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Polynomial Sequences of Binomial Type and Path Integrals*

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Abstract

Polynomial sequences $p_n(x)$ of binomial type are a principal tool in the umbral calculus of enumerative combinatorics. We express $p_n(x)$ as a *path integral* in the "phase space" $\mathbb{N} \times [-\pi, \pi]$. The Hamiltonian is $h(\phi) = \sum_{n=0}^{\infty} p_n'(0)/n!\, e^{in\phi}$ and it produces a Schrödinger type equation for $p_n(x)$. This establishes a bridge between enumerative combinatorics and quantum field theory.

Contents

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> Under the inspiring guidance of Feynman, a short-hand way of expressing—and of thinking about—these quantities have been developed.
>
> Lewis H. Ryder [19, Chap. 5].

1 Introduction on Polynomial Sequences of Binomial Type

The umbral calculus [11, 13, 14, 16, 17, 18] in enumerative combinatorics uses polynomial sequences of binomial type as a principal ingredient. A polynomial sequence $p_n(x)$ is said to be of *polynomial type* [13, p. 102] if $p_0(x) = 1$ and if it satisfies the *binomial identity*,

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \tag{1}$$

for all n, x, and y.

Example 1 Among many examples [13, § 5] of polynomial sequences of binomial type we mention only very few:

1. The power monomials $p_n(x) = x^n$.
2. The rising factorial sequence $p_n(x) = x(x+1)\cdots(x+n-1)$.
3. The falling factorial sequence $p_n(x) = x(x-1)\cdots(x-n+1)$.
4. The Abel polynomials $A_n(x) = x(x-an)^{n-1}$.
5. The Laguerre polynomials

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (-x)^k. \tag{2}$$

Using *delta functionals* one could show that [13, p. 106]

$$p_n(0) = 0, \qquad n > 0. \tag{3}$$

For a polynomial sequence $p_n(x)$ we could consider its coefficients $a_{n,k}$:

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

then $p_n(x)$ is of binomial type if and only if [13, Prop. 4.3]

$$\binom{i+j}{i} a_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} a_{k,i} a_{n-k,j}. \tag{4}$$

It follows from (3) that $a_{n,0} = \delta_{n,0}$. Moreover one could observe from (4) that the knowledge of the sequence $a_{n,1}$, $n = 0, 1, \ldots$ allows to reconstruct all $a_{n,k}$ by a recursion. One may state the following

Problem 2 Give a direct expression for a polynomial sequence $p_n(x)$ of binomial type from the sequence of its coefficients $a_{n,1} = p'_n(0)$, $n = 0, 1, \ldots$.

We are going to present a solution to this problem which is probably not the best from the computation point of view but has another advantages. We will express $p_n(x)$ as *path integral* (19) in the "phase space" $\mathbb{N} \times [-\pi, \pi]$. The Fourier transform $h(\phi) = \sum_{n=0}^{\infty} \frac{a_{n,1}}{n!} e^{in\phi}$ of the sequence $a_{n,1}/n! = p'_n(0)/n!$ plays the rôle of a *Hamiltonian*. Thus we establish a connection between enumerative combinatorics and quantum field theory. This connection could be useful in both areas. It is interesting to note that quantum mechanical commutation relations $PQ - QP = I$ was applied to umbral calculus by J. Cigler [2]. Existence of a "yet-to-be-discovered stochastic process" connected with Laguerre polynomials was conjectured by S.M. Roman and G.-C. Rota [13]. Note also that the approach to the umbral calculus developed in [11, 13, 14, 16] could be a prototype for a general construction of wavelets in Banach spaces [9].

2 Preliminaries on Path Integrals

Let a physical system has the *configuration space* $\mathbb{Q}$. This means that we could label states of the system at any time $t_0 \in \mathbb{R}$ by points of $\mathbb{Q}$. For a quantum system the principal quantity is the *propagator* [5, § 2.2], [19, § 5.1] $K(q_2, t_2; q_1, t_1)$—a complex valued function defined on $\mathbb{Q} \times \mathbb{R} \times \mathbb{Q} \times \mathbb{R}$. It is a probability amplitude for a transition from a state q_1 at time t_1 to q_2 at time t_2. The *probability* of this transition is

$$P(q_2, t_2; q_1, t_1) = |K(q_2, t_2; q_1, t_1)|^2 .$$

Let us fix any t_i, $t_1 < t_i < t_2$. We assume that

$$K(q_2, t_2; q_1, t_1) = \int_{\mathbb{Q}} K(q_2, t_2; q_i, t_i) K(q_i, t_i; q_1, t_1)\, dq_i, \tag{5}$$

i.e. there exists a measure dq on $\mathbb{Q}$ with the following property. The system could be at time t_i at any point q_i. The transition amplitude $q_1 \to q_2$ is a result of all possible transitions amplitudes $q_1 \to q_i \to q_2$ integrated over dq_i. Another assumption on propagator is its skew symmetry

$$K(q_2, t_2; q_1, t_1) = \overline{K(q_1, t_1; q_2, t_2)}. \tag{6}$$

It is typical to consider $K(q_2, t_2; q_1, t_1)$ as a kernel for the associated integral operator $K : L_2(\mathbb{Q}, dq) \to L_2(\mathbb{Q}, dq)$ defined as

$$[K\phi](q, t) = \int_{\mathbb{Q}} K(q, t; q_i, t_i)\phi(q_i, t_i)\, dq_i, \tag{7}$$

with t_i is fixed. Operator K is defined on functions $\phi(q, t) \in L_2(\mathbb{Q}, dq)$ such that the right-hand side of (7) does not depend on t_i. Then (5) is read as $K^2 = K$ and (6) implies $K^* = K$. By other word K is orthoprojection[1] from $L_2(\mathbb{Q}, dq)$ to its subspace.

There are special functions $\phi(g_2, t_2)$ which belongs to image of operator K and thus they are fixed points for K:

$$\phi(q, t) = [K\phi](q, t) = \int_{\mathbb{Q}} K(q, t; q_i, t_i)\phi(q_i, t_i)\, dq_i. \tag{8}$$

These are *wave functions*. It follows from (5) that all functions $\kappa(q, t) = K(q, t; q_0, t_0)$ for arbitrary fixed $q_0 \in \mathbb{Q}$, $t_0 \in \mathbb{R}$ are wave functions. The value $|\phi(q, t)|^2$ gives the probability distribution over $\mathbb{Q}$ to observe the system at time t.

R. Feynman developing ideas of A. Einstein, Smoluhovski and P.A.M. Dirac proposed an expression for the propagator via the "integral over all possible paths". In a different form it could be written as follows [19, (5.13)]

$$K(q_2, t_2; q_1, t_1) = \int \frac{\mathcal{D}q\,\mathcal{D}p}{h} \exp \frac{i}{\hbar} \left(\int_{t_1}^{t_2} dt\, (p\dot{q} - H(p, q)) \right). \tag{9}$$

Here $H(p, q)$ is the classical Hamiltonian [1, Chap. 9] of the system; the inner integral is taken over a path in *phase space*; and the first integral is taken over "all possible paths with respect to a measure $\mathcal{D}q\,\mathcal{D}p$ on paths in the phase space". Points in phase space have a double set of coordinates (q, p), where q is points of the configuration space $\mathbb{Q}$ and p is a coordinate in the space of momentums.

[1] For a model of relativistic quantization generated by two orthoprojections see [8].

From reading of physics textbooks [5, 19] one could obtain an impression that the measure $\mathcal{D}q\,\mathcal{D}p$ could be well defined from an unspecified complicated mathematical construction[2]. But this is *not* true up to now. There are different approaches of regularization of path integrals, see [4, 20] for recent accounts.

It is known that path integrals give fundamental solutions of evolutionary equations [3]. Particularly wave functions (8) are solutions to the Schrödinger equation [19, § 5.2], [5, § 4.1]

$$i\hbar\frac{\partial\phi(q,t)}{\partial t}=\widehat{H}(q,p)\phi(q,t), \tag{10}$$

where $\widehat{H}(q,p)$ is an operator which is the quantum Hamiltonian. It was mentioned that wave functions $\kappa(q,t)=K(q,t;q_0,t_0)$ are wave functions and thus are solutions to (10). In fact it follows from (8) that they are *fundamental solutions*. Thus the propagator $K(q_2,t_2;q_i,t_i)$ contains the complete information on dynamics of a quantum system.

3 Polynomial Sequences from Path Integrals

Let the configuration space $\mathbb{Q}$ be a semigroup with an operation denoted by $+$ and let a propagator $K(q_2,t_2;q_1,t_1)$ be homogeneous, i.e.

$$K(q_2+q,t_2+t;q_1+q,t_1+t)=K(q_2,t_2;q_1,t_1),$$

for all $q,q_1,q_2\in\mathbb{Q}$, $t,t_1,t_2\in\mathbb{R}$. Such a propagator corresponds to a free time-shift invariant system. Then $K(q_2,t_2;q_1,t_1)$ is completely defined by its values $S(q_2-q_1,t_2-t_1)=K(q_2,t_2;q_1,t_1)$ and (5) takes the form [3]

$$S(q,t_1+t_2)=\int_{\mathbb{Q}} S(q_i,t_1)S(q-q_i,t_2)\,dq_i. \tag{11}$$

The equation (11) was taken in [7] as the definition of *tokens* between two semigroups (convoloids) $\mathbb{Q}$ and $\mathbb{R}^+$. It is also known [7] that for a polynomial sequence $p_n(x)$ of binomial type the polynomial sequence $q_n(x)=p_n(x)/n!$ is

[2] The editor preface to the Russian translation of [5] tells: "There is no a rigorous definition of the path integral in the book of Feynman and Hibbs…However it is not very important for a physicist in most cases; he only need an assurance that a rigorous proof could be obtained". The recent textbook [19] which were already reprinted 9 times in two editions refereed for a rigorous path integral construction to the old paper [6] (it is known that this approach is not complete [3], [12, Notes X.11]).

token between convoloids $\mathbb{N}$ and $\mathbb{R}$, i.e. from (1) follows a realization of (11) in the form

$$q_n(x+y) = \sum_{k=0}^{\infty} q_k(y) q_{n-k}(x). \tag{12}$$

If polynomial sequence $q_n(x)$ represents a particular propagator for a configuration space $\mathbb{Q} = \mathbb{N}$ then one may wish to obtain a path integral representation in the form (9) for it. We will do it in a way parallel to the path integral deduction in [19, § 5.1].

Let $x \in \mathbb{R}$ be fixed and $N \in \mathbb{N}$ be an arbitrary positive integer. Then repetitive use of (12) allows to write

$$q_n(x) = \sum\sum\cdots\sum_{k_1+k_2+\cdots+k_N=n} q_{k_1}\left(\frac{x}{N}\right) q_{k_2}\left(\frac{x}{N}\right)\cdots q_{k_N}\left(\frac{x}{N}\right). \tag{13}$$

From (3) we could write

$$q_{k_2}\left(\frac{x}{N}\right) = \delta_{k_2 0} + q'_{k_2}(0)\frac{x}{N} + o\left(\frac{x}{N}\right).$$

We use the Pontryagin duality between $\mathbb{Z}$ and $[-\pi,\pi]$ to construct an integral resolution of the delta of Kronecker:

$$\delta_{k0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ipk}\,dp. \tag{14}$$

It is also convenient to introduce the Fourier transform for numerical sequence $q'_k(0)$:

$$h(p) = \sum_{k=0}^{\infty} q'_k(0) e^{ipk}. \tag{15}$$

Then we could write

$$\begin{aligned} q_k\left(\frac{x}{N}\right) &= \delta_{k0} + q'_k(0)\frac{x}{N} + o\left(\frac{x}{N}\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ipk}\,dp + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ipk} h(p)\,dp\frac{x}{N} + o\left(\frac{x}{N}\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ipk}\left(1 + h(p)\frac{x}{N}\right)dp + o\left(\frac{x}{N}\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(ipk + h(p)\frac{x}{N}\right)dp + o\left(\frac{x}{N}\right) \end{aligned} \tag{16}$$

Substituting (16) to (13) we obtain

$$q_n(x) = \sum\sum\cdots\sum_{k_1+k_2+\cdots+k_N=n} q_{k_1}\left(\frac{x}{N}\right) q_{k_2}\left(\frac{x}{N}\right)\cdots q_{k_N}\left(\frac{x}{N}\right)$$

$$
\begin{aligned}
&= \sum_{k_1+\cdots+k_N=n} \cdots \sum \left(\frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \exp\left(ip_1k_1 + h(p_1)\frac{x}{N} \right) dp_1 \right. \\
&\qquad \left. \times \cdots \times \int_{-\pi}^{\pi} \exp\left(ip_Nk_N + h(p_N)\frac{x}{N} \right) dp_N + o\Big(\frac{x}{N}\Big) \right) \\
&= \sum_{k_1+\cdots+k_N=n} \cdots \sum \left(\frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left(ip_1k_1 + h(p_1)\frac{x}{N} \right) \right. \\
&\qquad \left. \times \cdots \times \exp\left(ip_Nk_N + h(p_N)\frac{x}{N} \right) dp_1 \ldots dp_N + o\Big(\frac{x}{N}\Big) \right) \\
&= \sum_{k_1+\cdots+k_N=n} \cdots \sum \left(\frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{\sum_{l=1}^{n}\left(ip_lk_l + h(p_l)\frac{x}{N}\right)} dp_1 \ldots dp_N \right. \qquad (17) \\
&\qquad \left. + o\Big(\frac{x}{N}\Big) \right)
\end{aligned}
$$

At this point we say together with physicists magic words and (17) "could be symbolically written for $N \to \infty$" as

$$
q_n(x) = \int \mathcal{D}k\, \mathcal{D}p \exp \int_0^x dt(ipk' - h(p)), \tag{18}
$$

where the first integration is taken over all possible paths $k : [0, x] \to \mathbb{N}$ and $p : [0, x] \to [-\pi, \pi]$, such that $k(0) = 0$ and $k(x) = n$. It is enough to consider $k(t)$ with monotonic grow—other paths make the zero contributions ($q_l(x) \equiv 0$ for $l < 0$ by an agreement). In the exponent $k' = k'(t)$ means the derivative of the path $k(t)$ in the distributional sense.

In fact there is no magic in the above path integral. Due to the property $q_n(x) \equiv 0$ for $n < 0$ paths $k(t)$ take values on the finite set $k = 0, \ldots, n$, otherwise they make the zero contribution to the path integral. Under such a condition the situation is equivalent to a finite Markov process with a well-defined measure on the path space according to Kolmogorov theorem, see for example [3, § 1].

Now we could give a solution to Problem 2 using the relation $p_n(x) = n!q_n(x)$ between $p_n(x)$ and $q_n(x)$:

$$
p_n(x) = n! \int \mathcal{D}k\, \mathcal{D}p \exp \int_0^x dt(ipk' - h(p)), \qquad h(p) = \sum_{k=0}^{\infty} \frac{p'_k(0)}{k!} e^{ipk}. \tag{19}
$$

Example 3 There is a list of Hamiltonians producing the polynomial sequences from Example 1

1. The power monomials $p_n(x) = x^n$ are produced by $h(p) = e^{ip}$.

2. The rising factorial sequence $p_n(x) = x(x+1)\cdots(x+n-1)$ is produced by the Heaviside function $h(p) = \chi(p) = \sum_{k=0}^{\infty} \frac{e^{ipk}}{k}$.
3. The falling factorial sequence $p_n(x) = x(x-1)\cdots(x-n+1)$ is produced by the shifted Heaviside function $h(p) = \chi(p-\pi) = \sum_{k=0}^{\infty} \frac{(-1)^k e^{ipk}}{k} = \sum_{k=0}^{\infty} \frac{e^{i(p+\pi)k}}{k}$.
4. The Abel polynomials $A_n(x) = x(x-an)^{n-1}$ are produced by $h(p) = \sum_{k=0}^{\infty} \frac{(ak)^{k-1}}{k!} e^{ipk}$.
5. The Laguerre polynomials (2) are generated by the delta function $h(p) = \delta(p) = \sum_0^{\infty} e^{ipk}$.

Homogeneous propagators $S(q,t)$ (11) are wave functions (8) itself. Thus they should satisfy to the Schrödinger like equation (10). For polynomial sequences $q_n(x)$ this equation takes the form

$$\frac{\partial}{\partial x} q_n(x) = \widehat{H} q_n(x), \tag{20}$$

where the operator $\widehat{H}$ is of the pseudodifferential type [21, 22]

$$\begin{aligned} \widehat{H} a_n &= \mathcal{F}^{p\to n} h(p) \mathcal{F}^{n\to p} a_n = \int_{-\pi}^{\pi} e^{-inp} h(p) \sum_{k=0}^{\infty} a_n e^{inp}\, dp, \\ &= \sum_{k=0}^{\infty} h_k a_{n-k}, \end{aligned} \tag{21}$$

where $h(p)$ is the Hamiltonian from the path integral (18) and

$$h_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipn} h(p)\, dp.$$

It turns to be just a convolution on $\mathbb{N}$ because Hamiltonian $h(p)$ depends only from p and is independent from k. The equation (21) express the property [7] of tokens to intertwine shift invariant operators between two convoloids. In the present case the operators are the derivative with respect x (i.e. the convolution with $\delta'(x)$ on $\mathbb{R}$) and the convolution with h_n on $\mathbb{N}$.

Let $\widehat{M}$ be the operator on sequences $\widehat{M} : \{a_n\} \mapsto \{a_n/n!\}$. Then $q_n(x) = \widehat{M} p_n(x)$ and because $\widehat{M}$ and $\frac{\partial}{\partial x}$ commute we obtain for a polynomial sequence $p_n(x)$ of binomial type the equation

$$\frac{\partial}{\partial x} p_n(x) = \widehat{M}^{-1} \widehat{H} \widehat{M} p_n(x) \quad \Longleftrightarrow \quad \frac{\partial}{\partial x} p_n(x) = n! \sum_{k=0}^{\infty} h_k \frac{p_{n-k}(x)}{(n-k)!}. \tag{22}$$

Of course the above equation follows from the differentiation of identity (1) with respect to y at the point $y = 0$.

Example 4 The polynomial sequences from Examples 1 and 3 satisfy to the following realizations of equation (22):

1. The power monomials $p_n(x) = x^n$ satisfy to
$$p'_n(x) = n! \sum_{k=0}^{\infty} \delta_{k1} \frac{p_{n-k}(x)}{(n-k)!} = \frac{n!}{(n-1)!} p_{n-1}(x) = n p_{n-1}(x).$$
2. The rising factorial sequence $p_n(x) = x(x+1)\cdots(x+n-1)$ satisfies to
$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{1}{k} \frac{p_{n-k}(x)}{(n-k)!}.$$
3. The falling factorial sequence $p_n(x) = x(x-1)\cdots(x-n+1)$ satisfies to
$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \frac{p_{n-k}(x)}{(n-k)!}.$$
4. The Abel polynomials $A_n(x) = x(x-an)^{n-1}$ satisfy to
$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{(ak)^{k-1}}{k!} \frac{p_{n-k}(x)}{(n-k)!} = \sum_{k=0}^{\infty} \binom{n}{k} (ak)^{k-1} p_{n-k}(x).$$
5. The Laguerre polynomials (2) satisfy to
$$p'_n(x) = n! \sum_{k=0}^{\infty} \frac{p_{n-k}(x)}{(n-k)!}.$$

Remark 5 In our consideration we used primary the definition of token (11) and the Pontryagin duality in (14), (15). Thus formula (18) has the same form for many other tokens, which we shall not consider here however. It follows from general properties of path integrals that for any admissible function $h(p)$ formula (18) gives a token $q_n(x)$ and (19) produce a polynomial sequence $p_n(x)$ of binomial type.

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