Dedicated to M. Sh. Birman
on the occasion of his 70th birthday.

ANOTHER LOOK AT CWIKEL'S INEQUALITY

TIMO WEIDL
KUNGLIGA TEKNISKA HÖGSKOLAN
MATEMATISKA INSTITUTIONEN
S-10044 STOCKHOLM, SWEDEN
AND
UNIVERSITÄT REGENSBURG
NATURWISSENSCHAFTLICHE FAKULTÄT I - MATHEMATIK -
D-93040 REGENSBURG, GERMANY

ABSTRACT. We point out the basic bound on the singular values of integral operators of the type $a(x)b(D)$, from which Cwikel's inequality and its generalizations [7], [4], [16] can easily be derived.

1. INTRODUCTION

Cwikel's inequality provides an important source for estimates on the spectrum of certain integral operators. In its original form it states, that for the singular numbers $s_n\left(a(x)b(i\nabla)\right)$ of the operator $a(x)b(i\nabla)$ on $L_2(\mathbb{R}^d)$ the estimate [7]

$$n^{1/p}s_n\left(a(x)b(i\nabla)\right) \leq C(p,d)\|a\|_{L_p}\|b\|_{L_p^w}, \qquad 2 < p < \infty, \quad n \in \mathbb{N}, \tag{1.1}$$

holds. Here L_p^w denotes the weak L_p-space. In particular (1.1) implies the celebrated Cwikel-Lieb-Rosenblum bound on the number of negative eigenvalues of Schrödinger operators [7], [10], [11].

Various generalizations enhance the importance of Cwikel's inequality. Namely, in [6] it has been pointed out, that (1.1) carries over to sandwiched integral operators of the class $\mathcal{E}$, see also section 3. For instance, this fact proved to be useful for estimates on eigenvalues near the band edges of perturbed periodic Schrödinger operators [3].

Moreover, in [16] it has been shown that (1.1) is a special case of the bound

$$\|a(x)b(i\nabla)\|_{\Psi(\mathfrak{S}_2,\mathfrak{B})} \leq C(\Psi,d)\|h\|_{\Psi(L_2,L_\infty)}, \qquad h(x,y) = a(x)b(y), \tag{1.2}$$

where Ψ is a K-interpolation functor, acting on the left-hand side between the Hilbert-Schmidt operators $\mathfrak{S}_2$ and the bounded operators $\mathfrak{B}$ on $L_2(\mathbb{R}^d)$; and on the right-hand side between $L_2(\mathbb{R}^{2d})$ and $L_\infty(\mathbb{R}^{2d})$. Hence the inclusion of the symbol h in a certain appropriate function space implies the inclusion of $a(x)b(i\nabla)$ in the corresponding operator ideal.

We underline that the product structure of h is crucial for (1.2) to be true.

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The estimate (1.2) allows a "fine-tuning" of the interpolation functor Ψ to the distribution function for a given h. However, if the symbol h depends in a non-trivial way on certain parameters, the application of (1.2) might prove to be troublesome. Indeed, the optimal functor Ψ can now depend on the parameters as well, and an insufficient control on the constant $C(\Psi, d)$ prevents one from obtaining uniform bounds.

The main result of this note is the basic inequality (4.1) in Theorem 4.1. This bound and its consequences (4.2), (4.3) are in fact "dressed down" versions of (1.2), avoiding the introduction of interpolation functors. It is important that these inequalities invoke explicit (although probably not optimal) constants. They are of the same sharpness as (1.2) and (1.1), since they are based on the crucial Lemma 4.1. Moreover, (1.1) as well as concrete realizations of (1.2) can easily be derived from (4.2). For given h the direct evaluation of (4.1), (4.2) or (4.3) will effectively lead to the optimal realization of (1.2) for this concrete symbol with certain explicit constants.

We mention that (4.1), (4.2) and (4.3) are an easy consequence of the arguments in [7] and [16], where they have been used implicitly. Our aim is to bring these inequalities to the readers attention.

In this note we restrict ourself to the proof of (4.1), (4.2) and (4.3). We follow the constructions in [16].

To demonstrate the flexibility of (4.3), we provide a bound on the negative spectrum of the two-dimensional Schrödinger operator at the end of this paper. Such a bound cannot be derived from (1.1). Further applications shall be published elsewhere.

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2. Notation

Let H_1, H_2 be separable Hilbert spaces. By $\mathfrak{B} = \mathfrak{B}(H_1, H_2)$ we denote the Banach space of linear operators, which are bounded from H_1 to H_2 (below we omit the notation for the underlying spaces). For the respective operator norm we shall write $\|\cdot\|_{\mathfrak{B}}$.

Let $\mathfrak{K}_n$ be the set of operators $K \in \mathfrak{B}$, such that rank $K \leq n-1$, $n \in \mathbb{N}$. For $T \in \mathfrak{B}$ we consider the non-increasing non-negative sequence $\{s_n\}_{n\in\mathbb{N}}$ of its approximation numbers, that is

$$s_n(T) = \inf_{K\in\mathfrak{K}_n} \|T-K\|_{\mathfrak{B}}, \qquad n \in \mathbb{N}.$$

Then Ky-Fan's inequality holds:

$$s_{n+m-1}(T+S) \leq s_n(T) + s_m(S), \qquad S, T \in \mathfrak{B}, \quad n, m \in \mathbb{N}. \tag{2.1}$$

The operator $T \in \mathfrak{B}$ is compact, i.e. $T \in \mathfrak{S}_\infty = \mathfrak{S}_\infty(H_1, H_2)$, if and only if $s_n(T) \to 0$ as $n \to \infty$. Then the quantities $s_n(T)$ are also called the singular numbers of T. The class of Hilbert-Schmidt operators $\mathfrak{S}_2$ is defined by the norm $\|T\|_{\mathfrak{S}_2} = \left(\sum_n s_n^2(T)\right)^{1/2}$.

We shall also need a general class of weak operator ideals. Let $f = \{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers, such that $f_n \to \infty$ as $n \to \infty$. Assume there exists a $r > 0$, such that $f_{n+m}^r \leq f_n^r + f_m^r$ for all $m, n \in \mathbb{N}$. Then Σ_f

is defined as the set of all $T \in \mathfrak{S}_\infty(H_1, H_2)$, for which the quasi-norm

$$\|T\|_f = \sup_{n\in\mathbb{N}} s_n(T) f_n$$

is finite. The ideals Σ_f are complete, non-separable spaces with respect to $\|\cdot\|_f$.

Let $(\mathcal{M}, \mu)$ be a separable measure space and let h be a measurable function on it. Then

$$m_h(s) = \mu(x \in \mathcal{M} : |h(x)| > s), \qquad s > 0,$$

is the distribution function of h and

$$h^*(t) = \inf\{s > 0 : m_h(s) \le t\}, \qquad t > 0,$$

is the non-increasing rearrangement of h. The spaces $L_p^w(\mathcal{M})$ are equipped with the (quasi)-norm

$$\|h\|_{L_p^w} = \sup_{\tau>0} \tau^{1/p} h^*(\tau), \qquad p > 0.$$

If $h \in L_2 + L_\infty$, we shall use the notation

$$\langle h\rangle (t) = \left(t^{-1}\int_0^t (h^*(\tau))^2\, d\tau\right)^{1/2}, \qquad t > 0. \tag{2.2}$$

The function $\langle h\rangle$ is non-increasing as well.

3. Integral operators of the class $\mathcal{E}(C_0, C_1)$

Let $(\mathcal{M}, \mu), (\mathcal{N}, \nu)$ be separable spaces with σ-finite measures. What follows is substantial, if at least one of these spaces has infinite total measure. If $e \in L_\infty(\mathcal{M} \times \mathcal{N})$ and

$$\|e\|_{L_\infty(\mathcal{M}\times\mathcal{N})} = C_0, \tag{3.1}$$

then the integral operator

$$(Eu)(x) = \int_{\mathcal{N}} e(x, y)u(y)d\nu$$

is a bounded mapping from $L_1(\mathcal{N})$ to $L_\infty(\mathcal{M})$ and

$$\|Eu\|_{L_\infty(\mathcal{M})} \le C_0\|u\|_{L_1(\mathcal{N})} \quad \text{for all} \quad u \in L_1(\mathcal{N}). \tag{3.2}$$

In fact, the conditions (3.1) and (3.2) are equivalent, see [8] section VI.8.6. Following the notion in [4] we say, that E *is an operator of the class* $\mathcal{E}(C_0, C_1)$, if in addition to (3.2) E extends to a bounded mapping from $L_2(\mathcal{N})$ to $L_2(\mathcal{M})$ and

$$\|Eu\|_{L_2(\mathcal{M})} \le C_1\|u\|_{L_2(\mathcal{N})} \quad \text{for all} \quad u \in L_2(\mathcal{N}). \tag{3.3}$$

Let E be an operator of the class $\mathcal{E}(C_0, C_1)$. For suitable functions h we shall consider the integral operators

$$(E_h u)(x) := \int_{\mathcal{N}} e(x, y)h(x, y)u(y)d\nu. \tag{3.4}$$

If $h \in L_2(\mathcal{M} \times \mathcal{N})$, then by (3.1) E_h is a Hilbert-Schmidt operator. The linear mapping $\Pi_E : h \mapsto E_h$ is bounded from $L_2(\mathcal{M} \times \mathcal{N})$ to $\mathfrak{S}_2(L_2(\mathcal{N}), L_2(\mathcal{M}))$ as follows

$$\|E_h\|_{\mathfrak{S}_2} \le C_0\|h\|_{L_2(\mathcal{M}\times\mathcal{N})}. \tag{3.5}$$

Moreover, if $h(x,y) = a(x)b(y)$ with $a \in L_\infty(\mathcal{M})$ and $b \in L_\infty(\mathcal{N})$, then $E_h \in \mathfrak{B}\left(L_2(\mathcal{N}), L_2(\mathcal{M})\right)$ and the estimate

$$\|E_h\|_{\mathfrak{B}} \leq C_1 \|a\|_{L_\infty(\mathcal{M})} \|b\|_{L_\infty(\mathcal{N})} \tag{3.6}$$

holds.

Next let $L^\perp(\mathcal{M} \times \mathcal{N})$ be the set of measurable functions h, which permit a representation

$$h = \sum_{j \in \mathbb{N}} h_j, \qquad h_j(x,y) = a_j(x) b_j(y)$$

where

$$a_i a_j \equiv 0 \quad \text{and} \quad b_i b_j \equiv 0 \quad \text{for all} \quad i \neq j.$$

Further let $L_\infty^{\perp,\pi}(\mathcal{M} \times \mathcal{N})$ be the set of functions $g = \sum_{i \in \mathbb{N}} h^{(i)}$, such that $h^{(i)} \in L^\perp(\mathcal{M} \times \mathcal{N})$ and $\sum_i \|h^{(i)}\|_{L_\infty(\mathcal{M}\times\mathcal{N})} < \infty$. The linear space $L_\infty^{\perp,\pi}(\mathcal{M} \times \mathcal{N})$ is complete with respect to the norm

$$\|g\|_\infty^{\perp,\pi} := \inf \sum_i \|h^{(i)}\|_{L_\infty(\mathcal{M}\times\mathcal{N})},$$

where the lower bound shall be taken over all representations $\sum_i h^{(i)} = g$ with $h^{(i)} \in L^\perp(\mathcal{M} \times \mathcal{N})$ [16]. Obviously we have $L_\infty^{\perp,\pi}(\mathcal{M} \times \mathcal{N}) \hookrightarrow L_\infty(\mathcal{M} \times \mathcal{N})$. Moreover, (3.6) implies that $\Pi_E : h \mapsto E_h$ extends to a bounded mapping from $L_\infty^{\perp,\pi}(\mathcal{M} \times \mathcal{N})$ to $\mathfrak{B}(L_2(\mathcal{N}), L_2(\mathcal{M}))$ and that

$$\|E_h\|_{\mathfrak{B}} \leq C_1 \|h\|_{L_\infty^{\perp,\pi}(\mathcal{M}\times\mathcal{N})} \tag{3.7}$$

holds [16]. In fact, the functions $h \in L_\infty^{\perp,\pi}(\mathcal{M} \times \mathcal{N})$ are *multipliers* on the set of all bounded integral operators in the sense of [6]. The space $L_\infty^{\perp,\pi}$ is continuously embedded in the Banach space of these multipliers.

4. Cwikel type inequalities

We use the notation (2.2). Our main result is

Theorem 4.1. *Assume that $h(x,y) = a(x)b(y)$ and that h belongs to the space $(L_2 + L_\infty)(\mathcal{M} \times \mathcal{N})$. Then the series of estimates*

$$\begin{aligned} s_n(E_h) &\leq \inf_{t>0} \left(C_0 n^{-1/2} t^{1/2} \langle h \rangle (t) + 4C_1 h^*(t) \right) \\ &\leq \inf_{t>0} \left(C_0 n^{-1/2} t^{1/2} + 4C_1 \right) \langle h \rangle (t) \\ &\leq (1 + 4C_1) \langle h \rangle (C_0^{-2} n) \end{aligned} \tag{4.1}$$

holds.

Remark. If $\langle h \rangle (t) \to 0$ as $t \to \infty$, then $f = \{f_n\}_{n \in \mathbb{N}}$ with $f_n = 1/\langle h \rangle (C_0^{-2} n)$ automatically satisfies all necessary conditions to define a general weak operator ideal Σ_f. Then, if we are in the setting of Theorem 4.1, the last bound in (4.1) can be rewritten as

$$\|E_h\|_f \leq 1 + 4C_1, \qquad f = \{f_n\}_{n \in \mathbb{N}} \quad \text{with} \quad f_n = 1/\langle h \rangle (C_0^{-2} n). \tag{4.2}$$

Remark. Let Φ be the Fourier transform with the kernel $e(x,y) = e^{-ixy}/(2\pi)^{d/2}$ on $\mathcal{M} = \mathcal{N} = \mathbb{R}^d$ (with the Lebesgue measure). Then $\Phi, \Phi^* \in \mathcal{E}\left((2\pi)^{-d/2}, 1\right)$. Hence, for $a(x)b(i\nabla) := a\Phi^* b\Phi$ we have

$$s_n\left(a(x)b(i\nabla)\right) \leq 5\left\langle h\right\rangle\left((2\pi)^d n\right), \tag{4.3}$$

if $h(x,y) = a(x)b(y)$ belongs to $L_2 + L_\infty$.

Remark. Since for $p > 2$ we have

$$\left\langle h\right\rangle\left((2\pi)^d n\right) \leq \|h\|_{L_p^w}\left(\frac{1}{(2\pi)^d n}\int_0^{(2\pi)^d n}\frac{d\tau}{\tau^{2/p}}\right)^{1/2} = \frac{\|h\|_{L_p^w}}{\left((2\pi)^d n\right)^{1/p}\sqrt{1-\frac{2}{p}}},$$

the bound (1.1) follows immediately from (4.3) and the obvious estimate

$$\|h\|_{L_p^w(\mathbb{R}^{2d})} \leq \|a\|_{L_p(\mathbb{R}^d)}\|b\|_{L_p^w(\mathbb{R}^d)}.$$

A more careful investigation of (4.1) leads to (1.1) with the same constant as in [7].

Remark. The function $\left\langle h\right\rangle(t^2)$ gives a two-sided estimate for the K-function $K(t,h,L_2,L_\infty)$, see [1], [14]. This underlines the link between the estimates above and the interpolation result (1.2).

Theorem 4.1 is based upon the following technical key lemma. For its proof see Lemma 5.1 in [16].

Lemma 4.1. *Assume that $h(x,y) = a(x)b(y)$ and that $h \in L_2 + L_\infty$. Then for every fixed $t > 0$ there exists a decomposition $h = h_2 + h_\perp$ with $h_2 \in L_2(\mathcal{M}\times\mathcal{N})$, $h_\perp \in L_\infty^{\perp,\pi}(\mathcal{M}\times\mathcal{N})$, such that*

$$\|h_2\|^2_{L_2(\mathcal{M}\times\mathcal{N})} \leq \int_0^t \left(h^*(\tau)\right)^2 d\tau \quad \textit{and} \quad \|h_\perp\|_\infty^{\perp,\pi} \leq 4h^*(t).$$

Proof of Theorem 4.1. Fix $t > 0$ and let $h = h_2 + h_\perp$ be the decomposition in Lemma 4.1. Then by Ky-Fan's inequality we have

$$\begin{aligned} n^{1/2}s_n(E_h) &\leq n^{1/2}s_n(E_{h_2}) + n^{1/2}s_1(E_{h_\perp}) \\ &\leq \|E_{h_2}\|_{\mathfrak{S}_2} + n^{1/2}\|E_{h_\perp}\|_{\mathfrak{B}} \\ &\leq C_0\|h_2\|_{L_2(\mathcal{M}\times\mathcal{N})} + C_1 n^{1/2}\|h_\perp\|_\infty^{\perp,\pi} \\ &\leq C_0\left(\int_0^t \left(h^*(\tau)\right)^2 d\tau\right)^{1/2} + 4C_1 n^{1/2}h^*(t). \end{aligned}$$

Minimizing over $t > 0$ we arrive at the first inequality in (4.1). To pass from the first to the second bound we observe that h^* is monotone decreasing, and hence $h^*(t) \leq \left\langle h\right\rangle(t)$. Finally, for $t = C_0^{-1}n^{1/2}$ we find the third line in (4.1). □

5. An application to two-dimensional Schrödinger operators.

Let $A(V) = -\Delta - V(x)$ be the Schödinger operator on $L_2(\mathbb{R}^2)$. We shall assume that $V^*(t) \to 0$ as $t \to \infty$ and that

$$\int_0^1 V^*(t)(1+\ln_+ V(t))dt < \infty \quad \text{where} \quad \ln_+ t = \begin{cases}\ln t & t > 1\\ 0 & 0 < t \leq 1\end{cases}.$$

Then $A(V)$ is well-defined as a self-adjoint operator in the form sum sense. Let $N(\lambda, V)$ denote the rank of the spectral projection of $A(V)$ corresponding to the set $(-\infty, \lambda)$.

Cwikel's inequality (1.1) does not give estimates on the negative spectrum of $A(V)$. The inequality (4.3), however, can be applied to derive certain bounds on $N(-1,V)$.

To our knowledge the best estimate on this quantity is due to Solomyak [13]. It is stated in terms of a sequence of local averaged Orlicz norms with respect to the generating function $x(1+\ln_+ x)$. These norms appear due to the embedding theorems for critical exponents [13]. For related results see also [9], [5].

As a demonstration of the flexibility of (4.3) we prove the bound (5.1) below. Note that the assumption (5.1) is very close to the one in [13]. In our case (5.1) ensures that the average of the symbol of the Birman-Schwinger operator $V^{1/2}(x)(-\Delta+1)^{-1/2}$ is locally square integrable.

We point out that our estimate is *not* linear with respect to V, and thus does not lead to semi-classical asymptotics in the strong coupling limit; cf. with Proposition 5.2 in [13].

As an application of Theorem 4.1 we state

Theorem 5.1. *For $\beta\in(0,1)$ define $V_\beta(x)=V(x)$ for all x where $V(x)\geq\beta$ and $V_\beta(x)=0$ elsewhere. There exist finite constants $C(\beta)$, such that*

$$N(-1,V)\leq C(\beta)\left\{\|V_\beta \ln_+(V_\beta\|V_\beta\|_{L_1}^{-1})\|_{L_1}+\|V_\beta\|_{L_1}(1+\ln_+\|V_\beta\|_{L_1})\right\}. \tag{5.1}$$

Proof. Since

$$N(-1,V)\leq N(-1,(1-\beta)^{-1}V_\beta)+N(-1,\beta^{-1}(V-V_\beta))$$

and $N(-1,\beta^{-1}(V-V_\beta))=0$, we shall estimate the first term on the right-hand side. Put $a=V_\beta^{1/2}$, $b(\xi)=(|\xi|^2+1)^{-1/2}$ and $\mathcal{M}=\mathcal{N}=\mathbb{R}^2$. According to the Birman-Schwinger principle [2], [12] it holds

$$N(-1,(1-\beta)^{-1}V_\beta)=\operatorname{card}\left\{s_n\left(a(x)b(i\nabla)\right)>(1-\beta)^{1/2}\right\}. \tag{5.2}$$

We set $g(x,\xi)=a(x)\min\left\{1,|\xi|^{-1}\right\}$. Then it holds $g^*(t)\geq h^*(t)$ for all $t>0$; hence integrating (4.3) by part and applying (5.2) gives

$$\begin{aligned}(1-\beta)^{1/2}&<s_n\left(a(x)b(i\nabla)\right)\leq 5\left\langle h\right\rangle(4\pi^2 n)\leq 5\left\langle g\right\rangle(4\pi^2 n)\\&\leq 5\left\{(g^*(4\pi^2 n))^2+\frac{2}{4\pi^2 n}\int_{g^*(4\pi^2 n)}^{\infty} m_g(\tau)\tau d\tau\right\}^{1/2},\end{aligned} \tag{5.3}$$

where $n=N(-1,V)$.

The identity $m_g(s)=\pi s^{-2}\int_{|a(x)|>s}|a|^2dx$ implies

$$\pi^{-1}m_g(s)\geq s^{-2}\|a\|_{L_2}^2 \qquad \text{for} \quad s\geq\beta^{1/2},$$

and thus

$$g^*\left(\pi^{-1}t\right)\geq t^{-1/2}\|a\|_{L_2} \qquad \text{for} \quad t\geq m_a(\beta^{1/2}). \tag{5.4}$$

Note that

$$m_a(\beta^{1/2})\leq\beta^{-1}\|a\|_{L_2^w}^2\leq\beta^{-1}\|a\|_{L_2}^2. \tag{5.5}$$

Finally we have

$$m_g(s)\leq s^{-2}\pi\|a\|_{L_2}^2 \qquad \text{for} \quad s>0$$

and hence

$$g^*(\pi^{-1}t) \leq t^{-1/2}\|a\|_{L_2} \qquad \text{for} \quad t > 0. \tag{5.6}$$

The equation

$$\int_{g^*(\nu)}^{\infty} m_g(\tau)\tau d\tau = \pi \int a^2(x) \ln_+ \left(\frac{a(x)}{g^*(\nu)}\right) dx, \qquad \nu = 4\pi^2 n,$$

and (5.3) imply

$$\frac{1-\beta}{50} \leq (g^*(\nu))^2 + 2\nu^{-1}\pi \int a^2(x) \ln_+ \left(\frac{a(x)}{g^*(\nu)}\right) dx, \qquad \nu = 4\pi^2 n.$$

Inserting the estimates (5.4), (5.6) in view of (5.5) we conclude, that for appropriate finite constants $c_1(\beta)$, $c_2(\beta)$ and $c_3(\beta)$ at least one of the inequalities

$$n \leq c_1(\beta)\|a^2 \ln_+(a\|a\|_{L_2}^{-1})\|_{L_1}, \qquad n \leq c_2(\beta)\|a\|_{L_2}^2$$

or

$$\frac{n}{\ln 4\pi^3 + \ln n} \leq c_3(\beta)\|a\|_{L_2}^2$$

holds. Applying the monotone function $f(x) = x(1 + \ln_+ x)$ to the last bound and adding up the right-hand sides we complete the proof. $\square$

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E-mail address: weidl@math.kth.se