Dynamics of kinks in the Ginzburg-Landau equation: Approach to a metastable shape and collapse of embedded pairs of kinks

J. Rougemont

Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

Abstract

We consider initial data for the real Ginzburg-Landau equation having two widely separated zeros. We require these initial conditions to be locally close to a stationary solution (the "kink" solution) except for a perturbation supported in a small interval between the two kinks. We show that such a perturbation vanishes on a time scale much shorter than the time scale for the motion of the kinks. The consequences of this bound, in the context of earlier studies of the dynamics of kinks in the Ginzburg-Landau equation, [ER], are as follows: we consider initial conditions v_0 whose restriction to a bounded interval I have several zeros, not too regularly spaced, and other zeros of v_0 are very far from I. We show that all these zeros eventually disappear by colliding with each other. This relaxation process is very slow: it takes a time of order exponential of the length of I.

1. Introduction

This paper is a continuation of [ER], where a model of interface dynamics was analyzed. This model is based on the Ginzburg-Landau equation in an unbounded one-dimensional domain. A similar model had originally been studied on a finite interval subject to Neumann boundary conditions by J. Carr and R.L. Pego, [CP1,CP2]. For a physical motivation and a discussion of related models, see Bray, [B], and references therein. The interfaces are defined as the zeros of a solution $v(x,t)$ of the real Ginzburg-Landau evolution equation. These zeros are shown to have the following behavior: let their positions on the real line be denoted by $z_k(t)$, with $z_j(t) < z_{j+1}(t)$, $j = 0, \ldots, N-1$. When the zeros are sufficiently far from each other, their dynamics is approximately described by:

$$\partial_t z_j(t) \approx E\left(e^{-\alpha_c(z_{j+1}(t)-z_j(t))} - e^{-\alpha_c(z_j(t)-z_{j-1}(t))}\right) , \tag{1.1}$$

with E, α_c some numerical constants. After some time, two zeros might come close to each other. Then they annihilate over a short time scale. The shape of the function $v(x,t)$ is shown to be essentially determined by the locations of the zeros, assuming the initial condition $v(x,0)$ to have "the right shape". In particular, the interface (hereafter called "kink") corresponding to the zero at $x = z_k$ is very close to the function $\tanh\big(\pm(x-z_k)/\sqrt{2}\big)$. For more general equations than the Ginzburg-Landau equation, similar results hold, but E, α_c, and the local shape of the kinks are different.

In the present paper, we discuss the following problem left open in [ER]: suppose $v(x,0)$ has four zeros, $z_1, \ldots, z_4$. Suppose that at some time $t = t_1 < \infty$, z_2 and z_3 annihilate, by the mechanism explained above. Then $v(x,t_1)$ looks as follows (see Fig. 1): it has two zeros $z_1(t_1)$ and $z_4(t_1)$, it is (say) positive in-between and has a "bump" in the middle, where z_2 and z_3 have just annihilated. Does the evolution bring this system back to the situation where $v(x,t)$ has the "right" shape for Eq.(1.1) to hold? Namely, does the "bump" vanish sufficiently fast, so that one sees again two slowly moving kinks, which might again be shown to annihilate after some time? We will show below that this is indeed the case. This is different from the case of a dynamics in a bounded spatial domain, since in [CP2], the authors were only able to show that if one starts with N kinks, then after the collapse of a pair of them, the number of kinks will never be more than $N-2$, but they were unable to iterate this result.

Acknowledgments. This work was supported by the Fonds National Suisse. It is a pleasure to thank J.-P. Eckmann for useful discussions and encouragements.

2. Definitions and main result

Our results can easily be extended to any equation of the form discussed in [ER]. Here, however, we restrict ourselves to the following real Ginzburg-Landau equation which is the most explicit example:

$$\begin{aligned} \partial_t v(x,t) &= \partial_x^2 v(x,t) + v(x,t) - v^3(x,t)\,, \\ x \in \mathbf{R}\,, \quad t &\in \mathbf{R}^+\,, \quad v(x,t) \in [-1,1]\,. \end{aligned} \tag{2.1}$$

This equation has a few simple time-independent solutions which will be used in this paper: the trivial ones $v(x,t) = \pm 1$, the "kinks" $v(x,t) = \tanh(\pm x/\sqrt{2})$, and a one-parameter family of periodic solutions $v(x,t) = \varphi_D(x)$, where $D \in (\pi,\infty)$ is half the period of φ_D (see [ER], Proposition 1.1). We fix the definition of φ_D by requiring that $\varphi_D(x) < 0$ for $0 < x < D$. Note that translates of a solution are also solutions of the equation. Since $v(x,t) = \pm 1$ are solutions of Eq.(2.1), the maximum principle ([CE], Theorem 25.1) implies that if $|v(x,0)| \le 1$, then $|v(x,t)| \le 1$ for all $t > 0$. Hence the evolution Eq.(2.1) is well-defined.

Throughout the paper, we will use the following notations: $\|\cdot\|_p$ is the usual norm of $\mathrm{L}^p(\mathbf{R}, dx)$, where dx is Lebesgue measure. The scalar product of $\mathrm{L}^2(\mathbf{R}, dx)$ is denoted by $(\cdot,\cdot)$. If $A \subset \mathbf{R}$ is a Borel set, χ_A denotes its (sharp) characteristic function and Θ_A is a smooth version of it, *i.e.*, $\Theta_A(x) = 1$ for $x \in A$, $\Theta_A(x) = 0$ if $\mathrm{dist}(x,A) > 1$, and $\sum_{j=0}^{k} \|\partial_x^j \Theta_A\|_\infty \le C$ for some constant C independent of A and for a sufficiently large integer k.

Let $Z = \{z_1, z_2\} \in \mathbf{R}^2$, we define $|Z| \equiv z_2 - z_1$ and $m_1(Z) \equiv (z_1 + z_2)/2$. We will always assume $|Z| > \pi$. With any such $Z \in \mathbf{R}^2$ we associate a bounded smooth function u_Z as in [CP1,ER]:

$$u_Z(x) = \Theta_{\mathrm{L}}(x) \tanh\left(\frac{x - z_1}{\sqrt{2}}\right) + \Theta_{\mathrm{C}}(x)\varphi_{|Z|}(x - z_2) + \Theta_{\mathrm{R}}(x) \tanh\left(\frac{z_2 - x}{\sqrt{2}}\right)\,, \tag{2.2}$$

where $\mathrm{L} = (-\infty, z_1 - 1/2]$, $\mathrm{C} = [z_1 + 1/2, z_2 - 1/2]$, and $\mathrm{R} = [z_2 + 1/2, \infty)$.

We next introduce a class of functions containing the initial conditions we are interested in, see Fig. 1:

Definition 2.1. *We say that $f : \mathbf{R} \to \mathbf{R}$ is an $(\varepsilon, \alpha, \ell, \Gamma)$–admissible function if $\|f\|_\infty \leq 1$ and there is a $Z = \{z_1, z_2\}$ in $\mathbf{R}^2$ such that f can be written as $f = u_Z + w_1 + w_2$ with:*

– *The two kinks are far apart:*

$$z_2 - z_1 \equiv |Z| > 2\Gamma > 2\ell > \pi .$$

– *The large part of the perturbation has support in a relatively small interval, far from the kinks:*

$$\mathrm{supp}(w_2) \subset \left[m_1(Z) - \ell, m_1(Z) + \ell\right] \equiv Y(\ell) .$$

– *The remainder of the perturbation is very small:*

$$\max\{\|w_1\|_2, \|w_1\|_\infty\} \leq e^{-\alpha|Z|} .$$

– *The function is positive between the two kinks:*

$$f(x) > \varepsilon \quad \text{for all } x \in Y(\ell) .$$

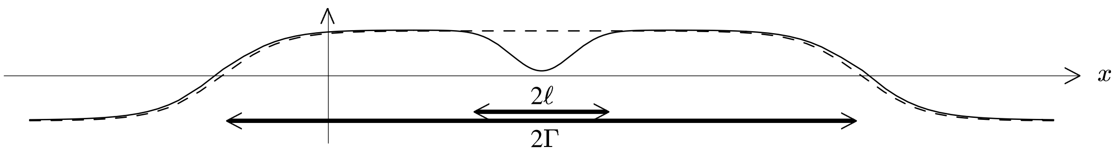

Fig. 1: An admissible function f (full line), with u_Z superimposed (dotted line).

The next lemma states that admissible functions have the following property: one can associate with them a function u_Z as given by Eq.(2.2) in such a way that the difference is "almost" in a stable subspace of the linearized evolution, see Lemma 5.1 below. Let

$$\tau_1(Z, x) = -\Theta_{(-\infty, m_1 - 1]}(x)\partial_x u_Z(x) , \quad \tau_2(Z, x) = -\Theta_{[m_1 + 1, \infty)}(x)\partial_x u_Z(x) .$$

Lemma 2.2. *For any positive $\alpha, \varepsilon, \ell$, for sufficiently large $\Gamma < \infty$, if f is an $(\varepsilon, \alpha, \ell, \Gamma)$–admissible function, then there is a unique $Z' \in \mathbf{R}^2$ with $\big(f - u_{Z'}, \tau_j(Z', \cdot)\big) = 0$, $j = 1, 2$. Moreover, Z' is a $\mathcal{C}^2$ function of f.*

Proof. Let $\mathcal{F}(u, Z) \in \mathbf{R}^2$, $\big(\mathcal{F}(u, Z)\big)_j = \big(u - u_Z, \tau_j(Z, \cdot)\big)$, $j = 1, 2$. Then $\mathcal{F}(u_Z, Z) = 0$ and $D_Z\mathcal{F}(u_Z, Z)$ is invertible, see [ER], Lemma 5.3. Let $\mathcal{B}(u_Z, \sigma)$ denote the ball of radius σ around u_Z in the topology $|||f|||_Z = \int |f||\partial_x u_Z|$. Then, by the Implicit Function Theorem, for sufficiently small σ, there is a $\mathcal{C}^2$ function $Z' : \mathcal{B}(u_Z, \sigma) \to \mathbf{R}^2$ such that $\mathcal{F}\big(u, Z'(u)\big) = 0$ for all $u \in \mathcal{B}(u_Z, \sigma)$. Note that there is a Γ such that any $(\varepsilon, \alpha, \ell, \Gamma)$–admissible function f is in this ball of radius σ. $\square$

Remark. We will use the following shorthands, to keep the notation simple: we will always write $Z(t)$ for $Z'\big(v(\cdot,t)\big)$. Similarly $m_1(Z)$, defined above as $m_1(Z) = (z_1 + z_2)/2$, is now a function of t also, denoted simply by $m_1(t)$. Throughout the paper, the same letter C will denote several numerical constants. We will often write the time variable as a subscript, *e.g.,* $v_t(\cdot) \equiv v(\cdot,t)$.

We next state the main technical result of the paper:

Theorem 2.3. *There are constants* $\alpha_c > 0$ *and* $K, M < \infty$ *such that for any positive* $\varepsilon < 1$, $\ell < \infty$, $\alpha \leq \alpha_c$, *for sufficiently large* $\Gamma = \Gamma(\varepsilon,\ell) < \infty$, *if* $v_0(x)$ *is an* $(\varepsilon,\alpha,\ell,\Gamma)$*–admissible function and* $v_t(x) = v(x,t)$ *is the corresponding solution of Eq.(2.1), then there is a* $T < K|Z(0)|$ *for which*
1) $|Z(T)| > |Z(0)|/2 > \Gamma$,
2) $\max\Big\{\|v_T - u_{Z(T)}\|_2, \|v_T - u_{Z(T)}\|_\infty\Big\} \leq Me^{-\alpha_c|Z(T)|}$.

Proof. See Section 5.

Remark 1. The constant α_c is the same as in Eq.(1.1) and is, for the equation considered in this paper, equal to $\sqrt{2}$. We use the constant $\alpha \leq \alpha_c$ in the proofs because we like to bound $C\exp(-\alpha_c|Z|)$ by $\exp(-\alpha|Z|)$ when some constant C appears.

Remark 2. The reader must view this result in the following context: we suppose that at some time $t_0 < 0$ in the past, v_{t_0} had four zeros $z_0, \ldots, z_3$. Under the evolution Eq.(2.1), these zeros have moved, until z_1 and z_2 (the central pair) annihilate. We suppose this happens at time $t = 0$, *i.e.,* u_t becomes strictly positive in the interval (z_0, z_3) when $t = 0$. Such a u_0 is the typical admissible function to which we want to apply Theorem 2.3. Theorem 2.3 imply that after a time T which is small compared to the time T' needed for a kink to move a large distance (typically a distance Γ needs a time $T' = \mathcal{O}(\Gamma\exp(\alpha_c|Z|)) \gg T = \mathcal{O}(|Z|)$), the distance (in the topologies of $\mathrm{L}^\infty(\mathbf{R}, dx)$ *and* of $\mathrm{L}^2(\mathbf{R}, dx)$) between the solution v_T of Eq.(2.1) and a two-kink state u_Z for some $Z \in \mathbf{R}^2$ will be smaller than any prefixed constant, provided $|Z(0)|$ is large enough. This shows that the local shape of u_t is restored by the evolution.

3. Dynamics of many kinks

In Section 2, we have restricted ourselves to the case of two kinks. In Section 4, we will extend Theorem 2.3 to the case of $N + 1$ kinks and work out some applications of this result. To do so we first generalize the definitions of Section 1 and recall some results proved in [ER].

Let N be an odd integer (the case of even N needs only minor modifications), let $\Omega_{N,\Gamma}$ be the set of all sequences of $N + 1$ kinks separated by a distance at least Γ:

$$\Omega_{N,\Gamma} = \Big\{ Z = \{z_0, \ldots, z_N\} \in \mathbf{R}^{N+1} \ : \ z_j - z_{j-1} > \Gamma \ , \ j = 1, \ldots, N \Big\} \ .$$

Let $\Gamma > \pi$, $Z \in \Omega_{N,\Gamma}$, $z_{-1} = -\infty$, and $z_{N+1} = +\infty$. We define the following numbers and intervals:

$$\begin{aligned}
\ell_j &= z_j - z_{j-1} \ , & j &= 0, \ldots, N+1 \ , \\
|Z| &= \min\{\ell_1, \ldots, \ell_N\} \ , & & \\
m_j &= \tfrac{1}{2}(z_j + z_{j-1}) \ , & j &= 0, \ldots, N+1 \ , \\
I_j &= \big(z_{j-1} + \tfrac{1}{2}, z_j - \tfrac{1}{2}\big) \ , & j &= 0, \ldots, N+1 \ .
\end{aligned}$$

We next construct the analogue of $u_Z(x)$, Eq.(2.2), for the case of $N + 1$ kinks:

$$\begin{aligned}
u_Z(x) \ = \ & \Theta_{I_0}(x) \tanh\left(\frac{x - z_0}{\sqrt{2}}\right) + \Theta_{I_{N+1}}(x) \tanh\left(\frac{z_N - x}{\sqrt{2}}\right) \\
& + \sum_{j=1}^{N} (-1)^j \Theta_{I_j}(x) \varphi_{\ell_j}(x - z_{j-1}) \ .
\end{aligned} \tag{3.1}$$

The following properties are readily verified: $u_Z \in \mathcal{C}^\infty(\mathbf{R})$, $\partial_x^2 u_Z(x) + u_Z(x) - u_Z^3(x) = 0$ for $|x - z_j| > 1/2$, $u_Z(z_j) = 0$ for $j = 0, \ldots, N$, and $(-1)^j \chi_{I_j}(x) u_Z(x) < 0$.

Below, we will extend the notion of admissible functions, Definition 2.1, to the case of $N + 1$ kinks. We first define a smaller set $\mathcal{T}_{N,\Gamma,\sigma}$ of nice functions, depending on the two parameters $\Gamma > \pi$ and $\sigma > 0$:

$$\begin{aligned}
\mathcal{T}_{N,\Gamma,\sigma} \ = \ & \Big\{ v \in L^\infty(\mathbf{R}) : \|v\|_\infty \leq 1, \\
& \max\big\{ \inf_{Z \in \Omega_{N,\Gamma}} \|\partial_x(v - u_Z)\|_2 \ , \ \inf_{Z \in \Omega_{N,\Gamma}} \|v - u_Z\|_2 \big\} < \sigma \Big\} \ .
\end{aligned} \tag{3.2}$$

We finally introduce a set of $N + 1$ functions, each of which "generates" the translation of one kink:

$$\tau_j(Z, x) \ = \ -\Theta_{M_j}(x) \partial_x u_Z(x) \ , \qquad j = 0, \ldots, N \ ,$$

where $M_j = [m_j + 1, m_{j+1} - 1]$.

In order to state the main results of [ER], we need to formulate a lemma, which summarizes several steps of the proofs presented in [ER]. We define $L_Z f \equiv \partial_x^2 f + \big(1 - 3u_Z^2\big) f$ (this is the r.h.s. of Eq.(2.1) linearized around $v = u_Z$).

Lemma 3.1. *For any integer $N < \infty$, for sufficiently large Γ and sufficiently small σ, there exists a unique $\mathcal{C}^2$ function $Z : \mathcal{T}_{N,\Gamma,\sigma} \to \Omega_{N,\Gamma}$ such that $\big(v - u_{Z(v)}, \tau_j(Z(v), \cdot)\big) = 0$ for $j = 0, \ldots, N$. Moreover, there is a constant $M > 1$ such that for any $v \in \mathcal{T}_{N,\Gamma,\sigma}$, one has:*

$$M \| L_{Z(v)} w \|_2^2 \; \geq \; -\big(w, L_{Z(v)} w\big) \; \geq \; \frac{1}{M} \|w\|_2^2 \, ,$$

where $w = v - u_{Z(v)}$.

The first part of Lemma 3.1 is the analogue of Lemma 2.2, with virtually the same proof. The second part is based on the spectral properties of the self-adjoint operator L_Z. It seems now legitimate to introduce the notation $-\big(f, L_Z f\big) \equiv \|f\|_Z^2$ for f in the orthogonal complement of $\mathrm{span}\{\tau_j(Z, \cdot), j = 0, \ldots, N\}$ in $\mathrm{L}^2(\mathbf{R}, dx)$.

It has been proved in [ER] that there exists a strictly positive function $g(Z)$, satisfying $g(Z) \to 0$ when $|Z| \to \infty$ such that the following holds:

Theorem 3.2. *Let*

$$\mathcal{Z}_{N,\Gamma} = \big\{ v \in \mathcal{T}_{N,\Gamma,\sigma} : \|w(v)\|_{Z(v)} < g\big(Z(v)\big) \big\} \, . \tag{3.3}$$

For any $N < \infty$, for sufficiently large Γ and sufficiently small σ, if $v_0 \in \mathcal{Z}_{N,\Gamma}$, then either the orbit v_t of v_0 under Eq.(2.1) lies in $\mathcal{Z}_{N,\Gamma}$ for all times $t > 0$, or there is a time $T < \infty$ and a $k \in \{1, \ldots, N\}$ such that $\ell_k(v_T) = \Gamma$.
Moreover there is an $\alpha_c > 0$ such that Eq.(1.1) holds for $Z(t) = Z(v_t)$ with $v_t \in \mathcal{Z}_{N,\Gamma}$, in the sense that the r.h.s. minus the l.h.s. is $\mathcal{O}(e^{-3\alpha_c |Z|/2})$, and there is an $s > 0$ such that the set

$$\mathcal{A}_{N,\Gamma} = \big\{ v \in \mathcal{T}_{N,\Gamma,\sigma} : \|w(v)\|_{Z(v)} < s \big\} \tag{3.4}$$

is exponentially attracted towards $\mathcal{Z}_{N,\Gamma}$.

It has also been shown that the above results can be extended to the case of infinitely many zeros, provided there are numbers k, N such that the intervals $[z_k, z_{k+1}]$ and $[z_{k+N}, z_{k+N+1}]$ are very large compared to $|Z^*|$, where $Z^* = \{z_{k+j}\}_{j=1,\ldots,N}$ (see also Section 4 below).

Consider an orbit v_t of Eq.(2.1) satisfying $v_t \in \mathcal{Z}_{N,\Gamma}$ for $t < T < \infty$ and $\ell_2(v_T) = \Gamma$, *i.e.,* the second case of the alternative of Theorem 3.2 holds with $k = 2$. Then, the following result was proved in [ER]:

Theorem 3.3. *For sufficiently large Γ, there are a $\Gamma_0 > \Gamma$ and a $T_0 > T$ such that if* $\min\{\ell_1(v_T), \ell_3(v_T)\} > \Gamma_0$, *then $|v_{T_0}(x)| > 0$ for $x \in [z_0 + \Gamma_0/2, z_2 - \Gamma_0/2]$.*

4. Applications of Theorem 2.3

The function v_{T_0} of Theorem 3.3 is *not* in $\mathcal{A}_{N-2,\Gamma}$. This was the main unsatisfactory point with the results of [ER]. In this section, we show that after a finite time and under some conditions on the position of the remaining kinks, it will get into $\mathcal{A}_{N-2,\Gamma}$.

First we state a condition which permits a generalization of Theorem 2.3 to the case of $N+1$ kinks using the set $\mathcal{Z}_{N,\Gamma}$ defined in Eq.(3.3).

Definition 4.1. *Let $f \in \mathrm{L}^\infty(\mathbf{R}, dx)$, $\|f\|_\infty \le 1$. We call f admissible if there is a $g \in \mathcal{Z}_{N,\Gamma}$, an $\ell < \infty$, and a $j \in \{1, \dots, N\}$ such that f can be written as $f = g + w$ where:*
1) w has support in $\big[m_j\big(Z(g)\big) - \ell, m_j\big(Z(g)\big) + \ell\big] \equiv Y_j(\ell)$,
2) $|f(x)| > \varepsilon$ for $x \in Y_j(\ell)$,
3) there is a $\beta > 1$ such that $\beta\ell < |Z|$.

Remark. Assumption 3) above is only stated for future reference. It is just a different formulation of the statement that for fixed ℓ, $|Z|$ must be larger than some $\Gamma = \Gamma(\ell)$, see Theorem 2.3.

Theorem 4.2. *Let $N < \infty$, let v_0 satisfy Definition 4.1. If $\Gamma > \pi$ and $\beta > 1$ are sufficiently large, then the conclusions of Theorem 2.3 hold for the corresponding solutions v_t of Eq.(2.1) (with $|Z|$ as defined in Section 3).*

Proof. The proof of Theorem 4.2 is easily worked out by combining Theorem 2.3, Lemma B.1, and a maximum principle as in Eq.(5.8) and Eq.(5.9) in the proof of Theorem 2.3. The details are left to the reader. □

Lemma 4.3. *Let v_T and α_c, be as in Theorem 2.3. There is a $C < \infty$ such that $\|v_T - u_{Z(T)}\|_{Z(T)} \le C \exp\big(-\alpha_c |Z(T)|\big)$, where $\|f\|_Z^2 = -\big(f, L_Z f\big) = -\big(f, f'' + (1 - 3u_Z^2) f\big)$.*

Proof. See Section 5.

Remark 1. Lemma 4.3 shows that the orbit of v_T enters the attracting neighborhood $\mathcal{A}_{N-2,\Gamma}$ of the invariant set $\mathcal{Z}_{N-2,\Gamma}$, after the collapse of an interval (see Eq.(3.3) and Eq.(3.4) for the definitions of these sets). In a way, it shows that the basin of attraction of the invariant cone $\mathcal{Z}_{N-2,\Gamma}$ is much larger than $\mathcal{A}_{N-2,\Gamma}$, and in fact contains points that have just come out of $\mathcal{Z}_{N,\Gamma}$ through the collapse of a pair of kinks. This is to be compared with the case of an evolution equation in a bounded spatial domain, see Proposition 4.3 in [CP2]. There it was shown that any orbit reaching the boundary of $\mathcal{Z}_{N,\Gamma}$ cannot ever re-enter it. This only shows that one will never see again a configuration with N kinks. But one still expects to see configurations with less kinks. With the result Lemma 4.3, we are able to show that there are initial configurations which "cascade" from $\mathcal{Z}_{N,\Gamma}$ to $\mathcal{Z}_{N-2,\Gamma}$ to $\mathcal{Z}_{N-4,\Gamma}$ and so on.

Remark 2. In [CP1,CP2], the authors use three small parameters in their proofs, and the game with these three parameters is quite involved. The first one, ρ in their notations, corresponds to $1/\Gamma$ with our definitions. This small parameter is the main ingredient of the whole proof. Their second small parameter is the diffusion constant ε (this *is not* the ε of Definition 2.1). Upon rescaling, this small parameter can be identified with the inverse of the size of the spatial domain in which the evolution is defined. It can be eliminated by working directly on the infinite line as was shown in [ER]. Finally the present paper shows that the constraint on the third parameter, called σ in [CP1], can be relaxed. This parameter measures the size of the allowed perturbations around the multi-kink state u_Z (this *is* the σ of Eq.(3.2)).

We next introduce a set of configurations of zeros which are quite general and for which we can control the dynamics of the kinks for arbitrarily long times. We begin with a construction involving only finitely many zeros, *i.e.,* we give ourselves a $Z \in \Omega_{N,\Gamma}$, with $N = 2M + 1$. We use the same notations as in Section 3: $Z = \{z_0, \ldots, z_N\}$, $\ell_j = z_j - z_{j-1}$, $j = 1, \ldots, N$, $|Z| = \min\{\ell_1, \ldots, \ell_N\}$. We will construct a discrete dynamics which approximate the behavior of the zeros of a solution of Eq.(2.1) by using only the following simple rule: at each time-step, erase the two nearest zeros and keep the other ones fixed (this *is* the model studied by Bray, Derrida, and Godrèche in [BDG]). Then we will state conditions on the initial configuration of zeros which imply that the continuous dynamics of Eq.(2.1) remain well-approximated by this discrete model for a long enough time.

We associate a labeled tree with the configuration Z. By a labeled tree we mean a set of vertices and edges. Each vertex is associated ("labeled") with a number or with ∞. The vertices are drawn on $M + 1$ levels numbered $0, \ldots, M$. On level k there are $N - 2k + 2$ vertices numbered $0, \ldots, N - 2k + 1$. Hence the $(j + 1)^{\text{th}}$ vertex from the left on the $(k + 1)^{\text{th}}$ level from the top is identified with $(k, j) \in \mathbf{Z}^2$. It is labeled with $v(k, j) \in \mathbf{R} \cup \{\infty\}$ which will be defined below. There are edges between some vertices of level k and some vertices of level $k + 1$ which will also be constructed below by iteration. We first define

$$\begin{aligned} v(k,0) \,=\, v(k, N-2k+1) \quad &= \infty \;, \quad k \,=\, 0, \ldots, M \;, \\ v(0,j) \quad &= \ell_j \;, \quad\; j \,=\, 1, \ldots, N \;. \end{aligned}$$

We next construct level $k + 1$ from level k, $0 \le k \le M - 1$. We define $j_{\text{min}}(k)$ by

$$v\left(k, j_{\text{min}}(k)\right) \,=\, \min\{v(k,j) : j = 1, \ldots, N - 2k\} \;.$$

We suppose here that there is a unique such $j_{\text{min}}(k)$. (In Definition 4.4 below we will restrict ourselves to configurations for which this is true.) The edges are drawn according to the following rule:

1) There are three edges going from the vertices $(k, j_{\text{min}}(k))$, $(k, j_{\text{min}}(k) + 1)$, $(k, j_{\text{min}}(k) - 1)$ to the single vertex $(k + 1, j_{\text{min}}(k) - 1)$.
2) There is an edge between (k, j) and $(k + 1, j)$ (if $j < j_{\text{min}}(k) - 1$) or between (k, j) and $(k + 1, j - 2)$ (if $j > j_{\text{min}}(k) + 1$).

It remains to define the numbers $v(k + 1, j)$:

$$v(k+1,j) \,=\, \sum_{m:(k,m)\to(k+1,j)} v(k,m) \;,$$

where $a \to b$ means "there is an edge between a and b." If one element of the above sum is ∞, then the sum is ∞. This construction is iterated from $k = 0$ to $k = M - 1$. We also define sequences $Z^{(k)} = \{z_0^{(k)}, \ldots, z_{N-2k}^{(k)}\}$ of real numbers: for $k = 0$ we simply let $Z^{(0)} = Z$. For $k > 0$ we first define $z_0^{(k)}$ by the following procedure: starting from the vertex $(k, 1)$ one goes up the tree following always the leftmost possible edge, until one reaches vertex $(0, j_0)$. More precisely

$$j_0 = \min\bigl\{j \in \{1, \ldots, N\} : (k, 1) \text{ is connected to } (0, j) \text{ by edges}\bigr\} .$$

We let $z_0^{(k)} = z_{j_0-1}^{(0)}$ and $z_{j+1}^{(k)} = z_j^{(k)} + v(k, j+1)$, $j = 0, \ldots, N-2k-1$.

Definition 4.4. *We say that $Z \in \Omega_{N,\Gamma}$ is (N, γ_1)–non-degenerate if the corresponding tree has labels $v(k, j)$ which satisfy: for each $k = 0, \ldots, M-1$, let*

$$d_1(k) = v(k, j_{\min}(k)) \quad d_2(k) = \min\bigl\{v(k, j) : j \in \{1, \ldots, N-2k\} \backslash j_{\min}(k)\bigr\} ,$$

then

$$\gamma_1 d_1(k) < d_2(k) , \tag{4.1}$$

with $\gamma_1 > 1$.

The discrete dynamics is now easy to formulate: $Z^{(k)}$ is the configuration of zeros after k steps of the discrete-time dynamics. The dynamics ends when there are only two zeros left, namely after M steps. This discrete dynamics is a good approximation for the continuous dynamics of Eq.(2.1) in the sense that if one starts with an initial condition v_0 with a set of zeros equal to $Z^{(0)}$, then there are times $t_1 < t_2 < \ldots < t_n$ such that $v(\cdot, t_k)$ has zeros approximately given by the set $Z^{(k)}$. In terms of the continuous dynamics Definition 4.4 means that two successive collapse times are never too close.

Example. We take $N = 7$, $Z = \{0, 7, 27, 32, 33, 41, 44, 56\}$ (in units in which $\Gamma = 1$). Fig. 2 shows the corresponding tree. We obtain the following sets of zeros:

$$Z^{(1)} = \{0, 7, 27, 41, 44, 56\} , \quad Z^{(2)} = \{0, 7, 27, 56\} , \quad Z^{(3)} = \{27, 56\} .$$

Fig. 3 shows the zeros $z_j^{(k)}$ on the interval $[-1, 57]$ and the functions u_Z given by Eq.(3.1) with $Z = Z^{(k)}$. The numbers $d_j(k)$ are in this case given by:

$$\begin{array}{lll} d_1(1) = 1 , & d_1(2) = 3 , & d_1(3) = 7 , \\ d_2(1) = 3 , & d_2(2) = 7 , & d_2(3) = 20 . \end{array}$$

For each $k = 0, \ldots, 3$, we have $d_2(k) > 2d_1(k)$. Hence Z is $(7, 2)$–non-degenerate in the sense of Definition 4.4.

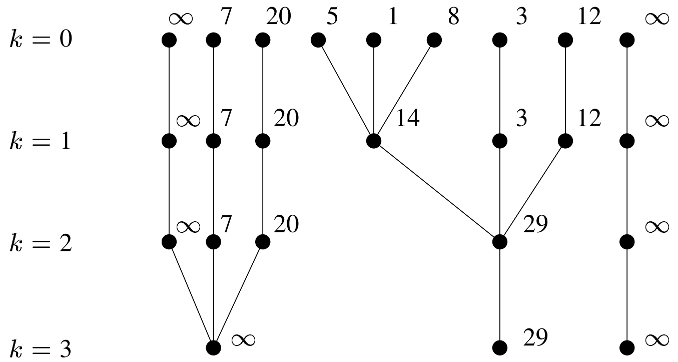

Fig. 2: The tree associated with the configuration $Z = \{0, 7, 27, 32, 33, 41, 44, 56\}$. The numbers are the labels of the vertices, *i.e.,* the distances between two successive zeros of the functions $u_{Z^{(k)}}$ shown in Fig. 3.

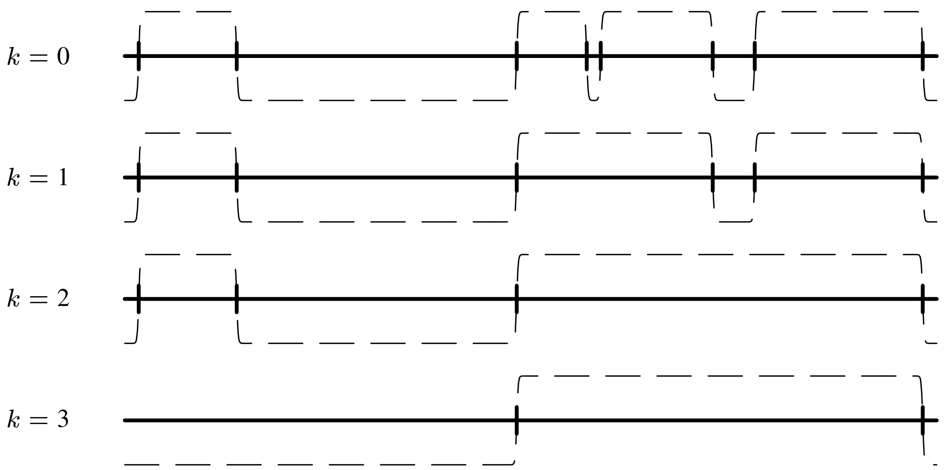

Fig. 3: The horizontal lines are 4 copies of the interval $[-1, 57]$. The vertical lines show the points $z_j^{(k)}$, $k = 0, \ldots, 3$, $j = 0, \ldots, 7 - 2k$, with j going from the left to the right and k from top to bottom. The superimposed dotted lines are the functions $u_{Z^{(k)}}$.

As will be shown in Theorem 4.6, for each $N < \infty$, for sufficiently large $\gamma_1 = \gamma_1(N)$, Definition 4.4 implies that the discrete model is a good approximation up to the time when all kinks have collapsed. Unfortunately, $\gamma_1(N) \to \infty$ when $N \to \infty$. Hence Definition 4.4 is not a sufficient condition to control infinitely many kinks. However, since all kinks will disappear in a finite time we can still make a condition on the remaining (infinitely many) kinks, so that nothing "invades" the small region we are looking at during this time. This is done by the following definition.

Definition 4.5. *We call $Z_\infty = \{z_j\}_{j\in\mathbf{Z}} \in \mathbf{R}^{\mathbf{Z}}$, a $(k, N, \gamma_1, \gamma_2)$–separable configuration of zeros if $\{z_k, \ldots, z_{k+N}\}$ is in $\Omega_{N,\Gamma}$, is (N, γ_1)–non-degenerate, and the following holds*

$$\log\left(\min\left\{\ell_k, \ell_{k+N+1}\right\}\right) \geq \gamma_2(z_{k+N} - z_k)\,, \tag{4.2}$$

with $\gamma_2 > 1$.

We next describe the "generic" behavior of zeros inside a finite region of the real line.

Theorem 4.6. *Let $N < \infty$. Let $\Gamma, \mathcal{Z}_{N,\Gamma}$ be as in Theorem 3.2. For sufficiently large $\gamma_2 = \gamma_2(N)$, $\gamma_1 = \gamma_1(N)$, for any $(k, N, \gamma_1, \gamma_2)$–separable configuration Z_∞, the following holds:*
Let $\mathcal{K} = [z_k - \Gamma, z_{k+N} + \Gamma]$ and let $v_t(x)$ be a solution of Eq.(2.1) for which
1) $\{x \in \mathbf{R} : v_0(x) = 0\} = Z_\infty$,
2) There exists a $v_0^* \in \mathcal{Z}_{N,\Gamma}$ for which $(v_0 - v_0^*)\chi_{\mathcal{K}} \equiv 0$, with $\mathcal{Z}_{N,\Gamma}$ as in Theorem 3.2.
Then there is a $T < \infty$ such that $|v_T(x)| > 0$ for all $x \in \mathcal{K}$.

Proof. Let v_t and v_t^* be the orbits of v_0 and v_0^* under Eq.(2.1). We suppose $k = 0$. For any $\delta > 0$, and for all $T < \exp(\gamma_2(z_N - z_0)) - \log(1/\delta)$, for sufficiently large $\gamma_2 = \gamma_2(N, \delta)$, Lemma B.1 and Definition 4.5 imply $\|\chi_{\mathcal{K}}(v_T - v_T^*)\|_\infty \le \delta$. Hence it suffices to prove the claim with v_t replaced by v_t^* and check that there is a $\gamma_2(N, \delta)$ such that T satisfies the above inequality with a sufficiently small δ.

By Definition 4.4, Eq.(1.1), and Theorem 3.3, there is a time $T_1 < C\exp(\alpha_c(z_N - z_0))$ such that $|v_{T_1}^*(x)| > 0$ if $x \in X_j \equiv [z_{j-1} - \Gamma, z_j + \Gamma]$ where $j = j_{\min}(0)$. Let $2\varepsilon = \inf_{x\in X_j} |v_{T_1}^*(x)|$, let $\beta = \gamma_1$. Then $\pm v_{T_1}^*$ satisfies Definition 4.1 hence by Theorem 4.2 and Lemma 4.3 there is a time $T_2 \le K|Z| \le K(z_N - z_0)$ such that $v_{T_1+T_2}^* \in \mathcal{A}_{N-2,\Gamma}$.

This argument can be repeated until all zeros of v_t^* have disappeared. The k^{th} step takes a time $T_1(k) + T_2(k)$, where $T_j(k)$ are as above, namely $T_1(k)$ is the time for the interval $\ell_{j_{\min}(k)}$ to collapse, and $T_2(k)$ is the time for v_t^* to enter $\mathcal{A}_{N-2k,\Gamma}$ after this collapse. Note that the following bound holds because of Definition 4.4:

$$\left|Z\big(T_1(k) + T_2(k)\big)\right| - d_1(k) \;\le\; \sum_{n=1}^{k} n\gamma_1^{-n} \le \frac{N}{\gamma_1 - 1}\ .$$

Assuming $\gamma_1(N) > 1 + 4N/\Gamma$ make the discrete model still valid up to this time.

By the above argument, for each k, $T_1(k) + T_2(k) \le C\exp(\alpha_c(z_N - z_0)) + K(z_N - z_0)$. This gives a total time $T_{\text{tot}} = \sum_k T_1(k) + T_2(k) \le 2N\exp(2\alpha_c(z_N - z_0))$. Assuming $\gamma_2(N, \delta) > N^{-1}\Gamma^{-1}(\log\log(1/\delta) + \log(2N)) + 2\alpha_c$ we have $T_{\text{tot}} \le \exp(\gamma_2(z_N - z_0)) - \log(1/\delta)$. Taking for example $\delta = 1/4$ completes the proof of Theorem 4.6. □

Remark. A separable configuration Z may contain many disjoint intervals $\mathcal{K}_n$, $n \in \mathcal{J}$ (with $\mathcal{J}$ finite or infinite denumerable), each one satisfying Definition 4.4 with a sufficiently large γ_1. In this case, Theorem 4.6 holds with $\mathcal{K}$ replaced by any of the $\mathcal{K}_n$, $n \in \mathcal{J}$. When $\mathcal{J}$ is finite, there is an open set $\mathcal{W} \subset \mathrm{L}^\infty(\mathbf{R}, dx)$ such that any orbit v_t of Eq.(2.1) with $v_0 \in \mathcal{W}$ satisfies the conclusions of Theorem 4.6 with $\mathcal{K}$ replaced by any $\mathcal{K}_n$, $n \in \mathcal{J}$.

5. Proofs of Theorem 2.3 and of Lemma 4.3

In this section, we consider the case $N = 1$ (*i.e.,* two kinks) as in Section 1. The general case is similar, see Section 4. In addition, we denote $\mathcal{L}(f)$ the r.h.s. of Eq.(2.1):

$$\mathcal{L}\big(f\big)(x) \;=\; \partial_x^2 f(x) + f(x) - f^3(x)\;, \tag{5.1}$$

and w_t is the perturbation of the pair of kinks, namely, $w_t(x) = v_t(x) - u_{Z(t)}(x)$. One has the equation:

$$\begin{aligned}\partial_t w_t(x) \;=\; & \mathcal{L}\big(u_{Z(t)}\big)(x) - \sum_{i=1,2} \partial_t z_i(t)\partial_{z_i} u_{Z(t)}(x) \\ & + \big(L_{Z(t)} w_t\big)(x) - 3u_{Z(t)}(x) w_t^2(x) - w_t^3(x)\;,\end{aligned} \tag{5.2}$$

where

$$\big(L_Z f\big)(x) \;=\; \partial_x^2 f(x) + \big(1 - 3u_Z^2(x)\big) f(x)\;.$$

The above differential expression defines a self-adjoint operator with domain $D(L_Z)$ dense in $\mathrm{L}^2(\mathbf{R}, dx)$. The same symbol, L_Z, will be used for this operator.

We will also use the notation $N(f,g)$ for the following polynomial appearing on the r.h.s. of Eq.(5.2):

$$N(f,g) \;=\; 1 - 3f^2 - 3fg - g^2\;. \tag{5.3}$$

The following results are taken from [ER].

Lemma 5.1. *There are constants $\alpha_c > 0$, $M^* > 0$, $C < \infty$ such that for sufficiently large $|Z|$, the following holds:*

1) $\|\mathcal{L}(u_Z)\|_\infty \le Ce^{-\alpha_c |Z|}$.

2) $\|L_Z \tau_k(Z,\cdot)\|_2 \le Ce^{-\alpha_c|Z|/2}$, *for* $k = 1, 2$.

3) For $k = 1, 2$, *if* $|x - z_k| \le |Z|/2$, *then*

$$\left| u_Z(x) - \tanh\Big(\frac{(-1)^k (z_k - x)}{\sqrt{2}}\Big)\right| \;\le\; Ce^{-\alpha_c|Z|}\;.$$

4) If $w \in D(L_Z)$ *satisfies* $\big(w, \tau_k(Z,\cdot)\big) = 0$, *for* $k = 1, 2$, *then*

$$\big(w, L_Z w\big) \;\le\; -M^*\big(w,w\big)\;.$$

5) For $k = 1, 2$, $M_k = \mathrm{supp}(\tau_k(Z,\cdot))$,

$$\|\chi_{M_k}\big(\partial_{z_k} u_Z - \tau_k(Z,\cdot)\big)\|_\infty \;\le\; Ce^{-\alpha_c|Z|}\;.$$

Remark. Statement 4) above is a direct consequence of the spectral analysis of L_Z performed in [ER] (see also Lemma 3.1, set $1/M \approx M^*$). In more intuitive words, statements 1) and 2) say that u_Z is almost a stationary solution of Eq.(2.1), statement 3) shows that the interface (locally) looks like the kink solution and statement 4) shows that the perturbation w, when defined as in Lemma 2.2 is (nearly) orthogonal to the unstable manifold of the point $|Z| = \infty$. Statement 5) shows that τ_j is close to the generator of the translation of the j^{th} kink.

The strategy of the proof of Theorem 2.3 is the following: first we show that solutions v_t of Eq.(2.1) that *remain* admissible for all $t \in [0, T]$ must satisfy: the speed of the kinks $\partial_t z_i(t)$ is very small, the "large part" of the perturbation (w_2 in the notation of Definition 2.1) decays uniformly and that the "small part" (w_1) remains small. Then we use the maximum principle and an inductive argument to show that admissible initial conditions remain admissible *and* converge to a small ball around $u_{Z(t)}$.

We begin with a bound on the speed of the kinks.

Proposition 5.2. *Let v_t be a solution of Eq.(2.1). For any $\alpha \le \alpha_c$, if v_{t^*} is $(\varepsilon, \alpha, \ell, \Gamma)$–admissible for some $t^* > 0$, $\ell < \infty$, $\varepsilon > 0$, and sufficiently large $\Gamma = \Gamma(\alpha)$, then $z_i(t^*)$ satisfies:*

$$|\partial_t z_i(t^*)| \le Ce^{-\alpha|Z(t^*)|} .$$

Proof. For simplicity, we write t for t^*. We also write $\tau_j(t)$ for the function $\tau_j(Z(t), \cdot)$. By the definition of $Z(t) \equiv Z(v_t)$, see Lemma 2.2, we have $\partial_t(w_t, \tau_j(t)) = 0$, or

$$\sum_{i=1,2} \partial_t z_i(t) \left\{ \big(\partial_{z_i} u_{Z(t)}, \tau_j(t)\big) - \big(w_t, \partial_{z_i} \tau_j(t)\big) \right\} = \big(\partial_t v_t, \tau_j(t)\big) .$$

If we define $\mathcal{S}_{ij} = \big(\partial_{z_i} u_{Z(t)}, \tau_j(t)\big) - \big(w_t, \partial_{z_i} \tau_j(t)\big)$ then the matrix $\mathcal{S} = \big(\mathcal{S}_{ij}\big)_{i,j=1,2}$ is invertible with uniformly bounded inverse (see [ER]). Thus we can write

$$\begin{aligned}
|\partial_t z_i(t)| &= \left| \sum_{j=1,2} \mathcal{S}^{-1}_{ij} \big(\partial_t v_t, \tau_j(t)\big) \right| \\
&= \left| \sum_{j=1,2} \mathcal{S}^{-1}_{ij} \big(\mathcal{L}(u_{Z(t)}) + L_{Z(t)} w_t - 3u_{Z(t)} w_t^2 - w_t^2, \tau_j(t)\big) \right| \\
&\le C\bigg(\|\mathcal{L}(u_{Z(t)})\|_2 \sup_j \|\tau_j(t)\|_2 + \sup_j \Big| \big(w_t, L_{Z(t)} \tau_j(t)\big) \Big| \\
&\qquad + \|(1 - \chi_{\mathcal{K}}) w_t\|_2^2 + \sup_j \|\chi_{\mathcal{K}} \tau_j(t)\|_\infty \bigg) ,
\end{aligned}$$

where $\chi_{\mathcal{K}}$ is the characteristic function of the interval $\mathcal{K} = [m_1(t) - \ell, m_1(t) + \ell]$, and $\mathcal{L}(\cdot)$ is given by Eq.(5.1). Using Lemma 5.1, one finds that each term in the above expression is bounded by $C \exp\big(-\alpha|Z(t)|\big)$. □

We next prove two lemmas which establish bounds on the evolution in the middle of the interval enclosed by the two kinks (first lemma) and outside and near the boundary of this interval (second lemma).

Lemma 5.3. *For any $\alpha \le \alpha_c$, $\varepsilon > 0$, $\ell < \infty$, and sufficiently large $\Gamma = \Gamma(\alpha, \varepsilon, \ell) > 0$, if v_t is an $(\varepsilon, \alpha, \ell, \Gamma)$–admissible solution of Eq.(2.1) for all $t \le 1$, then*

$$\|\chi_{\mathcal{K}}\big(u_{Z(T)} - v_T\big)\|_\infty \;\le\; C\big(e^{-\alpha|Z(0)|} + \|\chi_{\mathcal{K}}(u_{Z(0)} - v_0)\|_\infty e^{-\varepsilon T} + \sup_{0\le t\le T} \|\chi_\Delta w_t\|_\infty\big)$$

for $T \le 1$ and for any $\mathcal{K} = \big[m_1(0) - \ell^*, m_1(0) + \ell^*\big]$, $\Delta = \operatorname{supp}(\Theta'_{\mathcal{K}})$, where $\ell + 1 \le \ell^* \le \Gamma/2$.

Proof. By Proposition 5.2, we can take Γ so large that for all $t \le 1$, $|m_1(t) - m_1(0)| \le 1$ and $||Z(t)| - |Z(0)|| \le 1$. We use the notation $N(f, g)$ defined in Eq.(5.3). For $x \in \mathcal{K}^* \equiv \mathcal{K} \cup \Delta = \operatorname{supp}(\Theta_{\mathcal{K}})$, one has

$$\begin{aligned}
&N(u_{Z(t)}(x), w_t(x)) \\
&= 1 - u_{Z(t)}^2(x) - 2u_{Z(t)}(x)\big(u_{Z(t)}(x) + w_t(x)\big) - w_t(x)\big(u_{Z(t)}(x) + w_t(x)\big) \\
&\le 1 - (1 - Ce^{-\alpha\Gamma/2})^2 - 2\varepsilon(1 - Ce^{-\alpha\Gamma/2}) - \varepsilon \\
&\le -\varepsilon \,,
\end{aligned}$$

provided Γ is such that $C \exp\big(-\alpha\Gamma/2\big) \le \varepsilon/2$. We introduce the heat kernel

$$G_t(x) \;=\; \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t}\right) \,. \tag{5.4}$$

We next use Eq.(5.2), Lemma A.1, and Lemma 7 of [C] to obtain the following for $x \in \mathcal{K}$:

$$\begin{aligned}
&|w_T(x)| \;=\; |\Theta_{\mathcal{K}}(x) w_T(x)| \\
&= \Big|\int_{-\infty}^{\infty} dy\, G_T(x-y) w_0(y)\Theta_{\mathcal{K}}(y) + \int_0^T ds \int_{-\infty}^{\infty} dy\, G_{T-s}(x-y) \\
&\quad \times \Big(N\big(u_{Z(s)}(y), w_s(y)\big) w_s(y)\Theta_{\mathcal{K}}(y) - \Theta'_{\mathcal{K}}(y) w'_s(y) - \Theta''_{\mathcal{K}}(y) w_s(y) \\
&\qquad + \Theta_{\mathcal{K}}(y)\mathcal{L}\big(u_{Z(s)}\big)(y) - \sum_{i=1,2} \partial_t z_i(s) \Theta_{\mathcal{K}}(y) \partial_{z_i} u_{Z(s)}(y)\Big)\Big| \\
&\le \Big|\int_{-\infty}^{\infty} dy\, G_T(x-y)\Theta_{\mathcal{K}}(y) w_0(y) - \varepsilon \int_0^T ds \int_{-\infty}^{\infty} dy\, G_{T-s}(x-y)\Theta_{\mathcal{K}}(y) w_s(y)\Big| \\
&\quad + \Big|\int_0^T ds \int_{-\infty}^{\infty} dy\, G'_{T-s}(x-y)\Theta'_{\mathcal{K}}(y) w_s(y)\Big| + C \sup_{0\le s\le T} e^{-\alpha|Z(s)|} \\
&\le e^{-\varepsilon T} \left|\int_{-\infty}^{\infty} dy\, G_T(x-y)\Theta_{\mathcal{K}}(y) w_0(y)\right| + C \sup_{0<s<T} \|\chi_\Delta w_s\|_\infty + Ce^{-\alpha|Z(0)|} \,.
\end{aligned}$$

$\square$

Lemma 5.4. *For any $\alpha \le \alpha_c$, $\delta > 1$, $\varepsilon > 0$, $\ell < \infty$, and sufficiently large $\Gamma = \Gamma(\alpha, \ell, \delta) > 0$, if v_0 is an $(\varepsilon, \alpha, \ell, \Gamma)$–admissible function then the corresponding solution v_t of Eq.(2.1) satisfies:*

$$\max\left\{\|\chi_{\mathcal{B}}\big(u_{Z(t)} - v_t\big)\|_2 \,,\ \|\chi_{\mathcal{B}}\big(u_{Z(t)} - v_t\big)\|_\infty\right\} \le C \sup_{0\le s\le t}\big(e^{-\alpha_c|Z(s)|} + \|\chi_\Delta w_s\|_\infty\big)$$

for any $\mathcal{B} = \big(-\infty, m_1(0) - \hat{\ell}\big] \cup \big[m_1(0) + \hat{\ell}, \infty\big)$ where $\hat{\ell} \ge \ell + \delta$, $\Delta = \operatorname{supp}(\Theta'_{\mathcal{B}})$, and $t \le 1$.

Proof. We take Γ so large that, using Proposition 5.2, $|m_1(t) - m_1(0)| < (\delta - 1)$ for all $t \le 1$. We let $\mathcal{B}^* = \mathcal{B} \cup \Delta$.

Bound on $\|\cdot\|_2$. By Eq.(5.2), we have

$$\begin{aligned}
\tfrac{1}{2}\partial_t\|\Theta_{\mathcal{B}} w_t\|_2^2 \,=\, & \big(\Theta_{\mathcal{B}}\mathcal{L}(u_{Z(t)}), \Theta_{\mathcal{B}} w_t\big) - \sum_{i=1,2}\partial_t z_i(t)\big(\Theta_{\mathcal{B}}\partial_{z_i} u_{Z(t)}, \Theta_{\mathcal{B}} w_t\big) \\
& + \big(\Theta_{\mathcal{B}} w_t, \Theta_{\mathcal{B}} L_{Z(t)} w_t\big) - \big(\Theta_{\mathcal{B}}(3u_{Z(t)} + w_t)w_t^2, \Theta_{\mathcal{B}} w_t\big) \\
\le\, & C e^{-\alpha_c|Z(t)|}\|\Theta_{\mathcal{B}} w_t\|_2 + \big(\Theta_{\mathcal{B}} w_t, L_{Z(t)}\Theta_{\mathcal{B}} w_t\big) \\
& + \big(\Theta_{\mathcal{B}} w_t, -2\Theta'_{\mathcal{B}} w'_t - \Theta''_{\mathcal{B}} w_t\big) + 4\|\chi_{\mathcal{B}^*} w_t\|_\infty\|\Theta_{\mathcal{B}} w_t\|_2^2 \\
\le\, & C e^{-\alpha_c|Z(t)|}\|\Theta_{\mathcal{B}} w_t\|_2 + C\|\Theta_{\mathcal{B}} w_t\|_2\|\chi_\Delta w_t\|_\infty \\
& - \Big\{M^* - 4\|\chi_{\mathcal{B}^*} w_t\|_\infty - \sum_{i=1,2}\big(\partial_{z_i} u_Z, \Theta_{\mathcal{B}} w_t\big)^2\Big\}\|\Theta_{\mathcal{B}} w_t\|_2^2 \,,
\end{aligned} \tag{5.5}$$

using the spectral properties of the linear operator $L_{Z(t)}$, Lemma 5.1. Obviously (see Lemma B.1) there is a $K < \infty$ such that $\|\chi_{\mathcal{B}^*} w_t\|_\infty \le \exp(Kt)\|\chi_{\mathcal{B}^*} w_0\|_\infty$. Using Lemma 5.1 and taking Γ so large that for all $t \le 1$,

$$\begin{aligned}
& 4e^{Kt}\|\chi_{\mathcal{B}^*} w_0\|_\infty + \sum_{i=1,2}\big(\partial_{z_i} u_Z, \Theta_{\mathcal{B}} w_t\big)^2 \\
&= 4e^{Kt}\|\chi_{\mathcal{B}^*} w_0\|_\infty + \sum_{i=1,2}\big(\tau_i, (\Theta_{\mathcal{B}} - 1)w_t\big)^2 + C e^{-\alpha_c|Z|} \\
&\le 4e^{K - \alpha\Gamma} + C e^{-\alpha_c\Gamma} \le \frac{M^*}{2} \,,
\end{aligned}$$

we can integrate Eq.(5.5), and we get for all $t \le 1$,

$$\|\Theta_{\mathcal{B}} w_t\|_2 \le C \sup_{0\le s\le t}\big(e^{-\alpha_c|Z(s)|} + \|\chi_\Delta w_s\|_\infty\big) + e^{-M^* t/2}\|\Theta_{\mathcal{B}} w_0\|_2 \,. \tag{5.6}$$

Bound on $\|\cdot\|_\infty$. Let x be in $\mathcal{B}$, let $G_t(\cdot)$ be given by Eq.(5.4). We get

$$
\begin{aligned}
|w_t(x)| \le\ & \left|\int_{-\infty}^{\infty} dy\, \Theta_{\mathcal{B}}(y) G_t(x-y) w_0(y)\right| \\
& + C\left|\int_0^t ds \int_{-\infty}^{\infty} dy\, \big(\Theta_{\mathcal{B}}(y) G_{t-s}(x-y) + \Theta'_{\mathcal{B}}(y) G'_{t-s}(x-y)\big) w_s(y)\right| \\
& + C \sup_{0\le s\le t} e^{-\alpha_c |Z(s)|} \\
\le\ & C \sup_{0\le s\le t} \big(e^{-\alpha_c|Z(s)|} + \|\Theta_{\mathcal{B}} w_s\|_2 + \|\chi_\Delta w_s\|_2\big)\ .
\end{aligned}
\tag{5.7}
$$

We apply the bound (5.6) and Lemma 5.4 is proved. □

Proof of Theorem 2.3. Assume v_0 is $(\varepsilon, \alpha, \ell, \Gamma)$–admissible, for given ℓ, ε, $\alpha \le \alpha_c$, and for sufficiently large Γ. Let $Z(v_0) = \{z_1(v_0), z_2(v_0)\}$ be given by Lemma 2.2. There is a $Y = \{y_1, y_2, y_3, y_4\} \in \mathbf{R}^4$, $z_1(v_0) < y_1 < y_2 < y_3 < y_4 < z_2(v_0)$, with the following property: define the intervals $\mathrm{L} = (-\infty, y_1 - 1/2)$, $\mathrm{C}_j = (y_j + 1/2, y_{j+1} - 1/2)$, and $\mathrm{R} = (y_4 + 1/2, \infty)$. Then the function

$$
\begin{aligned}
v_0^*(x) =\ & \Theta_{\mathrm{L}}(x) \tanh\left(\frac{x-y_1}{\sqrt{2}}\right) + \Theta_{\mathrm{R}}(x) \tanh\left(\frac{y_4 - x}{\sqrt{2}}\right) \\
& + \sum_{j=1}^{3} (-1)^j \Theta_{\mathrm{C}_j}(x) \varphi_{y_j - y_{j-1}}(x - y_j)
\end{aligned}
\tag{5.8}
$$

lies strictly below v_0. By the maximum principle, [CE], the orbits v_t and v_t^* of Eq.(2.1) with initial conditions v_0 and v_0^* satisfy $v_t^*(x) < v_t(x)$ for all $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. Moreover, for sufficiently large Γ, Y can be chosen such that v_0^* satisfies the hypotheses of Theorem 3.2. It follows that there are positive constants C, B_1, B_2 and a function $Z^*(t) = \{z_1^*(t), \ldots, z_4^*(t)\} : [0, t^*] \to \mathbf{R}^4$, where $t^* = \exp\big(B_1 |z_3^*(0) - z_2^*(0)|\big)$ and $Z^*(0) = Y$ such that:

$$
\begin{aligned}
|z_j^*(t) - z_j^*(0)| &\le C e^{-B_2 |z_3^*(0) - z_2^*(0)|}\ , j = 1, \ldots, 4\ , \\
\|v_t^* - v_t^{**}\|_\infty &\le C e^{-B_2 |z_3^*(0) - z_2^*(0)|}\ ,
\end{aligned}
\tag{5.9}
$$

with v_t^{**} given by Eq.(5.8) replacing Y by $Z^*(t)$.

The above discussion shows that if there is an $\ell(t)$ such that v_t is $(\varepsilon, \alpha, \ell(t), \Gamma)$–admissible for $t \le t^*$, then $\ell(t) \le \ell_{\max} \equiv |z_3^*(0) - z_2^*(0)| + C \exp(-B_2 |z_3^*(0) - z_2^*(0)|)$, which is independent of Γ, α, and ε. We choose Γ such that $\ell_{\max} \le \Gamma/2$. Moreover, when using Lemma 5.4 and Lemma 5.3, we have the bound $\|\chi_\Delta w_t\|_\infty \le C \exp(-\alpha_c |Z(0)|/4)$ for all times $t \le t^*$.

Since v_0 is $(\varepsilon, \alpha, \ell, \Gamma)$–admissible, by continuity, there is a time $t > 0$ such that $v_s(x) > \varepsilon/2$ for $|x - m_1(0)| < \ell_{\max}$ and $s < t$. By Lemma 5.3, $v_t(x) > \varepsilon$ for $|x - m_1(0)| < \ell_{\max}$.

We repeat this argument until $t = T_1 \equiv |Z(0)|/\varepsilon$. It follows that for all $t \le T_1$, v_t is $(\varepsilon, \alpha, \ell_{\max}, |Z(0)| - \delta)$–admissible for some $\delta \le C T_1 \exp\big(-\alpha |Z(0)|\big) \le |Z(0)|/3$ (this bound follows from Proposition 5.2 if $\Gamma \le |Z(0)|$ is sufficiently large).

Using repeatedly Lemma 5.3 and Lemma 5.4, we obtain

$$\|u_{Z(T_1)} - v_{T_1}\|_2 \leq CT_1 \sup_{t \leq T_1} e^{-\alpha_c |Z(0)|/4} \quad \text{and} \quad \|u_{Z(t)} - v_t\|_\infty \leq CT_1 \sup_{t \leq T_1} e^{-\alpha_c |Z(0)|/4} .$$

We finally use Lemma 5.4 with $\mathcal{B} = \mathbf{R}$ (in fact equations (5.6) and (5.7), with $\chi_\Delta \equiv 0$ and $\Theta_\mathcal{B} \equiv 1$), to show that after a time T_2 of order $|Z(T_1)|$, we get the bound we claimed. The bound on $T = T_1 + T_2$ follows from Proposition 5.2. □

We finish this section with the

Proof of Lemma 4.3. We have trivially $\|f\|_Z^2 \leq \|f'\|_2^2 + 4\|f\|_2^2$, hence we need only bound $\|w_T'\|_2^2$. We decompose $w_T(x) = \chi_\mathrm{E}(x)w_T(x) + \chi_\mathrm{I}(x)w_T(x)$ where $\mathrm{I} = [m_1(0) - \ell_{\max}, m_1(0) + \ell_{\max}]$ and $\mathrm{E} = \mathbf{R}\backslash\mathrm{I}$, with ℓ as in Theorem 2.3. Let $G_t(x)$, $N(\cdot,\cdot)$, $\mathcal{L}(\cdot)$ be given by Eq.(5.4), Eq.(5.3), and Eq.(5.1) respectively. We compute first

$$\begin{aligned}
\|\chi_\mathrm{E} w_T'\|_2 \leq\ & \left\| \int_0^T ds \int_{-\infty}^{\infty} dy\, G_{T-s}'(x-y)\Theta_\mathrm{E}(y)\mathcal{L}\big(u_{Z(s)}\big)(y) \right\|_2 \\
&+ \left\| \int_{\infty}^{\infty} dy\, G_T'(x-y)\Theta_\mathrm{E}(y) w_0(y) \right\|_2 \\
&+ \left\| \int_0^T ds \int_{-\infty}^{\infty} dy\, G_{T-s}'(x-y) \right. \\
&\qquad \left. \times \Big(N\big(u_{Z(s)}(y), w_s(y)\big)\Theta_\mathrm{E}(y) w_s(y) - \big(\Theta_\mathrm{E}'(y) w_s(y)\big)' \Big) \right\|_2 \\
\leq\ & C \sup_{0 \leq s \leq T} e^{-\alpha|Z(s)|} + C \sup_{0 \leq s \leq T} \|\chi_{\mathrm{E}^*} w_s\|_2 \,,
\end{aligned}$$

where $\mathrm{E}^* = \mathbf{R}\backslash[m_1(0) - \ell_{\max} + 1, m_1(0) + \ell_{\max} - 1]$. The remaining term,

$$\|\chi_\mathrm{I} w_T'\|_2 \leq \sqrt{\ell_{\max}} \|\chi_\mathrm{I} w_T'\|_\infty \,,$$

can be bounded by a similar argument as in Lemma 5.3. □

Appendix A

Lemma A.1. *Let $f(x,t) \in \mathrm{L}^\infty\big(t \in [0,1] \to \mathrm{L}^2(\mathbf{R}, dx)\big)$, let $\lambda \in \mathbf{R}$, let $G_t(x)$ be as in Eq.(5.4). Then, the following holds:*

$$\int_{-\infty}^{\infty} dy\, G_t(x-y) f(y,0) + \lambda \int_0^t ds \int_{-\infty}^{\infty} dy\, G_{t-s}(x-y) f(y,s)$$
$$= e^{\lambda t} \int_{-\infty}^{\infty} dy\, G_t(x-y) f(y,0)\ .$$

Proof. Let A be the following operator, densely defined on $\mathrm{L}^\infty\big(t \in [0,1] \to \mathrm{L}^2(\mathbf{R}, dx)\big)$:

$$\big(Af\big)(x,t) \;=\; \int_0^t ds \int_{-\infty}^{\infty} dy\, G_{t-s}(x-y) f(y,s)\ .$$

It is easy to see that there is a constant C such that $\|A^n f\| \le C\|f\|/n!$ which implies that the series

$$f(x,t) \;=\; \sum_{n=0}^{\infty} \lambda^n \big(A^n g\big)(x,t) \tag{A.1}$$

converges and is a solution of the equation

$$f(x,t) \;=\; g(x,t) + \lambda\big(Af\big)(x,t)\ .$$

Substituting $g(x,t) = \int dy\, G_t(x-y) f(y,0)$ into Eq.(A.1) and using

$$\int_{\mathbf{R}^2} dy\, dz\, G_{t-s}(x-y) G_{s-\ell}(y-z) f(z,\ell) \;=\; \int_{\mathbf{R}} dy\, G_{t-\ell}(x-y) f(y,\ell)\ ,$$

one obtains that

$$f(x,t) \;=\; \sum_{n=0}^{\infty} \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \int_{-\infty}^{\infty} dy\, G_t(x-y) f(y,0)$$
$$=\; e^{\lambda t} \int_{-\infty}^{\infty} dy\, G_t(x-y) f(y,0)$$

is a solution of the equation

$$f(x,t) \;=\; \int_{-\infty}^{\infty} dy\, G_t(x-y) f(y,0) + \lambda \int_0^t ds \int_{-\infty}^{\infty} dy\, G_{t-s}(x-y) f(y,s)\ .$$

□

Appendix B

Lemma B.1. *Let* $\mathcal{K} = [-L, L]$*, let* $\varepsilon \in (0,1)$*, and let* v_t *and* w_t *be two solutions of Eq.(2.1), with* $\|\chi_{\mathcal{K}}(v_0 - w_0)\|_\infty \le \varepsilon$*. There are* $K_1, K_2 > 0$ *such that for any* δ, t, ℓ *satisfying*

$$1 \;<\; t \;<\; K_1\bigl(\min\{\ell, \log\varepsilon^{-1}\} - \log\delta^{-1}\bigr)\;, \quad \ell < L\;,$$

one has

$$\|\chi_{[-L+\ell, L-\ell]}(v_t - w_t)\|_\infty \;\le\; K_2(\delta + \varepsilon)\;.$$

Proof. Let $G_t(x)$ be given by Eq.(5.4). By Duhamel's principle, with $F(x,y) = 1 + x^2 + y^2 + xy$, we have

$$v_t(x) - w_t(x) \;=\; \bigl(G_t \star (v_0 - w_0)\bigr)(x) + \int_0^t ds\, \Bigl(G_{t-s} \star \bigl(F(v_s, w_s)(v_s - w_s)\bigr)\Bigr)(x)\;, \tag{B.1}$$

where $\star$ denotes convolution. We next consider $x \in [-L+\ell, L-\ell]$. For $t > 1$, the first term of Eq.(B.1) is easily bounded:

$$\bigl|\bigl(G_t \star (v_0 - w_0)\bigr)(x)\bigr| \;\le\; C_1(e^{-C_2\ell^2/t} + \varepsilon)\;. \tag{B.2}$$

We introduce the following notations: $\delta v_t(x) = v_t(x) - w_t(x)$, $\varphi(y) = \exp\Bigl(-2\sqrt{1+y^2}\Bigr)$, $\varphi_x(y) = \varphi(x - y)$. We first compute

$$\begin{aligned}
&\partial_t \int \varphi_x (\delta v_t)^2 \\
&= 2\int \varphi_x \delta v_t \Bigl(\partial_y^2(\delta v_t) + F\bigl(v_t, w_t\bigr)\delta v_t\Bigr) \\
&\le -\int \varphi_x\bigl(\partial_y(\delta v_t)\bigr)^2 + \int \varphi_x(\delta v_t)^2 \left(2\sup_{|x|\le 1, |y| \le 1} |F(x,y)| + \|\varphi_x'\varphi_x^{-1}\|_\infty^2\right) \\
&\le C_3 \int \varphi_x(\delta v_t)^2\;.
\end{aligned}$$

This shows that

$$\left(\int \varphi_x\bigl(\delta v_t\bigr)^2\right)^{\frac{1}{2}} \;\le\; e^{C_3 t}\left(\int \varphi_x(\delta v_0(y))^2\right)^{\frac{1}{2}} \;\le\; C_4 e^{C_3 t}\bigl(\varepsilon + e^{-\ell}\bigr)\;. \tag{B.3}$$

Combining Eq.(B.1), Eq.(B.3), and Eq.(B.2), we obtain the following bound:

$$
\begin{aligned}
&|v_t(x) - w_t(x)| \leq C_1\big(e^{-C_2\ell^2/t} + \varepsilon\big) \\
&+ \int_0^t ds \int_{\mathbf{R}} dy\, G_{t-s}(x-y)\varphi_x^{1/2}(y)\varphi_x^{-1/2}(y)\big|\delta v_s(y) F\big(v_s(y), w_s(y)\big)\big| \\
&\leq C_1\big(e^{-C_2\ell^2/t} + \varepsilon\big) \\
&+ C_5 \int_0^t ds \left(\int_{\mathbf{R}} dy\, \varphi_x^{-1}(y) G_{t-s}^2(x-y)\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}} dy\, \varphi_x(y)(\delta v_s(y))^2\right)^{\frac{1}{2}} \\
&\leq C_1\big(e^{-C_2\ell^2/t} + \varepsilon\big) + C_6 \int_0^t ds\, e^{C_2(t-s)}\big(\varepsilon + e^{-\ell}\big) \\
&\leq C_7\big(e^{-C_2\ell^2/t} + \varepsilon + e^{C_2 t}(\varepsilon + e^{-\ell})\big)\ .
\end{aligned}
$$

The claim follows easily. □

References

[B] Bray, A.J.: Theory of phase-ordering kinetics. Adv. Phys. **43** (3), 357–459 (1994).

[BDG] Bray, A.J., Derrida, B. and C. Godrèche: Non-trivial Algebraic Decay in a Soluble Model of Coarsening. Europhys. Lett. **27** (3), 175–180 (1994).

[C] Collet, P.: Thermodynamic limit of the Ginzburg-Landau equations. Nonlinearity **7** (4), 1175–1190 (1994).

[CE] Collet, P. and J.-P. Eckmann: *Instabilities and Fronts in Extended Systems*, Princeton, Princeton University Press (1990).

[CP1] Carr, J. and R.L. Pego: Metastable Patterns in Solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$. Comm. Pure Appl. Math. **XLII**, 523–576 (1989).

[CP2] Carr, J. and R.L. Pego: Invariant Manifolds for Metastable Patterns in $u_t = \varepsilon^2 u_{xx} - f(u)$. Proc. Roy. Soc. Edinburgh **116A**, 133–160 (1990).

[ER] Eckmann, J.-P. and J. Rougemont: Coarsening by Ginzburg-Landau dynamics. (To appear). Comm. Math. Phys.