, \frac{d}{dt}\, e^{it[H_0+V^s]}\, e^{-it\,H_0}\, \Phi'_{\bar{\mathbf{p}}} \right\| \le \int_{t_+}^{\infty} dt\, \left\| V^s\, e^{-it\,H_0}\, \Phi'_{\bar{\mathbf{p}}} \right\| \end{aligned} \tag{7.3}$$

and a similar term with t_- and $\Phi_{\bar{\mathbf{p}}}$. The integrand in(7.3) can be split into two terms

$$\begin{aligned} &\left\| V^s\, e^{-it\,H_0}\, \Phi'_{\bar{\mathbf{p}}} \right\| \\ &\le \| V^s\, F(|\mathbf{x}| > v(\bar{\mathbf{p}})\, t/2) \| + \| V^s \|\, \left\| F(|\mathbf{x}| \ge v(\bar{\mathbf{p}})\, t/2)\, e^{-it\,H_0}\, \Phi'_{\bar{\mathbf{p}}} \right\| \\ &=: h_1(v(\bar{\mathbf{p}})\, t) + h_2(v(\bar{\mathbf{p}})\, t) \end{aligned}$$

where $F(\cdot)$ here denotes the multiplication operator with the characteristic function of the indicated region in configuration space. The functions h_1 and h_2 are integrable due to (4.2) and (3.11), respectively. With the new variable $r := v(\bar{\mathbf{p}})\, t$ the integral (7.3) is bounded by

$$\frac{1}{v(\bar{\mathbf{p}})} \int_{r_+}^{\infty} dr\, [\, h_1(r) + h_2(r)\,] \le \frac{\varepsilon}{v(\bar{\mathbf{p}})}$$

for $r_+ \ge \rho(\varepsilon)$ large enough. This proves (7.2).

The kinematical **time scale of spreading** $T_{Sp}(\bar{\mathbf{p}})$ denotes the time after which spreading of wave packets becomes relevant in the time evolution. As

$$H_0(\mathbf{p})\, \Psi_{\bar{\mathbf{p}}} = H_0(\mathbf{p})\, e^{i\bar{\mathbf{p}}\mathbf{x}}\, \Psi_{\mathbf{0}} = e^{i\bar{\mathbf{p}}\mathbf{x}}\, H_0(\bar{\mathbf{p}} + \mathbf{p})\, \Psi_{\mathbf{0}}$$

we will expand the kinetic energy function around $\bar{\mathbf{p}}$

$$H_0(\bar{\mathbf{p}} + \mathbf{p}) =: H_0(\bar{\mathbf{p}}) + \nabla H_0(\bar{\mathbf{p}}) \cdot \mathbf{p} + H_2(\bar{\mathbf{p}},\, \mathbf{p}). \tag{7.4}$$

The first summand is a number giving an irrelevant phase, the second equals $\mathbf{v}(\bar{\mathbf{p}}) \cdot \mathbf{p}$ by (3.9). It is the dominant term which – as a multiple of the momentum operator – generates a translation of the wave packet without changing its shape. Only the third term H_2 (which is defined by (7.4)) is responsible for the spreading of the wave packet. In our examples of "power like" Hamiltonians this part of the Hamiltonian is weak compared to the translational component: On a compact subset of momentum space like $\mathbf{p} \in \operatorname{supp} \hat{\phi}_{\mathbf{0}}$

$$\frac{T_I(\bar{\mathbf{p}})}{T_{Sp}(\bar{\mathbf{p}})} \sim \frac{|H_2(\bar{\mathbf{p}},\, \mathbf{p})|}{v(\bar{\mathbf{p}})} \leq \frac{\text{const}}{|\bar{\mathbf{p}}|} \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} 0. \tag{7.5}$$

In the nonrelativistic case we have $H_2(\bar{\mathbf{p}},\, \mathbf{p}) = \mathbf{p}^2/2m$ which is independent of $\bar{\mathbf{p}}$ and bounded on $\operatorname{supp} \hat{\phi}_{\mathbf{0}}$.

$$\frac{|H_2(\bar{\mathbf{p}},\, \mathbf{p})|}{v(\bar{\mathbf{p}})} = \frac{\mathbf{p}^2/2m}{|\bar{\mathbf{p}}|/m} = \frac{\mathbf{p}^2/2}{|\bar{\mathbf{p}}|}\,.$$

In the relativistic case the denominator is bounded but the numerator decreases. With the shorthand $a := \sqrt{\bar{\mathbf{p}}^2 + m^2}$ we get for $|\bar{\mathbf{p}}|$ large enough

$$\frac{|H_2(\bar{\mathbf{p}},\, \mathbf{p})|}{v(\bar{\mathbf{p}})} = \frac{|a\{\sqrt{1 + 2\,(\bar{\mathbf{p}}\,\mathbf{p}/a^2) + (\mathbf{p}^2/a^2)} - 1 - \bar{\mathbf{p}}\,\mathbf{p}/a^2\}|}{|\bar{\mathbf{p}}|/a} \leq 2\frac{\mathbf{p}^2/2}{|\bar{\mathbf{p}}|}\,.$$

This proves (7.5). Therefore, the time $T_{Sp}(\bar{\mathbf{p}})$ is by a factor proportional to $|\bar{\mathbf{p}}|$ longer than $T_I(\bar{\mathbf{p}})$. For large $|\bar{\mathbf{p}}|$ we may choose times when the scattering due to the potential is over but the spreading has not yet really started. Alternatively, we may use radii for this splitting like the interaction radius $\rho(\varepsilon)$ of (7.2).

Usually, an interacting time evolution is complicated because the translation of a wave packet, its spreading, and the influence of the potential all occur at the same time. In the high-energy limit it is sufficient for the calculation of the scattering operator to treat translation of wave packets rather than their correct free evolution. Since in this limit spreading occurs only when the interaction is negligible, i.e. when the free and interacting time evolutions are almost the same, the effect of spreading is cancelled (becomes invisible) in the scattering operator. Thus, high energy scattering is simple and it can be inverted simply! The motion during the interaction is dominated by the translational part which is common to classical and quantum physics. The typical quantum effect of spreading of wave functions which results from the absence of localized states with sharp momentum is of lower order.

8 High Energy Scattering

The crucial uniformity of the estimate (7.2) enables us to interchange the limits $\pm t_\pm \to \infty$ and $|\bar{\mathbf{p}}| \to \infty$. This simplifies the remaining discussion very much. Actually, as to be expected, not the time but the separation from the region of a strong potential determines the quality of approximation. With correspondingly chosen variables

$r_\pm := t_\pm \, v(\bar{\mathbf{p}})$ we have

$$\lim_{|\bar{\mathbf{p}}|\to\infty} (\Phi'_{\bar{\mathbf{p}}},\ S\,\Phi_{\bar{\mathbf{p}}}) = \lim_{|\bar{\mathbf{p}}|\to\infty} \lim_{\pm t_\pm\to\infty} (\Phi'_{\bar{\mathbf{p}}},\ S(t_+,t_-)\,\Phi_{\bar{\mathbf{p}}})$$

$$= \lim_{|\bar{\mathbf{p}}|\to\infty} \lim_{\pm r_\pm\to\infty} \left(\Phi'_{\bar{\mathbf{p}}},\ S\left(\frac{r_+}{v(\bar{\mathbf{p}})}, \frac{r_-}{v(\bar{\mathbf{p}})}\right)\Phi_{\bar{\mathbf{p}}}\right)$$

$$= \lim_{\pm r_\pm\to\infty} \lim_{|\bar{\mathbf{p}}|\to\infty} \left(\Phi'_{\bar{\mathbf{p}}},\ S\left(\frac{r_+}{v(\bar{\mathbf{p}})}, \frac{r_-}{v(\bar{\mathbf{p}})}\right)\Phi_{\bar{\mathbf{p}}}\right). \tag{8.1}$$

As seen in (7.2) the asymptotic equality (8.1) remains true even after multiplication with $v(\bar{\mathbf{p}})$ which is a much stronger statement in the nonrelativistic case. To determine

$$(\Phi'_{\bar{\mathbf{p}}},\ S(t_+,t_-)\,\Phi_{\bar{\mathbf{p}}}) = \left(\Phi'_{\mathbf{0}},\ e^{-i\bar{\mathbf{p}}\mathbf{x}}\, S(t_+,t_-)\, e^{i\bar{\mathbf{p}}\mathbf{x}}\,\Phi_{\mathbf{0}}\right) \tag{8.2}$$

for large finite times and $\bar{\mathbf{p}}$ consider e.g. the second pair of factors in (5.2).

$$e^{-i\bar{\mathbf{p}}\mathbf{x}}\ e^{it_-[H_0+V^s]}\ e^{-it_- H_0}\ e^{i\bar{\mathbf{p}}\mathbf{x}}$$

$$= e^{it_-[H_0(\mathbf{p}+\bar{\mathbf{p}})+V^s(\mathbf{x})]}\ e^{-it_- H_0(\mathbf{p}+\bar{\mathbf{p}})}$$

$$= e^{it_-[H_0(\bar{\mathbf{p}})+\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})+V^s(\mathbf{x})]}\ e^{-it_-[H_0(\bar{\mathbf{p}})+\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})]}$$

$$= e^{it_-[\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})+V^s(\mathbf{x})]}\ e^{-it_-[\mathbf{v}(\bar{\mathbf{p}})\cdot\mathbf{p}+H_2(\bar{\mathbf{p}},\mathbf{p})]}$$

$$= e^{ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}+\{V^s(\mathbf{x})/v(\bar{\mathbf{p}})\}]}\ e^{-ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}]} \tag{8.3}$$

using again $t_\pm = r_\pm/v(\bar{\mathbf{p}})$ and the direction $\boldsymbol{\omega} = \mathbf{v}(\bar{\mathbf{p}})/v(\bar{\mathbf{p}})$ as in (3.9). Due to (7.5) the functions of the momentum operator

$$[\,\boldsymbol{\omega}\cdot\mathbf{p} + \{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}\,] \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \boldsymbol{\omega}\cdot\mathbf{p} \tag{8.4}$$

converge in strong resolvent sense and similarly for the other exponent. Therefore, for fixed r_- and large $|\bar{\mathbf{p}}|$ the following approximation is good:

$$e^{ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}+\{V^s(\mathbf{x})/v(\bar{\mathbf{p}})\}]}\ e^{-ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}]}$$

$$\approx e^{ir_-[\boldsymbol{\omega}\cdot\mathbf{p}+\{V^s(\mathbf{x})/v(\bar{\mathbf{p}})\}]}\ e^{-ir_-\boldsymbol{\omega}\cdot\mathbf{p}} \tag{8.5}$$

$$= \exp\left\{\frac{-i}{v(\bar{\mathbf{p}})}\int_{r_-}^{0} dr\, V^s(\mathbf{x}+\boldsymbol{\omega}\, r)\right\}. \tag{8.6}$$

The approximation (8.5) is the only approximation we have to make! If $\{H_2(\bar{\mathbf{p}},\mathbf{p})/v(\bar{\mathbf{p}})\}$ would commute with $\{V^s(\mathbf{x})/v(\bar{\mathbf{p}})\}$ then we would have exact cancellation and (8.5) would be an equality as well. A careful estimate of the correction terms can be given for all Hamiltonians considered here. It is uniform in r_- and when compared to $\{V^s(\mathbf{x})/v(\bar{\mathbf{p}})\}$ it has additional falloff like $1/|\bar{\mathbf{p}}|$ for $\bar{\mathbf{p}} \to \infty$ due to (7.5). For the proofs we refer to the papers cited in the theorems below.

Equation (8.6) is verified easily because as functions of r_- both expressions solve the same initial value problem

$$\frac{d}{dr_-}A(r_-) = A(r_-)\ \frac{-i}{v(\bar{\mathbf{p}})}\, V^s(\mathbf{x}+\boldsymbol{\omega}\, r_-), \quad A(0) = \mathbf{1}.$$

The same analysis of the first two factors in the expression (5.2) for the scattering operator yields analogously to (8.6) the factor

$$\exp\left\{\frac{-i}{v(\bar{\mathbf{p}})}\int_0^{r_+} dr\, V^s(\mathbf{x}+\boldsymbol{\omega}\, r)\right\}.$$

Combining this with (8.6) we obtain for large $|\bar{\mathbf{p}}|$

$$\begin{aligned}(\Phi'_{\bar{\mathbf{p}}}\,,\ S(t_+,t_-)\,\Phi_{\bar{\mathbf{p}}}) &= \left(\Phi'_{\mathbf{0}},\ e^{-i\bar{\mathbf{p}}\mathbf{x}}\, S\left(\frac{r_+}{v(\bar{\mathbf{p}})},\frac{r_-}{v(\bar{\mathbf{p}})}\right) e^{i\bar{\mathbf{p}}\mathbf{x}}\,\Phi_{\mathbf{0}}\right)\\ &\approx \left(\Phi'_{\mathbf{0}}\,,\ \exp\left\{\frac{-i}{v(\bar{\mathbf{p}})}\int_{r_-}^{r_+} dr\, V^s(\mathbf{x}+\boldsymbol{\omega} r)\right\}\ \Phi_{\mathbf{0}}\right).\end{aligned} \tag{8.7}$$

$$(\Phi'_{\bar{\mathbf{p}}}\,,\ S\,\Phi_{\bar{\mathbf{p}}}) \approx \left(\Phi'_{\mathbf{0}},\ \exp\left\{\frac{-i}{v(\bar{\mathbf{p}})}\int_{-\infty}^{\infty} dr\, V^s(\mathbf{x}+\boldsymbol{\omega} r)\right\}\ \Phi_{\mathbf{0}}\right). \tag{8.8}$$

These expressions confirm our intuitive expectation discussed in Section 3 that the limiting behavior of the scattering operator depends strongly on the growth or boundedness of $v(\bar{\mathbf{p}})$ as $|\bar{\mathbf{p}}| \to \infty$. We will discuss the two cases separately.

9 High Energy Limits of the Scattering Operator, the Short-Range Case

Next we give the limiting behavior of the scattering operator in several typical cases, $\Phi_{\bar{\mathbf{p}}}$, $\Phi'_{\bar{\mathbf{p}}}$, and $\bar{\mathbf{p}} \in \mathbb{R}^\nu$ as given in (3.8), (3.10). The integrals extend over the real line. In the quotations we include similar results obtained by other methods, sometimes under more restrictive assumptions.

Theorem 9.1 (scalar relativistic, short-range, [14])
For the scalar relativistic Hamiltonian

$$H = \sqrt{\mathbf{p}^2+m^2} + V^s(\mathbf{x})$$

with $v(\bar{\mathbf{p}}) \to 1$ one obtains

$$(\Phi'_{\bar{\mathbf{p}}},\ S\,\Phi_{\bar{\mathbf{p}}}) \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \left(\Phi'_{\mathbf{0}},\ \exp\left\{-i\int dr\, V^s(\mathbf{x}+\boldsymbol{\omega} r)\right\}\ \Phi_{\mathbf{0}}\right). \tag{9.1}$$

If, however, $v(\bar{\mathbf{p}}) \to \infty$ we can expand the exponential in (8.8)

$$\begin{aligned}&\exp\left\{-\frac{i}{v(\bar{\mathbf{p}})}\int dr\, V^s(\mathbf{x}+\boldsymbol{\omega} r)\right\}\\ &\quad\approx 1 - \frac{i}{v(\bar{\mathbf{p}})}\int dr\, V^s(\mathbf{x}+\boldsymbol{\omega} r) + \cdots\end{aligned} \tag{9.2}$$

which explains the following nonrelativistic result. The leading behavior of the scattering operator is the identity operator (no scattering). The next order correction depends on the potential.

Theorem 9.2 (nonrelativistic, short-range, [10], [4], [18], [6], [8])
For the Hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + V^s(\mathbf{x})$$

$$v(\bar{\mathbf{p}})\ (\Phi'_{\bar{\mathbf{p}}},\ i(S-\mathbf{1})\ \Phi_{\bar{\mathbf{p}}}) \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \int dr\ (\Phi'_{\mathbf{0}},\ V^s(\mathbf{x}+\boldsymbol{\omega} r)\ \Phi_{\mathbf{0}}). \tag{9.3}$$

This result is to be expected from the Born approximation. It holds also under the given weaker assumptions on the falloff of the potential where the validity of the Born approximation is not established.

The estimate (7.2) and the remark following (8.6) justify that multiplication with $v(\bar{\mathbf{p}}) \sim |\bar{\mathbf{p}}|$ is permitted. The approximation is better than $V^s/v(\bar{\mathbf{p}})$.

Remark
In all these limits there are **error bounds** for large but finite $|\bar{\mathbf{p}}|$ which are explicit. E.g. in equation (9.3) (and similarly in (10.5) etc.) we obtain

$$\left| v(\bar{\mathbf{p}})\ (\Phi'_{\bar{\mathbf{p}}},\ i(S-\mathbf{1})\ \Phi_{\bar{\mathbf{p}}}) - \int dr\ (\Phi'_{\mathbf{0}},\ V^s(\mathbf{x}+\boldsymbol{\omega} r)\ \Phi_{\mathbf{0}}) \right| \le \frac{\mathrm{const}(\Phi'_{\mathbf{0}},\ \Phi_{\mathbf{0}},\ V^s)}{|\bar{\mathbf{p}}|}.$$

Theorem 9.3 (Dirac equation, [12], [14])
For the Dirac Hamiltonian of a particle in a continuous short-range electromagnetic field

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m + V(\mathbf{x}), \quad V(\mathbf{x}) = (A_0 - \boldsymbol{\alpha}\cdot\mathbf{A})(\mathbf{x})$$

the high-energy limit of the scattering operator is

$$\lim_{|\bar{\mathbf{p}}|\to\infty} (\Phi'_{\bar{\mathbf{p}}},\ S\ \Phi_{\bar{\mathbf{p}}}) = \left(\Phi'_{\mathbf{0}},\ \exp\left\{-i\int dr\ (A_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\ \ \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}+\boldsymbol{\omega} r)\right\}\Phi_{\mathbf{0}}\right).$$

In the Fouldy-Wouthuysen representation let $S_\pm$ denote the scattering operator on the positive/negative energy subspaces and $\mathbf{x}_{NW}$ the Newton-Wigner position operator then

$$\operatorname*{s-lim}_{|\bar{\mathbf{p}}|\to\infty} e^{-i\,\bar{\mathbf{p}}\mathbf{x}_{NW}} S_\pm e^{i\,\bar{\mathbf{p}}\mathbf{x}_{NW}} = \exp\left\{-i\int dr\ (A_0 \mp \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\ \ \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW}+\boldsymbol{\omega} r)\right\}.$$

Similar results hold for other matrix valued potentials V.

10 High Energy Limits of the Scattering Operator, Long-Range Potentials, Magnetic Fields, etc.

The only long-range potential in physics is the Coulomb potential between electrically charged particles. The mathematical treatment includes more general potentials which may decrease slower and need not be centrally symmetric. A convenient class of long-range potentials are twice continuously differentiable functions with $V^\ell(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ and

$$\mathcal{V}^\ell = \left\{ V^\ell \ \middle|\ |\nabla V^\ell(\mathbf{x})| \le \mathrm{const}(1+|\mathbf{x}|)^{-(3/2)-\varepsilon},\right.$$

$$|\Delta V^{\ell}(\mathbf{x})| \leq \text{const}(1+|\mathbf{x}|)^{-2-\varepsilon},\ \varepsilon > 0\Big\}. \tag{10.1}$$

Local singularities may be treated with the short-range part. If long-range potentials are present the ordinary scattering operator (5.2), (5.3) no longer exits, the free time evolution has to be replaced by a better approximation generated by $H_0(t) := H_0 + V^{\ell}(t\,\mathbf{p}/m)$. The factor $e^{-it_- H_0}$ in (5.2) has to be replaced by

$$U^D(t_-, 0) := \exp\left\{-i\,t_-\,H_0 - i\int_{t_-}^{0} dt\; V^{\ell}(t\,\mathbf{p}/m)\right\} \tag{10.2}$$

and similarly $U^D(0, t_+)$ for the term with t_+. With this *Dollard correction* the modified scattering operator exists:

$$S^D(t_+, t_-) := [\,U^D(0, t_+)\,]^*\; e^{-i\,t_+[H_0+V^s+V^{\ell}]}\; e^{i\,t_-[H_0+V^s+V^{\ell}]}\; U^D(t_-, 0)\;, \tag{10.3}$$

$$S^D := \operatorname*{s-lim}_{\substack{t_+\to\infty\\ t_-\to-\infty}}\; S^D(t_+, t_-), \quad \text{satisfies} \quad S^D\,\Phi^- = \Phi^+. \tag{10.4}$$

The splitting of the potential into its short- and long-range parts is not unique. Different choices – as long as (4.2) and (10.1) are satisfied – correspond to different labellings of asymptotic configurations and are physically equivalent.

To define a modified scattering operator some knowledge of the long-range behavior of the potential is needed. A typical physical situation is that the total charge is known but the charge distribution is to be determined by scattering experiments.

Theorem 10.1 (nonrelativistic, long-range, [8])
For a splitting of the potential as given above let the Hamiltonian be

$$H = \frac{1}{2m}\mathbf{p}^2 + V^s(\mathbf{x}) + V^{\ell}(\mathbf{x})$$

and let S^D be the corresponding Dollard modified scattering operator then

$$\begin{aligned} &v(\bar{\mathbf{p}})\;(\Phi'_{\bar{\mathbf{p}}},\; i(S^D - \mathbf{1})\,\Phi_{\bar{\mathbf{p}}}) - \int dr\;(\Phi'_{\mathbf{0}},\;[\,V^{\ell}(\mathbf{x}+\boldsymbol{\omega}r) - V^{\ell}(\boldsymbol{\omega}r)\,]\;\Phi_{\mathbf{0}}) \\ &\xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \int dr\;(\Phi'_{\mathbf{0}},\; V^s(\mathbf{x}+\boldsymbol{\omega}r)\,\Phi_{\mathbf{0}}). \end{aligned} \tag{10.5}$$

Note that the integral $\int dr\;[\,V^{\ell}(\mathbf{x}+\boldsymbol{\omega}r) - V^{\ell}(\boldsymbol{\omega}r)\,]$ converges for every $\mathbf{x}$. Another version of the theorem yields at once the full potential by using the high-energy limit of $[p_1,\; S^D]$ for *any* modified scattering operator.

If one has a magnetic field described by a vector potential $\mathbf{A}$ in a nonrelativistic Schrödinger operator then the leading term in the interaction is $\mathbf{A}\cdot\mathbf{p}/m \approx \mathbf{A}\cdot\mathbf{v}(\bar{\mathbf{p}})$. Inserting this into (8.8) suggests that one obtains a nontrivial limit of the scattering operator although the kinematics is nonrelativistic. The other potentials show up as a correction to the leading behavior of lower order.

Theorem 10.2 (nonrelativistic with magnetic fields and potentials of short and long range, [15], [1], [3])
Let the vector potential $\mathbf{A}(\mathbf{x})$ *be a short-range multiplication operator and*

$$\begin{aligned} H &= \frac{1}{2m}\,[\,\mathbf{p} - \mathbf{A}(\mathbf{x})\,]^2 + V^s(\mathbf{x}) + V^\ell(\mathbf{x}) \\ &= \frac{1}{2m}\mathbf{p}^2 - \mathbf{A}\cdot\frac{\mathbf{p}}{m} + \frac{1}{2m}(i \text{ div}\,\mathbf{A} + |\mathbf{A}|^2) + V^s + V^\ell. \end{aligned} \tag{10.6}$$

The scattering operator itself has the limit

$$\underset{|\bar{\mathbf{p}}|\to\infty}{\text{s-lim}}\, e^{-i\,\bar{\mathbf{p}}\mathbf{x}} S e^{i\,\bar{\mathbf{p}}\mathbf{x}} = \exp\left\{ i \int dr\; \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}r) \right\}. \tag{10.7}$$

which yields a unique magnetic field $F(\mathbf{x}) = d\mathbf{A}(\mathbf{x})$ *for continuous* $\mathbf{A}(\mathbf{x})$*, and – after fixing a gauge – the vector potential* $\mathbf{A}(\mathbf{x})$*. In a second step one obtains as a lower order correction to the leading term*

$$\begin{aligned} &v(\bar{\mathbf{p}})\Big(\Phi'_{\bar{\mathbf{p}}},\; i\Big[S\, e^{-i\int dr\;\boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}r)} - \mathbf{1}\Big]\,\Phi_{\bar{\mathbf{p}}}\Big) - \text{known terms}(\mathbf{A}, V^\ell,\, \Phi'_{\mathbf{0}},\, \Phi_{\mathbf{0}}) \\ &\qquad \xrightarrow[|\bar{\mathbf{p}}|\to\infty]{} \int dr\; (\Phi'_{\mathbf{0}},\; V^s(\mathbf{x}+\boldsymbol{\omega}r)\;\Phi_{\mathbf{0}}). \end{aligned} \tag{10.8}$$

Clearly, only the magnetic field $d\mathbf{A}(\mathbf{x})$ ($=$ curl $A(\mathbf{x})$ in $\nu = 3$ dimensions) is a physical quantity. However, fixing a gauge and choosing a vector potential is needed to write down the Schrödinger operator (10.6) (choice of a representation). For homogeneous magnetic fields see [2], [3].

Let us now turn to nonrelativistic **multiparticle** scattering systems with short- and long-range potentials. As the interaction becomes weak for high speeds the incoming and outgoing scattering channels are the same in the high-energy limit. Excitations and rearrangements are of lower order.

Theorem 10.3 (nonrelativistic multiparticle systems)
In two-cluster scattering the limit of the channel scattering operator for high relative speed of the clusters yields the effective intercluster potential between the two bounded subsystems, [9].

In the totally free channel select a pair of particles with high relative speed and all other particles far away (or with even higher speeds). Then the pair potential of this pair is obtained from the limit. Multiparticle potentials can be obtained iteratively after the pair potentials are known, [7], [8].

The method has been extended and applied to other problems as well, see e.g. Stark effect Hamiltonians [20], time-dependent N-body Schrödinger operators [21], time-dependent Dirac operators [13], and nonlinear Schrödinger equations [22].

11 Reconstruction of the Potential

The condition $\nu \geq 2$ (*multi*dimensional inverse problem) enters here to obtain from the above limits reconstruction formulas and uniqueness. For bounded continuous (or more general) functions V^s the expression

$$W^s(\mathbf{x}, \boldsymbol{\omega}) := \int dr\; V^s(\mathbf{x}+\boldsymbol{\omega}r) \tag{11.1}$$

is the X-ray transform of V^s. In $\nu = 2$ dimensions lines and hyperplanes are the same. Therefore, (11.1) is the Radon transform as well. The latter is known to be uniquely invertible because the assumption (4.2) implies $V^s \in L^2(\mathbb{R}^2)$, see e.g. Theorem 2.17 in Chapter I of [11]. The inverse Radon transform yields the unique potential. In higher dimensions one fixes e.g. $x_3, \ldots, x_\nu$ and reconstructs the "slices" subsequently. In particular, it is sufficient to vary $\boldsymbol{\omega}$ in a two dimensional plane. For unbounded or discontinuous potentials the expectation value between states from a dense set of nice vectors (like those which satisfy (3.7)) effectively smoothes the potential. This is enough to reconstruct the potential as a multiplication operator, see e.g. [8].

The reconstruction of the magnetic field is similar but more complicated, see [1], [3], or [14] for details.

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