A comparison of two discrete mKdV equations

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Abstract— We consider here two discrete versions of the modified KdV equation. In one case, some solitary wave solutions, Bäcklund transformations and integrals of motion are known. In the other one, only solitary wave solutions were given, and we supply the corresponding results for this equation. We also derive the integrability of the second equation and give a transformation which links the two models.

Introduction— In a recent paper, Takeno [1] considered two nonlinear excitation transfer models in a d-dimensional version of the simple cubic lattice with nearest neighbour interactions. These models highlight two discrete versions of the mKdV equation

$$\dot{\phi_n} = (\phi_{n+1} - \phi_{n-1})(1+\phi_n^2), \tag{1}$$
$$\dot{\phi_n} = (\phi_{n+1} - \phi_{n-1})(1+\phi_n)^2. \tag{2}$$

The first equation which is integrable has been widely studied (see [2–4]) and is known to be a discrete mKdV equation. But, the second equation is also related to the continuous mKdV equation (see [1,5]). Indeed, in the continuous limit (the lattice spacing h becomes zero and $x = nh$), (1) and (2) become

$$\frac{\partial\Phi}{\partial t} = 2(1+\Phi^2)\frac{\partial\Phi}{\partial x} + \frac{1}{3}\frac{\partial^3\Phi}{\partial x^3}, \tag{3}$$
$$\frac{\partial\Phi}{\partial t} = 2(1+\Phi)^2\frac{\partial\Phi}{\partial x} + \frac{1}{3}\frac{\partial^3\Phi}{\partial x^3}, \tag{4}$$

respectively. Applying the transformation $\Phi(x,t) = \Psi(\xi,t) - 1$ where $\xi = x - 2t$, (4) becomes (3). Therefore, the continuous versions of (1) and (2) are linked by a simple Galilean transformation. But, this transformation can not be applied in the discrete case. One result of this paper is to find the transformation which links (1) and (2).

Travelling wave solutions— For completeness, we first review the known travelling wave solutions of (1) and (2) of the form $\phi_n(t) = \Phi(n - ct)$. Some solutions are given by Takeno [1,5] for both equations and Xiao [2] for eq.(2). Both equations have bright solitary wave solutions with a localised shape; for (1) and (2) respectively we have [1]

$$\phi_n = \pm\,\sinh(C)\,\mathrm{sech}[C(n-n_0) - \omega t], \tag{5}$$
$$\omega = -2\sinh C, \tag{6}$$
$$\phi_n = \sinh^2(C)\,\mathrm{sech}^2[C(n-n_0) - \omega t], \tag{7}$$
$$\omega = -\sinh 2C. \tag{8}$$

Besides, (2) has the singular solution [1]

$$\phi_n = -\sinh^2(C)\,\mathrm{cosech}^2[C(n-n_0) - \omega t].$$

We notice that the solitary wave solutions of (1) are closer to the continuous case which has sech-shaped solitary wave solutions. But, (2) is interesting because it has non vanishing waves with a sech^2 shape. Xiao [2] also gives solitary wave solutions with nonzero boundary conditions of (2) by using the real exponential approach

$$\phi_n = a_0 + (1+a_0)\,\sinh^2(C)\,\mathrm{sech}^2[C(n-n_0) - \omega t],$$
$$\phi_n = a_0 - (1+a_0)\,\sinh^2(C)\,\mathrm{cosech}^2[C(n-n_0) - \omega t],$$

where $\omega = -(1+a_0)^2\sinh 2C$. For (1), there is no known corresponding solutions with a sech shape with nonzero boundary conditions.

Lax pairs— We now consider the Lax pairs for (1) and (2). Recall that for discrete equation $\partial\phi_n/\partial t = F(\phi_n, \phi_{n-1}, \phi_{n+1})$, a Lax pair is a set of two matrices (M_n, N_n) which satisfy

$$V_{n+1} = M_n V_n,$$
$$\partial V_n/\partial t = N_n V_n,$$

where $N_n = \left((Font) LaTeX Font Warning : Command invalid in math mode 1012pt c \begin{matrix} A_n \\ B_n \\ C_n \\ D_n \end{matrix} \right)$ and $V_n = \left((Font) LaTeX Font Warning : Command invalid in math mode 1012pt c \begin{matrix} V_{n,1} \\ V_{n,2} \end{matrix} \right)$, and also, the compatibility condition

$$\frac{\partial}{\partial t}(SV_n) = S\frac{\partial V_n}{\partial t},$$

where S is the shift operator $SV_n = V_{n+1}$. In this case, this condition is $\partial M_n/\partial t = N_{n+1}M_n - M_nN_n$ and gives the dynamical equation for ϕ_n. For Equation (1), the Lax pair is given by Ablowitz and Segur [6]:

$$M_n = \left((Font) LaTeX Font Warning : Command invalid in math mode 1012pt c \begin{matrix} \eta \\ \phi_n \\ -\phi_n \\ 1/\eta \end{matrix} \right) \quad N_n = \left((Font) LaTeX Font Warning : Command invalid in math mode 1012pt c \begin{matrix} A_n \\ B_n \\ C_n \\ D_n \end{matrix} \right),$$

$$\text{where } A_n = \eta^2 + \phi_{n-1}\phi_n, \quad B_n = \phi_n\eta + \phi_{n-1}/\eta,$$
$$C_n = -\phi_{n-1}\eta - \phi_n/\eta, \quad D_n = \eta^{-2} + \phi_{n-1}\phi_n,$$

and η is the spectral parameter associated with the problem. It is straightforward to show that the Lax pair for

(2) is

$$M_n = \begin{pmatrix} (Font)LaTeXFontWarning: Commandinvalidinmathmode1012ptc & \begin{matrix} \eta \\ 1+\phi_n \\ -(1+\phi_n) \\ 0 \end{matrix} \end{pmatrix}, \quad N_n = \begin{pmatrix} (Font)LaTeXFontWarning: Commandinvalidinmathmode1012ptc & \begin{matrix} A_n \\ B_n \\ C_n \\ D_n \end{matrix} \end{pmatrix},$$

where $A_n = (1+\phi_n)(1+\phi_{n-1}) + \eta^2, \quad B_n = \eta(1+\phi_n).$
$C_n = -\eta(1+\phi_{n-1}), \quad D_n = (1+\phi_n)(1+\phi_{n-1}).$

This proves the complete integrability of both differential-difference equations.

Integrals of motion— We now consider integrals of motion, which are easy to obtain (see Ablowitz and Ladik [4], Homma and Takeno [5], Scharf and Bishop [7]) for an infinite 1D-lattice for (1)

$$N_1 = \sum_{n=-\infty}^{+\infty} \ln(1+\phi_n^2), \quad N_1' = \sum_{n=-\infty}^{+\infty} \arctan\phi_n. \tag{9}$$

In a similar way for eq.(2), we have

$$N_2 = \sum_{n=-\infty}^{+\infty} \ln(1+\phi_n), \quad N_2' = \sum_{n=-\infty}^{+\infty} \frac{\phi_n}{1+\phi_n}. \tag{10}$$

Other conserved quantities of (1) are given by Ablowitz and Ladik [4]: for instance,

$$C_1 = \sum_{n=-\infty}^{+\infty} \phi_n\phi_{n-1}, \tag{11}$$

$$C_2 = \sum_{n=-\infty}^{+\infty} \left\{\phi_n\phi_{n-2}(1+\phi_{n-1}^2) + \phi_n^2\phi_{n-1}^2/2\right\}. \tag{12}$$

Bäcklund transformations— The Bäcklund transformation for (1) was derived by Yang and Schmid [3], who give the two following Miura transformations

$$\phi_n = \frac{u_{n+1}/\eta - u_n\eta}{1+u_nu_{n+1}} = \Theta_1(u_n, u_{n+1}), \tag{13}$$

$$\phi_{n+1}' = \frac{u_n/\eta - u_{n+1}\eta}{1+u_nu_{n+1}} = \Theta_2(u_n, u_{n+1}). \tag{14}$$

where $u_n = V_{n,1}/V_{n,2}$ is a solution of the following differential-difference equation

$$\frac{\partial u_n}{\partial t} = \frac{u_{n+1} - u_n\left(\eta^2+\eta^{-2}\right) - u_n^3}{1+u_nu_{n+1}} - \frac{u_{n-1} - u_n\left(\eta^2+\eta^{-2}\right) - u_n^3}{1+u_nu_{n-1}}$$

Therefore, $\Phi' = (\Theta_2 \circ \Theta_1^{-1})\Phi$ defines a Bäcklund transformation for (1).

Applying the same method to (2), we find two Miura transformations which define a Bäcklund transformation for this equation

$$\phi_n = -1 - \eta\frac{v_n}{1+v_nv_{n+1}} = \tilde{\Theta_1}(v_n, v_{n+1}), \tag{15}$$

$$\phi_{n+1}' = -1 - \eta\frac{v_{n+1}}{1+v_nv_{n+1}} = \tilde{\Theta_2}(v_n, v_{n+1}). \tag{16}$$

where $v_n = V_{n,1}/V_{n,2}$ is a solution of the following differential-difference equation:

$$\frac{\partial v_n}{\partial t} = \eta^2 v_n^2\left(\frac{v_{n+1}}{1+v_nv_{n+1}} - \frac{v_{n-1}}{1+v_nv_{n-1}}\right)$$

Therefore, $\Phi' = (\tilde{\Theta_2} \circ \tilde{\Theta_1}^{-1})\Phi$ defines a Bäcklund transformation for (2).
These Bäcklund transformations link two solutions of the same differential-difference equation.

Miura-type transformation— Finally, we derive a transformation which links the two discrete mKdV equations under consideration, in the same way as their continuous versions are linked by a Galilean transformation. Unfortunately, some features of (2) are not similar to those of (1) (for instance, the travelling wave solutions $(5,6)$ or the integrals of motion $(7,8)$). So, the transformation which links both discrete equations is more difficult to obtain than the continuous case.
The idea is to link both equations with a discrete version of the KdV equation by two Miura transformations.
If we assume that

$$u_n = (1+i\phi_n)(1-i\phi_{n+1}), \tag{17}$$

(1) becomes by a straightforward calculation

$$\dot{u_n} = u_n(u_{n+1} - u_{n-1}). \tag{18}$$

This defines a Miura transformation $\hat{\Theta}_1$ which maps a solution of (13) into a solution of (1).
On the other hand, if we define

$$u_n = (1+\phi_n)(1+\phi_{n+1}), \tag{19}$$

(2) becomes (13). This also defines a Miura transformation $\hat{\Theta}_2$ which maps a solution of (13) into a solution of (2). Therefore, the transformation $\Phi' = (\hat{\Theta}_2 \circ \hat{\Theta_1}^{-1})\Phi$ maps a solution of (1) into a solution of (2). In theory, this transformation could be used to derive the results above, but such calculations do not appear to be straightforward.

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