Measures for lattice systems*

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Abstract

I review issues related with the presence or absence of Gibbsianness in measures describing random fields in lattices. After a brief exposition of the definition and properties of Gibbs measures, I discuss the phenomenon of non-Gibbsianness. The discussion involves three aspects: (1) Examples among renormalized measures and invariant measures of stochastic transformations; (2) probabilistic characterization of non-Gibbsian measures —lack of quasilocality and "wrong" large-deviation properties—, and (3) proposed clasification schemes and notions of generalized Gibbsianness. The review closes with comments on directions for further work.

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1 Introduction

Gibbs measures are the central objects of rigorous classical statistical mechanics. In the established formalism, due to Dobrushin, Lanford and Ruelle [8, 31], Gibbsianness is a property encoded in the finite-volume conditional expectations. A measure is Gibbsian if these expectations are determined by Hamiltonians defined by sums of local terms or, more precisely, of terms forming a summable interaction. The theory of Gibbs measures is so well established in physics and probability theory [52, 16] that in many instances a measure is assumed to be Gibbsian almost by default. Gibbsianness brings a package of useful properties: an efficient parametrization in terms of interactions and inverse temperatures, an extremal principle and its associated theory of large deviations, and a host of arguments and techniques developed during one century of work in statistical mechanics: contour arguments, cluster expansions, correlation inequalities, uniqueness criteria, ...

Over the last few years, however, a number of studies of equilibrium and dynamical classical spin systems showed the need to trascend the framework of Gibbsian theory. Some unexpected features were detected when rather simple and well known distributions —like the equilibrium measures of the Ising model— were subjected to natural transformations. The initial call to attention, due to Griffiths and Pearce [19, 20, 18], came from the study of renormalization transformations. It was soon understood [25] that these transformations [13, 17], designed to study the behavior of systems close to critical points, lead to probability measures that can not be described by any summable interaction, thus *non-Gibbsian measures*. Other instances of non-Gibbsianness were detected in the study of measures involving spin "contractions" [32, 11] and lattice projections [55], and, not unexpectedly, among the stationary measures of stochastic time evolutions [34]. These works can be associated to an initial stage of the study of non-Gibbsian measures, centered in the *symptomatology* of the phenomenon.

The second stage of this study —the *diagnosis* stage— originated in the pioneer article by Israel [25], which was formalized and exploited only a decade later [63, 64, 65, 66, 67, 68, 60]. In this stage, the different known occurrences of non-Gibbsianness were systematized and some key probabilistic aspects were emphasized. The non-Gibbsianness of renormalized measures was traced to the lack of continuity (in an appropriate sense, see below), with respect to the external (or boundary) conditions, of some finite-volume conditional expectations. This continuity, also known as *quasilocality* or *almost-Markovianness*, is lost because of the existence of "hidden" degrees of freedom that develop long-range correlations. Changes in the exterior conditions occurring arbitrarily far away can propagate, via these "hidden" correlations, and alter the expectations around the origin, even in the absence of ("nonhidden") fluctuations in the intermediate regions. In these examples non-Gibbsianness is thus a manifestation of first-order phase transitions taking place in the system of "hidden" variables even when the "non-hidden" (or *block*) variables were fixed.

The initial studies of "contractions", "projections" and measures invariant under spin-flip or other types of dynamics [32, 11, 34, 55, 48], contained a complementary type of diagnosis, based on the existence of large-deviation probabilities that are either too large or too small for the measure to be Gibbsian. For some of these examples the diagnosis was later narrowed down to absence of quasilocality. A detailed exposition of these arguments is presented in the long monography [67]. As complementary references I mention [42] which focuses on stationary measures for interacting particle systems, and [12] which presents the phenomenon in general probabilistic terms.

The last stage of the study of non-Gibbsian measures corresponds to what could be called the *treatment* of the phenomenon. A number of classification schemes have been proposed aiming to establish "hierarchies" or "degrees" of non-Gibbsianness. One such a scheme considers the behavior upon further decimation [46, 47, 39] to distinguish the so-called *robust* non-Gibbsianness. A second scheme is based on the size of the set of external configurations where the discontinuities take place [12] and leads to the notion of *almost-Gibbsian*, or, more properly, *almost-quasilocal* measures. A third scheme focuses on the existence of almost-everywhere summable potentials to define the *weakly Gibbsian* measures [7, 44, 9, 4, 10]. Every almost-quasilocal measure is weak Gibbsian and the converse is false [53]. On the other hand there seems to be no relation between these two categories and robust non-Gibbsianness [60, 36, 70].

These schemes have been used as the basis for a more ambitious, and largely incomplete, program to develop a "generalized Gibbsian theory" that includes some non-Gibbs distributions. Actually, this effort was started rather early by people working in stochastic evolutions [34, and references therein]. See [53] for an updated analysis of the different attempts to extend Gibbsianness and its properties.

In this review I shall start with a brief presentation of the definition and main properties of Gibbsian measures, followed by an overview of the most representative examples of non-Gibbsianness and a discussion of the different "treatments" proposed for the phenomenon. I will close with a personal view of further directions of research. In the preparation of this article I have benefited from the concise and complete account presented in [60].

2 Gibbs measures

In this section I review some key facts about Gibbs measures emphasizing its role in the issues that follow. The default reference for this section is the treatise by Georgii [16], though in some cases I may provide more specific references.

2.1 Basic definition

Our systems will be formed by finite spins placed at the sites of $\mathbb{Z}^d$. The *configuration space* is therefore a product space $\Omega = \Omega_0^{\mathbb{Z}^d}$ where Ω_0 is a finite set. Its elements —the *configurations*— are families $\omega = \{\omega_x\}_{x\in\mathbb{Z}^d}$ where each ω_x takes values in Ω_0. Ising spins — $\omega_x = -1, +1$ — or Potts spins — $\omega_x = 1, \ldots, q$ — are typical examples.

The statistical mechanical description starts with "finite windows", that is finite regions $\Lambda \subset \mathbb{Z}^d$. The system there is described via probability distributions on the corresponding space of finite-volume configurations $\Omega_\Lambda := \Omega_0^\Lambda$, defined by the well known Boltzmann-Gibbs weights. These are proportional to $\mathrm{e}^{-\beta H_\Lambda}$, where β is the inverse temperature and H_Λ is the *Hamiltonian* for the region Λ. Two important requirements must be considered at this point.

1. The Hamiltonians must be sums of *local terms*, that is of terms depending on spins at finite sets of sites. A Hamiltonian for a larger region is obtained simply by adding new local terms to the Hamiltonian for a smaller region.

2. The exterior of each window Λ is taken to be frozen in some configuration σ_{Λ^c}. The corresponding Hamiltonian includes terms coupling spins inside and outside Λ. Suitable summability conditions are required for such an expression to be well-defined.

The first requirement implies that the basic objects in the construction of Boltzmann-Gibbs weights are not the Hamiltonians but the *interactions*, namely families $\Phi = \{\Phi_B\}_{B\subset\subset \mathbb{Z}^d}$, indexed by the finite subsets B of $\mathbb{Z}^d$, where each Φ_B is a real- (or complex-) valued function of the configurations, which depends only of the spins in B. Given Φ, the Hamiltonian on a finite region Λ with external configuration σ_{Λ^c} is the function on Ω_Λ defined by the sum

$$H(\omega_\Lambda|\sigma_{\Lambda^c}) \;:=\; \sum_{B\,:\,B\cap\Lambda\neq\emptyset} \Phi_B(\omega_\Lambda\sigma_{\Lambda^c})\;. \tag{1}$$

Here and in the sequel the notation $\omega_\Lambda\sigma_{\Lambda^c}$ stands for the configuration taking values ω_x for $x\in\Lambda$ and values σ_x for $x\notin\Lambda$. In order for (1) to be well defined it is natural to demand the uniform and absolute summability condition:

$$\sup_{x\in\mathbb{Z}^d}\sum_{B\ni x}\|\Phi_B\|_\infty \;<\; \infty\;. \tag{2}$$

Physically, this condition means that the overturning of a single spin produces a finite change in the total energy.

Given an interaction and a external condition, the Boltzmann-Gibbs prescription assigns to each configuration $\omega_\Lambda\in\Omega_\Lambda$ the probability weight

$$\frac{\exp[-\beta\, H(\omega_\Lambda|\sigma_{\Lambda^c})]}{\text{Norm.}}\;. \tag{3}$$

These weights describe *equilibrium* in finite volume. They imply the averaging prescription

$$\langle f\,|\,\sigma_{\Lambda^c}\rangle \;=\; \sum_{\omega_\Lambda\in\Omega_\Lambda} f(\omega_\Lambda\sigma_{\Lambda^c})\,\frac{\exp[-\beta\, H(\omega_\Lambda|\sigma_{\Lambda^c})]}{\text{Norm.}}\;, \tag{4}$$

for all *observables* f (i.e. measurable functions f).

To describe bulk properties it is necessary to pass to the limit $\Lambda\to\mathbb{Z}^d$ in some appropriate sense. In this limit the notion of Hamiltonian loses its meaning; one must consider instead the limit of the expectations (4). An equivalent (in the present setting) approach, introduced by Dobrushin, Lanford and Ruelle [8, 31], transcribes the fact that the infinite-volume analogue of the probability measures (4) should describe equilibrium in the full space. This means that each finite volume must be in equilibrium with the whole, that is, the finite-volume prescriptions must be weighted by the full-volume prescription. This leads to the definition that μ is a *Gibbs measure* (for a given interaction and inverse temperature) if

$$\int_\Omega f(\omega)\,\mu(d\omega) \;=\; \int_\Omega \langle f\,|\,\sigma_{\Lambda^c}\rangle\,\mu(d\sigma) \tag{5}$$

for each observable f. These are the celebrated *DLR equations* (for Dobrushin, Lanford and Ruelle).

[Mathematical correctness requires, at this point, the definition of the measure space where these measures are defined. That is, we must decide which events or functions (=thought experiments or measurements) are observable and which have to be left out of the theory. As Ω is uncountable we can not declare *all* events to be part of our theory without running into contradictions. Here, as usual in the theory of random fields, we consider the σ-algebra generated by the cylinder sets, that is, the sets determined by the spins at finitely many sites (microscopic observables).]

¿From a more probabilistic point of view, formula (5) means that the *conditional expectation* of μ on the finite region Λ given σ outside coincides with the Boltzmann-Gibbs average (4):

$$\mu(f \mid \sigma_{\Lambda^c}) \;=\; \langle f \mid \sigma_{\Lambda^c} \rangle \;. \tag{6}$$

Thus, Gibbs measures are defined in terms of its conditional distributions. The richness of statistical mechanics comes from the fact that, unlike marginal distributions (Kolmogorov theorem), conditional distributions do not necessarily determine a measure in a unique way. It is known that there is always at least one such a measure, but there may be more than one. The central problem in classical statistical mechanics is, precisely, the determination of all measures satisfying (5) for a given interaction.

Watching formula (5) [or (6)] one realizes that it is not altered if $\langle f \mid \sigma_{\Lambda^c} \rangle$ is modified, or even undefined, for a set of configurations σ of μ-measure zero. This observation is the genesis of the generalization proposed by Dobrushin to be discussed below.

2.2 Quasilocality. The characterization theorem

The best known spin systems —Ising and Potts models— have *finite range* interactions. That is, there exists an $r > 0$ —the range—- such that $\Phi_B = 0$ whenever $\mathrm{diam} B > r$. In this case, the finite-volume expectations $\langle f \mid \sigma_{\Lambda^c} \rangle$, defined in (4) have a Markovian property: They only depend on the value of the external configuration σ at sites at most a distance r from the set Λ. In the general case, where the interaction has an infinite range, but satisfies the summability condition (2), the expectations have instead an *almost*-Markov, or *quasi*-Markov, property: While expectations do depend on the values taken by σ at sites arbitrarily far away, this dependence goes to zero at infinity.

More formally, we say that a function f on Ω is *quasilocal* at a certain σ if

$$\sup_{\xi_{\Gamma^c}, \eta_{\Gamma^c}} \left| f(\sigma_\Gamma \xi_{\Gamma^c}) - f(\sigma_\Gamma \eta_{\Gamma^c}) \right| \longrightarrow 0 \tag{7}$$

as $\Gamma \to \mathbb{Z}^d$. Equivalently, f is *continuous* in the product topology of Ω. Let us call a measure μ on Ω *almost-Markovian* if one can find finite-volume conditional expectations $\mu(f \,|\, \sigma_{\Lambda^c})$ that are quasilocal at all σ for all finite regions Λ and all quasilocal functions f. It is rather straightforward to verify that *every Gibbsian measure is almost-Markovian.* The non-trivial part, due to Kozlov [28] and Sullivan [57], is the converse.

Theorem 1 *A measure μ is Gibbsian if and only if all the conditional probabilities $\mu(\omega_\Lambda \,|\, \sigma_{\Lambda^c})$ are*

- *quasilocal at all σ for all ω, and*
- *strictly positive for all ω and σ.*

[We have indulged in a common abuse of notation and denoted $\mu(\omega_\Lambda \,|\, \sigma_{\Lambda^c})$ instead of the more pedantic $\mu(\mathbf{1}\{\eta \in \Omega : \eta_\Lambda = \omega_\Lambda\} \,|\, \sigma_{\Lambda^c})$, where $\mathbf{1}\{\,\cdot\,\}$ is the characteristic function of the set $\{\,\cdot\,\}$.]

The easy part of this theorem (necessity) is behind the proof of non-Gibbsianness of renormalized measures, while the hard part (sufficiency) justifies the notion of weak Gibbsianness.

We see that the definition of Gibbs measure is a combination of probabilistic (conditional expectations) and topological (continuity=quasilocality) notions. The generalized theories discussed below can be interpreted as attempts to remove topological constraints so as to leave the theory in a purely probabilistic framework. Among the most immediate consequences of such attempts is the loss of one-to-one-ness of the map "measures $\to$ conditional probabilities". Indeed, being defined by integral equations, the conditional probabilities of a measure can be freely changed in sets of measure zero. A choice of conditional probabilities for each set Λ and configuration σ constitutes a *realization.* The multiplicity of realizations disappears if one adds the continuity requirement:

Theorem 2 *A measure has at most one quasilocal realization of its finite-volume conditional probabilities.*

In particular, this result, which is rather elementary from the point of view of probability theory, shows that a measure can not be simultaneously a Gibbs measure for different temperatures or interactions producing different Boltzmann-Gibbs weights. For translation-invariant interactions this implies that the pressure is a strictly convex function of any linear parameter in the interaction, if one modules-out interactions leading to the same Boltzmann-Gibbs weights ("physically equivalent interactions") [21]. The loss of this physically very appealing property could be a potential source of discomfort for generalized theories that dispose of topology altogether.

The multiplicity of realizations must be taken into account when trying to prove non-almost-Markovianness (and hence non-Gibbsianness). One must show that the violation of quasilocality at a given external configuration, happens for *every* possible realization of a particular conditional probability. A discontinuity of this type is termed *essential* in measure-theoretical jargon.

2.3 Large-deviation properties

Given two measures μ and ν on Ω, the *information gain* of μ relative to ν in a finite region Λ is

$$I_\Lambda(\mu|\nu) \;:=\; \sum_{\omega_\Lambda\in\Omega_\Lambda} \mu(\omega_\Lambda)\,\log\frac{\mu(\omega_\Lambda)}{\nu(\omega_\Lambda)}\;, \tag{8}$$

with the convention $0\log 0\equiv 0$ and allowing the value $+\infty$. When ν gives equal weight to each configuration, this differs in a sign (plus the log of a normalization factor) from what physics textbooks call the *entropy* of μ_Λ ($=\mu$ restricted to Ω_Λ). Roughly speaking, the number $I_\Lambda(\mu|\nu)$ gauges how different the two measures are when restricted to the window Λ. Indeed, on the one hand it is a positive number with the distance-like property of being zero if and only if the two measures coincide in Λ. On the other hand, large-deviation theory shows that, again roughly speaking, the ν-probability of generating a sample that looks, in Λ, as "typical" for μ, decreases exponentially with the size of the sample, the rate being precisely $I_\Lambda(\mu|\nu)$.

For statistical-mechanical measures, the thermodynamic limit of (8) is of little use, because it is usually infinite. Nevertheless, this divergence occurs at a rate not exceeding $|\Lambda|$:=cardinality of Λ. Hence, in this limit the object of interest is the *density of information-gain* of μ relative to ν:

$$i(\mu|\nu) \;:=\; \lim_{\Lambda\to\mathbb{Z}^d}\frac{1}{|\Lambda|}\,I_\Lambda(\mu|\nu)\;. \tag{9}$$

Part of the problem is, of course, to show that such a limit exists. This is indeed the case if both μ and ν are translation invariant and the latter is a Gibbs measure. The heuristic interpretation of the ensuing theory of large deviations is that

$$\mathrm{Prob}_\nu(\omega_\Lambda \text{ is “typical” for } \mu) \;\sim\; \mathrm{e}^{-|\Lambda|\, i(\mu|\nu)}\,. \tag{10}$$

In passing to the densities (9) one loses the distance-like property of being nonzero if μ and ν are different, as the following theorem shows.

Theorem 3 *Assume ν is a translation-invariant Gibbs measure. Then*

$$i(\mu|\nu) = 0 \quad\Longleftrightarrow\quad \mu(\,\cdot\,|\,\sigma_{\Lambda^c}) \;=\; \nu(\,\cdot\,|\,\sigma_{\Lambda^c}) \tag{11}$$

for all regions Λ and all configurations σ.

[In the right-hand side of (11), “=” actually means “can be chosen equal to”.]

In words, μ has zero density of information-gain relative to a Gibbsian ν (both measures being translation invariant) if and only if μ is also Gibbsian for the *same* temperature and (class of physically equivalent) interaction. Physically, this corresponds to the fact that an untypical “island” should have a probabilistic cost of the order of its boundary if it involves configurations typical of a different Gibbs state for the *same* interaction (think of an island of “$-$” in the “$+$”-state of the Ising model at zero field and low temperature), while otherwise its cost is of the order of the volume of the island (eg. an island of “$-$” for the Ising model with positive field).

Theorem 3 has played an important role in the detection of non-Gibbsianness. It has been applied in the following two complementary ways.

1. Suppose μ is a well-known *non*-Gibbsian measure —for instance a *frozen-state*, that is, a (Dirac) delta concentrated in a single configuration. Then every measure ν with $i(\mu|\nu) = 0$ is non-Gibbsian. In view of (10) one can say that some large deviations probabilities of ν are “too-large” for ν to be Gibbsian. Measures of this type were obtained as a result of spin contractions [32, 11] and as invariant measures of stochastic transformations [34, 48].

2. Suppose μ and ν are such that *if* ν admits quasilocal conditional probabilities, then (i) those are also conditional probabilities for μ —$\mu(\,\cdot\,|\,\sigma_{\Lambda^c}) \;=\; \nu(\,\cdot\,|\,\sigma_{\Lambda^c})$ for all σ—, and (ii) $i(\mu|\nu) > 0$. Then neither

μ nor ν are Gibbsian. Given (10) one can say that these measures have large deviations probabilities that are "too small" for Gibbsianness. This situation has been found in measures associated to lattice projections [55].

A third way to apply Theorem 3 is contained in the following Corollary.

Corollary 1 (Dichotomy corollary) *If two translation-invariant measures μ and ν are such that $i(\nu|\mu) = i(\mu|\nu) = 0$, then either (1) both are Gibbsian and yield the same finite-volume Boltzmann-Gibbs averages, or (2) both are non-Gibbsian.*

This dichotomy corollary has been used to prove that cell (or local) renormalization transformations at the level of interactions are never many-valued [67, Section 3.2].

In view of Theorem 3, it is tempting to use the density of information-gain to estimate somehow the "distance to Gibbsianness" of a measure. For some remarks in this direction, see [67, Section 5.1.2].

3 Transformations of measures

We shall consider transformations sending measures on a configuration space $\Omega = \Omega_0^{\mathbb{Z}^d}$ —the space of *original* or *object* configurations— to measures on a target space $\Omega' = (\Omega_0')^{\mathbb{Z}^{d'}}$ —the space of *image* or *block* configurations. The latter can coincide with the former. Two types of questions are usually posed regarding the action of these transformations:

1. What happens after a *single* application of the transformation to a Gibbsian measure. This is the point of view of renormalization transformations. In general the transformed measure has a coarser σ-algebra (fewer observables) and the transformation is interpreted as some sort of "noise" or "blur-out" of the original measure. In renormalization-group transformations this noise is introduced on purpose, to extract only the "blurred-out" information characterizing critical points. In image processing or sound recognition the noise is an unwanted feature and the objective is to reconstruct the information contained in the original measure. In both cases it is important to determine whether the transformed measure is Gibbsian. The Gibbsianness hypothesis is

built into renormalization-group theory, and it is the basis of important sampling and restoration procedures.

2. What happens with the invariant measures of the transformation. This is the issue of interest in the study of cellular automata or stochastic dynamics, where one investigates the result of infinitely many iterations of a transformation starting from an arbitrary initial state. These are models of systems out of equilibrium, hence there is no reason to expect Gibbsianness of their stationary measures. Nevertheless Gibbsiannes has been proven in certain regimes, and at any rate, if this is not the case, it is meaningful to wonder which properties of Gibbsianness are still present.

3.1 Deterministic transformations

In a *deterministic* transformation, the image configuration is fully determined by the original one. It is defined by a map

$$\begin{aligned} t : \Omega &\rightarrow \Omega' \\ \omega &\mapsto \omega' = t(\omega) \;, \end{aligned} \tag{12}$$

which in turns defines a map that to a measure μ on Ω associates a measure μ' on Ω' with averages

$$\int_{\Omega'} f'(\omega')\, \mu'(d\omega') \;=\; \int_{\Omega} f'(t(\omega))\, \mu(d\omega) \,. \tag{13}$$

Let us call the transformation *local*, or a *block-* or *cell-transformation*, if there exists a *finite* set $B_{0'} \subset\subset \mathbb{Z}^{d'}$ and some number b such that the sets $B_{x'} := B_{0'} + bx'$ —the *blocks* or *cells*— satisfy: (i) Their union is all of $\mathbb{Z}^{d'}$, and (ii) each $\omega'_{x'}$ depends only of the original spins in the corresponding block, that is, of $\omega_{B_{x'}}$.

Conspicuous examples of this type of transformations are:

1) Projection transformations. The transformation t is just the restriction to a subset S of $\mathbb{Z}^d$. The transformed measure applies only to functions depending on spins in this subset S and averages out all the other spins. The following two cases have been studied in some detail:

1.1) Decimation of spacing b. The subset S is formed by lattice points all whose coordinates are multiples of b. This subset is, in fact, isomorphic to the original $\mathbb{Z}^d$, hence $\Omega' = \Omega$ and $\omega_{x'} = \omega_{bx'}$.

1.2) Projection on a hyperplane [55]. The subset S is (isomorphic to) $\mathbb{Z}^{d-1}$ and it is identified with the hyperplane $\{(x_1, x_2, \ldots, x_{d-1}, 0) \in \mathbb{Z}^d\}$. Formally, $\Omega' = \Omega_0^{\mathbb{Z}^{d-1}}$ and $\omega'_{x'} = \omega_{(x',0)}$. This is *not* a cell transformation.

2) Block-average transformations. These are cell transformations defined by

$$\omega'_{x'} \;=\; \frac{1}{|B_{x'}|} \sum_{y \in B_{x'}} \omega_y \;, \tag{14}$$

where we are assuming that Ω_0 is formed by consecutive integers. This is an example where $\Omega'_0 \neq \Omega_0$.

3) Majority-rule transformation. For Ising spins $\Omega_0 = \{-1, 1\}$, let

$$\omega'_{x'} \;=\; \mathrm{sign}\Big(\sum_{y \in B_{x'}} \omega_y \Big) \;. \tag{15}$$

This is a local transformation. If the block-size is even, a rule is needed to decide ties. Often this rule is stochastic ($+1$ or -1 with equal probability). These would be the simplest example of stochastic transformation (see below).

4) Spin contractions. These are single-site transformations ($B_{x'} = \{x'\}$), where $d' = d$ but Ω'_0 is strictly a subset of Ω_0. I mention two well studied cases:

4.1) Sign fields. In these examples Ω_0 is a symmetric subset of the real numbers while $\Omega'_0 = \{-1, 1\}$ or $\Omega'_0 = \{-1, 0, 1\}$. The original and image lattices coincide, $d = d'$. The map is

$$\omega'_{x'} \;=\; \mathrm{sign}\, \omega_{x'} \;. \tag{16}$$

Two particular cases are:

(i) The sign-field of (an)harmonic crystals. This corresponds to $\Omega_0 = \mathbb{R}$. This field was studied in [32] in relation with the phenomenon of entropic repulsion, and in [11] in reference to the renormalization-group theory of the Ising model in dimensions larger than four.

(ii) The sign-field of the SOS model [36, 70]. Here $\Omega_0 = \mathbb{Z}$.

[Note that in both cases the original model has an infinite single-spin space and hence it exceeds, rigorously speaking, the framework adopted here.]

4.2) Fuzzy Potts model[43]. The original spins, with values in $\Omega_0 = \{1, 2, \ldots, q\}$, are contracted into a smaller number n of values, where n divides p: $\omega'_{x'}$ takes the value i if $(i-1)q/n \leq \omega_{x'} \leq iq/n$.

5) Momentum transformations. They are "almost-local" transformations. The image spins depend of all the initial spins, but this dependence tends to zero for far-away spins. More precisely, the transformation is defined by a law

$$\omega'_{x'} \;=\; \sum_y F(bx' - y)\, \omega_y \;, \tag{17}$$

where F is the Fourier transform of a smooth function $\widehat{F}(k)$ (representing a "soft cutoff").

Decimation, average, majority-rule and momentum transformations have been intensively used in the renormalization-group analysis of various systems. For references see [67, Section 3.1.2] or [20, Section 1].

3.2 Stochastic transformations

For these transformations, the procedure to obtain the image spins involves some randomness. Formally (let me consider only the case of cell transformations with "parallel updating"), there is a collection of weights $\{T(\omega'_{x'}|\omega_{B_{x'}})\}$ such that

$$\sum_{\omega_{B_{x'}}} T(\omega'_{x'}|\omega_{B_{x'}}) \;=\; 1 \;. \tag{18}$$

These weights describe the probability of obtaining a spin $\omega'_{x'}$ from the original configuration $\omega_{B_{x'}}$ of the block $B_{x'}$. Correspondingly, the transformed μ' of a measure μ is the measure that for each function f' depending on finitely many image spins yields an average

$$\int_{\Omega'} f'(\omega')\, \mu'(d\omega') \;=\; \int_{\Omega} f'(\omega') \prod_{x'} T(\omega'_{x'}|\omega_{B_{x'}})\, \mu(d\omega) \;. \tag{19}$$

[Of course, this transformations include the deterministic transformations defined above as a particular case.]

As examples I mention:

1) Kadanoff transformations. Defined, for Ising spins, $\Omega_0 = \{-1, +1\}$, by the weights

$$T(\omega'_{x'}|\omega_{B_{x'}}) \;=\; \frac{\exp\Big(p\, \omega'_{x'} \sum_{y \in B_{x'}} \omega_y\Big)}{\text{Norm.}} \;, \tag{20}$$

where $p > 0$ is a parameter. These transformations have been used to study the critical properties of the Ising model. They admit several generalizations and interesting limit cases, see [67, Section 3.1.2].

2) Stochastic smooth sign-fields. Used in [29] to study continuous-spin systems in the presence of a random field. These are single-site spin contractions with $\Omega_0 = \mathbb{R}$ and $\Omega_0' = \{-1, 1\}$, defined by the probabilities

$$T(\omega'_{x'}|\omega_{x'}) \;=\; \frac{1}{2}\Big(1 + \omega'_{x'} \tanh(a\, \omega_{x'})\Big) \;, \tag{21}$$

parametrized by the constant a. When $a \to \infty$ these transformations tend to the deterministic sign-field transformation defined by (16).

3) Transformations defining stochastic cellular automata. In this case the blocks $B_{x'}$ are usually overlapping, the image space is identical to the original one and it is interpreted as the latter at a later "time", and the numbers $T(\omega'|\omega)$ are thought as transition probabilities. A large number of such automata has been proposed and studied. In particular, I shall refer below to work done for the voter model [34], a Swendsen-Wang-like dynamics studied in [48] and a numerical study of the Toom model [45].

4 Non-Gibbsian measures: The symptoms

4.1 The "peculiarities"

Griffiths and Pearce [19, 20, 18] were the first to point out problems with the assumption of Gibbsianness of measures subjected to renormalization transformations. While their arguments were not fully rigorous, many of their ideas and observations have been later put on a rigorous footing. As an illustration let us consider their discussion of what they call "model I". Take the Kadanoff transformation with blocks of size one (i.e. where the original and image spin coincide), for the (nearest-neighbor ferromagnetic) Ising model. If the map (19)–(20) is applied to a finite region Λ, the distribution of *original* spins ω when the image spins ω' are all set equal to -1 corresponds to an Ising model with field $h - p$. Consider now the energy cost of flipping ω'_0 from -1 to $+1$:

$$\exp W'_{\{0\}}(+1|-1) \;\;:=\;\; \frac{\mu'_\Lambda(+'_0\, -'_{\Lambda\setminus\{0\}}\, |{-'_{\Lambda^c}})}{\mu'_\Lambda(-'_0\, -'_{\Lambda\setminus\{0\}}\, |{-'_{\Lambda^c}})}$$

$$
\begin{aligned}
&= \langle \mathrm{e}^{2p\sigma_0} \rangle_\Lambda^{h-p} \\
&= \cosh 2p + \langle \sigma_0 \rangle_\Lambda^{h-p} \sinh 2p \;, \qquad (22)
\end{aligned}
$$

where $\langle \cdot \rangle_\Lambda^{h-p}$ stands for the Ising Boltzmann-Gibbs factor for the region Λ with field $h-p$. At low temperature, the right-hand side has a multivalued thermodynamic limit for $h = p$. Griffiths and Pearce conclude that this indicates that there is not a well defined interaction behind the measure μ'.

While this does not constitute a mathematically complete non-Gibbsianness argument, it already shows that the "peculiarities" —as Griffiths and Pearce call them— are due to the existence of phase transitions of the system of *original* spins *constrained* by well-chosen image-spin configurations (they call this a *modified object system*). Therefore they need not happen at transition regions of the original system.

Griffiths and Pearce proposed a second scenario for these "peculiarities", in which the renormalized interaction would be well-defined, but would not be a smooth function of the parameters of the original model. Soon after a number of numerical studies appeared, suggesting the presence of multivaluedness and discontinuities in the transformations at the level of Hamiltonians (see references in [67, Section 1.1]. Within the framework of standard Gibbs theory, this scenario was, however, ruled out by later studies [67, Section 3]. Nevertheless, the multivaluedness can occur if the transformations are non-cell, for instance if they include projections to lower-dimensional manifolds [39], or if one relaxes the theoretical framework by allowing weakly Gibbsian measures [9, 10] (Section 6.4 below).

4.2 Entropic repulsion and contracted Gaussians

Almost ten years later than Griffiths and Pearce, Lebowitz and Maes [32] produced an example of a different nature. They considered harmonic crystals, that is, systems with spins $\varphi_x \in \mathbb{R}$ and with formal Hamiltonian of the form

$$
H(\varphi) \;=\; \sum_{\langle x\, y\rangle} V_{xy}(\varphi_x - \varphi_y) \qquad (23)
$$

where the functions V_{xy} are even and convex. They showed that, due to the shift-symmetry $\varphi_x \to \varphi_x + k$ of the system, the probabilistic cost of shifting the spins within a region is subexponential in the volume of this region (is like inserting a bubble configured in a different Gibbs state). Physically, these systems can be used to model the height of an interface. The result implies that any linear perturbation of the interaction ("soft wall") sends the interface to infinity (entropic repulsion). Mathematically, the measure obtained by taking the sign of the spins has zero relative entropy with respect to the delta-measure concentrated in the all-"+" configuration. The resulting measure is therefore non-Gibbsian by Theorem (3) ("too large" large deviations).

The same phenomenon was generalized by Dorlas and van Enter to the sign field of self-similar Gaussians [11], and later to anharmonic crystals (V_{xy} not-necessarily quadratic) [67, Section 4.4]. Assuming that, as believed, block-average transformations of the critical Ising model in $d \geq 5$ converge to a Gaussian fixed point, the results of [11] imply that, after a sequence of majority-spin transformations with larger and larger block-size, the critical Ising-model measure converges to a non-Gibbsian distribution.

Lebowitz and Schonmann showed that the extremal invariant measures of the voter model ($d \geq 3$) are non-Gibbsian because they have also too-large probabilities of having bubbles of spins frozen in the all-"+" configuration [34, formula (3.8)].

4.3 Projections on hyperplanes

Schonmann [55] provided the first example of non-Gibbsiannes manifested via "too-small" large deviations. He considered the projection of the two-dimensional Ising model onto the x-axis and showed that changing the spins far away along this axis one can pass from the projection of the "+"-measure to the projection of the "$-$"-measure. Therefore, if the projection of the "+"-state has quasilocal conditional probabilities, these must be also conditional probabilities for the projection of the "$-$"-state. But, on the other hand, there are large-deviation results showing that there can be at most one Gibbs translation-invariant projected state. Hence neither projection is Gibbsian. The result is valid all the way up to the critical temperature of the two-dimensional Ising model. The example was later generalized and studied in more detail in [12]. In particular it was shown there that projections of a d-dimensional Ising model, $d \geq 2$, onto a coordinate hyperplane are non-Gibbsian for temperatures smaller than the critical temperature of the initial d-dimensional model.

5 Non-Gibbsian measures: The diagnosis

5.1 Non-quasilocality for cell-renormalization transformations

Israel [25] provided a (practically) rigorous argument that proved the existence, and clarified the nature, of Griffiths' and Pearce's "peculiarities" for the decimation of the two-dimensional Ising model at low temperatures. He showed how a phase transition in the constrained system of original spins causes the lack of quasilocality of one-point conditional probabilities. By Theorem 1 this implies non-Gibbsianness. This confirmed Griffiths' and Pearce's first scenario —lack of summable renormalized interaction.

The essence of Israel's argument is rather simple. Consider decimation of 2×2-blocks and fix the image (=nondecimated) spins in the alternating configuration $\omega'_x = (-1)^{|x|}$. These constrained spins act as additional magnetic fields over the remaining original spins, but these fields have alternating signs and cancel out. Therefore, the constrained system is a decorated Ising model (Ising model with additional sites at the middle of each bond) which is equivalent to a standard Ising model at a higher temperature. The model has, thus, a phase transition at low temperatures and it is not hard to see that one can select one or the other phase by choosing the image spins all "+" or all "$-$" in a ring of unit thickness and diverging radius. This, in turns, changes the magnetization at the origin: At low-enough temperature there exists a constant $\varepsilon > 0$ such that for all square sets Γ sufficiently large and all image configurations η' and ξ':

$$\Big| \langle \omega'_0 \,|\, \sigma'^{\pm}_{\Gamma} \; +'_{\partial\Gamma} \; \eta'_{(\Gamma\cup\partial\Gamma)^c} \rangle - \langle \omega'_0 \,|\, \sigma'^{\pm}_{\Gamma} \; -'_{\partial\Gamma} \; \eta'_{(\Gamma\cup\partial\Gamma)^c} \rangle \Big| \; > \; \varepsilon \; , \qquad (24)$$

where $\sigma'^{(\pm)}$ denotes the alternating configuration. An important technical point: The fact that the inequality holds uniformly in the configurations η' and ξ' implies that the jump involves two sets of configurations that are open in the product topology, and hence of non-zero measure. It follows that the discontinuity at $\sigma'^{(\pm)}$ is essential.

The main ingredients of this argument were abstracted and exploited in [67]. The proof of the violation of quasilocality requires:

I To exhibit a special image configuration $\sigma'_{\rm spec}$ such that the resulting *constrained* system of *original* spins has more than one phase.

II To show that two of these phases can be selected by fixing the *image* spins arbitrarily far away in a suitable manner.

III Furthermore, the selection of these phases must be made via *open* sets of image configurations, so the lack of quasilocality becomes essential.

Once this was understood, it was relatively straightforward to obtain a large catalogue of transformations for the Ising model in dimensions $d \geq 2$ leading to non-Gibbsianness: decimation with arbitrary spacing, Kadanoff transformations for arbitrary block size and value of p, block-averaging for even block sizes, and some cases of majority rule ($d = 2$) [67, Section 4]. One must discover special configurations $\sigma'_{\rm spec}$ such that the constrained system exhibits a phase transition that can be treated rigorously, for instance via Pirogov-Sinai theory as explained in [67, Appendix B].

All these examples are at temperatures strictly below the Ising critical temperature. Some of them, though, involve non-zero magnetic fields, required to be low-enough for decimation and Kadanoff transformations in $d \geq 3$ but that could have arbitrary values for block-averaging. Soon other examples showed that any region of the phase diagram could be hit by the phenomenon. For instance, decimation for the high-q Potts model leads to non-Gibbsianness for an interval of temperatures higher than the critical [62]. Furthermore, for each fixed temperature there is a (perversely designed) transformation leading to non-Gibbsianness [59]. Griffiths' and Pearce's suspicions that "peculiarities might be a fairly general phenomenon" [18, page 64], were fully confirmed.

5.2 Non-quasilocality of projections to hyperplanes

The original argument [55] proving the non-Gibbsianness of the projection of the $2d$-Ising to the line is somehow delicate. Its first part, proving that if quasilocality were present then both the "+" and "−" projections would have the same conditional probabilities, resorts to percolation results that are specifically two dimensional. This casted some doubts on whether the example could be generalized to higher dimension, and, if this generalization were possible, on which would be the limit temperature for the existence of non-Gibbsianness. A natural candidate is the critical temperature of the initial Ising model, but the use of percolation arguments pointed towards the Peierls temperature, that is the temperature above which there is percolation of minority spins (in two dimensions the Peierls and critical temperatures coincide). The second part of the argument is, in my opinion, even more subtle. It states that *if* one of the projections were Gibbsian, then the relative density of information-gain between both projections *would* exist and be positive. As the projections turned out to be non-Gibbsian, the actual existence of this density of information-gain was not proven. In fact it remains unproven to date. The argument consists, thus, in exposing a *potential* smallness of large-deviation probabilities.

Alternative arguments show that in fact the non-Gibbsianness is due to lack of quasilocality and that it happens for any dimension $d \geq 2$. This was first proven in [67, Section 4.5.2], via a Peierls argument, for an alternating special configuration. In [67, 42, 12] this lack of quasilocality was related to the existence of a *wetting* phenomenon.

If the spins in a coordinate hyperplane are all "+", the state of the system in a half space is unique and independent of the external conditions used for the other hyperplanes. The all-"+" configuration causes the formation of a droplet of the corresponding state whose thickness diverges in the thermodynamic limit. This corresponds to a situation of *complete wetting*. Similarly, the all-"−" configuration produces complete wetting. On the other hand, there are configurations that lead to *partial wetting*: the width of the associated layer remains finite and the bulk phase is decided by the boundary conditions. It is at these configurations that the quasilocality of the projections is lost. When one of these configurations is surrounded by an arbitrary far layer of "+" spins, complete wetting leads to local averages that are different from those obtained for a layer of "−".

Furthermore, an inequality presented in [12] shows that the quasilocality of the projections fails whenever the surface tension of the wetting droplet is positive. This is known to happen for all dimensions $d \geq 2$ and for all temperatures lower than the critical [15]. Hence, the non-Gibbsianness of the projections happens up to this temperature, rather than the Peierls temperature.

5.3 Non-Gibbsianness of invariant measures

The work in [48, 42] brings additional insight into the non-Gibbsianness of stationary states for cellular automata. The first reference studies a non-local dynamics for a lattice gas (i.e. $\Omega_0 = \{0, 1\}$). For its invariant measure, the probability that a region be all filled with particles decreases only subexponentially in the volume of the region. The measure has, therefore, "too-large"

large deviations with respect to the delta-measure concentrated on the "all-occupied" configuration. Furthermore, the study exhibits the *mechanism* behind this fact: Once a ring of particles has been established, the dynamics will proceed to fill the interior of the ring with particles. The probabilistic cost of establishing an occupied region is, therefore, dictated by the formation of the boundary of the region.

Perhaps the most important result for invariant measures is a *dichotomy theorem* analogous to Corollary 1. The theorem requires two properties from the stochastic transformations:

Theorem 4 (Dichotomy theorem) *Assume that the transformation satisfies (i) the transition probabilities $T(\omega'_x|\omega_{B_{x'}})$ are all strictly positive, and (ii) there exists $R > 0$ such that the $\cup_{x' \in [-L,L]^d} B_{x'} \subset [-L-R, L+R]^d$ for all $L > 0$. Then the translation-invariant measures that are invariant for the transformation are either all Gibbsian or none Gibbsian.*

The theorem remains valid for spin-flip processes with positive (and local) rates. For the latter, the theorem was first obtained by Künsch [30]; the form stated here was proven in [42]. Most renormalization transformations fail to satisfy the second assumption. Indeed if blocks do not overlap the image spins within a square of size L come from internal spins in a square of size bL.

An immediate consequence of this theorem is that all invariant measures are Gibbsian for transformations satisfying detailed balance with respect to Bolzmann-Gibbs weights. For non-reversible probabilistic cellular automata, like the Toom model [58], the situation is less clear. From Theorem 3 and the previous dichotomy theorem we see that the appearance of two invariant measures with strictly positive relative density of information-gain would automatically imply the non-Gibbsianness of all the invariant measures. In [42] this observation is transcribed into the following heuristic test, related to the mechanism described in [48]: Take a typical configuration of one of the invariant measures, introduce a boundary typical of the other measure and observe whether the dynamics tends to fill the interior of the region with the phase dictated by the boundary. If not, this would be an indication that the probabilistic cost of a region of such "mistakes" grows exponentially with the volume, and hence that the relative information-gain is not zero. Both (and all) invariant measures would then be non-Gibbsian. This test has been recently numerically performed for the Toom model [45]. The results are not totally conclusive, but they give some evidence that the information-gain density between the plus and minus invariant Toom measures is zero, in agreement with the nonrigorous but plausible argument presented in [71].

5.4 Other instances of non-quasilocality

5.4.1 The random-cluster model

The random-cluster model, introduced by Fortuin and Kasteleyn [14] (see also the historical references listed in [22]), is a correlated bond-percolation model. To each bond configuration $\mathbf{n}$ the model assigns a (finite-volume) probability weight

$$p^{\mathcal{N}_1(\mathbf{n})}\,(1-p)^{\mathcal{N}_0(\mathbf{n})}\,q^{\mathcal{C}(\mathbf{n})}/\mathrm{Norm}. \tag{25}$$

where $\mathcal{N}_1(\mathbf{n})$ is the number of open bonds, $\mathcal{N}_0(\mathbf{n})$ the number of closed bonds, and $\mathcal{C}(\mathbf{n})$ the number of connected clusters. For $q=1$ one recovers independent percolation, while for $q \geq 2$ the model is related by identities to (is a "representation" of) the q-state Potts model. In the latter case, p is a function of the inverse temperature.

¿From (25) one can define, in the obvious way, conditional probabilities for various boundary conditions. As the q-dependence in (25) is highly non-local, it is not difficult to see that for $q \neq 1$ these conditional probabilities are not quasilocal [1, 67] for any $0<p<1$. In fact the analysis in [51, 22] reveals that the lack of quasilocality happens exactly at those bond configurations exhibiting more than one infinite cluster. Indeed, if one considers a finite region and chooses one such configuration as boundary condition, one can produce a change in the number of connected sets within the region —and hence in the probability distribution—, by changing the external configuration arbitrarily far away so to join two of the infinite clusters. In contrast, if the boundary configuration has a single or no infinite cluster, for any two sites there is a finite volume such that the connection between them does not depend on what happens outside such a volume.

5.4.2 Momentum transformations

A momentum transformation is defined by a cutoff function $\widehat{F}(k)$, defined on $[-\pi,\pi]^d$ which is zero when the norm of k exceeds a certain threshold k_0. The function is used to modulate the Fourier transform of finite-volume configurations, $\widehat{\omega}_k := \sum_{x\in\Lambda}\omega_x \mathrm{e}^{-ikx}$, by imposing $\widehat{\omega}'_{k'} := \widehat{f}(k'k_0)\,\widehat{\omega}_{k'k_0/\pi}$. Transforming

Fourier, this relation yields (17) in the thermodynamic limit with $b = \pi/k_0$. The cutoff $\widehat{F}$ must be "soft", i.e. sufficient smooth, to guarantee summability of F. In this case the image spins $\omega'_{x'}$ are bounded. Furthermore, it is expected that the dependence of F on far-away spins decay fast with the distance, so that there is little difference with a (real-space) cell-transformation of blocks of size b (see, e.g. [50, Section 4.2]). It is, therefore, reasonable to expect no essential difference between momentum and cell transformations in relation to non-quasilocality. Indeed, an example of a non-quasilocal momentum-renormalized measure has already been constructed [61], where $\widehat{F}(k)$ is the identity except in one direction where it goes to zero at k_0 as $\cos^2[(k\pi/(2k_0)]$.

6 Non-Gibbsian measures: The treatment

The in-depth study of the properties of non-Gibbsian measures is only at the beginning. I review here some important attempts, which can be grouped in two categories:

1. *Classification schemes.* Three schemes have been devised to gauge how far a measure is from being Gibbsian. They give rise, respectively, to the notions of *robust non-Gibbsian*, *almost quasilocal* and *weakly Gibbsian* measures. [This nomenclature is still not fully established.]

2. *Generalized Gibbsian theory.* Efforts have been made to extend parts of the standard theory of Gibbs measures into the non-Gibbsian realm. There are some partial results regarding the existence of the relative density of information-gain, the validity of a variational principle, and the generalization of the characterization theorem (Theorem 1).

6.1 Gibbsianness preservation

For completeness let me start with an account of transformations known to preserve Gibbsianness. Regarding renormalization transformations there are by now classical results showing Gibbsianness for decimation and Kadanoff transformations of the high-field or high-temperature lattice-gas and Ising-spin systems with uniformly absolutely summable interactions [20, 25, 26, 5], and for the averaging transformation for high-temperature summable Ising systems [26].

More recently, for the Ising model at nonzero field it was shown [46] that at low enough temperatures there exists a spacing $b \sim 1/h$ such that decimation with block of this size or larger preserves Gibbsianness. The interpretation is the following: The non-decimated spins act as boundary conditions for the remaining spins, hence, in order not to trigger a phase transition the former must not compete with "bulk" effects. This happens if the non-decimated spins are at distances larger than the size of a critical droplet of the "wrong" phase. There seems to be, therefore, a relation between non-Gibbsianness and metastability effects. In [47] similar results were obtained for cell transformations followed by decimations with large-enough block size.

In [24, 27] Gibbsianness was established for a number of transformations of the Ising model at temperatures that include a neighborhood of the critical temperature. The later case corresponds to (i) 2×2-decimation of the bidimensional model for temperatures $T > T_{\rm c}/1,36$ —almost complementing the interval $T < T_{\rm c}/1.73$ where non-Gibbsianness was asserted [25]—, and (ii) Kadanoff transformations of the model in the triangular lattice, for some intervals of p. The central tool of the method used in [46, 47, 24, 27], is a theorem showing that suitable mixing properties of the constrained systems implies Gibbsianness of the transformed measure. In [24, 27] this mixing behavior is proven using a uniqueness condition due to Dobrushin.

The sufficient mixing condition can, in fact, be subjected to numerical studies. This has been the basis of not-totally rigorous but highly suggestive analyses giving evidence for the Gibbsianness of the majority-rule and block-average transformation of the Ising model at the critical temperature (see [49, 6] and references therein).

In [38] the mixing behavior of constrained systems is controlled via the so-called *disagreement percolation*. In this way, all deterministic cell renormalization transformations of systems with $|\Omega_0| < \infty$ and nearest-neighbor interactions are proven to lead to Gibbsian measures at high-enough temperatures. The threshold temperature depends on the transformation. The particular case of decimations of Potts models was previously obtained in [35] using more abstract techniques.

Recent work has shown that if some of the harmonic crystals of Section 4.2, are subjected to a single-site double-well potential, then the sign field can become Gibbsian [29]. More precisely, this Gibbsianness has been proven for the smooth stochastic spin-fields, transformation (21), with suitably chosen parameter a, of ferromagnetic continuous spins (e.g. Φ^4-models). The result holds even in the presence of a (not necessarily uniform) magnetic field that is small in absolute value. The double-well potential breaks the shift-symmetry $\varphi_x \to \varphi_x + k$ which was behind the nonGibbsian large deviation properties of the harmonic crystals.

Schonman's projection on a coordinate axis is known now to be Gibbsian throughout the whole uniqueness region, except possibly at the critical point itself [39, 35]. The Gibbsianness at nonzero field and low temperature is not believed to hold for higher-dimensional projections to hyperplanes (Basuev phenomena) [35]. On the other hand, the projection to the axis followed by sufficiently spaced decimation preserves the Gibbsianness of both the "+" and "−" Ising measures *but* leading to *different*, nonequivalent interactions [39].

Among the measures that are invariant under stochastic dynamics and are known to be Gibbsian, are those satisfying detailed balance for spin-flip processes (see above), and the invariant measures of probabilistic cellular automata in the high-noise regime [33]. More recently, transformations involving rates that are not too sensitive to the past, in variational-distance sense, were shown to preserve Gibbsianness of measures defined by nearest-neighbor interactions and finite single-spin spaces [38].

6.2 Decimation and robustness

A measure is said *robustly non-Gibbsian* if this non-Gibbsianness persists under transformation by decimations of any (finite) block size. This notion originates in the observation [46, 47] that decimation can, in some cases, restore Gibbsianness. Two reasons can be invoked for the relevance of this notion. First, decimation does not change the partition function, and hence the free energy. Thus, measures that are not robustly non-Gibbsian may admit a useful thermodynamic description. Second, every transformed measure can be seen as the decimation of the product measure $\mu \times T$ on $\Omega \times \Omega'$ (see e.g. [67, pages 987–90]). Thus, (non) robustness can conceivably yield information over general renormalization transformations. The sign-field of harmonic crystals and of a SOS model, and the invariant measures for the voter model in $d \geq 3$ and the model in [48] provide examples of measures that are robustly non-Gibbsian in the sense that they remain non-Gibbsian after application of a Kadanoff transformation combined with one or several decimations [69, 36, 70].

6.3 Almost quasilocality

It is natural to judge the level of lack of quasilocality of a measure μ through the size of the set

$$\begin{aligned} \Omega_{\mathrm{q}}^{\mu} := \Big\{ \sigma \in \Omega : \exists \Lambda \subset\subset \mathbb{Z}^d \text{ and } \omega_\Lambda \in \Omega_\Lambda \\ \text{such that } \mu(\omega_\Lambda | \cdot\,) \text{ is essentially not quasilocal at } \sigma \Big\} . \end{aligned} \tag{26}$$

A measure μ is *almost quasilocal* if $\Omega_{\mathrm{q}}^{\mu}$ has μ-measure zero. [Rigorously speaking, almost quasilocality is a property of the conditional probabilities, rather than the measure.]

In [12] it is shown that for the Ising model in the region of uniqueness decimations and projections lead to measures that are either Gibbsian or almost-quasilocal. The status at the coexistence region is as yet an interesting open problem. Likewise, the sign-field of a SOS model at temperatures below the roughening transition defines an almost-quasilocal measure. Also the random-cluster model of Section 5.4.1 is almost quasilocal because the set of configurations with no or a single infinite cluster has full measure. Nevertheless, if one considers a tree instead of the lattice $\mathbb{Z}^d$, for certain values of the parameters p and q there is a measure for which this set of configurations has measure zero [23]. Hence one can pass from almost-quasilocality to almost-sure non-quasilocality just by changing the underlying lattice.

Another important example of almost-sure non-quasilocality is provided by (non-trivial) convex combinations of Gibbs measures for different, non-equivalent interactions [69]. In particular, this happens for the combination of the Gibbsian measures obtained in [39] through the projection to a line plus decimation of the "+" and "−" Ising measures.

6.4 Weak Gibbsianness

Griffiths noted that his and Pearce's "peculiarities" happened for configurations that were atypical for the original measure. Hence, he contended that "it is at least plausible [...] that an approximate scheme which was, so-to speak, 'unaware' of the existence of peculiarities might produce an approximation to H' which would give a reasonable estimate for the probabilities [of typical configurations]" [18, pag. 66]. One can see here hints of the idea of "weak Gibbsianness" whose main advocate was Roland Dobrushin. Inspired by the statistical mechanics of systems with unbounded spins and long-range interactions, he proposed to extend the definition of Gibbsianness by disposing of the "sup" in (2) and considering summability in restricted sets of configurations, hopefully of full measure. The constructions in [40, 41] can be considered early attempts in this direction. After Dobrushin's first (and, unfortunately, last) formal presentation at the workshop in Renkum [7], his ideas took hold and were later developed by him and Shlosman [9, 10] and by two other research groups [44, 4]. They succeeded in showing that practically all the non-quasilocal measures obtained via renormalization transformations and projections fit into this generalized framework.

Following [37], let us call a measure μ *weak Gibbsian* (the expression "partly defined Gibbs" is used in [9, 10]) if there exists a set of configurations $\Omega^{\mu}_{\mathrm{w}}$ and an interaction Φ satisfying

$$\sum_{B \ni x} |\Phi_B(\sigma)| \;<\; \infty\;, \quad \text{for all } \sigma \in \Omega^{\mu}_{\mathrm{w}}\;, \tag{27}$$

such that $\mu(\Omega^{\mu}_{\mathrm{w}}) = 1$ and μ satisfies (6) for all $\sigma \in \Omega^{\mu}_{\mathrm{w}}$, and hence (5) with the integration restricted to $\Omega^{\mu}_{\mathrm{w}}$. The set $\Omega^{\mu}_{\mathrm{w}}$ is assumed to be *measurable at infinity*, that is, if σ belongs to it, then every modification of σ in finitely many spins also belongs.

The Belgian [44] and Russian [9, 10] teams showed the weak Gibbsianness of Schonmann's projections of the "+" and "−" states of the Ising model on a coordinate axis. The first group characterized the sets $\Omega^{\mu^+}_{\mathrm{w}}$ and $\Omega^{\mu^-}_{\mathrm{w}}$ in terms of empirical magnetizations sufficiently far from zero, and used disagreement percolation to control the summability. A nice formula is presented relating finiteness of relative energy densities with the decay of correlations for the constrained system. This establish an analogy with disordered systems, where also the mixing properties are only estimated for "typical" values of the disorder. The Russian group takes a more straightforward, and technically more involved, approach: A system of projected conditional probabilities is defined in a natural way (see [12]), and a lattice-gas interaction is obtained via Moebius' inversion formula. The hard part is to analyze the summability properties of such a interaction, for that the authors use cluster-expansion methods in terms of contours. Likewise, the Belgian-Finnish group [4] defines the "good" configurations and the interaction in terms of contours. They apply renormalization techniques developed to study disordered systems, and prove the weak Gibbsianness of most cell transformations of the Ising model, including decimation, majority rule, Kadanoff for large p and, with some adaptations, block-averaging.

6.5 Relations between classification schemes

It is still too early to determine the relative interest of the different schemes. Robustness could be relevant in relation to thermodynamic descriptions, weak Gibbsianness is an appealing notion from the physical point of view, and almost-quasilocality is a natural property from the probabilistic point of view.

There does not seem to exist an obvious relation between robustness and the other two classification schemes. For instance, the sign-field of the SOS model is robustly non-Gibbsian but at the same time almost-quasilocal [36, 70]. The opposite can also occur, as illustrated by van den Berg's "avalanche" example [44]: $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$, and μ_p the transformed of a Bernoulli measure with density $p < 1/2$ via the map $\omega'_x = \omega_x \, \omega_{x+1}$. The non-Gibbsianness of μ_p is not at all robust —a decimation of alternating spins makes it a product measure— but the measure is completely non-quasilocal [44] —the conditional probabilities are discontinuous at *all* external configurations— and furthermore it is not weakly Gibbsian [53].

The relation between almost- and weak Gibbsianness has been nicely clarified in [53]: Every almost-quasilocal measure is weakly Gibbsian, and the converse is not true. It would be very interesting to determine, for instance by incorporating ideas of [12], which of the weak Gibbsian measures analyzed in [9, 44, 4, 10] are in fact almost-quasilocal.

6.6 Generalized Gibbsian theory

For practical purposes, qualifiers like "*weak*-Gibbs" or "*almost* quasilocal" are not very informative unless there is some knowledge of which features of Gibbs measures extend to the more general categories. Little has been done in this regard.

The issue had an early start in [34], where FKG inequalities are used to construct level-1 large deviation principles and to define the pressure for non-null invariant measures of attractive interacting particle systems. These measures need not be Gibbsian (e.g. invariant measures of the voter model).

Other known results related to the "thermodynamic" description are:

1. If μ and ν are such that $i(\mu|\nu)$ exists, then so does $i(\mu'|\nu')$ where μ' and ν' are the respective images under cell renormalization transformations [69, and references therein]. Also, the existence of the density

of information-gain has been established for random-cluster measures [56].

2. Cell renormalizations of Gibbsian measures satisfy, under some mild conditions, the usual identity between pressure, energy and entropy densities [37]. Nevertheless, no characterization of these measures in terms of a variational principle has been proven.

Another result pertaining to Gibbsian properties is the "generalized Kozlov theorem" mentioned above: Every almost-quasilocal measure is weakly Gibbsian [53].

7 Conclusions

Non-Gibbsian statistical mechanical measures have come a long way in the last decade. A lot has been learned regarding its occurrence and detection, and there is even a promisory theoretical framework —weak Gibbsianness— that comes close to cover all the cases of interest. Nevertheless, these have been mostly advances at the theoretical level and it is still not clear what is the actual relevance of the phenomenon beyond mathematical finesse. Are these non-Gibbsian measures a genuinely new breed of measures, requiring new techniques and intuitions, or are their differences with Gibbsian measures almost imperceptible in practice?

Fortunately, recent publications hint a turning point in the research on non-Gibbsianness. More than contributing to the flow of "witty examples", as put in [10], recent papers point in the direction of a true theory of non-Gibbsianness and of the boundaries of Gibbsian intuition. The advances are limited, and the enterprise may not be so easy, but it is worthwhile. I end this overview commenting on directions for future work.

7.1 How Gibbsian are the generalized Gibbsian measures

It is very encouraging to discover that many non-Gibbsian measures admit, in fact, some sort of Boltzmann-Gibbs description. But this does not prevent their theory from having drastic differences with the theory of Gibbs measures. It is necessary to clarify these differences and see to which extent they force a retooling of the intuition and existing mathematical and numerical approaches. Let me mention some points in this regard (I am using as a reference the list of properties given in [37, Remark 2.5]).

1. Without quasilocality it is not necessarily true that limits of finite-volume Boltzmann-Gibbs distributions lead to infinite-volume measures satisfying the (generalized) DLR equations. This rules out the equivalence between the DLR-approach and the more physical approach based on limits of correlations (see, for instance, the comment below Proposition 2.23 in [67]). Which is, then, the right approach? In particular the theorem of existence of at least one infinite-volume measure breaks down.

2. Without quasilocality, it is not clear how to select a canonical realization of the conditional probabilities (c.f. Theorem 2). Moreover, if configurations outside the support of each measure are ignored, one can put together any two mutually singular measures —like Gibbs measures for different temperatures or different values of the magnetic field— to assemble a single "weakly Gibbsian" system. This possibility is physically unnatural and must be limited in some way. A related problem is to establish a theory of uniqueness for weak Gibbsian, or almost quasilocal, measures. What is needed, in fact, is a carefully designed notion of "physical equivalence" for almost-everywhere defined interactions.

3. Can one define a simplex-like structure for weak-Gibbsian measures? If not, what is the definitions of "macrostate"(=extremal state) and the associated definition of phase diagram?

4. Is there a variational principle and a large-deviation theory for generalized Gibbs measures? Less ambitiously, when does the relative density of information-gain exist for non-Gibbsian measures? If this object is purely probabilistic, then it should not make much difference on whether measures are quasilocal of almost quasilocal. Is it true that if ν is translation invariant and almost quasilocal the density $i(\mu|\nu)$ exists for all translation-invariant μ?

7.2 The unfinished homework: Concrete manifestations

Perhaps the less developed aspect of the theory of non-Gibbsianness is the study of its concrete, e.g. numerical, manifestations. Does there exist an experimental (in particular, numerical) situation where the differences between Gibbsianness and non-Gibbsianness become noticeable? There have been a couple of brave isolated attempts [54, 45] and some as yet unexploited arguments [67, Section 5.1], but no definite conclusive evidence. The topic is probably delicate, because non-Gibbsianness involves very rare events. A related question would be: Suppose one applies some sampling procedure designed for Gibbsian measures to a non-Gibbsian one. Would there be any observable consequence?

There is also a debt with the renormalization-group practitioners. I think we owe them answers to the following natural questions:

1. We know now that their assumption of (strong) Gibbsianness may be false. Does this invalidate some of the calculations? The answer seems to be "hardly". Is this really so? Weak Gibbsianness does not seem to help much in this regard, because it is not clear how to relate the contour-based interactions obtained for instance in [4], with the small number of coupling constants followed in usual renormalization-group calculations.

2. Several numerical groups have reported apparent discontinuities in the transformations in presence of first-order phase transitions [67, Section 1.1]. Is this an acceptable scenario in the framework of weak Gibbsianness or almost quasilocality? For the Schonmann projection, it seems to be so [9, 10].

3. Can our knowledge of non-Gibbsianness help us to design more efficient renormalization transformations? History tells us that these transformations work better when defined on geometrical, nonlocal objects, like contours or polymers [67, Section 6.1.3]. Let me mention also Griffiths' and Pearce's call for taking more seriously successful "approximate" transformations (like Kadanoff's bond shifting) that avoid "peculiarities" by slightly altering the final probability distribution: "...the peculiarities may arise from taking [the transformed Hamiltonian] too "literally," and [...] a modified H' which closely reproduces

the probabilities of the more likely configurations and changes those of the less likely configurations, and thus has the "right physics," could be produced by an approximate transformation lacking the pathologies discussed above." [19, page 919]. The formalization of this idea could prove to be worthwhile.

7.3 A test case: Chains

Many of the issues discussed above may play a role in simpler probabilistic systems. In particular, I believe they could be of interest in the study of chains. A chain is a stationary stochastic process labelled by the integers, $X = (X_n)_{n\in\mathbb{Z}}$ with a finite alphabet, $X_n \in A$ with $|A| < \infty$. It is defined by a transition matrix $P(a|x)$, $a \in A$, $x = (x_j)_{j\leq -1}, x_j \in A$, determining the probability of obtaining $X_0 = a$ when the past is given by x. Quite a lot is known for the so-called *chains with complete connections*. These are chains whose transitions satisfy the continuity condition

$$\sup_{\substack{a\,,x,y\,:\,x_j=y_j\,,\\ -1\geq j\geq -n}} \left|\frac{P(a|x)}{P(a|y)} - 1\right| \;\leq\; \gamma_n \;, \tag{28}$$

where $\lim_{n\to\infty}\gamma_n = 0$. Because of personal involvement I mention two results that apply when $\sum_n \gamma_n < \infty$:

1. The chains can be approximated in the Ornstein $\overline{d}$-topology by the k-step canonical Markov approximation at a rate γ_k (see [2] for definitions, references and results).

2. The chains relax, from an arbitrary initial configuration $(x_j)_{j\leq -1}$ to the invariant measure, at a rate γ_n^* which is exponential (respectively power law) if the rate γ_n is [3].

To understand better the consequences of passing from quasilocality to almost-quasilocality it could be interesting to study what happens with these two results, and other properties of the process, if condition (28) is made non-uniform and valid only for almost-all histories (with respect to the process).

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