Effective Dynamics for a Mechanical Particle Coupled to a Wave Field

Alexander Komech[1]

Department of Mechanics and Mathematics of
Moscow State University, Moscow 119899, Russia
email: komech@sci.lpi.ac.ru

Markus Kunze[2]

Mathematisches Institut der Universität Köln
Weyertal 86, D-50931 Köln, Germany
email: mkunze@mi.uni-koeln.de

Herbert Spohn

Zentrum Mathematik, TU München
D-80290 München, Germany
email: spohn@mathematik.tu-muenchen.de

Abstract

We consider a particle coupled to a scalar wave field and subject to the slowly varying potential $V(\varepsilon q)$ with small ε. We prove that if the initial state is close, order ε^2, to a soliton (=3Ddressed particle), then the solution stays forever close to the soliton manifold. This estimate implies that over a time span of order ε^{-1} = the radiation losses are negligible and that the motion of the particle is governed by the effective Hamiltonian $H_{\mathrm{eff}}(q,P) = 3DE(P) + V(\varepsilon q)$ with energy-momentum relation $E(P)$.

[1]Supported partly by French-Russian A.M.Liapunov Center of Moscow State University, by research
grants of RFBR (96-01-00527) and o= f Volkswagen-Stiftung.

[2]Supported by DAAD and NSF (through a grant of Ch. Jones) during a stay at Brown University.

1 Introduction

When a particle interacts with a field its mechanical properties are renormalized, e.g. the particle acquires an effective mass. In the contex= t of charges interacting with the Maxwell field such an effective energy-mo= mentum relation is discussed at length already in the classical work of Abraham [1] and Lorentz [16] with the implicit understanding that thi= s relation determines how the particle responds to external forces. Kramers [14] emphasizes the distinction between bare (appearing in the equat= ion of motion) and physical (observable by outside means) parameters of a charge. His vision has been implemented through the renormalization of qu= antum electrodynamics. To our knowledge, even on the classical level, it has ne= ver been properly settled in which sense and on what scale the dynamics gover= ned by the effective energy-momentum relation is an approximation to the true solution of the coupled equations of motion. To gain some understanding w= e study here the arguably simplest model, namely a single particle interacting with a scalar wave field.

Our second source of interest lies in the, by now, long list of examples = we have for the emergence of an effective dynamics, to mention only the Boltzmann and Vlasov equation, hydrodynamics [21], homogenization i= n periodic and random environments [3, 8], interface and vortex dynamics in Ginzburg-Landau theories [11], quantum systems weakly coupled to = a heat bath [6], and a quantum particle in the semiclassical limit =¸ ite Ha,Ro,Sprl. Their common thread is a separation of space-time scales together with some sort of local stationarity in such a way that the slow= ly varying dynamical variables are governed by an effective dynamics. Howeve= r, the detailed mechanisms differ notably from case to case. Here we add a novel item to the list. It is not covered by the mathematical techniques developed so far.

We consider a scalar wave field $\phi(x)$, in three-dimensional space, co= upled to a particle with position q, momentum p, governed by

$$\begin{aligned} &\dot\phi(x,t) = 3D\pi(x,t), && \dot\pi(x,t) = 3D\Delta\phi(x,t) - \rho(x-q(t =)), \\ &\dot q(t) = 3Dp(t)/(1+p^2(t))^{1/2}, && \dot p(t) = 3D\int d =^3 x\, \phi(x,t)\nabla\rho(x-q(t)). \end{aligned} \tag{1.1}$$

This is a Hamiltonian system with the Hamiltonian functional

$$\mathcal{H}(\phi,\pi,q,p) \;= 3D \quad (1+p^2)^{1/2} + \frac{1}{2}\int d^3x\Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big)$$

$$+ \quad \int d^3x\phi(x)\rho(x-q). \tag{1.2}$$

We have set the mechanical mass of the particle and the speed of wave propagation equal to one. In spirit the interaction term is simply $\phi(=q)$. This would result however in an energy which is not bounded from below. Therefore we smoothen out the coupling by the function $\rho(x)$. In anal= ogy to the Maxwell-Lorentz equations we call $\rho(x)$ the "charge distribution". We assume the function $\rho(x)$ to be smooth, radial, an= d of compact support

$$\rho \in C_0^\infty(\mathbb{R}^3)\,, \quad \rho(x) = 3D\rho_r(|x|)\,, \quad \rho(x) == 3D0 \quad \text{for} \quad |x| \geq R_\rho\,. \tag{C}$$

The system (1.1) has solutions traveling with constant velocity v, $|v| < 1$. They are given by

$$S_v(t) = 3D(\phi_v(x-q-vt), \pi_v(x-q-vt), q+vt, p_v)\,, \quad p_v = 3Dv/\sqrt{=}1-v^2 \tag{1.3}$$

with

$$\phi_v(x) = 3D-\int \frac{\rho(y)d^3y}{4\pi((1-v^2)(y-x)^2+(v\cdot(y-x))^2)^{1/2}}, \quad \bar{p}i_v(x) = 3D-v\cdot\nabla\phi_v(x)\,. \tag{1.4}$$

To have a short name we call $S_v(t)$ the *soliton* with velocity v centered at $q(t) = 3Dq + vt$. We define the energy of a soliton as

$$E_s(v) = 3D\mathcal{H}(S_v) - \mathcal{H}(S_0)\,, \tag{1.5}$$

$S_v = 3DS_v(0)$, which, using the rotational invariance of ρ, is give= n by

$$E_s(v) = 3D(1-v^2)^{-1/2} - 1 + 3m_e\left[\frac{2-v^2}{2(1-v^2)} - \frac{1}{2|v|}\log\frac{1+|v|}{1-|v|}\right]. \tag{1.6}$$

Here $3m_e = 3D - \langle\rho, \Delta^{-1}\rho\rangle$ with $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{R}^3)$. Since the system (1.1) is invariant under spatial translations, the total momentum,

$$\mathcal{P}(\phi,\pi,q,p) = 3Dp - \int d^3x\,\pi(x)\,\nabla\phi(x), \tag{1.7}$$

is conserved. Inserting S_v, the total momentum of a soliton is given by

$$P_s(v) = 3D\mathcal{P}(S_v) = 3Dv(1-v^2)^{-1/2} + 3m_e v\left[\frac{1}{2v^2(1-v^2)} - \frac{1}{4|v|^3}\log\frac{1+|v|}{1-|v|}\right]. \tag{1.8}$$

The map $v \mapsto P_s(v)$ is invertible from $\mathcal{V} = 3D\{v \in \mathrm{I\!R}^3 : |v| < 1\}$ onto $\mathrm{I\!R}^3$ with the inverse $v_s(P)$ [12]. Therefore we obtain the *effective energy-momentum relation*

$$E(P) = 3DE_s(v_s(P))\,. \tag{1.9}$$

$E(P)$ is radial. In the nonrelativistic limit, v small, we have

$$E_s(v) = 3D\frac{1}{2}(1+m_e)v^2 \quad \text{and} \quad P_s(v) = 3D(1+m_e)v \quad \text{for} \quad |v| \ll 1\,. \tag{1.10}$$

Thus m_e is the additional mass acquired by the particle through the coupling to the field. For large $|P|$ we have the relativistic dependenc= e $E(P) \sim |P|$.

Now let us assume that, at some time t, we have the soliton $S_v(t)$ centered at $q(t)$, $v = 3D\dot{q}(t)$, and that an external force is actin= g on the particle. This force changes the velocity to $v' \neq v$ and $S_v(t)$ is no longer a solution to the system (1.1). However, if the force is small, so is the difference $v' - v$ and, if the force is slowly varying, the wave field has enough time to reestablish a soliton with the new velocity v'. In fact this happens essentia= lly with the speed of wave propagation (one in our case). Geometrically in phase space, we have the 6-dimensional manifold $\mathcal{S}$ of solitons labeled by their center q and velocity v. For zero external force eac= h point in this manifold moves on an orbit $t \mapsto (q + vt, v)$. Under a wea= k, slowly varying force, the true solution should remain close to the solito= n manifold thereby inducing on it an effectively 6-dimensional motion.

With this picture in mind, we add to $\mathcal{H}$ in (1.2) the slowly varying potential $V(\varepsilon q)$, $\varepsilon \ll 1$,

$$\begin{aligned} \mathcal{H}_\varepsilon(\phi,\pi,q,p) &= 3D \quad (1+p^2)^{1/2} + V(\varepsilon q) = 20 + \frac{1}{2}\int d^3x\Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big) \\ &+ \quad \int d^3x \phi(x)\rho(x-q)\,. \end{aligned} \tag{1.11}$$

For the potential V we require

$$V \in C^2(\mathrm{I\!R}^3)\,, \quad \inf_{q\in\mathrm{I\!R}^3} V(q) > -\infty \tag{P}$$

and

$$\sup_{q\in\mathrm{I\!R}^3}\Big(|\nabla V(q)| + |\nabla\nabla V(q)|\Big) < \infty\,. \tag{U}$$

We remark that, using the conservation of energy, condition (U) can be replaced by

$$V(q) \to \infty \quad \text{as} \quad |q| \to \infty, \tag{U'}$$

i.e., by V to be confining. In the sequel we study the Hamiltonian dynamics generated by (1.11),

$$\begin{aligned} &\dot{\phi}(x,t) = 3D\pi(x,t), && \dot{\pi}(x,t) = 3D\Delta\phi(x,t) - \rho(x-q(t)), \\ &\dot{q}(t) = 3Dp(t)/(1+p^2(t))^{1/2}, && \dot{p}(t) = 3D - \varepsilon\nabla V(\varepsilon q(t)) + \int d^3x \phi(x,t)\nabla\rho(x-q(t)). \end{aligned} \tag{1.12}$$

The derivatives in (1.12) and below are understood in the sense o= f distributions. We consider the Cauchy problem for the system (1.12) with initial conditions

$$(\phi(x,0), \pi(x,0), q(0), p(0)) = 3D(\phi^0(x), \pi^0(x), q^0, p^0)\,. \tag{1.13}$$

Under our assumptions, the global solution to the Cauchy problem (1.12), (1.13) exists and is unique for initial data with finite energy. The solution depends on ε through the potential and possibly also through the initial conditions. Not to overburden our notation, we mostly suppress this dependence.

We assume the initial state to be close to a soliton. Since the force is slowly varying, near the particle such a wave field should persist. Indeed, we prove that

$$\|(\phi(q(t)+x,t), \pi(q(t)+x,t)) - (\phi_{v(t)}(x), \pi_{v(t)}(x))\|_R \le C_R\varepsilon\,, \quad \forall R > 0\,, \tag{1.14}$$

uniformly in $t \in \mathbb{R}$ with the norm $\|\cdot\|_R$ defined by the field energy in a ball of radius R, provided a smallness condition on ρ is satisfied. Presumably, this condition is an artifact of our method.

In (1.12) the external force is $\mathcal{O}(\varepsilon)$. So is the self-force, since according to (1.14) the field ϕ deviates = from the soliton only by $\mathcal{O}(\varepsilon)$. Then $\ddot{q}$ is of order ε, whereas $\dot{q}$ is of order 1. The effective energy-momentum relation should be visible on a time scale $\mathcal{O}(1)$. Therefore we define the comparison dynamics through the effective Hamilto= nian

$$H_{\text{eff}}(Q,P) = 3DE(P) + V(\varepsilon Q)$$

with the equations of motion,

$$\dot{Q}(t) = 3D\nabla E(P(t))\,, \qquad \dot{P}(t) = 3D - \varepsilon\nabla V(\varepsilon Q(t))\,, \tag{1.15}$$

suppressing the ε-dependence of $(Q(t), P(t))$. Since the energy-momentum relation $E(P)$ depends on the charge distribution only through m_e, the effective dynamics is a structure independent property of the coupled system particle+field in the sense of the Kramers [14].

The particle looses energy through radiation, which is proportional to $\ddot{q}^2$ and thus $\mathcal{O}(\varepsilon^2)$. Therefore the comparison dynamics should be a valid approximation over a time scale ε^{-1}, i.e. over *any* time interval of duration $\varepsilon^{-1}\tau$. At time t_0 the comparison dynamics is adjusted to the true solution through the initial conditions

$$Q(t_0) = 3Dq(t_0)\,, \qquad P(t_0) = 3DP_s(\dot{q}(t_0))\,. \tag{1.16}$$

Let $(Q(t), P(t))$ be the solution to (1.15) with these initial values. We then establish that, for $|t - t_0| = 3D\mathcal{O}(\varepsilon^{-1})$,

$$|q(t) - Q(t)| = 3D\mathcal{O}(1)\,, \quad |\dot{q}(t) - \dot{Q}(t)| = 3D\mathcal{O}(\varepsilon)\,, \quad |\ddot{q}(t) - \ddot{Q}(t)| = 3D\mathcal{O}(\varepsilon^2) \tag{1.17}$$

uniformly in t_0. This is our main result.

In the proof, we stick for a while to the traditional route. One solves the inhomogeneous wave equation and inserts the solution into the self-force. Thereby the force on the particle depends on its past history, but not on the field. If one expands this force at $q(t)$ up to order $\ddot{q}(t) =$, one recovers the term missing in the full energy-momentum relation. To justify such a procedure mathematically we have to know *a priori* that

$$|\ddot{q}(t)| \sim \varepsilon \qquad \text{and} \qquad |\dddot{q}(t)| \sim \varepsilon^2 \tag{1.18}$$

uniformly in t, which requires an estimate of the field difference ($\stackrel{\circ}{=}$ eadf) and a corresponding one to deal with $\dddot{q}(t)$. Our experience from the past is confirmed, namely a direct analysis of th= e exact delay equation for $q(t)$ is hopeless. To make progress one has to switch back and forth between particle and field.

2 Main results

To formulate our results precisely, we need some definitions. We introduce the phase space suitable for the Cauchy problem corresponding to (1.12) and (1.13).

Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with norm $|\cdot|$, and let $\dot{H}^1$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with norm $\|\phi(x)\| =3D|\nabla\phi(x)|$. Equivalently, using Sobolev's embedding theorem, $\dot{H}^1 =3D\{\phi(x)\in L^6(\mathbb{R}^3) : |\nabla\phi(x)| \in L^2\}$; see [15]. Let $|\phi|_R$ denote the norm in $L^2(B_R)$ for $R>0$, where $B_R =3D\{x\in \mathbb{R}^3 : |x|\le R\}$. Then the seminorms $\|\phi\|_R =3D|\nabla\phi|_R + |\phi|_R$ are continuous on $\dot{H}^1$.

Definition 2.1 *i) The phase space $\mathcal{E}$ is the Hilbert space $\dot{H}^1\oplus L^2\oplus\mathbb{R}^3\oplus\mathbb{R}^3$ of states $Y =3D(\phi,\pi,q,p)$ with finite norm*

$$\|\,Y\|_{\mathcal{E}} =3D\|\phi\| + |\pi| + |q| + |p|\,.$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|Y\|_R =3D\|\phi\|_R + |\pi|_R + |q| + |p|,\ \ \forall R>0\,.$$

iii) $\mathcal{F}$ is the Hilbert space $\dot{H}^1\oplus L^2$ of the fields $\Phi =3D(\phi,\pi)$ with finite norm

$$\|\,\Phi\|_{\mathcal{F}} =3D\|\phi\| + |\pi|\,.$$

iv) $\mathcal{F}_F$ is the space $\mathcal{F}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|\Phi\|_R =3D\|\phi\|_R + |\pi|_R,\ \ \forall R>0\,.$$

A point in phase space is referred to as state. We write the Cauchy probl= em (1.12), (1.13) in the form

$$\dot{Y}(t) =3D\mathbf{F}(Y(t))\,,\quad t\in\mathbb{R}\,,\quad Y(0) =3DY^0\,, \tag{2.1}$$

where $Y(t) =3D(\phi(t),\pi(t),q(t),p(t))$ and $Y^0 =3D(\phi^0,\pi^0,q^=0,p^0)$. As already mentioned, we mostly suppress the ε-dependence of the solutions, of the vector field $\mathbf{F}$, and of the initial conditio= ns.

The following lemma is proved analogously to the corresponding result in [13].

Lemma 2.2 *Let (C), (P), and (U), resp. (U′), hold. Then for every $Y^0 \in \mathcal{E}$, $|\varepsilon| \leq 1$, the Cauchy problem (2.1) has a unique solution $Y \in C(\mathbb{R}, \mathcal{E})$ with speed bounded as*

$$\sup_{t\in\mathbb{R}} |\dot{q}(t)| \leq \overline{v} < 1\,. \tag{2.2}$$

The bound $\overline{v} = 3D\overline{v}(Y^0)$ is uniform in $|\varepsilon| \leq 1$ and for initial values Y^0 in bounded subsets of $\mathcal{E}$.

If the effective dynamics is approximately valid, then the field should be close to the soliton centered at $q(t)$ with velocity $v(t) = 3D\dot{q}(t)$. We therefore consider the difference

$$Z(x,t) = 3D\Phi(x,t) - \Phi_*(x,t), \tag{2.3}$$

where

$$\Phi(x,t) = 3D(\phi(x,t), \pi(x,t))\,, \quad \Phi_*(x,t) = 3D\Phi_{v(t)}(x - q(t))$$

and $\Phi_v(x) = 3D(\phi_v(x), \pi_v(x))$ is the field part of the soliton. Defining $\overline{\rho}(x) = 3D(0, \rho(x))$ and $A(\phi, \pi) = 3D(\pi, \Delta\phi)=$, it follows that Φ and Z obey the equations of motion

$$\dot{\Phi}(x,t) \;\; = 3D \quad A\Phi(x,t) - \overline{\rho}(x - q(t)), \tag{2.4}$$

$$\dot{Z}(x,t) \;\; = 3D \quad AZ(x,t) - B(x,t)\,, \quad B(x,t) = 3D\dot{p}(t) \cdot \nabla_{=}p\Phi_{v(t)}(x - q(t)), \tag{2.5}$$

where, according to the chain rule,

$$\nabla_p \Phi_v = 3D\nabla_v \Phi_v dv(p), \quad v(p) = 3D\frac{p}{\sqrt{1+p^2}}. \tag{2.6}$$

Theorem 2.3 *Let the conditions of Lemma 2.2 hold and let $|\rho|$ be sufficiently small, $|\rho| \leq \delta(\overline{v})$. Then for every $R > 0$ there exists C_R such that*

$$\sup_{t\in\mathbb{R}} \|Z(\cdot + q(t), t)\|_R \leq C_R(\|Z(0)\|_{\mathcal{F}} + \varepsilon)\,. \tag{2.7}$$

For the unperturbed, $\varepsilon = 3D0$, system our theorem states that the distance between the true solution and the soliton manifold

$$\mathcal{S} = 3D\{(\phi_v(x-q), \pi_v(x-q), q, p_v) : \, q \in \mathbb{R}^3, \, v \in \mathcal{V}\} \tag{2.8}$$

remains bounded in time. This property is called orbital stability, which has been established for the system (1.1) in [12] and for related equations in [7, 2] using the Liapunov method in com= bination with energy and momentum conservation. For $\varepsilon > 0$ such an argument breaks down, since the Hamiltonian vector field is no longer parallel to $\mathcal{S}$. To have a stability result as (2.7) we therefore need to exploit that through radiation damping the solution is "pushed" towards $\mathcal{S}$. In other words, through the free wave equation a small deviation from the soliton is transported to infinity, which also shows that we are not allowed to replace the local energy norm in (2.7) by the global one. An adequate mathematical argument is provided by the nonautonomous integral equation method [4, 5, 19, 20], which has been used to prove the convergence to the soliton manifold in the context of the nonlinear Schrödinger equation.

If we assume that initially $\|Z(0)\|_{\mathcal{F}} \leq C\varepsilon$, then according to (2.7) the solution remains $\mathcal{O}(\varepsilon)$ close to $\mathcal{S}$ for all times. Thus it remains to characterize the motion along $\mathcal{S}$ as given by the particle trajectory $q(t)$. To obtain its approximate equation of motion we have to estimate the self= -force. By Theorem 2.3 it is of $\mathcal{O}(\varepsilon)$. To control the error, $\mathcal{O}(\varepsilon^2)$, the solution has to be slowly varying in time with outgoing fields, which we formalize through the notion of an *adiabatic family* of solutions $Y_\varepsilon(t)$.

We denote by $U(t)$ the dynamical group on $\mathcal{F}$ generated by the free wave equation and set

$$U(t)\Phi^0 = 3D\Phi_0(x,t) = 3D(\varphi_0(x,t), \psi_0(x,t)). \tag{2.9}$$

Definition 2.4 *A family of solutions $Y_\varepsilon(t) \in C(\mathbb{R}, \mathcal{E})$, $0 < \varepsilon \leq 1$, to the system (1.12) is called adiabatic, if there exist the constants $a, T_0 > 0$, and $\overline{v} < 1$, such that the following bounds hold,*

$$\sup_{t\in\mathbb{R}} |v(t)| \leq \overline{v}\,, \tag{2.10}$$

$$\sup_{t\in\mathbb{R}} |\dot{v}(t)| \leq a\varepsilon\,, \tag{2.11}$$

$$\sup_{t\in\mathbb{R}} |\ddot{v}(t)| \leq a\varepsilon^2\,, \tag{2.12}$$

$$| < \varphi_0(x,t), \nabla\rho(x-q) > | \leq a\varepsilon^2 \quad \textit{for} \quad |q| < |t| - T_0\,. \tag{2.13}$$

The definition is time-invariant, i.e. a family of solutions $Y_\varepsilon(t+T)$ is adiabatic for any $T \in \mathbb{R}$ if it is for some T.

Our main result is the following

Theorem 2.5 *Let the assumptions of Theorem 2.3 hold and l= et $Y_\varepsilon(t) = 3D = EFY(t) \in C(\mathrm{I\!R}, \mathcal{E})$ be an adiabatic family of solutions to (1.12). Let $(Q(t), P(t))$ be the comparison dynamics (1.15) with initial v= alues (**??**). Then for any $\tau > 0$ there exists $C = 3DC(\tau)$ such that for $|t - t_0| \le \varepsilon^{-1}\tau$*

$$|q(t) - Q(t)| \le C\,, \quad |\dot{q}(t) - \dot{Q}(t)| \le C\varepsilon\,, \quad |\ddot{q}(t) - \ddot{Q}(t)| \le C\varepsilon^2\,. \tag{2.14}$$

The constant $C(\tau)$ can be chosen independently of t_0.

Of course, we still need a criterion on the family of initial states, whi= ch=20 ensures that the corresponding family of solution trajectories is adiabatic. The following theorem provides sufficient conditions, which=20 in particular show that any initial soliton $(\phi_v(x - q^0), \pi_v(x - q^0), q^0, p_v)$ defines=20 an adiabatic family of solutions and that=20 the set of adiabatic families of solutions is=20 open in an appropriate topology. =20

We set $(\varphi^0(x), \psi^0(x)) = 3DZ^0(x) = 3DZ(x, 0)$ with Fourier transforms $(\hat{\varphi}^0(k), \hat{\psi}^0(k))$.

Theorem 2.6 *Let there exist $a^0 > 0$ such that for the initial states $Y_\varepsilon^0 = 3DY^0 = 3D(\phi^0, \pi^0, q^0, p^0) \in \mathcal{E}$, $0 < \varepsilon \le 1$, the following bounds hold,*

$$\begin{aligned} \|Y^0(x)\|_{\mathcal{E}} &\le a^0\,, & (2.15)\\ \|Z^0(x)\|_{\mathcal{F}} &\le a^0\varepsilon\,, & (2.16)\\ \|\nabla Z^0(x)\|_{\mathcal{F}} &\le a^0\varepsilon^2\,, & (2.17)\\ \int d^3k\, \Big(|k||\nabla\hat{\varphi}^0(k)| + |\nabla\hat{\psi}^0(k)|\Big)\, |\hat{\rho}(k)| &\le a^0\varepsilon\,. & (2.18)\\ \int d^3k\, \Big(|k||\hat{\varphi}^0(k)| + |\hat{\psi}^0(k)|\Big)\, |\hat{\rho}(= k)| &\le a^0\varepsilon^2\,, & (2.19) \end{aligned}$$

Then family of solutions $Y_\varepsilon(t) \in C(\mathrm{I\!R}, \mathcal{E})$ to the Cauchy problem (2.1) is adiabatic.

Theorem 2.6, in essence, requires that the deviation from the soliton has sufficient smoothness and decay.

Our paper is organized as follows. Theorem 2.3 is proved in Section 3, and Theorem 2.6 is established in Section 4. In Section 5 we compute the self-force, and in Section 6 we complete the proof of Theorem 2.5. Section 7 concerns the translation invariant system (1.1). In Appendix A we collect Fourier space computations. Finally, in Appendix B, we list some remarks on the Hamiltonian structure.

3 Stability of soliton manifold

We prove Theorem 2.3 and establish first the required bound for $R = 3DR_\rho$ from (C).

Lemma 3.1 *Under the assumptions of Theorem 2.3, the bound (2.7) holds for $R = 3DR_\rho$,*

$$\|Z(\cdot + q(t), t)\|_{R_\rho} \le C(\|Z(0)\|_{\mathcal{F}} + \varepsilon)\,. \tag{3.1}$$

Proof: Eq. (2.5) implies the mild solution representation

$$Z(t) = 3DU(t)Z(0) - \int_0^t U(t-s)[\dot{p}(s)\cdot\nabla_p\Phi_{v(s)}(\cdot - q(s))]\,ds \tag{3.2}$$

with $U(t)$ the group generated by the free wave equation in $\dot{H}^1 \oplus L^2$. By conservation of energy for the wave equation

$$\|[U(t)Z(0)](\cdot + q(t))\|_{R_\rho} \le \|[U(t)Z(0)](\cdot + q(t))\|_{\mathcal{F}} \le C\|Z(0)\|_{\mathcal{F}}\,. \tag{3.3}$$

Denote by $\varphi(x,t) = 3D\phi(x,t) - \phi_{v(t)}(x - q(t))$ the first comp= onent of $Z(x,t)$ and observe that $\langle\phi_v(x), \nabla\rho(x)\rangle = 3D0=$ for $|v| < 1$ because the soliton (1.3) is a solution to (1.1). T= hen (1.12) implies

$$\dot{p}(t) = 3D - \varepsilon\nabla V(\varepsilon q(t)) + \langle\varphi(x + q(t), t), \nabla\rho(x)\rangle\,. \tag{3.4}$$

Thus with assumption (U) we obtain,

$$|\dot{p}(t)| \le C\Big(\varepsilon + \|Z(\cdot + q(t), t)\|_{R_\rho}\,\|\rho\|\Big)\,. \tag{3.5}$$

Let us denote $\overline{\pi}_v = 3D\nabla_p\pi_v$, $\overline{\phi}_v = 3D\nabla_p\phi_v$, $S_{t-s}(x) = 3D\{y :\ |y - x| = 3Dt - s\}$, and

$$(\overline{\phi}(\cdot, t, s), \overline{\pi}(\cdot, t, s)) = 3DU(t-s)[\nabla_p\Phi_{v(s)}(\cdot - q(s))].$$

Then Kirchhoff's formula for $U(t-s)$ implies the representation

$$\begin{aligned}\nabla\overline{\phi}(x,t,s) = 3D &\sum_{|\alpha|\le 1}(t-s)^{|\alpha|-2}\int_{S_{t-s}(x)} d^2y\ a_\alpha(x-y)\partial_y^\alpha\overline{\pi}_{v(s)}(y - q(s))\\ &+ \sum_{|\alpha|\le 2}(t-s)^{|\alpha|-3}\int_{S_{t-s}(x)} d^2y\ b_\alpha(x-y)\partial_y^\alpha\overline{\phi}_{v(s)}(y - q(s))\end{aligned} \tag{3.6}$$

and a similar representation for $\overline{\pi}(x,t,s)$. The coefficients $a_\alpha(\cdot)$, $b_\alpha(\cdot)$ are bounded and the sums run over the multiindices $\alpha = 3D(\alpha_1,\alpha_2,\alpha_3)$ with integers $\alpha_j \geq 0$. Therefore $\nabla\overline{\phi}(x+q(t))$ and $\overline{\pi}(x+q(t))$ can be represented as integrals of type (3.6) over the shifted sphere $S_{t-s}(x+q(t))$. If $|x| \leq R_\rho$, we have on this sphere

$$\begin{aligned} |y-q(s)| &= 3D \quad |(y-x-q(t))+(x+q(t)-q(s))| \\ &\geq \quad (t-s)-|x|-\overline{v}(t-s) \geq (1-\overline{v})(t-s)-R_\rho \quad (3.7) \end{aligned}$$

by the bound (2.2) on $\dot{q}(t)$. On the other hand, the integral representation (1.4) yields

$$\sup_{x\in \mathbb{R}^3,\, |v|\leq \overline{v}} \Big[|x||\overline{\phi}(x)| + |x|^2(|\nabla\overline{\phi}(x)| + |\overline{\pi}(x)|) + |x|^3(|\nabla\nabla\overline{\phi}(x)| + |\nabla\overline{\pi}(x)|)\Big] \leq C(\overline{v},\rho) < \infty\,. \qquad (3.8)$$

Inserting (3.8) and (3.7) in Kirchhoff's formula for $\nabla\overline{\phi}(x+q(t),t,s)$, we obtain the bound=20

$$\begin{aligned} |\nabla\overline{\phi}(x,t,s)| &\leq \sum_{|\alpha|\leq 1}(t-s)^{|\alpha|-2}\frac{C_1(\overline{v},\rho)(t-s)^2}{(1+|t-s|)^{|\alpha|+2}} \\ &+ \sum_{|\alpha|\leq 2}(t-s)^{|\alpha|-3}\frac{C_1(\overline{v},\rho)(t-s)^2}{(1+|t-s|)^{|\alpha|+1}} \leq \frac{C_2(\overline{v},\rho)}{1+(t-s)^2}\,. \quad (3.9) \end{aligned}$$

for $|x| \leq R_\rho$ and provided $t-s \gg R_\rho/(1-\overline{v})$, while for small $t-s$ it is obvious. (3.9) implies, together with similar bound for $\overline{\pi}(x+q(t),t,s)$,

$$\|(\overline{\phi}(x+q(t),t,s), \overline{\pi}(x+q(t),t,s)\|_{R_\rho} \leq \frac{C_3(\overline{v},\rho)}{1+(t-s)^2}\,. \qquad (3.10)$$

Therefore, (3.5) and (3.10) imply

$$\begin{aligned} &\|\dot{p}(s)\cdot(\overline{\phi}(x+q(t),t,s),\overline{\pi}(x+q(t),t,s)\|_{R_\rho} \\ &\quad \leq C_4(\overline{v},\rho)\,\frac{\varepsilon + \|Z(\cdot+q(s),s)\|_{R_\rho}\,\|\rho\|}{1+(t-s)^2}\,. \end{aligned}$$

and combining (3.2) and (3.3) we arrive at

$$\begin{aligned} &\|Z(\cdot+q(t),t)\|_{R_\rho} \qquad (3.11) \\ &\leq C\|Z(0)\|_{\mathcal{F}} \;+\; C_5(\overline{v},\rho)\int_0^t \frac{\varepsilon+\|Z(\cdot+q(s),s)\|_{R=\rho}\,\|\rho\|}{1+(t-s)^2}\,ds \quad \text{for} \quad t\geq 0. \end{aligned}$$

Thus, denoting $M(t) = 3D \max_{0\le s\le t} \|Z(q(s)+x,s)\|_{R_\rho}$, we have

$$M(t) \le C\|Z(0)\|_{\mathcal{F}} + C_6(\overline{v},\rho)(\varepsilon + \|\rho\| M(t)).$$

We choose now $\|\rho\|$ so small that $C_6(\overline{v},\rho)\|\rho\| < 1$. Then (3.1) follows for $t \ge 0$.
□

We claim that the bound (3.1) implies (2.7) for any $R > 0$.= Indeed, (3.10)-(3.11) hold with the norm $\|\cdot\|_R$ instead of $\|\cdot\|_{R_\rho}$ on the *left* hand sides and with $C_i(\overline{v},\rho,R)$ instead of $C_i(\overline{v},\rho)$ on the right hand sides. Then (3.11) with this generalization and (3.1) imply (2.7).

4 Adiabatic solutions

We prove Theorem 2.6. The bound (2.10) is the assertion of Lemma 2.2. Let us prove (2.13). We have

$$U(t)\Phi^0 = 3DU(t)\Phi_{v(0)} + U(t)Z^0. \tag{4.1}$$

$[U(t))\Phi_{v(0)}](x) = 3D0$ for $|x-q^0| < |t| - R_\rho$ by Kirchhoff's formula, since $\Phi_{v(0)}$ has the representation

$$\Phi_{v(0)} = 3D\int_{-\infty}^0 U(-s)\overline{\rho}(\cdot - q^0 - v(0)s)\,ds\,. \tag{4.2}$$

Therefore with the choice $T_0 = 3D2R_\rho + |q^0|$ (2.13) holds for the first component of $[U(t))\Phi_{v(0)}](x)$. With the choice $T_0 = 3D0$, (2.19) implies (2.13) for the first component of $U(t)Z^0$, as can be seen in Fourier space representation. Thus it remains to prove (2.11) and (2.12).

Proposition 4.1 *i) Let (2.15) and (2.16) hold. Then*

$$\sup_{t\in\mathbb{R}} |\dot v(t)| \le \varepsilon C(\overline{v},\rho)\,. \tag{4.3}$$

ii) Let in addition (2.17) and (2.18) hold. Then also

$$\sup_{t\in\mathbb{R}} |\ddot v(t)| \le \varepsilon^2 C(\overline{v},\rho)\,. \tag{4.4}$$

Proof: *ad i)* (4.3) follows from (2.7) and (3.5).
ad ii) For the bound (4.4) we differentiate (3.4),

$$\ddot{p}(t) = 3D - \varepsilon^2 v(t) \cdot \nabla \nabla V(\varepsilon q(t)) + M(t)$$

where $M(t) = 3D\langle L(t)\varphi(x+q(t),t), \nabla\rho(x)\rangle$ and $L(t) = 3D\partial_t + v(t)\cdot\nabla$. Then (U) implies

$$|\ddot{p}(t)| \leq C(\varepsilon^2 + |M(t)|)\,. \tag{4.5}$$

Therefore (4.4) is a consequence of

Lemma 4.2 *Let (2.15)-(2.18) hold. Then*

$$|M(t)| \leq \varepsilon^2 C(\overline{v}, \rho)\,. \tag{4.6}$$

Proof: We extend the method of the previous section. Denoting $\Xi(x,t) = 3DL(t)Z(x,t)$, we have $M(t) = 3D\langle \Xi(x,t), \nabla\rho_*(x-q(t))\rangle$. To obtain an equation for $\Xi(t)$ we apply $L(t)$ to (2.5),

$$\dot{\Xi}(x,t) = 3DA\Xi(x,t) - L(t)B(x,t) + \dot{v}(t)\cdot\nabla Z(x,t)\,,$$

which implies the integral equation

$$M(t) = 3D\langle U(t)\Xi(0), \nabla\rho_*(x-q(t))\rangle - \int_0^t \langle U(t-s)L(s)B(s), \nabla\rho_*(x-q(t))\rangle ds$$
$$+ \int_0^t \dot{v}(s)\cdot\langle U(t-s)\nabla Z(s)\,, \nabla\rho_*(x-q(t))\rangle\, ds\,. \tag{4.7}$$

We analyze the three summands one by one.

(i) For the first summand we prove the bound

$$\sup_{t\geq 0} |\langle U(t)\Xi(0), \nabla\rho_*(x-q(t))\rangle| \leq= q\varepsilon^2 C(\overline{v}, \rho)\,. \tag{4.8}$$

Let us consider $\Xi(0)$. With the definition of Z in (2.3), Eq. (2.5) implies

$$\Xi(x,t) = 3DA\Phi(x,t) - \overline{\rho}(x-q(t)) + v(t)\cdot\nabla\Phi(x,t) - \dot{p}(t)\cdot\nabla_p\Phi_{v(t)}(x-q(t))\,.$$

Therefore $\Xi(x,0) = 3D\Xi_1(x) + \Xi_2(x)$, where

$$\begin{aligned}\Xi_1(x) &= 3D \quad A\Phi(x,0) - \overline{\rho}(x-q^0) + v(0)\cdot\nabla\Phi(x,0),\\ \Xi_2(x) &= 3D \quad -\dot{p}(0)\cdot\nabla_p\Phi_{v^0}(x-q^0)\,.\end{aligned}$$

The bound (2.17) implies $\|\Xi_1(x)\|_{\mathcal{F}} \leq C\varepsilon^2$. Energy conservation yields then the uniform bound

$$\sup_{t\geq 0} \|U(t)\Xi_1\|_{\mathcal{F}} \leq C\varepsilon^2 \,. \tag{4.9}$$

On the other hand, for $\|U(t)\Xi_2(x)\|_{\mathcal{F}}$ we only have a bound of order ε, and this is all one can expect. Fortunately, a stronger bound as in (4.9) is not necessary for a proof of (4.8). We only need

$$\sup_{t\geq 0} |\langle U(t)\Xi_2, \nabla\rho_*(x-q(t))\rangle| \leq \varepsilon^2 C(\overline{v},\rho)\,. \tag{4.10}$$

To prove this bound we first note that, similarly to (3.10),

$$\Big\| \, [U(t)\Xi_2](\cdot + q(t))\Big\|_{R_\rho} \leq \frac{\varepsilon C(\overline{v},\rho)}{1+t^2}\,. \tag{4.11}$$

The second crucial observation is the identity

$$\langle U(t)\Xi_2, \nabla\rho_*(x - v^0 t)\rangle = 3D - \dot{p}(0)\cdot\nabla_p\langle U(t)\Phi_{v^0}(x-q^0), \nabla\rho_*(x-q^0-v^0t)\rangle = 3D0\,, \tag{4.12}$$

since $U(t)\Phi_{v^0}(x-q^0) = 3D\Phi_{v^0}(x-q^0-v^0t)$. Combining (4.11) and (4.12) we obtain (4.10), since $|q(t)-q^0-v^0t| \leq C\varepsilon t$ by (4.3). Now (4.9) and (4.10) imply (4.8).

(ii) For the second summand in (4.7) we estimate as

$$\begin{aligned} &\left|\int_0^t \langle U(t-s)L(s)B(s)\,, \nabla\rho_*(x-q(t))\rangle \, ds\right| \\ &\leq C(\overline{v},\rho)\, \|\rho\| \int_0^t \frac{\varepsilon^2 + \|\rho\|\, |M(s)|}{1+(t-s)^2}\, ds\,, \quad t\geq 0. \end{aligned} \tag{4.13}$$

Indeed, we observe that

$$L(t)B(x,t) = 3D\ddot{p}(t)\cdot\nabla_p\Phi_{v(t)}(x-q(t)) + (\dot{p}(t)\bar{c}dot\nabla_p)^2\Phi_{v(t)}(x-q(t))\,.$$

Then (4.5) and (4.3) result in $L(t)B(x,t) = 3De(x,t) + m(x,t) =$, where

$$\sup_{t\geq 0}\|e(x,t)\|_{\mathcal{F}} \leq \varepsilon^2 C(\overline{v},\rho=), \quad \|m(x,t)\|_{\mathcal{F}} \leq C\|\rho\|\,|M(t)|\,. \tag{4.14}$$

Therefore (4.13) follows by repeating the argument in (3.10)-(3.11).

(iii) For the third summand in (4.7) we prove

$$\left|\int_0^t \dot v(s)\cdot\langle U(t-s)\nabla Z(s)\,,\nabla\rho_*(x-q(t))\rangle\,ds\right|\le\varepsilon^2C(\overline{v},\rho)\,,\quad t\ge 0. \quad (4.15)$$

Taking the gradient of (3.2) yields

$$U(t-s)\nabla Z(s)=3DU(t)\nabla Z(0)-\int_0^s\dot p(\tau)\cdot U(t-\tau)\nabla[\nabla_p\Phi_{v(\tau)}(\cdot-q(\tau)=)]\,d\tau\,. \quad (4.16)$$

For the first term, by partial integration in the Fourier representation, (2.18) implies that

$$|\langle U(t)\nabla Z(0),\nabla\rho_*(x-q(t))\rangle|\le\varepsilon C(\rho)(1+t)^{-1}\,.$$

To bound the second term we note, similarly to (3.10),

$$\|U(t-\tau)\nabla[\nabla_p\Phi_{v(\tau)}(\cdot-q(\tau))]\|_{R_\rho}\le\frac{C(\overline{v},\rho)}{1+(t-\tau)^3}\,. \quad (4.17)$$

since the bounds of type (3.8) hold for $\nabla\nabla_p\Phi_v(x)$ with an additional power of $|x|$ on the left hand side. Then (4.16)-(4.17) and (4.3) imply (4.15).

Finally, we substitute (4.8), (4.14), and (4.15) into (4.7) and obtain the integral inequality

$$|M(t)|\le C\varepsilon^2+C(\overline{v},\rho)\,\|\rho\|\int_0^t\frac{\varepsilon^2+\|\rho\|\,|M(s)|}{1+(t-s)^2}\,ds\,.$$

Therefore (4.6) for $t\ge 0$ follows provided that $\|\rho\|\ll 1$. □

5 Inertial representation for self-force

We study the self-action term

$$F_s(t)=3D\int d^3x\,\phi(x,t)\,\nabla\rho(x-q(t))\,.$$

Let us denote by $T_1=3D2R_\rho(1-\overline{v})^{-1}$, where $\overline{v}<1$ is the bound from (2.10), and by $\overline{T}=3D\max(T_0,T_1)$ with T_0 from (2.13). We also introduce the field part of the total momentum,

$$P_{\rm f}(v)=3DP_s(v)-p_v\,, \quad (5.1)$$

compare with (1.8), (1.3). The corresponding "effective mass" is

$$m_{\mathrm{f}}(v) = 3DdP_{\mathrm{f}}(v)$$

as Jacobian matrix.

Lemma 5.1 *Let the assumptions of Theorem 2.5 hold. Then*

$$F_s(t) = 3D - m_{\mathrm{f}}(\dot q(t))\ddot q(t) + f_s(t)\,, \quad |f_s(t)| \le C\varepsilon^2 \quad \textit{for} \quad |t| \ge \overline{T}\,. \tag{5.2}$$

Proof: We note that by (1.12), $\phi(x,t) = 3D\phi_r(x,t) + \phi_0(x,t)$, where $\ddot\phi_0 = 3D\Delta = i_0$ with initial values $\phi_0(0) = 3D\phi^0$, $\dot\phi_0(0) = 3D\pi^0$, an= d where ϕ_r is the retarded potential

$$\phi_r(x,t) = 3D - \frac{1}{4\pi}\int_0^t \frac{ds}{t-s}\int_{|x-y|=3Dt-s} d^2y\,\rho(y-q(s))\,. \tag{5.3}$$

We decompose accordingly, $F_s(t) = 3DF_r(t) + F_0(t)$ with

$$F_r(t) = 3D\langle\phi_r(x,t), \nabla\rho(x-q(t))\rangle\,, \quad F_0(t) = 3D\langle\phi_0(x,t), \nabla\rho(x-q(t))\rangle\,. \tag{5.4}$$

¿From (2.10) we conclude that $|q(t)-q^0| \le \overline{v}t$, and therefo= re

$$F_0(t) = 3D\mathcal{O}(\varepsilon^2) \quad \text{for} \quad t \ge T_0 \tag{5.5}$$

by (2.13) and since the solution is adiabatic. Hence

$$F_s(t) = 3DF_r(t) + \mathcal{O}(\varepsilon^2) \quad \text{for} \quad t \ge T_0 = \,. \tag{5.6}$$

Eqs. (5.3) and (5.4) imply

$$F_r(t) = 3D - \frac{1}{4\pi}\int_0^t \frac{ds}{t-s}\int d^3x \int_{|x-y|=3Dt-s} d^2y\,\rho(y-q(s))\,\nabla\rho(x-q(t))\,. \tag{5.7}$$

Next we remark that for all $t, T \ge T_1$ the $\int_0^t ds(\ldots)$-integral in (5.7) may be changed to a $\int_{t-T}^t ds(\ldots)$-integral, since

$$\rho(y-q(s))\,\nabla\rho(x-q(t)) = 3D0 \quad \text{if} \quad |x-y| = 3Dt-s \ge T_1. \tag{5.8}$$

Indeed, $\rho(y-q(s))\,\nabla\rho(x-q(t)) \neq 0$ implies $|y-q(s)| < R_\rho$ and $|x-q(t)| < R_\rho$. Therefore $|x-y| < 2R_\rho + \overline{v}(t-s)$, since $|q(t)-q(s)| \leq \overline{v}(t-s)$ by (2.2). Substituting $|x-y|$ by $t-s$ we obtain $t-s < 2R_\rho/(1-\overline{v}) = 3DT_1$.

Let us fix now $t, T \geq T_1$ and substitute in (5.7) the Taylor expansion $q(s) = 3Dq(t) - \dot{q}(t)(t-s) + (1/2)\ddot{q}(t)(t-s)^2 + \mathcal{O}(\varepsilon^2)$ according to (2.11)-(2.12). T= hen

$$F_r(t) \quad = 3D \quad -\frac{1}{4\pi}\int\limits_{t-T}^{t} \frac{ds}{t-s} \int d^3x \int\limits_{|x-y|=3Dt-s} d^2y\, \rho\Big(y - q(t) + \dot{q}(t)(t-s)$$

$$-\frac{1}{2}\ddot{q}(t)(t-s)^2 + \mathcal{O}(\varepsilon^2)\Big)\nabla\rho(x-q(t)).$$

Combining with (5.6) we finally obtain

$$F_s(t) = 3D - \frac{1}{4\pi}\int\limits_{t-T}^{t} \frac{ds}{t-s} \int d^3x \int\limits_{|x-y|=3Dt-s} d^2y\,\Big[\rho(y-q(t)+\dot{q}(t)(t-s))$$
$$-\frac{1}{2}(t-s)^2\,\ddot{q}(t)\cdot\nabla\rho(y-q(t)+\dot{q}(t)(t-s))\Big]\nabla\rho(x-q(t)) + f_s(t) \tag{5.9}$$

with $f_s(t)$ satisfying (5.2). The integral does not depend on T provided $T, t > T_1$, which reflects the strong Huyghen's principle. We will show in Appendix A that the integral in (5.9) in fact equals $-m_{\mathrm{f}}(\dot{q})\ddot{q}$. Then (5.2) follows for $t \geq \overline{T}$. □

6 Adiabatic limit

We complete the proof of Theorem 2.5. Let us first ensure the existence of the effective dynamics.

Lemma 6.1 *Let $E(P)$ be given by (1.9) and let the potential V satisfy (U). Then for every initial state $(Q(0), P(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$ the Hamiltonian system*

$$\dot{Q}(t) = 3D\nabla E(P(t)), \quad \dot{P}(t) = 3D - \varepsilon\nabla V(\varepsilon Q(t)) \tag{6.1}$$

has a unique solution $(Q(t), P(t)) \in C(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$. Moreover, $|\ddot{Q}(t)|$ and $|\dddot{Q}(t)|$ are bounded uniformly in t.

Proof: Both $\nabla\nabla E(P)$ and $\nabla\nabla V(Q)$ are bound= ed and $H_{\text{eff}}(P,Q)$ is bounded from below. $\square$

Let $m(v) = 3DdP_s(v)$. From Lemma 5.1, together with definitions (1.8), (5.1) and the equations of motion (1.12), one concludes that

$$m(\dot{q}(t))\ddot{q}(t) = 3D - \varepsilon\nabla V(\varepsilon q(t)) + f_s(t). \tag{6.2}$$

We want to rewrite (6.2) in a Hamiltonian form. For this purpose we introduce $\Pi(t) = 3DP_s(\dot{q}(t))$, which yields $m(\dot{q}(t))\ddot{q}(t) = 3D\dot{\Pi}(t)$. Secondly, we need $\dot{q}$ as a function of Π, i.e., we have to invert the map $v \mapsto P_s(v)$.

Lemma 6.2 *The inverse function to $P_s(v)$ is given by*

$$v_s(P) = 3D\nabla E(P)\,. \tag{6.3}$$

Proof: Using the chain rule Eq. (9.1) states

$$v = 3D\nabla E_s(v)(dP_s(v))^{-1} = 3D\nabla E(P_s(v))\,. \qquad \square$$

With these definitions (6.2) becomes

$$\dot{q}(t) = 3D\nabla E(\Pi(t))\,, \quad \dot{\Pi}(t) = 3D - \varepsilon\nabla V(\varepsilon q(t)) + f_s(t)\,. \tag{6.4}$$

Let $q^\varepsilon(t) = 3D\varepsilon q(\varepsilon^{-1}t), Q^\varepsilon(t) = 3D\varepsilon Q(\varepsilon^{-1}t)$ and $\Pi^\varepsilon(t) = 3D\Pi(\varepsilon^{-1}t), P^\varepsilon(t) = 3DP(\varepsilon^{-1}t)$. Then (**??**) and (6.1) read

$$\begin{aligned} \dot{q}^\varepsilon(t) &= 3D\nabla E(\Pi^\varepsilon(t))\,, & \dot{\Pi}^\varepsilon(t) &= 3D - \nabla V(\varepsilon q^\varepsilon(t)) + \varepsilon^{-1} f_s(\varepsilon = t), \\ \dot{Q}^\varepsilon(t) &= 3D\nabla E(P^\varepsilon(t))\,, & \dot{P}^\varepsilon(t) &= 3D - \nabla V(\varepsilon Q^\varepsilon(t)). \end{aligned}$$

Since $\nabla\nabla E$ and $\nabla\nabla V$ are bounded, and $|f_s(\varepsilon t)| \le C\varepsilon^2$ for $|t| \ge \varepsilon\overline{T}$, from a Gronwall argument for $r(t) = 3D|q^\varepsilon(t) - Q^\varepsilon(t)| + |\Pi^\varepsilon(t) - P^\varepsilon(t)|$, we conclude that

$$r(t) \le C(r_0 + \varepsilon)e^{C|t-t_0|} \tag{6.5}$$

Here $r_0 := 3Dr(t_0) = 3D0$ due to (**??**), if $|t_0| > \overline{T}$, otherwise $r_0 := 3Dr(\pm\varepsilon\overline{T}) = 3D\mathcal{O}(\varepsilon)$, since $q^\varepsilon(t), Q^\varepsilon(t), \Pi^\varepsilon(t), P^\varepsilon(t)$ change by $\mathcal{O}(\varepsilon)$ over the time interval $|t| \le \varepsilon\overline{T}$. Therefore, (6.5) implies the first two bounds of (2.14). The third bound follows from the second order equation (6.2) for $\ddot{q}$ and a similar equation for $\ddot{Q}$. $\square$

7 Translation invariant case

For $V = 3D0$ the velocity $\dot{q}(t)$ of the particle should, after a transient period, stabilize at some definite v dressed by the corresponding soliton field. Such a result was established in [12], where we only had to assume the Wiener condition $\hat{\rho}(k) > 0$. The technique developed here avoids this condition at the prize of $\|\rho\| \ll 1$ and obtains even a bound on the rate of convergence. Let us denote by $Z(0) = 3D(\varphi^0(x), \psi^=0(x))$.

Proposition 7.1 *Let $\|\rho\|$ be sufficiently small, $\|\rho\| \le \delta(\overline{v})$, and let for some $\sigma \in (0,1]$*

$$|\varphi^0(x)| + |x|(|\nabla\varphi^0(x)| + |\psi^0(x)|) + |x|^2(|\nabla\nabla\varphi^0(x)| + |\nabla\psi^0(x)|) \\ = 3D\mathcal{O}(|x|^{-\sigma}) \quad as \quad |x| \to \infty \,. = 20 \tag{7.1}$$

Then the solution to the system (1.1) satisfies

$$\|Z(\cdot + q(t), t)\|_R \le C_R(1+|t|)^{-1-\sigma}, \quad \forall R > 0. \tag{7.2}$$

Corollary 7.2 *Under the same assumptions the acceleration is bounded as*

$$|\ddot{q}(t)| \le C(1+|t|)^{-1-\sigma}. \tag{7.3}$$

Therefore, the limits $\lim_{t\to\pm\infty} \dot{q}(t) = 3Dv_\pm \in \mathcal{V}$ exist and

$$|\dot{q}(t) - v_\pm| \le C(1+|t|)^{-\sigma}. \tag{7.4}$$

Proof: Eqs. (7.1) and (3.11) with $\varepsilon = 3D0$ imply

$$\|Z(\cdot + q(t), t)\|_{R_\rho} \le C(1+|t|)^{-1-\sigma} + C(\overline{v}, \rho)\|\rho\| \int_0^t \frac{\|Z(q(s)+x, s)\|_{R_\rho}}{(1+|t-s|)^2}\, ds\,, t \ge 0.$$

Therefore, setting $M(t) = 3D\max_{0\le s\le t}(1+|t|)^{1+\sigma}\|Z(q(s)+x,s)\|_{R_\rho}$, we have

$$M(t) \le C + C(\overline{v}, \rho)\|\rho\| I_\sigma M(t)\,,$$

where

$$I_\sigma = 3D\sup_{t\ge 0}(1+|t|)^{1+\sigma}\int_0^t \frac{(1+|s|)^{-1-=ma}}{(1+|t-s|)^2}\, ds < \infty \text{ for } \sigma \in (0,1]\,.$$

It remains to choose $C(\overline{v},\rho)\|\rho\|I_\sigma < 1$, then (7.2) with $R = 3DR_\rho$ follows for $t \geq 0$. The corollary is a consequence of (3.4) with $\varepsilon = 3D0$. $\square$

Remark Soliton-like asymptotics are established in [17] for some translation invariant 1D completely integrable equations, in [4, 5] for small perturbations of soliton solutions to 1D translation invariant nonlinear Schrödinger equations, and in [19, 20] for $U(1)$-invariant 2D and 3D nonlinear Schrödinger equations with a potential term decaying like a power decay at infinity. [9] studies soliton-like asymptotics for 1D translation invariant nonlinear reaction systems.

8 Appendix A. Fourier integrals

As usual, we denote by $\hat{f}(k) = 3D(2\pi)^{-3/2}\int d^3x \exp(ikx)f(x)$ the Fourier transf= orm of $f(x)$.

Solitons: The soliton (1.4) has Fourier transform

$$\begin{cases} \hat{\phi}_v(k) = 3D - \dfrac{\hat{\rho}(k)}{k^2-(k\cdot v)^2}\,, \\ \\ \hat{\pi}_v(k) = 3D - \dfrac{ik\cdot v\ \hat{\rho}(k)}{k^2-(k\cdot v)^2}\,. \end{cases} \tag{8.1}$$

Energy-momentum relation: Inserting (8.1) in (1.2) and (1.7), the energy and the total momentum of a soliton with velocity v are, respectively,

$$\mathcal{H}(S_v) \ = 3D \quad (1-v^2)^{-1/2} + \frac{1}{2}\int d^3k\,|\hat{\rho}(k)|^2 \frac{3(k\cdot v)^2-k^2}{[k^2-(k\cdot v)^2]^2}\,, \tag{8.2}$$

$$P_s(v) \ = 3D \quad v(1-v^2)^{-1/2} + \int d^3k\,|\hat{\rho}(k)|^2 \frac{k\cdot v}{[k^2-(k\cdot v)^2]^2}k\,. \tag{8.3}$$

In both equations we first perform a rotational average in k. This yields (1.6) and (1.8).

Field mass: Eq. (8.3) implies that the effective mass due to the coupling to the field is given by

$$m_{\mathrm{f}}(v) = 3DdP_{\mathrm{f}}(v) = 3D\int d^3k\,|\hat{\rho}(k)|^2\,\frac{k^2+3(k\cdot v)^2}{[k^2-(k\cdot v)^2]^3}\,k\otimes k\,, \quad |v|<1\,. \tag{8.4}$$

Self-force: We compute the integral (5.9) by switching to Fourier space. The wave propagator in Fourier space is multiplication by $|k|^{-1}\sin|k|t$. Hence

$$\begin{aligned} F_s(t) &= 3D \quad \int d^3k\,|\hat\rho(k)|^2 ik \int_{t-T}^t ds\, e^{-ik\cdot\dot q(t)(t-s)}\Big[1-\frac{1}{2}\ddot q(t)\cdot(-ik)(t-s)^2\Big] \\ &\quad \times|k|^{-1}\sin|k|(t-s)+f_s(t). \end{aligned} \tag{8.5}$$

We evaluate this integral by taking the limit as $T\to\infty$, recalling that the integral does not depend on T provided $T\geq T^1$. We set $F_s(t)=3DI_1(T)+I_2(T)+f_s(t)$.

In (8.5) we integrate over s. Setting $v=3D\dot q(t)$ and $k_\pm=3D-k\cdot v\pm|k|$, the first integral reads

$$\begin{aligned} I_1(T) &= 3D \quad \int d^3k\,|\hat\rho(k)|^2 ik\int_{t-T}^t ds\,e^{-ik\cdot\dot q(t)(t-s)}\,\frac{\sin|k|(t-s)}{|k|} \\ &= 3D \quad i\int d^3k\,|\hat\rho(k)|^2\left[\frac{k}{k^2-(k=otv)^2}-\frac{k}{2|k|}\left(\frac{e^{ik_+T}}{k_+}-\frac{e^{ik_-T}}{k_-}\right)\right] \\ &= 3D \quad -i\int d^3k\,|\hat\rho(k)|^2\,\frac{k}{2|k|}\left(\frac{e^{ik_+T}}{k_+}-\frac{e^{ik_-T}}{k_-}\right)=3D: I_1^+(T)+I_1^-(T)\,. \end{aligned}$$

Introducing polar coordinates $\nu=3D|k|$, $\theta=3Dk/|k|$ we have

$$\begin{aligned} I_1^+(T) &= 3D \quad -i\int d^3k\,|\hat\rho(k)|^2\,\frac{k}{2|k|}\,\frac{e^{ik_+T}}{k_+} \\ &= 3D \quad -\frac{i}{2}\int_{|\theta|=3D1} d^2\theta\,\frac{\theta}{\theta\cdot v+1}\int_0^\infty d\nu\,\nu\,|\hat\rho(\nu\theta)|^2\,e^{i(\theta\cdot v+1)\,\nu T}\,. \end{aligned}$$

an integrating by parts twice in the ν-integral yields $|I_1^+(T)|\leq CT^{-2}$ because $|v|=3D|\dot q(t)|<1$. The same argument applies to $I_1^-(T)$ and it follows that

$$|I_1(T)|\leq CT^{-2}\to 0 \quad \text{as} \quad T\to\infty\,. \tag{8.6}$$

The second integral reads

$$I_2(T) = 3D \quad -\frac{1}{2}\int d^3k\,|\hat\rho(k)|^2 k\;\ddot q(t)\cdot k\int_{t-T}^t ds\,e^{-ik\cdot\dot q(t)(t-s)}(t-s)^2\,\frac{\sin|k|(t-s)}{|k|}$$

$$= 3D \quad -\int d^3k\, |\hat\rho(k)|^2\, k\ \ddot q(t)\cdot k\ \bar{b}igg[\frac{k^2+3(k\cdot v)^2}{(k^2-(k\cdot v)^2)^3} + \frac{1}{2|k|}\Big(\frac{e^{ik_+T}}{k_+^3} - \frac{e^{ik_-T}}{k_-^3}\Big)$$
$$- \quad \frac{iT}{2|k|}\Big(\frac{e^{ik_+T}}{k_+^2} - \frac{e^{ik_-T}}{k_-^2}\Big) - \frac{T^2}{4|k|}\Big(\frac{e^{ik_+T}}{k_+} - \frac{e^{ik_-T}}{k_-}\Big)\Big].$$

The integrals containing T are again oscillatory and vanish as $T \to \infty$. Therefore, comparing with (8.4), we conclude

$$I_2(T) \to -m_{\mathrm{f}}(\dot q(t))\ddot q(t) \quad \text{as} \quad T \to \infty\,. \tag{8.7}$$

Hence (5.2) follows from (8.6) and (8.7).

9 Appendix B. Hamiltonian structure

Energy-momentum relation: In Section 6 we used the identity

$$v\, dP_s(v) = 3D\nabla E_s(v)\,, \quad |v| < 1\,. \tag{9.1}$$

While it follows from the explicit expressions (8.2), (8.3), resp. (1.6), (1.8), this identity should be understood as a direct consequence of the conservation of total momentum, i.e., of the translation invariance of the system (1.1).

Our argument uses the canonical transformation

$$\begin{aligned} \mathcal{T} : (\phi,\pi,q,p) \ \mapsto \ & (\Phi(x),\Pi(x),Q,P) \\ & = 3D(\phi(q+x),\pi(q+x),q,p-<\pi(x),\nabla\phi(x)>)\,. = 20 \end{aligned}$$

In the new variables the Hamiltonian (1.2) reads

$$\begin{aligned} \mathcal{H}_P(\Phi,\Pi) \ &= 3D \quad \mathcal{H}\Big(\Phi(x-Q),\Pi(x-Q),Q,P+<\Pi(x),\nabla\Phi(x)>\Big) \\ &= 3D \quad \int d^3x\,\Big(\frac{1}{2}|\Pi(x)|^2 + \frac{1}{2}|\nabla\Phi(x)|^2 + \Phi(x)\rho(x)\Big) \\ &\qquad + \Big(1 + \Big(P+<\Pi(x),\nabla\Phi(x)>\Big)^2\Big)^{1/2}\,. \end{aligned}$$

$\mathcal{H}_P$ is bounded from below and has its unique minimum at the point (ϕ_v,π_v), the soliton at velocity $v = 3Dv_s(P)$, with minimal value $\mathcal{H}_P(\phi_v,\pi_v) =$

$3DE_s(v) + \mathcal{H}(S_0)$ [12]. Differentiating in v we obtain

$$\nabla E_s(v) \quad = 3D \quad \langle \frac{\delta \mathcal{H}_P}{\delta \Phi}(\phi_v, \pi_v), \nabla_v \phi_v \rangle + \langle \frac{\delta \mathcal{H}_P}{\delta \Pi}(\phi_v, \pi_v), \nabla_v \pi_v \rangle$$

$$+\nabla_P \mathcal{H}_P(\phi_v, \pi_v)\, dP_s(v) = 3Dv\, dP_s(v)\,,$$

since (ϕ_v, π_v) is a critical point of $\mathcal{H}_P$ and the first two terms vanish, while $v = 3D\dot{Q} = 3D\nabla_P \mathcal{H}_P(\phi_v, \pi_v)$ because $\mathcal{T}$ is a canonical transformation.

Correspondence of Hamiltonian structures: The definitions (1.5), (1.8) and (1.9) imply that the Hamiltonian functional $\mathcal{H}_\varepsilon$ of (1.11) restricted to the soliton $S_v = 3D(\phi_v(x-q), \pi_v(x-q), q, p)$ becomes

$$\mathcal{H}_\varepsilon(S_v) = 3DE(P) + V(\varepsilon q) + \mathcal{H}(S_0) = 3DH_{\text{eff}}(q, P) + \mathcal{H}(S_0) \qquad (9.2)$$

with $P = 3DP_s(v)$. Thus the effective Hamiltonian can be understood as the restriction of $\mathcal{H}_\varepsilon$ to the soliton manifold. We need in addition the appropriate choice of the canonical variables to write the Hamilton's equations in standard form (1.15). On general grounds one expects the conserved quantities to play a distinguished role. In our case this suggests P and q as canonical variables. The next lemma gives an inherent geometrical meaning to this choice, which might be valuable in a more general context.

Lemma 9.1 *The canonical structure $P\, dq$ on the soliton manifold $\mathcal{S}$ is the restriction of the full canonical form $p\, dq + < \phi, d\pi >$, i.= e.*

$$P\, dq = 3D(p\, dq + \langle \phi, d\pi \rangle)\Big|_{\mathcal{S}}\,.$$

Proof: We have $p\, dq + \langle \phi, d\pi \rangle = 3DP\, dQ + \langle \Phi, d\Pi \rangle$, since $\mathcal{T}$ is a canonical transformation, and

$$\langle \Phi, d\Pi \rangle\Big|_{\mathcal{S}} = 3D\langle \phi_v, d\pi_v \rangle = 3D\langle \phi_v, \nabla_v \pi_v \rangle\, dv = 3D0$$

by antisymmetry in Fourier space and $|\hat{\rho}(-k)| = 3D|\hat{\rho}(k)|$. □

References

[1] M. Abraham, "Theorie der Elektrizität", Band 2: Elektromagnetische Theorie der Strahlung. Teubner, Leipzig, 1905.

[2] D. Bambusi, L. Galgani, Some rigorous results on the Pauli-Fierz model of classical electrodynamics, *Ann. Inst. H. Poincaré, Phys. Theor.* **58** (1993), 155-171.

[3] A. Bensoussan, J.L. Lions, G. Papanicolaou, "Asymptotic Analysis for Periodic Structures", Studies in Mathematics and its Applications, Vol. 5. North-Holland, Amsterdam, 1978.

[4] V.S. Buslaev, G.S. Perelman, On nonlinear scattering of states which are close to a soliton, pp. 49-63 in: Méthodes Semi-Classiques, Vol.2 Colloque International (Nantes, juin 1991), *Asterisque* **208** (1992).

[5] V.S. Buslaev, G.S. Perelman, On the stability of solitary waves for nonlinear Schrödinger equations, *Trans. Amer. Math. Soc.* **164** (1995), 75-98.

[6] E.B. Davies, "Quantum Theory of Open Systems". Academic Press, London, 1976.

[7] M. Grillakis, J. Shatah, W.A. Strauss, Stability theory of solitary waves in the presence of symmetry I and II, *J. Func. Anal.* **74** (1987), 160-197; **94** (1990), 308-348.

[8] A. De Masi, P.A. Ferrari, S. Goldstein, W.D. Wick, An invariance principle for reversible Markov processes. Application to random environments, *J. Stat. Phys.* **55** (1989), 787-855.

[9] J. Fleckinger, A. Komech, On soliton-like asymptotics for 1D nonlinear reaction systems, *Russian J. Math. Phys.* **5** (1997), 295-307.

[10] G.A. Hagedorn, A time dependent Born-Oppenheimer approximation, *Comm. Math. Phys.* **77** (1980), 1-19.

[11] R.L. Jerrard, H.M. Soner, Dynamics of Ginzburg-Landau Vortices, preprint, 1995.

[12] A. Komech, H. Spohn, Soliton-like asymptotics for a classical particle interacting with a scalar wave field, *Nonlinear Anal.* **33** (1998), 13-24.

[13] A. Komech, H. Spohn, M. Kunze, Long-time asymptotics for a classical particle interacting with a scalar wave field, *Comm. Partial Differential Equations* **22** (1997), 307-335.

[14] H.A. Kramers, Non-relativistic Quantum-Electrodynamics and correspondence Principle, pp. 241-265 in: Solvay Conference 1948, Rapport et Discussions, Bruxelles, 1950 (pp. 845-869 in: H.A. Kramers, Collected Scientific Papers. North-Holland, Amsterdam, 1956).

[15] J.L. Lions, "Problèmes aux Limites dans les Equations aux Dérivées Partielles". Presses de l'Univ. Montréal, Montréal, 1962.

[16] H.A. Lorentz, "Theory of Electrons", 2nd edition 1915. Reprinted by Dover, New York, 1952.

[17] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, "Theory of Solitons: The Inverse Scattering Method". Consultants Bureau, 1984.

[18] D. Robert, "Autour de l'Approximation Semi-Classique", Progress in Mathematics, Vol. 68. Birkhäuser, Basel, 1987.

[19] A. Soffer, M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, *Comm. Math. Phys.* **133** (1990), 119-146.

[20] A. Soffer, M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data, *J. Differential Equations* **98** (1992), 376-390.

[21] H. Spohn, "Large Scale Dynamics of Interacting Particles'= '. Springer, Berlin, 1991.

[22] H. Spohn, Long time asymptotics for quantum particles in a periodic potential, *Phys. Rev. Lett.* **77** (1996), 1198-1201.