THE SPECTRA OF NONNEGATIVE INTEGER MATRICES VIA FORMAL POWER SERIES

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ABSTRACT. We characterize the d-tuples of nonzero complex numbers $(\lambda_1, \lambda_2, \dots, \lambda_d)$ which can occur as the nonzero part of the spectrum of a matrix with nonnegative integer/rational entries. These results follow easily from our main theorem: a characterization of the possible nonzero portions of spectra of primitive integer matrices (the integer case of Boyle and Handelman's Spectral Conjecture). For the proof of the main theorem we use polynomial matrices to reduce the problem of realizing a candidate spectrum $(\lambda_1, \lambda_2, \dots, \lambda_d)$ to factoring the polynomial $\prod_{i=1}^{d}(1-\lambda_i t)$ as a product $(1-r(t))\prod_{i=1}^{n}(1-q_i(t))$ where the q_i's are polynomials in $t\mathbb{Z}_+[t]$ satisfying some technical conditions and r is a formal power series in $t\mathbb{Z}_+[[t]]$. To obtain the factorization, we present a hierarchy of estimates on coefficients of power series of the form $\prod_{i=1}^{d}(1-\lambda_i t)/\prod_{i=1}^{n}(1-q_i(t))$ to ensure non-positivity in nonzero degree terms.

1. INTRODUCTION

An old problem in matrix theory is to determine the n-tuples of complex numbers which can occur as the spectrum of a matrix with nonnegative entries (see [?, chapter 4] or [?, chapter VII]). Authors have studied the case where the n-tuple is comprised of real numbers [?, ?, ?, ?, ?, ?, ?, ?], the case where the matrices under consideration are symmetric [?, ?], and the general problem [?, ?, ?]. Various necessary conditions and sufficient conditions have been provided, but a complete characterization is known for real n-tuples only for $n \leq 4$ [?, ?] and for complex n-tuples only for $n \leq 3$ [?].

Motivated by symbolic dynamics, Boyle and Handelman refocused attention on the *nonzero part* of the spectrum by conjecturing the following [?, ?] (see also [?, §8] and [?, Chapter 11]).

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Spectral Conjecture (Boyle, Handelman). *Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ be a d-tuple of nonzero complex numbers and let S be a unital subring of $\mathbb{R}$. There is a primitive matrix over S with characteristic polynomial $t^m \prod_{i=1}^d (t - \lambda_i)$ for some $m \geq 0$ if and only if*

(1) *the coefficients of the polynomial $\prod_{i=1}^d (t - \lambda_i)$ all belong to S,*
(2) *there exists λ_j in Λ such that $\lambda_j > |\lambda_i|$ for all $i \neq j$,*
(3) *if $S = \mathbb{Z}$, then $tr_n(\Lambda) \geq 0$ for all $n \geq 1$,*
(3′) *if $S \neq \mathbb{Z}$, then for all $n \geq 1$, $tr(\Lambda^n) \geq 0$ and for all $k \geq 1$, $tr(\Lambda^n) > 0$ implies $tr(\Lambda^{nk}) > 0$.*

Above,

$$tr(\Lambda^n) = \sum_{i=1}^{d} (\lambda_i)^n \qquad \text{and} \qquad tr_n(\Lambda) = \sum_{k|n} \mu\left(\frac{n}{k}\right) tr\left(\Lambda^k\right)$$

where μ is the Möbius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^e & \text{if } n \text{ is the product of } e \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by } k^2 \text{ for some } k \geq 2. \end{cases}$$

Boyle and Handelman proved the Spectral Conjecture in many cases, including the case $S = \mathbb{R}$ [**?**, **?**]. In this paper we resolve an important remaining case by proving the Spectral Conjecture for $S = \mathbb{Z}$ (Theorem 2.2). The Spectral Conjecture for $S = \mathbb{Q}$ (Corollary 2.3) follows from this result as do characterizations of the possible nonzero spectra of irreducible and general nonnegative matrices over $\mathbb{Z}$ and $\mathbb{Q}$ (Corollaries 2.5, 2.6).

It is not difficult to see the necessity of the conditions in the Spectral Conjecture for $S = \mathbb{Z}$. Suppose A is a primitive matrix over $\mathbb{Z}$ with $\det(t\mathrm{I} - A) = t^m \prod_{i=1}^d (t - \lambda_i)$. Then condition (1) is clearly satisfied. Condition (2) follows from Perron-Frobenius Theory, the spectral radius of primitive matrix (a matrix with nonnegative entries such that some power has all positive entries) is always an eigenvalue of multiplicity one for that matrix. Condition (3) follows when we interpret A as the adjacency matrix for a directed graph G_A. In this case, $tr(\Lambda^n)$ represents the number of loops of length n in the graph, and $tr_n(\Lambda)$ represents the number of loops of length n which are not formed by concatenating a single loop of shorter length some number of times.

Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ satisfy the conditions of the Spectral Conjecture for $\mathbb{Z}$. We outline our construction of a matrix A over $\mathbb{Z}_+$ with nonzero spectrum Λ. Firstly, we note that there is a primitive matrix

$A(t)$ over $t\mathbb{Z}_+[t]$ (the ring of polynomials with nonnegative integer entries and no constant term) such that $\det(\mathrm{I} - A(t)) = \prod_{i=1}^{d}(1 - \lambda_i t)$ if and only if there is a primitive integer matrix A with nonzero spectrum Λ (see [**?**, §5]). In Lemma 2.9, we use this fact to essentially reduce the problem of constructing a polynomial matrix $A(t)$ as above to finding polynomials $q_1, q_2, \dots, q_n \in t\mathbb{Z}_+[t]$ and a power series $r \in t\mathbb{Z}_+[[t]]$ such that

$$\prod_{i=1}^{d}(1 - \lambda_i t) = (1 - r(t)) \prod_{i=1}^{n}(1 - q_i(t)).$$

In particular, we show that there are integers $o(n) \geq 0$, $n_0 \geq 1$ and a polynomial $q \in t\mathbb{Z}_+[t]$ such that the nth coefficient of

$$\frac{\prod_{i=1}^{d}(1 - \lambda_i t)}{(1 - q(t)) \prod_{n=1}^{n_0}(1 - t^n)^{o(n)}}$$

is non-positive for all $n \geq 1$. We obtain the desired inequality by giving estimates for the nth coefficient of $\prod_{n=1}^{n_0}(1 - t^n)^{o(n)}$ in certain linear, polynomial, subexponential, exponential ranges of n_0. For sufficiently large n_0, these estimates imply the non-positivity of coefficients of the above quotient up to a degree where the additional factor $1 - q(t)$ takes over.

With the results here and in [**?**], a complete characterization of the spectra of primitive matrices over $\mathbb{R}$ or $\mathbb{Z}$ would follow from sharp bounds on the size of the realizing matrix. However, it seems that bounds of this type will be quite difficult to pin down. In particular, it follows from an inequality in [**?**, **?**] that given any N one can construct a 3-tuple satisfying the Boyle-Handelman conditions for $\mathbb{R}$ such that the size of a realizing matrix must be at least $N \times N$ (see [**?**]). This is in dramatic contrast to the symmetric case where Johnson, Laffey and Loewy showed that if a d-tuple $(\lambda_1, \lambda_2, \dots, \lambda_d)$ of real numbers is the nonzero spectrum of a symmetric matrix then it is the nonzero spectrum of a symmetric matrix of size no more than $d(d+1)/2$ [**?**].

We propose that a variant of the size bound problem may be more approachable: Given $(\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfying the Boyle-Handelman conditions, produce sharp bounds on the size of a polynomial matrix $A(t)$ with $\det(\mathrm{I} - A(t)) = \prod_{i=1}^{d}(1 - \lambda_i t)$. In some respects, bounds of this type would be more useful than bounds on ordinary matrices since one can exhibit a much wider range of behavior with polynomial matrices of a fixed size. Moreover, it is conceivable that this line of study could lead to a solution to the original problem. For an example of the freedom afforded by polynomial matrices, Perrin showed that for every λ which occurs as the spectral radius of a primitive integer matrix, there is a

2×2 matrix $A(t)$ over $t\mathbb{Z}_+[t]$ with $\det(I-A(t)) = (1-\lambda t)\prod_{i=1}^{d}(1-\lambda_i t)$ and $\lambda > |\lambda_i|$ for all i [?] (see also [?, §5]).

Our characterization of the possible nonzero spectra of nonnegative/irreducible/primitive matrices over $\mathbb{Z}$ translates into a characterization of the possible zeta functions for general/irreducible/mixing shifts of finite type (SFTs) in symbolic dynamics. In this setting our work follows Lind's classification of entropies of SFTs [?], Boyle and Handelman's classification of zeta functions for finitely presented systems [?] and various authors' development of polynomial matrices as tools in symbolic dynamics [?, ?, ?, ?]. In general, there is a deep connection between symbolic dynamics and the asymptotic algebra of nonnegative matrices (see [?, ?, ?]). Boyle and Handelman's Generalized Spectral Conjecture (see [?, §8]) concerns the matrix relation, strong shift equivalence over S, which in the case $S = \mathbb{Z}_+$ corresponds to conjugacy between associated SFTs. Strong shift equivalence over S is defined as the transitive closure of the elementary relation $\sim_S$ where $A \sim_S B$ if there exist matrices U, V over S such that

$$A = UV \qquad \text{and} \qquad VU = B.$$

Strong shift equivalence seems to be the strongest asymptotic equivalence relation [?, ?]. In particular strong shift equivalent matrices have the same nonzero spectrum. Boyle and Handelman conjecture that a matrix A over a unital ring $S \subseteq \mathbb{R}$ is strong shift equivalent to a primitive matrix if and only if the nonzero spectrum of A satisfies the conditions of the Spectral Conjecture.

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2. Main Results

2.1. **Statements.** For a unital ring $S \subseteq \mathbb{R}$ we will often refer to the conditions of the Spectral Conjecture for S as the *Boyle-Handelman conditions for S*. We will also clean up notation a bit by noting that for a matrix A, there is an $m \geq 0$ such that $\det(t\mathrm{I}-A) = t^m \prod_{i=1}^{d}(t-\lambda_i)$ if and only if $\det(\mathrm{I}-At) = \prod_{i=1}^{d}(1-\lambda_i t)$.

Definition 2.1. A matrix A over $\mathbb{R}_+$ is *primitive* there is an $n > 0$ such that all of the entries of A^n are positive.

Main Theorem 2.2. *Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ be a d-tuple of nonzero complex numbers with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$. There exists a primitive integer matrix A such that* $\det(\mathrm{I}-At) = \prod_{i=1}^{d}(1-\lambda_i t)$ *if and only if*

(1) *the polynomial* $\prod_{i=1}^{d}(1-\lambda_i t)$ *has integer coefficients,*

(2) $\lambda_1 > |\lambda_i|$ *for* $i = 2, 3, \ldots, d$,
(3) $tr_n(\Lambda) \geq 0$ *for all* $n \geq 1$.

As shown in [?], the Spectral Conjecture for $\mathbb{Q}$ follows from our main result.

Corollary 2.3. *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ *be a n-tuple of nonzero complex numbers with* $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$. *There is a primitive rational matrix* A *with* $\det(I - At) = \prod_{i=1}^{d}(1 - \lambda_i t)$ *if and only if*

(1) *the polynomial* $\prod_{i=1}^{d}(1 - \lambda_i t)$ *has rational coefficients,*
(2) $\lambda_1 > |\lambda_i|$ *for* $i = 2, 3, \ldots, d$,
(3) *for all* $n \geq 1$, $tr(\Lambda^n) \geq 0$ *and*
for all $k \geq 1$, $tr(\Lambda^n) > 0$ *implies* $tr(\Lambda^{nk}) > 0$.

We briefly describe how a characterization of the spectra of irreducible and nonnegative matrices follow from the primitive case (see also [?]).

Definition 2.4. A matrix A over $\mathbb{R}_+$ is *irreducible* if for all (i, j) there is an $n \geq 0$ such that the (i, j) entry of A^n is positive.

If A is irreducible then there is a $p \geq 1$ and a primitive matrix B such that $\det(\mathrm{I} - At) = \det(\mathrm{I} - Bt^p)$. This leads to the following.

Corollary 2.5. *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ *be a d-tuple of nonzero complex numbers. There exists an irreducible matrix* A *over* $\mathbb{Z}$ $(\mathbb{Q})$ *such that* $\det(\mathrm{I} - At) = \prod_{i=1}^{d}(1 - \lambda_i t)$ *if and only if there exist an integer* $p > 0$ *and a partition of* Λ *into subtuples* $\{\Lambda_k : 0 \leq k < p\}$ *such that*

(1) $\Lambda_{k+1} = e^{2\pi i/p}\Lambda_k$ *for* $0 \leq k < (p-1)$,
(2) $(\Lambda_0)^p$ *satisfies the Boyle-Handelman conditions for* $\mathbb{Z}$ $(\mathbb{Q})$.

If A is a nonnegative matrix then there are irreducible matrices $A_1, A_2, \ldots, A_m$ such that $\det(\mathrm{I} - At) = \prod_{j=1}^{m} \det(\mathrm{I} - A_j t)$.

Corollary 2.6. *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ *be a d-tuple of nonzero complex numbers. There exists a nonnegative matrix* A *over* $\mathbb{Z}$ $(\mathbb{Q})$ *such that* $\det(\mathrm{I} - At) = \prod_{i=1}^{d}(1 - \lambda_i t)$ *if and only if there exist an integer* $n > 0$, *integers* $p(1), p(2), \ldots, p(n) > 0$ *and a partition of* Λ *into subtuples* $\{\Lambda_{(j,k)} : 1 \leq j \leq n,\ 0 \leq k < p(j)\}$ *such that*

(1) $\Lambda_{(j,k+1)} = e^{2\pi i/p(j)}\Lambda_{(j,k)}$ *for all* $0 \leq k < (p(j) - 1)$ *and* $1 \leq j \leq n$,
(2) $(\Lambda_{(j,0)})^{p(j)}$ *satisfies the Boyle-Handelman conditions for* $\mathbb{Z}$ $(\mathbb{Q})$ *for all* $1 \leq j \leq n$.

Remark 2.7. A characterization of zeta functions for SFTs follows from Corollary 2.6. Suppose A is a nonnegative matrix A and S_A is the corresponding SFT. If $Fix((S_A)^n)$ is the set of fixed points of the map $(S_A)^n$ then we have the following definition and result.

$$\zeta_{(S_A)}(t) = \exp\left(\sum_{i=1}^{\infty} \frac{\#Fix((S_A)^n)}{n} t^n\right) = \frac{1}{\det(\mathrm{I} - At)}.$$

Therefore, a power series in $\mathbb{Z}[[t]]$ is a zeta function if and only if it is equal to $\prod_{i=1}^{d}(1-\lambda_i t)^{-1}$ where $(\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfies the conditions of Corollary 2.6. Moreover, since irreducible matrices give rise to irreducible SFTs and primitive matrices give rise to mixing SFTs, Theorem 2.2 and Corollary 2.5 give finer characterizations for zeta functions associated to SFTs in those classes.

2.2. **Proof Scheme for Main Theorem.** Proving the sufficiency of the Boyle-Handelman conditions for $\mathbb{Z}$ will come down to producing a certain factorization of the polynomial $\prod_{i=1}^{d}(1-\lambda_i t)$ in the ring of formal power series with integer coefficients. To show this, we must first examine the behavior of coefficients of large degree in a rational expression.

Lemma 2.8. *Suppose $p, q \in t\mathbb{Z}[t]$ and $r \in \mathbb{Z}[[t]]$ such that*

$$\frac{1-p(t)}{1-q(t)} = r(t) = \sum_{i=0}^{\infty} r_n t^n.$$

Suppose that $1 - q(t) = (1-\alpha t)\prod_{i=1}^{d}(1-\alpha_i t)^{j_i}$ where the α_i are all distinct, $j_i \geq 1$ and $\alpha > |\alpha_i|$ for $i \geq 1$. Then there exist constants $K, k > 0$ such that

$$\left|\frac{r_n}{\alpha^n} - \frac{\alpha(1-p(1/\alpha))}{q'(1/\alpha)}\right| < Ke^{-kn}.$$

Proof. First we consider the power series $S(t) = (1-q(t))^{-1}$. Using a partial fraction decomposition we may write

$$S(t) = \frac{a}{1-\alpha t} + \sum_{i=1}^{d}\sum_{j=1}^{j_i} \frac{a(i,j)}{(1-\alpha_i t)^j} \tag{2.1}$$

where $a(i,j)$ are constants. We obtain the formula $a = \alpha/q'(1/\alpha)$ by multiplying both sides of equation 2.1 by $1-q(t)$ and plugging in $1/\alpha$.

Since $(1-\alpha_i t)^{-j} = \sum_{n=1}^{\infty}\binom{n+j-1}{n}(\alpha_i t)^n$ we have a formula for the nth coefficient of $S(t)$

$$s_n = \alpha^n\left(a + \sum_{i=1}^{d}\sum_{j=1}^{j_i}\binom{n+j-1}{n} a(i,j)\,(\alpha_i/\alpha)^n\right).$$

Therefore, there exist constants $C, c > 0$ such that

$$\left|s_n\alpha^{-n} - \alpha/q'(1/\alpha)\right| = \left|s_n\alpha^{-n} - a\right| < Ce^{-cn}.$$

The formula for r_n, the nth coefficient of $S(t)(1-p(t))$, is given by the following

$$r_n = s_n - \sum_{i=1}^{d} p_i s_{n-i} = s_n\left(1 - \sum_{i=1}^{d} p_i s_{n-i}/s_n\right).$$

By the estimates on s_n, the ratio s_{n-i}/s_n is exponentially close to α^{-i}. This gives $r_n \approx s_n(1-p(1/\alpha))$ with an exponentially small error estimate. Plugging in the estimate for s_n, we obtain the desired result. □

If, as in the previous lemma, a polynomial $1-q(t)$ has a factorization of the form $(1-\alpha t)\prod_{i=1}^{d}(1-\alpha_i t)^{j_i}$ where $j_i \geq 1$ and $\alpha > |\alpha_i|$ for all i then we will call α the *Perron value* of $1-q(t)$. If $q \in t\mathbb{Z}_+[t]$ then $1-q(t)$ has such a factorization as long as the degrees in which nonzero coefficients occur are relatively prime.

For the remainder of the paper, suppose $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ satisfies the Boyle-Handelman conditions for $\mathbb{Z}$ and $1-p(t) = \prod_{i=1}^{d}(1-\lambda_i t)$. As noted in the Introduction, one can produce a primitive matrix A with entries in $\mathbb{Z}_+$ such that $\det(\mathrm{I}-At) = 1-p(t)$ if and only if there is a primitive polynomial matrix $A(t)$ with entries in $t\mathbb{Z}_+[t]$ such that $\det(\mathrm{I}-A(t)) = 1-p(t)$ (see [**?**] for details). Thus to prove the remaining direction of the main theorem it suffices to show that there exist polynomials $q_j \in t\mathbb{Z}_+[t]$ and a power series $r \in t\mathbb{Z}_+[[t]]$ such that the hypothesis of the following are satisfied.

Lemma 2.9 (Main Reduction)**.** *Let $p \in \mathbb{Z}[t]$. Suppose there exist polynomials $q_1, q_2, \ldots, q_m \in t\mathbb{Z}_+[t]$ and a formal power series $r \in t\mathbb{Z}_+[[t]]$ such that*

(1) *there is a Perron value α for the polynomial $\prod_{j=1}^{m}(1-q_j(t))$,*
(2) *$1-p(1/\alpha) < 0$,*
(3) *$1-p(t) = (1-r(t))\prod_{j=1}^{m}(1-q_j(t))$ in the ring of formal power series.*

Then there exists a matrix $A(t)$ with entries in $t\mathbb{Z}_+[t]$ such that

$$\det(\mathrm{I}-A(t)) = 1-p(t).$$

Proof. Assume that the polynomials q_i are numbered so that $1/\alpha$ is a root of the first polynomial $1-q_1(t)$. By Lemma 2.8, and hypotheses 1 and 2 there is a $N > 0$ such that for all $1 \leq k \leq m$, the nth coefficient of $(1-p(t))\prod_{j=1}^{k}(1-q_j(t))^{-1}$ is negative for all $n>N$. Select this N to exceed the degree of $p(t)$ and the degree of all polynomials $q_j(t)$.

Proceeding by induction, fix $0\le k<m$ and assume

$$\frac{1-p(t)}{\prod_{j=1}^{k}(1-q_j(t))}=1-a_k(t)-\sum_{i=1}^{k}\frac{b_i(t)}{\prod_{j=i}^{k}(1-q_j(t))} \tag{2.2}$$

where $a_k(t)$ is a polynomial of degree $2N$ and the $b_i(t)$'s are polynomials in $t^{2N+1}\mathbb{Z}_+[t]$. The above clearly holds for $k=0$, the base step of our induction.

Let $a_{k+1}\in t\mathbb{Z}[t]$ where $1-a_{k+1}(t)$ is the sum of the first $2N$ terms of $(1-p(t))\prod_{j=1}^{k+1}(1-q_j(t))^{-1}$. Let $b_{k+1}(t)$ be the polynomial such that

$$1-a_k(t)=(1-a_{k+1}(t))(1-q_{k+1}(t))-b_{k+1}(t).$$

Notice that all coefficients of $b_{k+1}(t)$ in degrees $2N$ and less are zero. In degrees greater than $2N$, the coefficients of $b_{k+1}(t)$ are nonnegative since they must cancel with the coefficients of $a_{k+1}(t)q_{k+1}(t)$.

We plug the above expression for $1-a_k(t)$ into equation 2.2 and divide both sides by $(1-q_{k+1}(t))$. We obtain the equation

$$\frac{1-p(t)}{\prod_{j=1}^{k+1}(1-q_j(t))}=1-a_{k+1}(t)-\sum_{i=1}^{k+1}\frac{b_i(t)}{\prod_{j=i}^{k+1}(1-q_j(t))}.$$

Thus, by induction we have

(2.3)

$$1-p(t)=(1-a_m(t))\prod_{j=1}^{m}(1-q_j(t))-\sum_{i=1}^{m}\left[b_i(t)\prod_{j=1}^{i-1}(1-q_j(t))\right]$$

where the degree of a_m is $2N$ and $b_i\in t^{2N+1}\mathbb{Z}_+[t]$ for all i. But by the hypotheses of the lemma, $a_m(t)$ is equal to the first $2N$ terms of $r(t)$, therefore all of the coefficients of $a_m(t)$ are nonnegative.

Now let $A(t)$ be the following polynomial matrix over $t\mathbb{Z}_+[t]$.

$$A(t)=\begin{pmatrix} a_m(t) & 1 & 0 & \dots & 0\\ b_m(t) & q_m(t) & 1 & \ddots & \vdots\\ b_{m-1}(t) & 0 & q_{m-1}(t) & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & 1\\ b_1(t) & 0 & \dots & 0 & q_1(t)\end{pmatrix}.$$

Using equation 2.3, we find $\det(\mathrm{I}-A(t))=1-p(t)$.

In order to have all entries of $A(t)$ in $t\mathbb{Z}_+[t]$, we may replace each occurrence of a 1 with a t and each $b_i(t)$ with $b_i(t)/t^{m+1-i}$. $\square$

How do we find a factorization of $1 - p(t)$ as in the Main Reduction Lemma? We first note that for the polynomial $1 - p(t)$ factors into an infinite product

$$1 - p(t) = \prod_{i=1}^{\infty}(1 - t^i)^{o(i)} \tag{2.4}$$

where $o(i) = tr_i(\Lambda)/i$. To see this, take the logarithm of both sides of equation 2.4 and compare ith coefficients of the resulting power series. We obtain the formula $tr(\Lambda^i) = \sum_{k|i} ko(k)$. Noting the relation $tr(\Lambda^i) = \sum_{k|i} tr_k(\Lambda)$, we obtain $o(i) = tr_i(A)/i$ by induction.

As we will see, we can choose an n_0 so that the coefficients of the power series $(1 - p(t)) \prod_{i=1}^{n_0}(1 - t^i)^{-o(i)}$ are non-positive up to an arbitrarily high degree. However, as we saw in Lemma 2.8 this will not give the non-positivity for all nonzero term coefficients if for example, $1-p(1) > 0$. In general, we are forced to introduce an additional factor $1 - q(t)$ with an appropriate Perron value to control the coefficients in high degree.

We conclude this section with the outlines of the two different arguments we will use to determine the sign of coefficients of $1 - p(t)$ times a power series. Let λ denote the candidate Perron eigenvalue $\lambda = \lambda_1$. Let $C(t) = \sum_{i=0}^{\infty} c_i t^i$ denote an arbitrary power series with $c_i \geq 0$ for all i.

Argument 1 (Differences Argument)**.** *If* $(c_{n-1}\lambda - c_n)$ *is positive and large compared to* $\left| (c_{n-i}\lambda^i - c_{n-i+1}\lambda^{i-1}) - (c_{n-i+1}\lambda^{i-1} - c_{n-i+2}\lambda^{i-2}) \right|$ *for* $i = 0, 1, \ldots, d$ *then the* n*th coefficient of* $(1 - p(t))C(t)$ *is negative.*

Let D_i denote the first order differences, $D_i = c_{n-i}\lambda^i - c_{n-i+1}\lambda^{i-1}$ for $1 \leq i \leq d$. Then

$$c_{n-i}\lambda^i = c_n + \sum_{j=1}^{i} D_j = c_n + iD_1 + \sum_{j=1}^{i}(i-j)(D_{j+1} - D_j).$$

Our assumption is that the second order differences $(D_{j+1} - D_j)$ are small in comparison to the first order difference D_1. Ignoring the second order differences, the nth coefficient of $(1 - p(t)) \sum_{n=1}^{\infty} c_n t^n$ is

$$c_n - \sum_{i=1}^{d} p_i c_{n-i} \approx c_n \left(1 - \sum_{i=0}^{d} p_i \lambda^{-i}\right) - D_1 \sum_{i=0}^{d} i p_i \lambda^{-i} = -D_1 p'(1/\lambda).$$

We know that $p'(1/\lambda) > 0$ by the Perron condition. Therefore, if the second order differences are a small fraction of the $D_1 = (\lambda c_{n-1} - c_n)$ then we will obtain an upper bound on the nth coefficient of $(1 - p(t))C(t)$ of the form $-K(\lambda c_{n-1} - c_n)$ where $K > 0$ is a constant.

For the second argument, fix $\lambda' \in (|\lambda_2|, \lambda)$ such that $t \in [1/\lambda, 1/\lambda')$ implies $1 - p(t) < 0$ and $p'(t) > 0$.

Argument 2 (Ratio Argument)**.** *If there exists a number $\alpha \in (\lambda', \lambda)$ such that $|c_{n-i}/c_n - \alpha^{-i}|$ is small compared to $|\alpha - \lambda|$ for $i = 1, 2, \ldots, d$ then the nth coefficient of $(1 - p(t))C(t)$ is negative.*

This follows since the nth coefficient of $(1 - p(t))C(t)$ is equal to $c_n - \sum_{i=0}^{d} p_i c_{n-i}$. Factoring out c_n, we obtain

$$n\text{th coefficient } = c_n \left(1 - \sum_{i=0}^{d} p_i (c_{n-i}/c_n)\right).$$

If all ratios c_{n-i}/c_n are sufficiently close to α^{-i} then this nth coefficient will be close to $c_n(1 - p(1/\alpha)) < 0$.

More specifically, since $-p'(1/\lambda) < 0$, there is a constant $K > 0$ such that $1 - p(t) \leq -K|\lambda - 1/t|$ for $t \in (1/\lambda, 1/\lambda')$. So if the difference $|c_{n-i}/c_n - \alpha^{-i}|$ is some small fraction of $|\lambda - \alpha|$ then we will obtain an upper estimate on the nth coefficient of $(1 - p(t))C(t)$ of the form $-Kc_n|\alpha - \lambda|/2$.

3. Example

The polynomial $1 - p(t) = 1 - 4t + 6t^2 - 6t^3$ has a factorization $\prod_{i=1}^{3}(1 - \lambda_i t)$ where $\lambda_1 \approx 2.57474$ and $\lambda_2 = \overline{\lambda_3} \approx .71263 + 1.35000i$. Lind and Marcus [**?**, chapter 11] give $(\lambda_1, \lambda_2, \lambda_3)$ as an example of an n-tuple satisfying the Boyle-Handelman conditions, but one for which there was no known primitive matrix with the corresponding nonzero spectrum. We will need a polynomial matrix $A(t)$ of at least size 4×4 to realize $1 - p(t)$. To see this, suppose $A(t)$ is a 3×3 matrix with entries in $t\mathbb{Z}_+[t]$,

$$A(t) = \begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix}$$

such that $\det(I - A(t)) = 1 - p(t)$. Equating coefficients, $a_1 + e_1 + i_1 = 4$ and $a_1e_1 + a_1i_1 + e_1i_1 \geq 6$. This is impossible since all coefficients are nonnegative integers.

Following our factorization scheme, we begin by dividing $1 - p(t)$ by powers of $1 - t^i$. The additional factor $1 - q(t)$ will not be necessary here since $1 - p(1) < 0$. A calculation shows that for $j = 1, 2, 3$ the coefficients of the power series $(1 - p(t))(1 - t)^{-j}$ are non-positive in degrees 3 to 6. Moreover, the coefficients of $(1 - p(t))(1 - t)^{-3}$ are non-positive in degrees 1 to 6. Following the scheme in the Main Reduction

Lemma, we produce the matrix

$$A(t) = \begin{pmatrix} t + 2t^3 + 7t^4 + 15t^5 & t & 0 & 0 \\ 15t^5 & t & t & 0 \\ 8t^4 & 0 & t & t \\ 3t^3 & 0 & 0 & t \end{pmatrix}$$

with $\det(I - A(t)) = 1 - 4t + 6t^2 - 6t^3$. Notice that $A(t)^3$ has no nonzero entries i.e., $A(t)$ is primitive. To point out the advantages of the polynomial matrix approach, we note that the primitive integer matrix corresponding to the above is 194×194.

4. Initial estimates on $S(t)$

For the remainder of the paper, fix $(\lambda_1, \lambda_2, \dots, \lambda_d)$ satisfying the Boyle-Handelman conditions and let $1 - p(t) = \prod_{i=1}^{d}(1 - \lambda_i t)$. Fix n_0 and let $S(t) = \sum_{n=1}^{\infty} s_n t^n = \prod_{i=1}^{n_0}(1 - t^i)^{-o(i)}$.

In this section, we show that for any $k \geq 1$, if n_0 is sufficiently large then the nth coefficient of the power series $S(t)(1 - p(t))$ will be non-positive for $n_0 < n \leq (n_0)^k$ (Polynomial Range Lemma). We begin by giving estimates for s_n when $n_0 < n \leq 2n_0 + \log^2 n_0$.

We use the term "constant" in the following lemmas to mean a number which is independent of the choice of n_0.

Lemma 4.1. *There exists a constant $K > 0$ such that if n_0 is sufficiently large and $n_0 < n \leq n_0 + \log^2 n_0$ then*

$$\left| \frac{s_{n+1}}{\lambda s_n} - 1 \right| < K \frac{\log^2 n_0}{n_0}.$$

Proof. We have the following expansions of $S(t)$.

$$S(t) = \prod_{i=1}^{n_0}(1 - t^i)^{-o(i)} = \prod_{i=n_0+1}^{\infty}(1 - t^i)^{o(i)}/(1 - p(t)).$$

Let c_n denote the nth coefficient of $(1-p(t))^{-1}$. If $n_0 < n \leq n_0 + \log^2 n_0$ then

$$s_n = c_n - \sum_{i=n_0+1}^{n} c_{n-i} o(i).$$

Let $a = \lambda / p'(1/\lambda)$. By Lemma 2.8 there exist constants $C_1, c > 0$ such that

$$\left| c_n \lambda^{-n} - a \right| < C_1 e^{-cn}$$

for all $n > 0$. Recalling the relationship between $o(n)$ and the nth net trace, we have constants $C_2, c > 0$ such that

$$\left|o(n)\lambda^{-n} - 1/n\right| < C_2 e^{-cn}$$

for all $n > 0$. In the estimates that follow, these will allow us to replace $c_n\lambda^{-n}$ with a and $o(n)\lambda^{-n}$ with $1/n$.

Using these estimates, if $n_0 < n \le n_0 + \log^2 n_0$ then

$$\lambda^{-n} \sum_{i=n_0+1}^{n} c_{n-i} o(i) < C_3 \log^2 n_0 / n_0$$

for some constant $C_3 > 0$.

Therefore, the ratio $s_{n+1}/\lambda s_n$ is within a constant times $\log^2 n_0/n_0$ of the ratio $\lambda^{-n-1}c_{n+1}/\lambda^n c_n$. Applying our estimates on c_n a second time, we find a constant $K > 0$ such that

$$\left|\frac{s_{n+1}}{\lambda s_n} - 1\right| < K\frac{\log^2 n_0}{n_0}.$$

□

Lemma 4.2. *There exist constants $k_1 > k_2 > 0$ such that if n_0 is sufficiently large and $n_0 + \log^2 n_0 < n \le 2n_0 + \log^2 n_0$ then*

$$\left(1 - \frac{k_1}{n_0}\right) < \frac{s_{n+1}}{\lambda s_n} < \left(1 - \frac{k_2}{n_0}\right).$$

Proof. Assume $n_0 < n \le 2n_0 + \log^2 n_0$. Using the notation of the previous lemma,

$$s_n = c_n - \sum_{i=n_0+1}^{n} c_{n-i}o(i) + \sum_{i=n_0+1}^{n/2} \binom{o(i)}{2} c_{n-2i} + \sum_{\substack{i,j>n_0 \\ i+j\le n}} o(i)o(j)c_{n-i-j}.$$

To obtain our estimates we will examine the difference of scaled coefficients $|s_n\lambda^{-n} - s_{n+1}\lambda^{-(n+1)}|$ in the above formula for s_n.

Assume that $n_0 + \log^2 n_0 < n \le 2n_0 + \log^2 n_0$. ¿From our estimates on c_i and $o(i)$ we obtain estimates for the last two terms of the form

$$\lambda^{-n} \sum_{i=n_0+1}^{n/2} \binom{o(i)}{2} c_{n-2i} < C_1 \left(\frac{\log^2 n_0}{n_0}\right)$$

and

$$\lambda^{-n} \sum_{\substack{i,j>n_0 \\ i+j\le n}} o(i)o(j)c_{n-i-j} < C_2 \left(\frac{\log^4 n_0}{n_0^2}\right)$$

for constants $C_1, C_2 > 0$. To obtain the above, notice each term in the above two summations is less than a constant times $1/n_0^2$.

Finally, we split the difference of the remaining summation terms into three parts. For each part we will use $c_n\lambda^{-n} \approx a$ and $o(n)\lambda^{-n} \approx 1/n$ with errors which are exponentially small in n_0. Let $n_1 = [\log^2 n_0]$, where $[x]$ is the greatest integer function. For the first part,

$$\left| \sum_{i=n_0+1}^{n-n_1} \frac{o(i)}{\lambda^i} \left(\frac{c_{n-i}}{\lambda^{n-i}} - \frac{c_{n+1-i}}{\lambda^{n+1-i}} \right) \right| < C_3 e^{-c\log^2 n_0}$$

for some constants $C_3, c > 0$. The second part is estimated as

$$\left| \sum_{i=n-n_1+1}^{n} \frac{c_{n-i}}{\lambda^{n-i}} \left(\frac{o(i)}{\lambda^i} - \frac{o(i+1)}{\lambda^{i+1}} \right) \right| < C_4 \frac{\log^2 n_0}{n_0^2}$$

for some constant $C_4 > 0$. This leaves only one term

$$c_{n_1} o(n+1-n_1)\lambda^{-n} = \left(\frac{a}{n+1-\log^2 n_0} \right) \left(1 \pm C_5 e^{-c\log^2 n_0} \right).$$

This last term is the largest in the scaled difference $s_n\lambda^{-n} - s_{n+1}\lambda^{-(n+1)}$. For $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$, its ratio with $1/n_0$ is bounded between two positive constants. Therefore, the difference of scaled s_n terms is within two constants of $1/n_0$.

Since the terms $s_n\lambda^{-n}$ are uniformly bounded in this range we can find two constants $k_1 > k_2 > 0$ such that if n_0 is chosen sufficiently large and $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$ then

$$\left(1 - \frac{k_1}{n_0} \right) < \frac{s_{n+1}}{\lambda s_n} < \left(1 - \frac{k_2}{n_0} \right).$$

□

To get further results for $S(t)$, we will approximate the coefficients of $S(t)$ with the coefficients of $\exp(L(t))$ where $L(t)$ is defined as follows. Let$\tau_n = \sum_{i=1}^{d} (\lambda_i)^n$,

$$L(t) = \sum_{i=1}^{n_0} (\tau_i/i)\, t^i \qquad L'(t) = \log(S(t)) - L(t).$$

To obtain formulas for $L(t)$ and $L'(t)$ we notice

$$\begin{aligned}\log(S(t)) &= \log\left(\prod_{i=1}^{n_0}(1-t^i)^{-o(i)}\right)\\ &= -\sum_{i=1}^{n_0} o(i)\log(1-t^i)\\ &= \sum_{n=1}^{n_0}\frac{\tau_n t^n}{n} + \sum_{n=n_0+1}^{\infty}\sum_{\substack{i\mid n\\ i\le n_0}}\frac{io(i)}{n}t^n.\end{aligned}$$

The above follows by expanding $\log(1-x)$ into a power series and gives us a formula for the coefficients L'_n. The next lemma gives estimates on the terms L'_n. This will help produce estimates on how well the coefficients of $S(t)$ approximate those of $\exp(L(t))$.

Lemma 4.3. *There exists a constant $K > 0$ such that if n_0 is sufficiently large and $n > n_0$ then*
(1) $L'_n \le K\lambda^{n/2}$
(2) $L'_n \le K\lambda^{n_0}$
(3) $[\exp L'(t)]_n \le K\lambda^{2n/3}$

Proof. If $n > n_0$ our estimates on $o(i)$ give

$$L'_n = \sum_{\substack{i\mid n\\ i\le n_0}} io(i)/n \le \sum_{i=1}^{n/2} io(i)/n < C_1\lambda^{n/2}$$

for some constant $C_1 > 0$ and

$$L'_n = \sum_{\substack{i\mid n\\ i\le n_0}} io(i)/n \le \sum_{i=1}^{n_0} io(i)/n < C_2\lambda^{n_0}$$

for some constant $C_2 > 0$.

For the third estimate, plug in the quantity $\lambda^{-2/3}$ into the function $L'(t)$.

$$L'(\lambda^{-2/3}) = \sum_{n=n_0+1}^{\infty} L'_n\lambda^{-2n/3} < \sum_{n=n_0+1}^{\infty} C_1\lambda^{n/2}\lambda^{-2n/3} < C_3\lambda^{-n_0/6}$$

for some constant $C_3 > 0$.

Therefore, if we take n_0 sufficiently large, $\exp(L'(\lambda^{-2/3}))$ will be close to one. Let d_n be the nth coefficient of $\exp(L'(t))$. For sufficiently large

n_0,

$$2 > \exp(L'(\lambda^{-2/3})) = 1 + \sum_{n=1}^{\infty} d_n \lambda^{-2n/3}.$$

Since all d_n are nonnegative, this implies $d_n < \lambda^{2n/3}$ for all n. □

Let

$$E(t) = \exp(L(t)) = \sum_{n=0}^{\infty} e_n t^n.$$

We will show in the next two lemmas that the ratio s_n/e_n is approximately one for $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$.

Lemma 4.4. *There exists a constant $K > 0$ such that if n_0 is sufficiently large, $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$ and $0 \leq m < n$ then*

$$\frac{s_m}{s_n} < K\lambda^{m-n}.$$

Proof. If $m > n_0 + \log^2 n_0$ then using Lemma 4.2,

$$\frac{\lambda^{n-m} s_m}{s_n} < \left(1 - \frac{k_1}{n_0}\right)^{m-n} < \left(1 - \frac{k_1}{n_0}\right)^{-n_0}.$$

This upper bound above approaches e^{k_1} as n_0 goes to infinity so we obtain a uniform bound on these ratios.

Let $n_1 = [\log^2 n_0]$. If $n_0 < m < n_0 + n_1$ then using Lemma 4.1 we obtain a constant $C_1 > 0$ such that

$$\frac{\lambda^{n_0+n_1-m} s_m}{s_{n_0+n_1}} < (1 + C_1 \log^2 n_0/n_0)^{\log^2 n_0}.$$

This upper bound approaches 1 as n_0 goes to infinity, so these ratios are also uniformly bounded.

If $m \leq n_0$ then there is a uniform bound on $\lambda^{n_0-m} s_m/s_{n_0}$ since in this range s_m is equal to the mth coefficient of the series $(1 - p(t))^{-1}$. Recall that the terms of $(1 - p(t))^{-1}$ look like $a\lambda^n(1 \pm C_2 e^{-cn})$ with $C_2, c > 0$.

The product of the bounds for each region will be a constant $K > 0$ which uniformly bounds $\lambda^{n-m} s_m/s_n$ for all m, n in the hypothesis. □

Lemma 4.5. *There exists a constant $K > 0$ such that if n_0 is sufficiently large and $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$ then*

$$\left| \frac{s_n}{e_n} - 1 \right| < K\lambda^{-n_0/3}$$

Proof. Recall $E(t) = S(t)\exp(-L'(t))$. Therefore

$$e_n = s_n + \sum_{i=n_0+1}^{n} s_{n-i}[\exp(-L'(t))]_i$$

¿From the estimates on $L'(t)$, there is a constant $C_1 > 0$ such that

$$|e_n/s_n - 1| \le \sum_{i=n_0+1}^{n} (s_{n-i}/s_n)\big|[\exp(-L'(t))]_i\big| < \sum_{i=n_0+1}^{n} C_1\lambda^{-i}\lambda^{2i/3}$$

This gives an upper bound of the form $K\lambda^{-n_0/3}$ for some $K > 0$. □

This shows that at least initially, e_n will give a good approximation for s_n. To continue we will examine the e_n terms. Notice that from $S(t) = E(t)\exp(L'(t))$, we get

$$e_n \le s_n \le e_n + \sum_{i=n_0}^{n} e_{n-i}\lambda^{2i/3}.$$

Therefore, as long as the ratios e_{n+1}/e_n are at least $\lambda^{(\epsilon+2)/3}$ for some ϵ, we will have an estimate on $s_n/e_n - 1$ which is exponentially small in n_0.

To obtain estimates on e_n, we will make heavy use of the following recursive formula which we get from $\frac{d}{dt}E(t) = \left(\sum_{i=1}^{n_0} \tau_i t^{i-1}\right)E(t)$,

$$ne_n = \sum_{i=1}^{n_0} \tau_i e_{n-i}.$$

The key to the next estimates on e_n will be to use this recursion formula to look at the ratio of terms e_{n+1}/e_n in the following way

$$\begin{aligned}
\frac{e_{n+1}}{e_n} &= \left(\frac{n}{n+1}\right)\frac{\sum_{i=1}^{n_0}\tau_i e_{n+1-i}}{\sum_{j=1}^{n_0}\tau_j e_{n-j}} \\
&= \left(\frac{n}{n+1}\right)\sum_{i=1}^{n_0}\left(\frac{\tau_i e_{n-i}}{\sum_{j=1}^{n_0}\tau_j e_{n-j}}\right)(e_{n+1-i}/e_{n-i}).
\end{aligned}$$

In this way the ratio e_{n+1}/e_n is $n/(n+1)$ times a weighted average of the previous n_0 ratios e_{n+1-i}/e_{n-i}. We proceed to find upper and lower bounds on ratios of consecutive e_n terms.

Lemma 4.6. *There exists a constant $k_3 > 0$ such that if n_0 is sufficiently large and $n_0 + \log^2 n_0 < n$ then*

$$\frac{e_{n+1}}{\lambda e_n} < \left(1 - \frac{k_3}{n_0}\right)$$

and

$$\frac{e_{n+1}}{\lambda e_n} < \left(\frac{3n_0}{n}\right)^{1/n_0}$$

Proof. By Lemmas 4.5 and 4.2, there exists a constant $C_1 > 0$ such that if $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$ then

$$\begin{aligned}\frac{e_{n+1}}{\lambda e_n} &< \frac{s_{n+1}}{\lambda s_n}\left(1 + C_1\lambda^{-n_0/3}\right) \\ &< \left(1 - \frac{k_1}{n_0}\right)\left(1 + C_1\lambda^{-n_0/3}\right) \\ &< \left(1 - \frac{k_3}{n_0}\right)\end{aligned}$$

for some $k_3 > 0$. Now by the recursion formula, our upper bound on $e_{n+1}/(\lambda e_n)$ for this string of n_0 consecutive ratios is an upper bound for all subsequent ratios.

For the second estimate, let r_n represent the ratio $e_{n+1}/(\lambda e_n)$ and w_i the weight so that $r_n = n/(n+1)\sum_{i=1}^{n_0} w_i r_{n-i}$.

Fix $n > 2n_0 + \log^2 n_0$. Let m_1 be the index of the largest of the previous n_0 ratios i.e., $r_{m_1} = \max_{1 \leq i \leq n_0} r_{n-i}$. Notice that

$$r_n \leq \left(\frac{n}{n+1}\right) r_{m_1}.$$

As long as $m_i > 2n_0 + \log^2 n_0$ define m_{i+1} to be the index of the largest ratio r_j where $m_i - n_0 \leq j \leq m_i - 1$. In other words, $r_{m_{i+1}} = \max_{1 \leq j \leq n_0} r_{m_i - j}$. Using the fact that $r_i < 1$ for all i, by induction we get

$$r_n \leq \left(\frac{n}{n+1}\right)\left(\frac{m_1}{m_1+1}\right)\left(\frac{m_2}{m_2+1}\right)\cdots\left(\frac{m_N}{m_N+1}\right)$$

for some N with $n_0 + \log^2 n_0 < m_N \leq 2n_0 + \log^2 n_0$.

To obtain an upper bound, let $q = [n/n_0]$. Take $q-1$ of the above factors $m_i/(m_i+1)$, one each between in_0 and $(i+1)n_0 - 1$ for $i = 2, 3, \ldots, q$. For $in_0 \leq n \leq (i+1)n_0 - 1$ we see

$$\frac{n}{(n+1)} \leq \frac{(i+1)n_0 - 1}{(i+1)n_0}.$$

This gives an upper bound of

$$\begin{aligned}\log(r_n) &\leq \log\left(\prod_{i=3}^{(q+1)}\left(1-\frac{1}{in_0}\right)\right)\\ &= \sum_{i=3}^{(q+1)}\log\left(1-\frac{1}{in_0}\right)\\ &< -\sum_{i=3}^{(q+1)}\frac{1}{in_0}\\ &< -(1/n_0)\log(n/(3n_0))\end{aligned}$$

Exponentiating, we obtain our upper bound for r_n

$$r_n < \left(\frac{3n_0}{n}\right)^{1/n_0}$$

□

We use a similar technique to establish a lower bound for the ratio $e_{n+1}/(\lambda e_n)$.

Lemma 4.7. *There exists a constant $b > 0$ such that if $r > 0$ is sufficiently small and n_0 is sufficiently large then $n_0 + \log^2 n_0 < n \leq e^{rn_0}$ implies*

$$\frac{e_{n+1}}{(\lambda e_n)} > \left(\frac{1}{n}\right)^{b/n_0}.$$

Proof. We obtain a lower bound for the initial coefficients as we did in Lemma 4.6. If n_0 is sufficiently large and $n_0 + \log n_0 < n \leq 2n_0 + \log n_0$ then

$$\frac{e_{n+1}}{\lambda e_n} > \frac{s_{n+1}}{\lambda s_n}\left(1 - C_1\lambda^{-n_0/3}\right) > \left(1 - \frac{k_4}{n_0}\right)$$

for some constant $k_4 > 0$. Suppose we have chosen n_0 large enough to ensure

$$(1 - k_4/n_0)^{n_0/3} > 1/n_0.$$

The left hand side approaches $e^{-k_4/3}$ so this is true for large n_0. Then for $n_0 + \log^2 n_0 < n \leq 2n_0 + \log^2 n_0$,

$$e_{n+1}/(\lambda e_n) > (1 - k_4/n_0) > (1/n_0)^{3/n_0}.$$

We will show by induction that this is a lower bound for $e_{n+1}/(\lambda e_n)$ for all $n > 2n_0 + \log^2 n_0$.

Fix $n > 2n_0 + \log^2 n_0$ and assume $e_{m+1}/(\lambda e_m) \geq (1/m)^{3/n_0}$ for all $(n - n_0) \leq m < n$. Then using the notation of the previous lemma,

$$r_n = \left(\frac{n}{n+1}\right) \sum_{i=1}^{n_0} w_i r_{n-i} \geq \left(\frac{n}{n+1}\right) \sum_{i=1}^{n_0} w_i \left(\frac{1}{n-i}\right)^{3/n_0}.$$

We would now like to replace all of the weights w_i in the above expression with $(1/n_0)$ making the weighted average into a flat average. If doing so would produce a smaller estimate then the following argument would finish the proof.

For sufficiently large n_0,

$$\begin{aligned}
\frac{1}{n_0} \sum_{i=1}^{n_0} \left(\frac{n}{n-i}\right)^{3/n_0} &= \frac{1}{n_0} \sum_{i=1}^{n_0} \left(1 + \frac{i}{n-i}\right)^{3/n_0} \\
&> \frac{1}{n_0} \sum_{i=1}^{n_0} \left(1 + \frac{3i}{n_0(n-i)}\right) \\
&> 1 + \frac{3}{n_0^2} \sum_{i=1}^{n_0} \frac{i}{n} \\
&= 1 + \frac{3n_0(n_0-1)}{2nn_0^2} \\
&> \frac{n+1}{n}.
\end{aligned}$$

Rearranging,

$$\left(\frac{n}{n+1}\right) \frac{1}{n_0} \sum_{i=1}^{n_0} \left(\frac{1}{n-i}\right)^{3/n_0} > \left(\frac{1}{n}\right)^{3/n_0}.$$

It remains then to examine the consequence of replacing w_i with $(1/n_0)$.

If the weights w_i and the terms $(n-i)^{-3/n_0}$ are both increasing in i for all i, then the flat average is less than the weighted average. The idea is that the weighted average puts more weight on the biggest terms than does the flat average, therefore the weighted average is more. (This is made precise in Lemma A.1.)

Recall that the weight w_i is equal to $\tau_i e_{n-i} / \sum_{j=1}^{n_0} \tau_j e_{n-j}$. The weights may not be increasing for all i, but there is a constant k such that for sufficiently large n_0, the weights w_i are increasing for $i \geq k \log n_0$. This follows since there exist constants $C_1, c > 0$ such that for $i \geq k \log n_0$

$$\frac{w_{i+1}}{w_i} = \frac{\tau_{i+1}}{\lambda \tau_i} \frac{\lambda e_{n-i}}{e_{n-i-1}} > \left(1 - C_1 e^{-ci}\right) \left(1 - \frac{k_3}{n_0}\right)^{-1} > 1 + \frac{k_3}{n_0} - \frac{C_1}{n_0^{ck}}$$

Let $n_1 = [k \log n_0]$ and $W = \sum_{i=n_1}^{n_0} w_i$. By Lemma A.1

$$\sum_{i=1}^{n_0} w_i r_{n-i} \geq W \sum_{i=n_1+1}^{n_0} \frac{w_i}{W} \left(\frac{1}{n-i}\right)^{3/n_0} \geq \frac{W}{n_0 - n_1} \sum_{i=n_1+1}^{n_0} \left(\frac{1}{n-i}\right)^{3/n_0}.$$

We now wish to examine the difference between the term above and $(1/n_0)\sum_{i=1}^{n_0}(n-i)^{-3/n_0}$ which we had previously.

First we show that W is approximately one.

$$|W - 1| = \sum_{i=1}^{n_1} w_i = \sum_{i=1}^{n_1} \tau_i e_{n-i} / \sum_{i=1}^{n_0} \tau_i e_{n-i}.$$

Approximating the numerator, we have $e_{n-i} \leq \lambda^{n_1-i} e_{n-n_1}$ for all $i \leq n_1$, and $\lambda^{-i}\tau_i$ is uniformly bounded. Therefore,

$$\lambda^{-n} \sum_{i=1}^{n_1} \tau_i e_{n-i} \leq C_2 \lambda^{n_1-n} e_{n-n_1} n_1$$

for some constant $C_2 > 0$. Approximating the denominator, we use the fact that the terms w_i are increasing for $i \geq n_1$. Then

$$\begin{aligned} \lambda^{-n} \sum_{i=1}^{n_0} \tau_i e_{n-i} &\geq \lambda^{-n} \sum_{i=n_1}^{n_0} \tau_i e_{n-i} \\ &\geq \lambda^{-n} \tau_{n_1} e_{n-n_1} (n_0 - n_1) \\ &\geq C_3 \lambda^{n_1-n} e_{n-n_1} n_0 \end{aligned}$$

for some constant $C_3 > 0$. Thus we get an estimate for W of the form

$$|W - 1| \leq C_4(n_1/n_0)$$

for some constant $C_4 > 0$. Using this, we get the following estimate.

$$\left[\frac{W}{n_0 - n_1} - \frac{1}{n_0}\right] \sum_{i=1}^{n_0} \left(\frac{1}{n-i}\right)^{3/n_0} \leq C_5(\log n_0/n_0)$$

for some constant $C_5 > 0$.

Finally, we get a constant $C_6 > 0$ such that

$$\begin{aligned} r_n &= \sum_{i=1}^{n_0} w_i r_{n-i} \\ &\geq \frac{W}{n_0 - \log n_0} \sum_{i=\log n_0}^{n_0} \left(\frac{1}{n-i}\right)^{3/n_0} \\ &\geq \frac{1}{n_0} \sum_{i=1}^{n_0} \left(\frac{1}{n-i}\right)^{3/n_0} - C_6(\log n_0/n_0) \\ &\geq n^{-3/n_0} - C_6(\log n_0/n_0). \end{aligned}$$

Fix $r > 0$ and suppose $\hat{b} > 2C_6 e^{3r}$. If n_0 is sufficiently large and $n_0 < n \leq e^{rn_0}$ then

$$1 - C_6 n^{3/n_0} \frac{\log n_0}{n_0} > 1 - \frac{\hat{b} \log n_0}{2n_0} > n_0^{-\hat{b}/n_0} > n^{-\hat{b}/n_0}.$$

The above follows since for large x, we have $x^{k/x} \approx 1 + k \log x/x$. Letting $b = \hat{b} + 3$, the inequality follows. $\square$

We are now ready to show that the difference between the ratio s_n/e_n and 1 is exponentially small in n_0 for $n_0 < n \leq e^{rn_0}$.

Lemma 4.8. *There exist constants $r > 0$ and $K, k > 0$ such that if n_0 is sufficiently large and $n_0 < n \leq e^{rn_0}$ then*

$$\left| \frac{s_n}{e_n} - 1 \right| < Ke^{-kn}.$$

Proof. Assume $b > 0$ and $0 < r < \log \lambda/4b$ such that the estimates in Lemma 4.7 hold. If n_0 is sufficiently large and $n_0 + \log^2 n_0 < n \leq e^{rn_0}$ then

$$\frac{\lambda e_n}{e_{n+1}} < n^{b/n_0} \leq e^{rb} < \lambda^{1/4}.$$

Therefore, we can ensure that

$$\frac{e_{n-i}}{e_n} < \lambda^{-3i/4}$$

for all $n_0 + \log^2 n_0 < (n-i) < n \leq e^{rn_0}$. Using the relationship

$$s_n = e_n \left(1 + \sum_{i=1}^{n_0} \frac{e_{n-i}}{e_n} [\exp(L'(t))]_i\right)$$

and the fact that $\exp(L'(t))_i < K\lambda^{i/3}$, we get the desired estimate on s_n/e_n. $\square$

The next two lemmas show that the second order differences of ratios $e_{n+1}/(\lambda e_n)$ are small compared to the first order differences.

Lemma 4.9. *Fix any $k \geq 1$. There exists constants $K_1, K_2 > 0$ such that if n_0 is sufficiently large and $n_0 + \log^2 n_0 < n \leq n_0^k$ then*

$$\frac{K_1}{n_0} < \left(\frac{\lambda e_n}{e_{n+1}} - 1\right) < \frac{K_2 \log(n_0)}{n_0}$$

Proof. ¿From Lemma 4.6 we have $k_3 > 0$ such that

$$\frac{\lambda e_n}{e_{n+1}} > \left(1 - \frac{k_3}{n_0}\right)^{-1} > 1 + \frac{K_1}{n_0}$$

for some constant $K_1 > 0$.

For the other inequality, for sufficiently large n_0 we can choose a constant $K_2 > 0$ such that

$$\frac{\lambda e_n}{e_{n+1}} < n^{b/n_0} \leq \exp\left[\frac{bk}{n_0} \log n_0\right] < 1 + K_2 \frac{\log n_0}{n_0}.$$

□

Lemma 4.10. *Fix any $k \geq 1$. There exists a constant $K > 0$ such that if n_0 is sufficiently large and $2n_0 + \log^2 n_0 < n \leq n_0^k$ then*

$$\left| \left(\frac{\lambda e_{n+1}}{e_{n+2}} - 1\right) - \left(\frac{\lambda^2 e_n}{e_{n+2}} - \frac{\lambda e_{n+1}}{e_{n+2}}\right) \right| < K \left(\frac{\log n_0}{n_0^2}\right).$$

Proof. Fix n and let R_i denote $\lambda^{n+2-i} e_i / e_{n+2}$. ¿From the recursion formula on e_n

$$R_n = \frac{1}{n} \sum_{i=1}^{n_0} \lambda^{-i} \tau_i R_{n-i}.$$

Using this we get

$$\begin{aligned}
&(R_{n+1} - 1) - (R_n - R_{n+1}) \\
&\qquad = \frac{1}{n} \sum_{i=1}^{n_0} \lambda^{-i} \tau_i \left[(R_{n+1-i} - R_{n+2-i}) - (R_{n-i} - R_{n+1-i})\right] \\
&\qquad\quad - \frac{2}{n(n+1)} \sum_{i=1}^{n_0} \lambda^{-i} \tau_i \left[R_{n+1-i} - R_{n+2-i}\right] \\
&\qquad\quad + \frac{2}{n(n+1)(n+2)} \sum_{i=1}^{n_0} \lambda^{-i} \tau_i R_{n+2-i}
\end{aligned}$$

Let $D_i = R_i - R_{i+1}$. By the previous lemma, each D_i is less than a constant times $\log n_0/n_0$. We also know that the terms $\lambda^{-i}\tau_i$ are exponentially close to one. These estimates give the following.

For some constant $C_1 > 0$ we get an upper bound on the first term

$$\begin{aligned}
&\frac{1}{n}\left|\sum_{i=1}^{n_0}\lambda^{-i}\tau_i(D_{n+1-i}-D_{n-i})\right| \\
&\quad = \frac{1}{n}\left|\lambda^{-1}\tau_1 D_n + \sum_{i=2}^{n_0}(\lambda^{-i}\tau_i - \lambda^{-i+1}\tau_{i-1})D_{n+1-i} - \lambda^{-n_0}\tau_{n_0}D_{n-n_0}\right| \\
&\quad < C_1\frac{\log n_0}{n_0^2}.
\end{aligned}$$

For some constant $C_2 > 0$ we get an upper bound on the second of the form

$$\frac{2}{n(n+1)}\left|\sum_{i=1}^{n_0}\lambda^{-i}\tau_i D_{n-i}\right| \leq C_2\frac{\log n_0}{n_0^2}.$$

Using the fact that the R_i's are bounded, for some $C_3 > 0$ we get an upper bound of the form

$$\frac{2}{n(n+1)(n+2)}\sum_{i=1}^{n_0}\lambda^{-i}\tau_i R_{n-i} \leq C_3\frac{1}{n_0^2}.$$

Altogether, this gives an upper bound of the form $K\log n_0/n_0^2$ on the second order differences. $\square$

Taking the previous two lemmas together with some estimates on the intimal coefficients of $S(t)$ gives us the non-positivity of coefficients of $(1-p(t))S(t)$ for $n_0 < n \leq n_0^k$.

Lemma 4.11 (Polynomial Range)**.** *Given any $k \geq 1$ there exist a constant $K > 0$ such that for all sufficiently large n_0 if $0 < n \leq n_0^k$ then the nth coefficient of $(1-p(t))\prod_{i=1}^{n_0}(1-t^i)^{-o(i)}$ is less than $-Ks_n/n_0$ where s_n is the nth coefficient of $\prod_{i=1}^{n_0}(1-t^i)^{-o(i)}$.*

Proof. Notice that

$$(1-p(t))\prod_{i=1}^{n_0}(1-t^i)^{-o(i)} = \prod_{i=n_0+1}^{\infty}(1-t^i)^{o(i)}.$$

Therefore, for $1 \leq n \leq n_0$, the nth coefficient of this series is zero.

Fix n such that $n_0 < n \leq 2n_0 + \log^2 n_0$. We have the following formula for the nth coefficient of this series.

$$n\text{th coefficient } = -o(n) + \sum_{\substack{i+j=n \\ i,j>n_0}} o(i)o(j) + \binom{o(n/2)}{2}.$$

(The last term occurs only if n is even.) We get estimates on these terms similar to those in Lemma 4.6. There exist $C_1, C_2, C_3 > 0$ such that

$$o(n) > C_1\lambda^{n_0}/n_0$$
$$\sum_{\substack{i+j=n \\ i,j>n_0}} o(i)o(j) < C_2\lambda^{n_0}\log^2 n_0/n_0^2$$
$$\binom{o(n/2)}{2} < C_3\lambda^{n_0}/n_0^2.$$

Therefore, for sufficiently large n_0 if $n_0 < n \leq 2n_0 + \log^2 n_0$ then the nth coefficient is negative.

Now fix n such that $2n_0 + \log^2 n_0 < n \leq n_0^k$. We have a formula for the nth coefficient of $(1 - p(t))\prod_{i=1}^{n_0}(1 - t^i)^{-o(i)} = (1 - p(t))S(t)$

$$n\text{th coefficient } = s_n\left(1 - \sum_{i=1}^{d} p_i(s_{n-i}/s_n)\right).$$

By Lemma 4.8 we have constants $C_4, c > 0$ such that

$$\sum_{i=1}^{d} p_i\left|(s_{n-i}/s_n) - (e_{n-i}/e_n)\right| < C_4 e^{-cn_0}.$$

Now, by Lemmas 4.9 and 4.10, we may apply the Differences Argument to show that there is a constant $C_5 > 0$ such that

$$1 - \sum_{i=1}^{d} p_i(e_{n-i}/e_n) < -C_5/n_0.$$

The error from replacing the ratios s_{n-i}/s_n with e_{n-i}/e_n in the formula for the nth coefficient is negligible compared to the above estimate. Therefore, we have an upper bound on the nth coefficient of $(1-p(t))\prod_{i=1}^{n_0}(1-t^i)^{o(i)}$ of the form $-Ks_n/n_0$ for some constant $K > 0$. $\square$

5. Application of Matrix Methods to the Recursion Formula

In this section, we show that there is a choice of $r > 0$ such that if n_0 is sufficiently large then the nth coefficient of the power series $S(t)(1-p(t))$ will be non-positive for $n_0^{14} < n \leq e^{rn_0}$ (Polynomial to Exponential Range Lemma). In order to do so, we make further use of the recursion formula for e_n,

$$e_n = (1/n)\sum_{i=1}^{n_0} \tau_i e_{n-i}$$

to get estimates on the e_n terms for $n > n_0^k$. We consider the matrix

$$A_n = \begin{pmatrix} \tau_1/n & \tau_2/n & \tau_3/n & \cdots & \tau_{n_0}/n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

If we multiply A_n by the column vector comprised of a string of coefficients $(e_{n-1}, e_{n-2}, \ldots, e_{n-n_0})^{\mathrm{T}}$ then we obtain the string of coefficients $(e_n, e_{n-1}, \ldots, e_{n-n_0+1})^{\mathrm{T}}$. We will get estimates on the coefficients e_n for $n > n_0^k$ by considering ratios of the entries of $A_{n+N}A_{n+N-1}\cdots A_n\mathbf{v}$ where $\mathbf{v}$ is an arbitrary vector with nonnegative entries.

We begin by getting estimates on the Perron eigenvalue of A_n. Fix $n > n_0$ and let $f(t)$ be the characteristic polynomial of A_n,

$$f(t) = t^{n_0} - \tau_1 t^{n_0-1}/n - \ldots - \tau_{n_0}/n.$$

Note that there exists a unique positive real root α of $f(t)$ whose modulus exceeds that of all other roots.

Lemma 5.1. *If n_0 is sufficiently large and $n > 2n_0$ then*

$$\left(\frac{1}{2n}\right)^{1/n_0} < \frac{\alpha}{\lambda} < \left(\frac{2n_0}{n}\right)^{1/n_0}.$$

Proof. Choose n_0 such that

$$2n_0 > \sum_{i=1}^{n_0} \lambda^{-i}\tau^i \quad \text{and} \quad \lambda^{-n_0}\tau_{n_0} > 1/2.$$

We first see that $0 < \alpha < \lambda$. This follows since α is the only positive real root of $f(t)$, and we have $f(0) = -\tau_{n_0}/n < 0$ and

$$f(\lambda) = \lambda^{n_0} - (1/n)\sum_{i=1}^{n_0} \tau_i \lambda^{n_0-i} > \lambda^{n_0}(1 - 2n_0/n) > 0.$$

For the upper bound on α,

$$\alpha^{n_0} = (1/n)\sum_{i=1}^{n_0}\tau_i\alpha^{n_0-i} < (1/n)\sum_{i=1}^{n_0}\tau_i\lambda^{n_0-i} < \lambda^{n_0}(2n_0/n).$$

For the lower bound on α,

$$\alpha^{n_0} = (1/n)\sum_{i=1}^{n_0}\tau_i\alpha^{n_0-i} > (1/n)\tau_{n_0} > \lambda^{n_0}(1/2n).$$

□

Next, we would like an upper bound on the modulus of the other roots of the polynomial $f(t)$. We begin with some general lemmas about polynomials whose coefficients satisfy various conditions.

Lemma 5.2. *Let $q(t) = \sum_{i=0}^{d} q_i t^i$ be a polynomial with all coefficients nonnegative and suppose $\alpha > 0$. If $z \in \mathbb{C}$ with $|z| < \alpha$ then*

$$\left|\frac{q(z) - q(\alpha)}{z - \alpha}\right| \le q'(\alpha).$$

Moreover, the inequality is strict as long as $q_i > 0$ for some $i \ge 2$.

Proof. Let $r(t)$ be the polynomial $(q(t) - q(\alpha))/(t - \alpha)$. Then $r(t)$ has all nonnegative real coefficients since

$$r(t) = \frac{\sum_{i=1}^{d} q_i(t^i - \alpha^i)}{(t - \alpha)} = \sum_{i=i}^{d} q_i(t^{i-1} + \alpha t^{i-2} + \cdots + \alpha^{i-2}t + \alpha^{i-1}).$$

Therefore,

$$\left|\frac{q(z) - q(\alpha)}{z - \alpha}\right| = |r(z)| \le r(|z|) = \frac{q(|z|) - q(\alpha)}{|z| - \alpha}.$$

By the Mean Value Theorem in $\mathbb{R}$,

$$\frac{q(|z|) - q(\alpha)}{|z| - \alpha} = q'(\beta)$$

for some $|z| < \beta < \alpha$. Since $q'(t)$ has all nonnegative coefficients, $q'(\beta) \le q'(\alpha)$, proving the inequality. If $q'(t)$ is not constant then $q'(\beta) < q'(\alpha)$ giving a strict inequality. □

Lemma 5.3. *Suppose $f(t) = t^{n_0} - \sum_{i=1}^{n_0} f_i t^{n_0-i}$ where $n_0 > 3$ and $f_i \ge 0$ for all $1 \le i \le n_0$. Let α be the real root of $f(t)$ whose modulus exceeds the modulus of all other roots. If $\beta \ne \alpha$ is a root of $f(t)$ then $|\alpha - \beta| > \alpha/n_0$.*

Proof. Let $q(t) = \sum_{i=1}^{n_0} q_i t^{n_0-i}$ be the polynomial $q(t) = f(t)/(t-\alpha)$. Equating coefficients, we see that $q_{n_0}\alpha = f_{n_0}$ and $\alpha q_{i-1} = f_{i-1} + q_i$ for $2 \leq i \leq n_0$. In particular, this shows that $q_i \geq 0$ for all i. Therefore by Lemma 5.2,

$$\left|\frac{q(\beta) - q(\alpha)}{\beta - \alpha}\right| < q'(\alpha).$$

Since $q(\beta) = 0$, after rearranging this becomes

$$|\alpha - \beta| > \frac{q(\alpha)}{q'(\alpha)}.$$

But $q(\alpha) = f'(\alpha)$ and $q'(\alpha) = f''(\alpha)/2$, thus $|\alpha - \beta| > 2f'(\alpha)/f''(\alpha)$.

Now estimating $2n_0 f'(\alpha)$,

$$\begin{aligned} 2n_0 f'(\alpha) &= 2n_0^2 \alpha^{n_0-1} - 2n_0 \sum_{i=1}^{n_0-1} (n_0 - i) f_i \alpha^{n_0-i-1} \\ &= 2n_0^2 \sum_{i=1}^{n_0} f_i \alpha^{n_0-i-1} - 2n_0 \sum_{i=1}^{n_0-1} (n_0 - i) f_i \alpha^{n_0-i-1} \\ &= 2n_0 \sum_{i=1}^{n_0} i f_i \alpha^{n_0-i-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \alpha f''(\alpha) &= n_0(n_0-1)\alpha^{n_0-1} - \sum_{i=1}^{n_0-2} (n_0 - i)(n_0 - i - 1) f_i \alpha^{n_0-i-1} \\ &= \sum_{i=1}^{n_0-2} \left[2in_0 - i^2 - i\right] f_i \alpha^{n_0-i-1} \\ &\qquad + n_0(n_0-1) f_{n_0-1} + \alpha^{-1} n_0(n_0-1) f_{n_0}. \end{aligned}$$

By comparing the coefficients of α^{n_0-i-1} in each expression we see $2n_0 f'(\alpha) > \alpha f''(\alpha)$. Therefore,

$$|\beta - \alpha| > 2f'(\alpha)/f''(\alpha) > \alpha/n_0.$$

□

Lemma 5.4. *Let $n_0 > 0$ be an even integer and let $f_1, f_2, \ldots, f_{n_0}$ be positive real numbers. Assume there exist constants $k_1, k_2 > 0$ such that for all i,*

$$k_1 < f_{i+1}/f_i < k_2.$$

Let $\alpha > 0$ be a real number and β be a complex number with $\alpha > |\beta|$ and $|\alpha - \beta| > \alpha/n_0$. Then there is a constant $K > 0$ depending only on α, k_1 and k_2 such that

$$\left|\frac{\sum_{i=1}^{n_0} f_i\beta^{n_0-i}}{\sum_{i=1}^{n_0} f_i\alpha^{n_0-i}}\right| < 1 - K/n_0^2.$$

Proof. We begin with the inequality

$$\begin{aligned}\left|\sum_{i=1}^{n_0} f_i\beta^{n_0-i}\right| &\leq \sum_{i=1}^{n_0/2} \left|f_{2i-1}\beta^{n_0-2i+1} + f_{2i}\beta^{n_0-2i}\right| \\ &< \sum_{i=1}^{n_0/2} f_{2i-1}\alpha^{n_0-2i}|\beta + f_{2i}/f_{2i-1}|.\end{aligned}$$

We have a similar expression for $\sum_{i=1}^{n_0} f_i\alpha^{n_0-i}$. Let $r_i = f_{2i}/f_{2i-1}$. We proceed by comparing $|\alpha + r_i|$ to $|\beta + r_i|$.

If θ is the angle which β makes with the positive real axis then

$$(\alpha/n_0)^2 < |\alpha - \beta|^2 = \alpha^2 + |\beta|^2 - 2\alpha|\beta|\cos\theta.$$

The above gives an upper bound for $2|\beta|\cos\theta$ which we plug into the following

$$\begin{aligned}|\beta + r_i|^2 &= |\beta|^2 + (r_i)^2 + 2|\beta|r_i\cos\theta \\ &< |\beta|^2 + (r_i)^2 + \alpha r_i + |\beta|^2 r_i/\alpha - \alpha r_i/n_0^2.\end{aligned}$$

Using the fact that $|\beta| < \alpha$, we obtain

$$|\beta + r_i|^2 < \alpha^2 + (r_i)^2 + 2\alpha r_i - \alpha r_i/n_0^2 = (\alpha + r_i)^2 - \alpha r_i/n_0^2.$$

Therefore, there exists a $K > 0$ depending only on α, k_1 and k_2 such that

$$|\beta + r_i| < |\alpha + r_i|(1 - K/n_0^2).$$

The desired inequality follows. □

Next we will apply the previous three lemmas to the polynomial $f(t) = t^{n_0} - (1/n)\sum_{i=1}^{n_0} \tau_i t^{n_0-i}$. This will give us bounds on the second largest eigenvalue of the matrix A_n.

Lemma 5.5. *There is a constant $r > 0$ such that if n_0 is sufficiently large and $n_0^4 < n \leq e^{rn_0}$ then $|\beta| < (1 - 1/n_0^4)\alpha$ where $\beta \neq \alpha$ is a root of $f(t) = t^{n_0} - (1/n)\sum_{i=1}^{n_0} \tau_i t^{n_0-i}$.*

Proof. Since α and β are both roots of $f(t)$

$$\left|\frac{\beta^{n_0}}{\alpha^{n_0}}\right| = \left|\frac{\sum_{i=1}^{n_0} \tau_i\beta^{n_0-i}}{\sum_{i=1}^{n_0} \tau_i\alpha^{n_0-i}}\right|. \tag{5.1}$$

Assume $|\beta| \geq (1 - 1/n_0^4)\alpha$. Then for some $C_1 > 0$ we have

$$|\beta/\alpha|^{n_0} > (1 - 1/n_0^4)^{n_0} > 1 - C_1/n_0^3.$$

We will show that the right hand side of equation 5.1 has an upper bound of the form $1 - C_2/n_0^2$ giving us a contradiction for sufficiently large n_0. We cannot directly apply the previous lemma since we do not have good estimates on τ_{i+1}/τ_i when i is small.

¿From our estimates on τ_i,

$$\sum_{i=1}^{n_0} \tau_i t^{n_0 - i} = \sum_{i=1}^{n_0} \lambda^i t^{n_0 - i} + \sum_{i=1}^{n_0} \lambda^i a_i t^{n_0 - i} \tag{5.2}$$

where $|a_i| < C_1 e^{-c_1 i}$ for some constants $C_1, c_1 > 0$. We wish to show that this second sum in equation 5.2 is negligible compared to the first for $t = \alpha, \beta$. First note that

$$\left| \sum_{i=1}^{n_0} \lambda^i a_i \alpha^{n_0 - i} \right| < C_1 \alpha^{n_0} \sum_{i=1}^{n_0} e^{-c_1 i} \left(\frac{\lambda}{\alpha} \right)^i$$

Using the upper bound on α from Lemma 5.1 and the fact that $n > n_0^4$ we have $\alpha^{n_0} < (2n_0/n)\lambda^{n_0} < 2\lambda^{n_0}/n_0^3$. If we select $0 < r < c_1/2$ then using the lower bound from Lemma 5.1 and the fact that $n < e^{rn_0}$ for sufficiently large n_0, we have $\lambda/\alpha < 2^{1/n_0} e^r < e^{2c_1/3}$. Therefore the summation is a convergent geometric series. Using the fact that $n > n_0^4$ there is a constant $C_2 > 0$ such that

$$\left| \sum_{i=1}^{n_0} \lambda^i a_i t^{n_0 - i} \right| < C_2 \alpha^{n_0} < C_2 \lambda^{n_0}/n_0^3$$

for $t = \alpha, \beta$.

For a lower estimate on the first term, notice that

$$\left| \sum_{i=1}^{n_0} \lambda^i \beta^{n_0 - i} \right| = \lambda^{n_0} \left| \frac{1 - (\beta/\lambda)^{n_0}}{1 - (\beta/\lambda)} \right|.$$

The numerator is at least $1 - (\alpha/\lambda)^{n_0}$ which is at least $1 - 2/n_0^3$. The denominator is no more than 2. Thus we get a lower bound on the term $|\sum_{i=1}^{n_0} \lambda^i t^{n_0 - i}|$ of the form $C_3 \lambda^{n_0}$ for some constant $C_3 > 0$.

Thus there is a constant $C_4 > 0$ such that

$$\left| \frac{\sum_{i=1}^{n_0} \tau_i \beta^{n_0 - i}}{\sum_{i=1}^{n_0} \tau_i \alpha^{n_0 - i}} \right| = \left| \frac{\sum_{i=1}^{n_0} \lambda^i \beta^{n_0 - i}}{\sum_{i=1}^{n_0} \lambda^i \alpha^{n_0 - i}} \right| (1 \pm C_4/n_0^3)$$

Now, for the remaining term use Lemma 5.4. We obtain a constant $C_5 > 0$ such that

$$\left| \frac{\sum_{i=1}^{n_0} \lambda^i \beta^{n_0-i}}{\sum_{i=1}^{n_0} \lambda^i \alpha^{n_0-i}} \right| < 1 - C_5/n_0^2.$$

Therefore, there exists a constant $C_6 > 0$ such that

$$1 - C_1/n_0^3 < \left| \frac{\beta^{n_0}}{\alpha^{n_0}} \right| = \left| \frac{\sum_{i=1}^{n_0} \tau_i \beta^{n_0-i}}{\sum_{i=1}^{n_0} \tau_i \alpha^{n_0-i}} \right| < 1 - C_6/n_0^2.$$

For sufficiently large n_0, this is a contradiction. □

Now we have estimates from Lemma 5.1 on the Perron eigenvalue of A_n and from Lemma 5.5 on the second largest modulus of a root. We would like to use these to show that for any vector $\mathbf{v}$ with nonnegative entries if $N \geq n_0^6$ then $(A_n)^N v$ is asymptotically close to the eigenvector for A_n. It will be convenient for us to use the matrix B_n where B_n is A_n scaled by the Perron eigenvalue α i.e.,

$$B_n = \begin{pmatrix} \tau_1/n & \tau_2/(n\alpha) & \tau_3/(n\alpha^2) & \cdots & \tau_{n_0}/(n\alpha^{n_0-1}) \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

It will also be convenient for us to use the ∞-norm in the following lemmas. I.e., for $\mathbf{u} = (u_1, u_2, \ldots, u_{n_0})$, $\|\mathbf{u}\| = \max |u_j|$.

Lemma 5.6. *There exist constants $r, K, k > 0$ such that if n_0 is sufficiently large and $n_0^4 < n \leq e^{rn_0}$ then $\mathbf{v} \in (\mathbb{R}_+)^{n_0}$ implies*

$$\left| \frac{\left[(B_n)^{n_0^6}\mathbf{v}\right]_{j+1}}{\left[(B_n)^{n_0^6}\mathbf{v}\right]_j} - 1 \right| < Ke^{-kn_0}.$$

Proof. Fix $n_0^4 < n \leq e^{rn_0}$, let $B = B_n$. Suppose $\mathbf{c}$ is the column eigenvector for B corresponding to the eigenvalue one, $\mathbf{c} = (1, 1, \ldots, 1)^{\mathrm{T}}$. By Lemma 5.5, all other eigenvalues have modulus less than $(1 - C_1/n_0^4)$. Let W denote the span of the complementary generalized eigenvectors. We wish to write the vector $\mathbf{v}$ as $a\mathbf{c} + \mathbf{w}$ where $\mathbf{w} \in W$. Then

$$B^N \mathbf{v} = a\mathbf{c} + B^N \mathbf{w}.$$

To produce the lower bound for a, we note that a row eigenvector $\mathbf{r}$ corresponding to the eigenvalue 1 is perpendicular to all of W. To see this suppose $(\mathrm{I}\beta - A)^m\mathbf{x} = 0$. Since 1 is an eigenvalue of multiplicity one, the following equation shows $\mathbf{rx} = 0$

$$\mathbf{r}(\mathrm{I}\beta - A)^m\mathbf{x} = \sum_{j=0}^{m}\binom{m}{j}\beta^j\mathbf{r}A^{m-j}\mathbf{x} = \sum_{j=0}^{m}\binom{m}{j}\beta^j\mathbf{rx} = (\beta-1)^m\mathbf{rx}.$$

Take $\mathbf{r} = (r_0, r_1, \ldots, r_{n_0-1})$ to be the row eigenvector for 1, where $r_j = 1 - \sum_{i=1}^{j}\tau_i\alpha^{-i}/n = \sum_{i=j+1}^{n_0}\tau_i\alpha^{-i}/n$.

Using the fact that $\mathbf{r}^{\mathrm{T}} \in W^{\perp}$ we can write

$$\mathbf{v} = \left(\tfrac{\mathbf{rv}}{\mathbf{r}\cdot\mathbf{r}}\right)\mathbf{r}^{\mathrm{T}} + \mathbf{w}_1 \quad \text{where } \mathbf{w}_1 \in W$$
$$\mathbf{c} = \left(\tfrac{\mathbf{rc}}{\mathbf{r}\cdot\mathbf{r}}\right)\mathbf{r}^{\mathrm{T}} + \mathbf{w}_2 \quad \text{where } \mathbf{w}_2 \in W.$$

Thus we may write

$$\mathbf{v} = \left(\tfrac{\mathbf{rv}}{\mathbf{rc}}\right)\mathbf{c} + \mathbf{w} \quad \text{where } \mathbf{w} \in W.$$

Without loss of generality, we may assume that $\|\mathbf{v}\| = 1$. In this case, there is a constant $C_1 > 0$ such that

$$\mathbf{rv} \geq \min r_j \geq (1/n)\tau_{n_0}\alpha^{-n_0} > C_1/n_0.$$

The above follows from the bounds on α/λ from Lemma 5.1 and the fact that τ_n grows like λ^n. For the denominator,

$$\mathbf{rc} = (1/n)\sum_{i=1}^{n_0} i\tau_i\alpha^{-i} < n_0/n\sum_{i=1}^{n_0} i\tau_i\alpha^{-i} = n_0.$$

Therefore we can write $\mathbf{v}$ as $a\mathbf{c} + \mathbf{w}$ where $\mathbf{w} \in W$ and $a > C_1/n_0^2$.

To estimate $\|\mathbf{w}\|$, we simply note that $\mathbf{rv} < \mathbf{rc}$ therefore

$$\|\mathbf{w}\| \leq 1 + a\|\mathbf{c}\| \leq 2.$$

Now we would like an upper bound for $\|B^{n_0^6}\mathbf{w}\|$. To obtain this estimate, we triangulate the matrix corresponding to $B|_W$ and keep track of the size of entries of the resulting matrix. Following the triangulization scheme in [**?**, §6.1], there are $(n_0 - 1)$ linearly independent vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_{n_0-1}\}$ in $\mathbb{C}^{n_0}$ such that every element of W can be written as a $\mathbb{C}$-linear combination of the $\mathbf{u}_i$'s and for all $k = 1, 2, \ldots, (n_0 - 1)$, $A\mathbf{u}_k \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Moreover, by Gram-Schmidt, we may assume that these vectors are orthonormal. Therefore, if C is the $n_0 \times (n_0 - 1)$ matrix with columns $\mathbf{u}_j$ then $C^{\mathrm{T}}BC$ is an $(n_0-1)\times(n_0-1)$ triangular matrix D. The diagonal elements of D are the eigenvalues corresponding to W, so each has modulus less than $(1 - 1/n_0^4)$. For all j, $\|B\mathbf{u}_j^{\mathrm{T}}\|_\infty < 1$ therefore $\|\mathbf{u}_k B\mathbf{u}_j^{\mathrm{T}}\|_\infty < n_0$ for all k and each off-diagonal entry of D has modulus less than n_0. This implies that the entries of D^N are each less than $(1 - 1/n_0^4)^N {n_0}^{n_0} \binom{N+n_0-1}{N}$ in modulus (see Lemma A.2).

Altogether, this gives a bound on $\|B^N \mathbf{w}\| = \|CD^N C^{\mathrm{T}} \mathbf{w}\|$ of the form

$$\|B^N \mathbf{w}\| \leq (1 - 1/n_0^4)^N {n_0}^{n_0} \binom{N + n_0 - 1}{n_0}$$

where $C_3 > 0$ is a constant. Letting $N = n_0^6$, there exist constants $C_4, C_5, c_1, c_2 > 0$ such that

$$(1 - 1/n_0^4)^{n_0^6} \leq C_4 e^{-c_1 n_0^2}$$

and

$$n_0^{n_0} \binom{n_0^6 + n_0 - 1}{n_0} < C_5 n_0^{n_0} {n_0}^6 (1 + n_0^5)^{n_0} < C_5 e^{c_2 n_0 \log n_0}$$

(see Lemma A.3). Thus we have constants $C_6, C_7, c_3 > 0$ and an upper bound on $\|B^{n_0^6} \mathbf{w}\|$ of the form

$$\|B^{n_0^6} \mathbf{w}\| < C_6 e^{-c_1 n_0^2 + c_2 n_0 \log n_0} < C_7 e^{-c_3 n_0}$$

for sufficiently large n_0. This tells us that the jth coordinate of $B^{n_0^6} \mathbf{v}$ is $a \pm C_7 e^{-c_1 n_0}$ which gives the desired result. $\square$

It remains to show that the difference between using $(A_n)^{n_0^6}$ and the product $A_{n+n_0^6-1} A_{n+n_0^6-2} \cdots A_n$ is negligible.

For the following, fix $n_0^{14} < n \leq e^{r n_0}$. Let α denote the Perron eigenvalue of the first matrix A_n. Let B_{n+k} denote the matrix A_{n+k} scaled by α.

Lemma 5.7. *There exists a constant $r > 0$ such that if n_0 is sufficiently large and $n_0^k < n \leq e^{r n_0}$ then for $\mathbf{v} \in (\mathbb{R}_+)^{n_0}$,*

$$\|(B_n)^{n_0^6} \mathbf{v} - B_{n_0+n_0^6-1} B_{n_0+n_0^6-2} \cdots B_n \mathbf{v}\|_\infty < n_0^{12-k} \|\mathbf{v}\|_\infty.$$

Proof. Note that with the ∞-norm, for any $1 \leq i \leq n_0^6$

$$\|B_n \mathbf{v} - B_{n+i} \mathbf{v}\| = \|(B_n - B_{n+i})\mathbf{v}\| \leq \frac{i}{n+i} \|\mathbf{v}\| \leq n_0^{6-k} \|\mathbf{v}\|$$

and

$$\|B_{n+i} \mathbf{v}\| \leq \|\mathbf{v}\|.$$

Therefore,

$$\|B_n^{n_0^6} \mathbf{v} - B_{n_0+n_0^6-1} B_{n_0+n_0^6-2} \cdots B_n \mathbf{v}\| \leq n_0^6 n_0^{6-k} \|\mathbf{v}\| \leq n_0^{12-k} \|\mathbf{v}\|.$$

$\square$

Combining the lemmas of this section, we may select an $r > 0$, then an n_0 such that there is an $\alpha < \lambda$ with $|e_{n+1}/e_n - \alpha|$ as small as we like compared to $\lambda - \alpha$. This gives the following lemma.

Lemma 5.8 (Polynomial to Exponential Range)**.** *There are constants $K, r > 0$ such that if n_0 is sufficiently large and $n_0^{14} < n \leq e^{rn_0}$ then the nth coefficient of $(1 - p(t)) \prod_{i=1}^{n_0} (1 - t^i)^{-o(i)}$ is less than $-Ks_n/n_0$.*

Proof. Lemma 5.1 provides bounds on the Perron eigenvalue α of the matrix A_n. Using this lemma, we may choose an $r > 0$ such that if $2n_0 < n \leq e^{rn_0}$ then $1 - p(1/\alpha) < -C_1|\lambda - \alpha|$ for some constant $C_1 > 0$. If necessary, decrease this value of r so that the estimates from Lemmas 4.8, 5.6 and 5.7 all hold for $n_0^{14} < n \leq e^{rn_0}$.

Fix $n_0^{14} < n \leq e^{rn_0}$. Let $m = n - n_0^6$, and let α denote the Perron eigenvalue of the matrix A_m. Let $\mathbf{u} = (u_0, u_1, \ldots, u_{n_0-1})$ denote the vector

$$\mathbf{u} = (B_m)^{n_0^6}(e_m/\alpha^m, e_{m-1}/\alpha^{m-1}, \ldots, e_{m-n_0+1}/\alpha^{m-n_0+1})^{\mathrm{T}}.$$

By Lemma 5.7, there is a constant $C_2 > 0$ such that

$$\left|(\alpha^i e_{n-i}/e_n) - (u_i/u_0)\right| < C_2/n_0^2$$

for all $0 \leq i < n_0$.

By Lemma 5.6, there are constants $C_3, c_1 > 0$ such that

$$|(u_i/u_0) - 1| < C_3 e^{-c_1 n_0}$$

for all $0 \leq i < n_0$. By Lemma 4.8 we have constants $C_4, c_2 > 0$ such that

$$|(s_n/e_n) - 1| < C_4 e^{-c_2 n_0}.$$

Altogether, this gives a constant $C_5 > 0$ such that

$$\left|(s_{n-i}/s_n) - \alpha^{-i}\right| < C_5/n_0^2.$$

By Lemma 5.1 there is a constant $C_6 > 0$ such that

$$|\lambda - \alpha| > C_6 \log n_0/n_0.$$

Therefore, by the Ratios Argument, there is a constant $K > 0$ such that if n_0 is sufficiently large, and $n_0^{14} < n \leq e^{rn_0}$ then

$$n\text{th coefficient} < -Ks_n/n_0.$$

□

6. Managing the Product

Now, in order to complete the proof, we multiply $(1-p(t))S(t)$ by a power series of the form $(1-q(t))^{-1}$ where $q_i > 0$ for all i. We wish to have the nth coefficient of $(1-q(t))^{-1}$ closely approximate β^n for some β.

Fix $r > 0$ to make the Polynomial to Exponential Range Lemma hold. Let $\beta = \lambda e^{-r/2}$ and let

$$g(t) = t^{n_0^3} - \sum_{i=n_0^2}^{n_0^3} \left[\frac{\beta^i}{n_0^3 - n_0^2}\right] t^{n_0^3 - i}$$

where $[x]$ denotes the greatest integer function. Let γ denote the unique positive real root of $g(t)$ whose modulus exceeds that of all other roots.

Lemma 6.1. *There is a constant $K > 0$ such that if n_0 is sufficiently large then $|\gamma - \beta| < K\beta^{-n_0^2}$.*

Proof. Let $n_1 = n_0^2$ and $n_2 = n_0^3$.

Notice that $\gamma < \beta$. If $\gamma < z < \beta$ then

$$\begin{aligned} g'(z) &= n_2 z^{n_2 - 1} - \sum_{i=n_1}^{n_2} (n_2 - i) \left[\frac{\beta^i}{n_2 - n_1}\right] z^{n_2 - i - 1} \\ &\geq \sum_{i=n_1}^{n_2} i \left[\frac{\beta^i}{n_2 - n_1}\right] z^{n_2 - i - 1}. \end{aligned}$$

Taking only the last term in the sum, if n_0 is sufficiently large then we have a lower bound on $g'(z)$ of the form $\beta^{n_2 - 1}$.

Replacing β^{n_2} with $\sum_{i=n_1}^{n_2} \beta^i (n_2 - n_1)^{-1} \beta^{n_2 - i}$ we obtain an upper bound on $g(\beta)$ from

$$g(\beta) = \beta^{n_2} - \sum_{i=n_1}^{n_2} \left[\frac{\beta^i}{n_2 - n_1}\right] \beta^{n_2 - i} < \sum_{i=n_1}^{n_2} \beta^{n_2 - i}.$$

Therefore, by the Mean Value Theorem there is a $z \in (\gamma, \beta)$ such that

$$|\gamma - \beta| = g(\beta)/g'(z) < \beta \sum_{i=n_1}^{n_2} \beta^{-i} < K\beta^{-n_1}.$$

$\square$

Let $1 - q(t)$ denote the polynomial $t^{n_2} g(1/t)$. The following lemmas give estimates on the coefficients of $R(t) = \sum_{i=0}^{\infty} r_i t^i = (1-q(t))^{-1}$.

Lemma 6.2. *There exist constants $K, k > 0$ such that if n_0 is sufficiently large and $n > n_0^{21}$ then*

$$\left|\frac{r_{n+1}}{\gamma r_n} - 1\right| < Ke^{-kn_0^3}$$

Proof. Again let $n_1 = n_0^2$ and $n_2 = n_0^3$. The recursion relation which gives r_n is simply

$$r_n = \sum_{i=n_1}^{n_2} q_i r_{n-i}.$$

Therefore we may directly apply the "matrix methods" from the previous section.

The string of coefficients $(r_{n+i}, r_{n+i-1}, \dots, r_{n+i-n_2+1})$ is obtained by multiplying the vector $(r_n, r_{n-1}, \dots, r_{n-n_2+1})$ by a recursion matrix raised to the ith power. We have bounds on the ratio of consecutive coefficients

$$\beta - \left(\frac{n_2 - n_1}{\beta^i}\right) < \frac{q_{i+1}}{q_i} < \beta\left(1 - \frac{n_2 - n_1}{\beta^i}\right)^{-1}.$$

Therefore using Lemmas 5.2, 5.3, and 5.4 if $\zeta \neq \gamma$ is a root of $g(z)$ then there is a constant $C_1 > 0$ such that

$$\left|\frac{\zeta^{n_2}}{\gamma^{n_2}}\right| = \left|\frac{\sum_{i=1}^{n_2} q_i \zeta^{n_2-i}}{\sum_{i=1}^{n_2} q_i \gamma^{n_2-i}}\right| < 1 - C_1/n_2^2.$$

Reasoning in a manner similar to Lemma 5.5, for sufficiently large n_0,

$$|\zeta| < (1 - 1/n_2^4)\gamma.$$

Following the argument in Lemma 5.6, there exist constants $K, k > 0$ such that if we multiply an arbitrary vector $\mathbf{v}$ with nonnegative entries by the recursion matrix raised to the n_2^6 power, the result will be a vector $\mathbf{u}$ whose entries satisfy $|u_{j+1}/u_j - \gamma| < Ke^{-kn_2}$. This gives the desired estimate on ratios of coefficients r_{n+1}/r_n. □

Lemma 6.3. *The coefficient of t^n in $R(t) = (1 - r(t))^{-1}$ is less than $n\beta^n$ for all n.*

Proof. We consider the nth coefficient of $(1 - q(t))^{-1} = \sum_{i=0}^{\infty} (q(t))^i$. The result follows from the fact that the coefficient of t^n from $(q(t))^i$ is always less than β^n. This holds since the number of ways to select i coefficients of $q(t)$ whose indices sum to n is less than or equal to $(n_0^3 - n_0^2)^i$. □

We now have the necessary estimates on the nth coefficient of the power series $R(t) = (1-q(t))^{-1}$ for all n and on the nth coefficient of the series $S(t) = \prod_{i=1}^{n_0}(1-t^i)^{-o(i)}$ as long as $n \leq e^{rn_0}$. The final estimates will be for the nth coefficient of $S(t)$ where $n > e^{rn_0}$.

Lemma 6.4. *There exists a $\delta > 0$ such that if n_0 is sufficiently large and $n > e^{3rn_0/4}$ then $s_n < (\beta - \delta)^n$.*

Proof. We begin by finding estimates on $\binom{o(i)+n/i-1}{n/i}$ which is the nth coefficient of the series $(1-t^i)^{-o(i)}$ as long as i divides n. We use the formula

$$\binom{o(i)+n/i-1}{n/i} < n\left(1+\frac{io(i)}{n}\right)^{n/i}\left(1+\frac{n}{io(i)}\right)^{o(i)}$$

from Lemma A.3. We will fix $n > e^{3rn_0/4}$ and find estimates on the logarithm of each term on the right hand side.

Suppose $i < \log n_0$. Then there exist constants K, k such that if n_0 is sufficiently large then $io(i) < K(\log n)^k$ since there is a constant $C_1 > 0$ such that

$$io(i) < C_1\lambda^{\log n_0} = C_1 n_0^{\log\lambda} < K(\log n)^{\log\lambda}.$$

Using the fact that $\log(1+x) < x$ for $x > 0$ we can certainly ensure

$$\frac{n}{i}\log\left(1+\frac{io(i)}{n}\right) < o(i) < K(\log n)^k.$$

For the other term,

$$o(i)\log\left(1+\frac{n}{io(i)}\right) < o(i)\log(n+1) < 2K(\log n)^{k+1}.$$

Altogether, this shows that we can select a $\delta > 0$ such that for sufficiently large n_0,

$$\log\binom{o(i)+n/i-1}{n/i} < n\log(\beta - 2\delta).$$

Now suppose $i \geq \log n_0$. Then for one term we have an upper bound of the form

$$o(i)\log\left(1+\frac{n}{io(i)}\right) < \frac{n}{i} \leq \frac{n}{\log n_0}.$$

For the other term we have two cases. If $io(i) < n$ then

$$\frac{n}{i}\log\left(1+\frac{io(i)}{n}\right) < \frac{n}{i}\log 2 \leq \frac{n\log 2}{\log n_0}.$$

If $io(i) \geq n$, since $i \geq \log n_0$ we may assume $io(i) < 2\lambda^i$. Therefore

$$\frac{n}{i} \log\left(1 + \frac{io(i)}{n}\right) < \frac{n}{i} \log\left(\frac{4\lambda^i}{n}\right) = n\left(\log\lambda + \frac{\log 4}{i} - \frac{\log n}{i}\right)$$

Recalling $\log n > 3rn_0/4$, this term has an upper bound of the form

$$\frac{n}{i} \log\left(1 + \frac{io(i)}{n}\right) < n\left(\log\lambda + \frac{\log 4}{\log n_0} - \frac{3r}{4}\right).$$

Since $\log\beta = \log\lambda - r/2$, in either case we can choose a $\delta > 0$ such that for sufficiently large n_0,

$$\log\binom{o(i) + n/i - 1}{n/i} < n\log\left(\beta - 2\delta\right).$$

Therefore, we may assume that the nth coefficient of $(1 - t^i)^{-o(i)}$ is at most $(\beta - 2\delta)^n$ for $n > e^{3rn_0/4}$. The nth coefficient in the product $\prod_{i=1}^{n_0}(1 - t^i)^{-o(i)}$ is made up of at most $\binom{n+n_0-1}{n}$ terms each of which is less than $(\beta - 2\delta)^n$. Using the upper bound $\binom{n+n_0-1}{n} < (2n)^{n_0}$, this tells us that if n_0 is sufficiently large and $n > e^{3rn_0/4}$ then we have an estimate of the nth coefficient of $(1 - t^i)^{-o(i)}$ of the form

$$n\text{th coefficient} < (2n)^{n_0}(\beta - 2\delta)^n < (\beta - \delta)^n$$

□

We are now ready to finish the main theorem.

Lemma 6.5 (Final Factorization). *There is a choice of β and n_0 such that if*

$$S(t) = \prod_{i=1}^{n_0}(1 - t^i)^{-o(i)} \text{ and } R(t) = \left(1 - \sum_{i=n_0^2}^{n_0^3}\left[\frac{\beta^i}{n_0^3 - n_0^2}\right] t^i\right)^{-1}$$

then the nth coefficient of the product $(1-p(t))S(t)R(t)$ is non-positive for all $n \geq 1$.

Proof. **Region I:** The nth term of the product $(1-p(t))S(t)R(t)$ looks like

$$\sum_{i=0}^{n} s_i r_{n-i} - p_1 \sum_{i=0}^{n-1} s_i r_{n-i-1} - \cdots - p_d \sum_{i=0}^{n-d} s_i r_{n-i-d}$$

We know from the results of section 4 that there exists a constant $C_1 > 0$ such that given any $\epsilon > 0$ and any $k \geq 1$, if n_0 is sufficiently large, $n_0 < n \leq n_0^k$ and $0 \leq i \leq n$ then $s_i/s_n < C_1(\lambda - \epsilon)^{i-n}$. Thus

$$\sum_{i=0}^{d} s_i r_{n-i}/s_n < C_1 n \left(\frac{\beta}{\lambda - \epsilon}\right)^n$$

Therefore, the nth term of $(1 - p(t))S(t)R(t)$ looks like

$$\sum_{i=d}^{n} s_i r_{n-i} - p_1 \sum_{i=d-1}^{n-1} s_i r_{n-i-1} - \cdots - p_d \sum_{i=0}^{n-d} s_i r_{n-i-d} + e(n)$$

where $e(n)$ is an error term whose absolute value has an upper bound of the form

$$|e(n)| < C_2 n \left(\frac{\beta}{\lambda - \epsilon}\right)^n s_n.$$

Notice that the error term is zero if $n < n_0^2$. Rearranging the terms of the nth coefficient and ignoring the error term, we obtain

$$\sum_{i=d}^{n} s_i r_{n-i}(1 - p_1 s_{i-1}/s_i - \cdots - p_d s_{i-d}/s_i).$$

By the Polynomial Range Lemma, each term in this sum is non-positive for $n_0 < n \leq n_0^k$. In fact, for $n \geq n_0^2$ we have an upper bound on the last term in the sum of the form

$$s_n(1 - p_1 s_{n-1}/s_i - \cdots p_d s_{n-d}/s_n) < -s_n C_3/n_0$$

for some constant $C_3 > 0$. This shows that for sufficiently large n_0, the error term will not affect the non-positivity of the nth coefficient in this range.

Region II: We consider the range $n_0^{k-1} < n \leq e^{rn_0}$ and use practically the same argument as in Region I. Let $n_1 = n_0^2$ and $n_2 = n_0^3$. Given any $\epsilon > 0$ if n_0 is sufficiently large and $0 \leq i \leq n_1$ then the ratio $s_i/s_{n_1} < C_1(\lambda - \epsilon)^{i-n_1}$.

¿From the estimates in this section, given any $\epsilon > 0$, if n_0 is sufficiently large and $n_0^{21} < i < n$ then

$$r_n/r_i < (\beta + \epsilon)^{n-i}.$$

By selecting $\epsilon > 0$ small enough, there is a constant $c_1 > 0$ such that $(\beta + \epsilon)/(\lambda - \epsilon) < e^{c_1}$. Therefore, for some constant $C_2 > 0$,

$$\frac{\sum_{i=0}^{d} s_i r_{n-i}}{s_{n_1} r_{n-n_1}} < C_2 e^{-c_1 n_1}.$$

Now again, we look at the nth term of $(1-p(t))S(t)R(t)$ as

$$\sum_{i=d}^{n} s_i r_{n-i}(1 - p_1 s_{i-1}/s_i - \cdots - p_d s_{i-d}/s_i) + e(n)$$

where $e(n)$ is an error term whose absolute value has an upper bound of the form

$$e(n) < s_{n_1} r_{n-n_1} C_2 e^{-c_1 n_1}.$$

By the Polynomial Range Lemma and the Polynomial to Exponential Range Lemma, each term in the sum is non-positive. Moreover, we have an upper bound on the term

$$s_{n_1} r_{n-n_1}(1 - p_1 s_{n_1-1}/s_{n_1} - \cdots - p_d s_{n_1-d}/s_{n_1}) < -s_{n_1} r_{n-n_1} K/n_0.$$

Thus for sufficiently large n_0, the error term will not affect the non-positivity of the nth coefficient.

Region III: We consider the range $n > e^{3rn_0/4} + n_0^{21}$. For $n > e^{3rn_0/4}$ we have $s_n < (\beta - \delta)^n$. Let $n_1 = n_0^2$ and $n_3 = n_2^7 = n_0^{21}$. By the results of this section for any $\epsilon > 0$ by choosing n_0 sufficiently large we have $r_n > (\beta - \epsilon)^{n-n_3} r_{n_3}$. By selecting ϵ small enough, there is a constant $c_2 > 0$ such that $(\beta - \delta)/(\beta - \epsilon) < e^{-c_2}$.

If $n > e^{3rn_0/4} + n_3$ then there exists a constant $C_3 > 0$ such that

$$\frac{s_n + \sum_{i=n_1}^{n_3+d} s_{n-i} r_i}{r_n} < C_3 n_3 e^{-c_2(n-n_3)}.$$

Now looking at the nth term of $(1-p(t))S(t)R(t)$ as

$$\sum_{i=0}^{n-n_3-d} s_{n-i} r_i (1 - p_1 r_{i-1}/r_i - \cdots p_d r_{i-d}/r_i) + e(n)$$

where $e(n)$ is an error term whose absolute value is at most

$$e(n) < r_n C_3 n_3 e^{-c_2(n-n_3)}.$$

Each term in the sum here is non-positive using Lemma 6.2 and the ratio argument with the r_n coefficients. Moreover, we have an upper bound on the first term of the summation

$$r_n(1 - p_1 r_{n-1}/r_n - \cdots p_d r_{n-d}/r_n) < -r_n K/n_0.$$

Therefore, by choosing n_0 sufficiently large, the error term will not affect the non-positivity of the coefficient.

Therefore, by selecting n_0 sufficiently large, we can ensure that the polynomial $(1-p(t))S(t)R(t)$ has all non-positive coefficients in degrees greater than zero. $\square$

The above shows that we may factor $1 - p(t)$ in the form appearing in the Main Reduction Lemma. This proves the remaining direction of the theorem.

Appendix A. Lemmas on Binomial Coefficients and Matrix Theory

These results are probably known to experts in the appropriate areas but we do not have convenient references.

Lemma A.1. *Suppose $w_1 \leq w_2 \leq \cdots \leq w_n$ with $\sum_{i=1}^n w_i = 1$. If $r_1 \leq r_2 \leq \cdots \leq r_n$ then*

$$\sum_{i=1}^n w_i r_i \geq (1/n) \sum_{i=1}^n r_i.$$

Proof. Since the weights w_i are increasing there is an index j such that $w_i < 1/n$ for all $1 \leq i < j$ and $w_i \geq 1/n$ for all $j \leq i \leq n$.

Notice that

$$\sum_{i=j}^n (w_i - 1/n) = \sum_{i=1}^{j-1} (1/n - w_i).$$

Therefore

$$\sum_{i=j}^n (w_i - 1/n) r_i \geq \sum_{i=1}^{j-1} (1/n - w_i) r_i.$$

Rearranging we obtain the desired inequality. □

Lemma A.2. *If A is an $n_0 \times n_0$ triangular matrix over $\mathbb{R}_+$ with diagonal entries less than β and off-diagonal entries less than βK then the entries of A^N are all less than $K^{n_0} \binom{N+n_0+1}{N} \beta^N$.*

Proof. The entry $A^N(i,j)$ is the sum of all possible products of the form $A(i,k_1)A(k_1,k_2) \cdots A(k_{N-1},j)$. Since A is triangular, only products where $i \leq k_1 \leq k_2 \leq \cdots \leq k_{N-1} \leq j$ are nonzero. There are at most $\binom{N+n_0-1}{N}$ nonzero products in the sum. We have the bounds $A(k_n, k_{n+1}) < K\beta$ for $k_n < k_{n+1}$ and $A(k_n, k_{n+1}) < \beta$ for $k_n = k_{n+1}$. There are at most n_0 terms $A(k_n, k_{n+1})$ in each product with $k_n < k_{n+1}$. Therefore each term $A^N(i,j)$ is less than $K^{n_0} \binom{N+n_0-1}{N} \beta^N$. □

Lemma A.3. *For all $n, k \geq 1$,*

$$\binom{n+k-1}{k} < n(1 + k/n)^n (1 + n/k)^k.$$

Proof. We have the following estimates on $\log(n!) = \sum_{i=1}^{n} \log i$

$$\int_1^n \log x dx < \log(n!) < \int_1^{n+1} \log x dx.$$

Plugging these in we obtain

$$\log \binom{n+k-1}{k} < (n+k)\log(n+k) - k\log k - (n-1)\log(n-1) - 2.$$

Using the inequality $(n-1)\log(n-1)+2 > (n-1)\log n$ and combining terms,

$$\log \binom{n+k-1}{k} < n\log(1+k/n) + k\log(1+n/k) + \log n.$$

This gives the inequality. □

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