Correlations in the impenetrable electron gas

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Abstract

We consider non-relativistic electrons in one dimension with infinitely strong repulsive delta function interaction. We calculate the long-time, large-distance asymptotics of field-field correlators in the gas phase. The gas phase at low temperatures is characterized by the ideal gas law. We calculate the exponential decay, the power law corrections and the constant factor of the asymptotics. Our results are valid at any temperature. They simplify at low temperatures, where they are easily recognized as products of free fermionic correlation functions with corrections arising due to the interaction.

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Introduction

Non-relativistic electrons in one dimension, interacting strictly locally through a delta potential constitute one of the important exactly solvable many-body systems with applications in solid state physics. Historically this was the first system to be solved exactly by nested Bethe ansatz [1, 2]. Soon after an exact formulation of its thermodynamics was obtained [3]. In this article we address the problem of the exact calculation of its correlation functions. Our work is based on a recently derived determinant representation for the field-field correlators [4, 5] valid in the case of infinite repulsion. The determinant representation gives the field-field correlators as functions of space and time variables x and t in a grand canonical description, i.e. the correlators depend on temperature T, chemical potential μ and on the external magnetic field B. We study the the long-time, large-distance asymptotics of the field-field correlators. We shall consider the limit $t \to \infty$ for fixed ratio $k_0 = x/2t$.

Let us write the Hamiltonian in terms of canonical anticommuting quantum fields $\psi_\alpha(x,t)$ ($\alpha = \uparrow, \downarrow$),

$$H = \int_{-\infty}^{\infty} dx \left\{ \partial_x \psi_\alpha^\dagger \partial_x \psi_\alpha + c : \left(\psi_\alpha^\dagger \psi_\alpha \right)^2 : - \mu\, \psi_\alpha^\dagger \psi_\alpha + B(\psi^\dagger \sigma^z \psi) \right\} \quad . \tag{1}$$

Here σ^z is a Pauli matrix, and c is the coupling constant. In the following we shall consider infinite repulsion $c = +\infty$. Then there is an explicit expression for the pressure as a function of temperature, chemical potential and magnetic field [3],

$$P = \frac{T}{2\pi} \int_{-\infty}^{\infty} dk \ln \left(1 + e^{(\mu + B - k^2)/T} + e^{(\mu - B - k^2)/T} \right) \quad , \tag{2}$$

which may serve as thermodynamic potential. Note that this expression is formally the same as for a free Fermi gas with effective chemical potential $\beta T = \mu + T \ln(2\mathrm{ch}(B/T))$. Hence there are two different zero temperature phases depending on whether $\lim_{T \to 0+} \beta T = \mu + |B|$ is positive or negative. If $\mu + |B| > 0$, the density $D = \partial P / \partial \mu$ has a positive limit as the temperature goes to zero. If $\mu + |B| < 0$, the density at zero temperature vanishes. This is the case we are interested in. We shall assume throughout the remainder of this article that $\beta < 0$. We call the phase determined by this condition the gas phase, since in this phase pressure and density at low temperatures are related by the ideal gas law,

$$P = DT \quad . \tag{3}$$

As the temperature increases the density becomes large, and the ideal gas law does not hold anymore. There is a macroscopic number of electrons in the system, which may carry a current. At strictly zero temperature there are no particles in the system, and the system is an insulator.

We shall consider the following two-point correlation functions of local fields,

$$G^+_{\alpha\beta}(x,t) = \frac{\mathrm{tr}\left(e^{-H/T}\,\psi_\alpha(x,t)\psi^\dagger_\beta(0,0)\right)}{\mathrm{tr}\left(e^{-H/T}\right)} \quad , \tag{4}$$

$$G^-_{\alpha\beta}(x,t) = \frac{\mathrm{tr}\left(e^{-H/T}\,\psi^\dagger_\alpha(x,t)\psi_\beta(0,0)\right)}{\mathrm{tr}\left(e^{-H/T}\right)} \quad , \tag{5}$$

where $\alpha, \beta =\uparrow, \downarrow$. Due to conservation of the total spin $G^\pm_{\uparrow\downarrow}(x,t) = G^\pm_{\downarrow\uparrow}(x,t) = 0$. Furthermore, due to the invariance of the Hamiltonian (1) under the transformation $\uparrow \rightleftharpoons \downarrow$, $B \rightleftharpoons -B$, we have the identity $G^\pm_{\downarrow\downarrow}(x,t) = G^\pm_{\uparrow\uparrow}(x,t)|_{B\rightleftharpoons -B}$. Hence, we can restrict our attention to the correlation functions $G^\pm_{\uparrow\uparrow}(x,t)$.

Determinant representation

Let us recall the determinant representation for the correlation functions $G^\pm_{\uparrow\uparrow}(x,t)$, which was derived in [4, 5]. We shall basically follow the account of [6]. Yet, it turns out to be very useful for further calculations to rescale the variables and the correlation functions. The rescaling

$$x_r = -\sqrt{T}x/2 \quad , \quad t_r = Tt/2 \quad , \tag{6}$$

$$g^\pm = G^\pm_{\uparrow\uparrow}/\sqrt{T} \quad , \tag{7}$$

$$h = B/T \tag{8}$$

removes the explicit temperature dependence from all expressions. Furthermore, it will allow us to make close contact with results which were obtained for the impenetrable Bose gas [7, 8, 9]. The index 'r' in (6) stands for 'rescaled'. For the sake of simplicity we shall suppress this index in the following three sections. We shall come back to physical space and time variables only in the last section, where we consider the low temperature limit.

The rescaled correlation functions g^+ and g^- in the rescaled variables can be expressed as [6],

$$g^+(x,t) = \frac{-e^{2\mathrm{i}t(\beta-h-\ln(2\mathrm{ch}(h)))}}{2\pi} \int_{-\pi}^{\pi} d\eta \, \frac{F(\gamma,\eta)}{1-\cos(\eta)} \, b_{++}\mathrm{det}\left(\hat{I}+\hat{V}\right) \quad , \tag{9}$$

$$g^-(x,t) = \frac{e^{-2\mathrm{i}t(\beta-h-\ln(2\mathrm{ch}(h)))}}{4\pi\gamma} \int_{-\pi}^{\pi} d\eta \, F(\gamma,\eta) B_{--}\mathrm{det}\left(\hat{I}+\hat{V}\right) \quad . \tag{10}$$

Here γ and $F(\gamma,\eta)$ are elementary functions,

$$\gamma = 1+e^{2h} \quad , \tag{11}$$

$$F(\gamma,\eta) = 1+\frac{e^{\mathrm{i}\eta}}{\gamma-e^{\mathrm{i}\eta}}+\frac{e^{-\mathrm{i}\eta}}{\gamma-e^{-\mathrm{i}\eta}} \quad . \tag{12}$$

$\mathrm{det}(\hat{I}+\hat{V})$ is the Fredholm determinant of the integral operator $\hat{I}+\hat{V}$, where $\hat{I}$ is the identity operator, and $\hat{V}$ is defined by its kernel $V(\lambda,\mu)$. λ and μ are complex variables, and the path of integration is the real axis. In order to define $V(\lambda,\mu)$ we have to introduce certain auxiliary functions. Let us define

$$\tau(\lambda) = \mathrm{i}(\lambda^2 t+\lambda x) \quad , \tag{13}$$

$$\vartheta(\lambda) = \frac{1}{1+e^{\lambda^2-\beta}} \quad , \tag{14}$$

$$E(\lambda) = \mathrm{p.v.}\int_{-\infty}^{\infty} d\mu \, \frac{e^{-2\tau(\mu)}}{\pi(\mu-\lambda)} \quad , \tag{15}$$

$$e_-(\lambda) = \sqrt{\frac{\vartheta(\lambda)}{\pi}}e^{\tau(\lambda)} \quad , \tag{16}$$

$$e_+(\lambda) = \frac{1}{2}\sqrt{\frac{\vartheta(\lambda)}{\pi}}e^{-\tau(\lambda)}\left\{(1-\cos(\eta))e^{2\tau(\lambda)}E(\lambda)+\sin(\eta)\right\} \quad . \tag{17}$$

Note that $\vartheta(\lambda)$ is the Fermi weight. $V(\lambda,\mu)$ can be expressed in terms of e_+ and e_-,

$$V(\lambda,\mu) = \frac{e_+(\lambda)e_-(\mu)-e_+(\mu)e_-(\lambda)}{\lambda-\mu} \quad . \tag{18}$$

Denote the resolvent of $\hat{V}$ by $\hat{R}$,

$$\left(\hat{I}+\hat{V}\right)\left(\hat{I}-\hat{R}\right) = \left(\hat{I}-\hat{R}\right)\left(\hat{I}+\hat{V}\right) = \hat{I} \quad . \tag{19}$$

Then $\hat{R}$ is an integral operator with symmetric kernel,

$$R(\lambda,\mu) = \frac{f_+(\lambda)f_-(\mu) - f_+(\mu)f_-(\lambda)}{\lambda-\mu} \quad , \tag{20}$$

which is of the same form as $V(\lambda,\mu)$. The functions $f_\pm$ are obtained as the solutions of the integral equations

$$f_\pm(\lambda) + \int_{-\infty}^{\infty} d\mu\, V(\lambda,\mu) f_\pm(\mu) = e_\pm(\lambda) \quad . \tag{21}$$

We may now define the potentials

$$B_{ab} = \int_{-\infty}^{\infty} d\lambda\, e_a(\lambda) f_b(\lambda) \quad , \quad C_{ab} = \int_{-\infty}^{\infty} d\lambda\, \lambda e_a(\lambda) f_b(\lambda) \tag{22}$$

for $a, b = \pm$. B_{--} enters the definition of $g^-(x,t)$, equation (10). b_{++} in (9) is defined as

$$b_{++} = B_{++} - G(x,t) \quad , \tag{23}$$

where

$$G(x,t) = \frac{1-\cos(\eta)}{2\pi} \int_{-\infty}^{\infty} d\lambda\, e^{-2\tau(\lambda)} = \frac{(1-\cos(\eta))e^{-\mathrm{i}\pi/4}}{2\sqrt{2\pi t}} e^{\mathrm{i}x^2/2t} \quad . \tag{24}$$

The remaining potentials B_{ab} and C_{ab} will be needed later.

It is instructive to compare the determinant representation (9) for the correlation function $g^+(x,t)$ with the corresponding expression for impenetrable Bosons (cf e.g. page 345 of [9]). The main formal differences are the occurrence of the η-integral in (9) and the occurrence of η in the definition of e_+. As can be seen from the derivation of (9) in [5], the η-integration is related to the spin degrees of freedom. For $\eta = \pm\pi$ the expression $-\frac{1}{2}e^{2\mathrm{i}\beta t} b_{++} \det(\hat{I} + \hat{V})$ agrees with the field-field correlator for impenetrable Bosons (remember, however, the different physical meaning of β).

Differential equations

As in case of impenetrable Bosons [7, 9] it is possible to derive a set of integrable nonlinear partial differential equations for the potentials b_{++} and B_{--} and to express the logarithmic derivatives of the Fredholm determinant $\det(\hat{I} + \hat{V})$ in terms of the potentials B_{ab} and C_{ab}.

The functions $f_\pm$ satisfy linear differential equations with respect to the variables x, t, and β,

$$\hat{L}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \hat{M}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \hat{N}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = 0 \quad . \tag{25}$$

The Lax operators $\hat{L}$, $\hat{M}$ and $\hat{N}$ are given as

$$\begin{aligned} \hat{L} &= \partial_x + \mathrm{i}\lambda\sigma^z - 2\mathrm{i}Q \quad , & (26)\\ \hat{M} &= \partial_t + \mathrm{i}\lambda^2\sigma^z - 2\mathrm{i}\lambda Q + \partial_x U \quad , & (27)\\ \hat{N} &= 2\lambda\partial_\beta + \partial_\lambda + 2\mathrm{i}t\lambda\sigma^z + \mathrm{i}x\sigma^z - 4\mathrm{i}tQ - 2\partial_\beta U \quad , & (28) \end{aligned}$$

where the matrices Q and U are defined according to

$$Q = \begin{pmatrix} 0 & b_{++} \\ B_{--} & 0 \end{pmatrix} \quad , \quad U = \begin{pmatrix} -B_{+-} & b_{++} \\ -B_{--} & B_{+-} \end{pmatrix} \quad . \tag{29}$$

Mutual compatibility of the linear differential equations (25) leads to a set of nonlinear partial differential equations for the potentials b_{++} and B_{--}. Instead of using b_{++} and B_{--}, it is convenient to introduce the following notations (cf. p. 352 of the book [9]),

$$g_+ = e^{2\mathrm{i}t\beta} b_{++} \quad , \quad g_- = e^{-2\mathrm{i}t\beta} B_{--} \quad , \tag{30}$$

$$n = g_+ g_- = b_{++} B_{--} \quad , \quad p = g_- \partial_x g_+ - g_+ \partial_x g_- \quad . \tag{31}$$

The space and time evolution is driven by the separated nonlinear Schrödinger equation,

$$\begin{aligned} -\mathrm{i}\partial_t g_+ &= 2\beta g_+ + \tfrac{1}{2}\partial_x^2 g_+ + 4g_+^2 g_- \quad , & (32)\\ \mathrm{i}\partial_t g_- &= 2\beta g_- + \tfrac{1}{2}\partial_x^2 g_- + 4g_-^2 g_+ \quad . & (33) \end{aligned}$$

p and n satisfy the continuity equation

$$-2\mathrm{i}\partial_t n = \partial_x p \quad . \tag{34}$$

The equations containing derivatives with respect to β are

$$\begin{aligned} -\mathrm{i}\partial_t \varphi + 4\partial_\beta p &= 0 \quad , & (35)\\ \partial_x \varphi + 8\partial_\beta n + 2 &= 0 \quad . & (36) \end{aligned}$$

Here the function φ is defined as

$$\varphi(x,t,\beta) = \frac{\partial_\beta \partial_x g_+}{g_+} = \frac{\partial_\beta \partial_x g_-}{g_-} \quad . \tag{37}$$

To describe the correlation functions (9) and (10) one has to relate the Fredholm determinant $\det(\hat{I}+\hat{V})$ and the potentials B_{ab} and C_{ab}. Let us use the abbreviation $\sigma(x,t,\beta) = \ln\det(\hat{I}+\hat{V})$. The logarithmic derivatives of the Fredholm determinant with respect to x, t and β are

$$\begin{aligned}
\partial_x \sigma &= -2\mathrm{i} B_{+-} \quad , &(38)\\
\partial_t \sigma &= -2\mathrm{i}(C_{+-} + C_{-+} + G(x,t)B_{--}) \quad , &(39)\\
\partial_\beta \sigma &= -2\mathrm{i}t\partial_\beta(C_{+-} + C_{-+} + G(x,t)B_{--}) - 2\mathrm{i}x\partial_\beta B_{+-} - 2(\partial_\beta B_{+-})^2 \\
&\quad -2\mathrm{i}t(B_{--}\partial_\beta b_{++} - b_{++}\partial_\beta B_{--}) + 2(\partial_\beta b_{++})(\partial_\beta B_{--}) \quad . &(40)
\end{aligned}$$

For the calculation of the asymptotics of the Fredholm determinant we further need the second derivatives of σ with respect to space and time,

$$\begin{aligned}
\partial_x^2 \sigma &= 4B_{--}b_{++} \quad , &(41)\\
\partial_x \partial_t \sigma &= 2\mathrm{i}(B_{--}\partial_x b_{++} - b_{++}\partial_x B_{--}) \quad , &(42)\\
\partial_t^2 \sigma &= 2\mathrm{i}(B_{--}\partial_t b_{++} - b_{++}\partial_t B_{--}) + 8B_{--}^2 b_{++}^2 + 2(\partial_x B_{--})(\partial_x b_{++}) . &(43)
\end{aligned}$$

Note that

$$\lim_{\beta\to-\infty} \sigma = 0 \quad . \tag{44}$$

This follows from $\lim_{\beta\to-\infty}\vartheta(\lambda) = 0$ and is important for fixing the integration constant in the calculation of the asymptotics of the determinant.

The remarkable result of this section is that the differential equations for b_{++} and B_{--} as well as the expressions (38)-(40) for the logarithmic derivative of the Fredholm determinant are of the same form as in case of impenetrable Bosons [7, 9].

Asymptotics of the correlation functions

The differential equations (32), (33) have many solutions. b_{++} and B_{--} are the particular solution of (32) and (33) which is fixed by the integral equations

(21). Alternatively b_{++} and B_{--} can be obtained by solving a Riemann-Hilbert problem, which was formulated in [6]. The Riemann-Hilbert problem is the appropriate tool to obtain the leading asymptotics of b_{++}, B_{--}, $\partial_x\sigma$ and $\partial_t\sigma$. In fact, using the method first suggested by Manakov [10] and further developed in [11] and in [8], it was shown in [6] that the following estimates hold,

$$b_{++} = \mathcal{O}\left(t^{-\frac{1}{2}}\right) \quad , \quad B_{--} = \mathcal{O}\left(t^{-\frac{1}{2}}\right) \quad , \tag{45}$$

and

$$\partial_x\sigma = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \, \mathrm{sign}(\lambda-\lambda_0)\ln(\varphi(\lambda,\beta)) + \mathcal{O}\left(t^{-\frac{1}{2}}\right) \quad , \tag{46}$$

$$\partial_t\sigma = \frac{2}{\pi}\int_{-\infty}^{\infty} d\lambda \, \lambda \, \mathrm{sign}(\lambda-\lambda_0)\ln(\varphi(\lambda,\beta)) + \mathcal{O}\left(t^{-\frac{1}{2}}\right) \quad . \tag{47}$$

The function $\varphi(\lambda,\beta)$ in the above expressions is defined as

$$\varphi(\lambda,\beta) = 1 + \vartheta(\lambda)\left(e^{-\mathrm{i}\eta\,\mathrm{sign}(\lambda-\lambda_0)} - 1\right) \quad . \tag{48}$$

λ_0 is the stationary point of the phase $\tau(\lambda)$, $\tau'(\lambda_0)=0$, i.e.

$$\lambda_0 = -\frac{x}{2t} \quad . \tag{49}$$

Equations (45)-(49) came out of the direct asymptotic analysis of the Riemann-Hilbert problem. As we shall explain in the following these equations, together with (38)-(43), fix the complete asymptotic expansion of the correlation functions $g^+(x,t)$ and $g^-(x,t)$ up to a single unknown function.

It follows from (45) that b_{++} and B_{--} are a decaying solution of the separated nonlinear Schrödinger equation (32), (33). The form of the complete asymptotic decomposition of the decaying solutions of the separated nonlinear Schrödinger equation is known [12, 13],

$$b_{++} = t^{-\frac{1}{2}}\left(u_0 + \sum_{n=1}^{\infty}\sum_{k=0}^{2n}\frac{\ln^k 4t}{t^n}u_{nk}\right)\exp\left\{\frac{\mathrm{i}x^2}{2t} - \mathrm{i}\nu\ln 4t\right\} \quad , \tag{50}$$

$$B_{--} = t^{-\frac{1}{2}}\left(v_0 + \sum_{n=1}^{\infty}\sum_{k=0}^{2n}\frac{\ln^k 4t}{t^n}v_{nk}\right)\exp\left\{-\frac{\mathrm{i}x^2}{2t} + \mathrm{i}\nu\ln 4t\right\} \quad . \tag{51}$$

Here the quantities u_0, v_0, u_{nk}, v_{nk} and ν are functions of $\lambda_0 = -x/2t$ and of β and η. Inserting the above asymptotic expansions for B_{--} and b_{++} into the differential equations (32), (33) we obtain expressions for u_{nk}, v_{nk} and ν in terms of u_0 and v_0. In other words, the two unknown functions u_0 and v_0 determine the whole asymptotic expansion (50), (51). In particular,

$$u_0 v_0 = -\frac{\nu}{4} \quad . \tag{52}$$

This relation can be used to calculate ν. (41), (50), (51) and (52) imply that

$$\partial_x^2 \sigma = -\frac{\nu}{t} + \mathcal{O}\left(\frac{\ln^2 4t}{t^2}\right) \quad . \tag{53}$$

Differentiating (46) with respect to x and comparing the result with the right hand side of (53) we find

$$\nu = -\frac{1}{2\pi} \ln\left(1 - 2(1 - \cos(\eta))\vartheta(\lambda_0)(1 - \vartheta(\lambda_0))\right) \quad , \tag{54}$$

such that the product $u_0 v_0$ is fixed, and we are indeed left with only one unknown function, say, u_0.

The second logarithmic derivatives of the Fredholm determinant with respect to space and time are given as functions of b_{++} and B_{--} by the equations (41)-(43). Hence, their complete asymptotics, too, is determined by the expansions (50) and (51). We may integrate (41)-(43) to obtain $\partial_x \sigma$ and $\partial_t \sigma$. The integration constant, which is a function of β, is fixed by the leading asymptotics (46), (47). Then, integrating (38)-(40) yields σ up to a numerical constant, which follows from the asymptotic condition (44). The calculation is the same as for the impenetrable Bose gas and can be found on pages 455 - 457 of the book [9]. It results in the asymptotic formula

$$\begin{aligned} \sigma \; = \; & \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, |x + 2\lambda t| \ln(\varphi(\lambda, \beta)) + \frac{\nu^2}{2} \ln 4t + \frac{\nu^2}{2} \\ & + 2\mathrm{i} \int_{-\infty}^{\beta} d\beta \, (u_0 \partial_\beta v_0 - v_0 \partial_\beta u_0) \\ & + \frac{1}{2\pi^2} \int_{-\infty}^{\beta} d\beta \, \left(\partial_\beta \int_{-\infty}^{\infty} d\lambda \, \mathrm{sign}(\lambda - \lambda_0) \ln(\varphi(\lambda, \beta)) \right)^2 \\ & + \mathcal{O}\left(\frac{\ln^4 4t}{t}\right) \quad . \end{aligned} \tag{55}$$

So far the equations (50), (51), (52), (53) and (55) determine the asymptotics of the correlation functions $g^+(x,t)$ and $g^-(x,t)$ up to an unknown function u_0. In order to calculate u_0 one has to reexamine the Riemann-Hilbert problem. The asymptotic calculation of [6] has to be refined. For impenetrable Bosons a similar refinement was achieved in [8]. Fortunately, the result of [8] depends only on some general features of the functions entering the Riemann-Hilbert problem in its canonical form. It turns out that it also applies in our case. We can use equation (6.21) of [8] to obtain

$$u_0 = \sqrt{\frac{\pi}{2}}\,\frac{e^{-\mathrm{i}\frac{3\pi}{4}+\mathrm{i}\Psi_1-\frac{\pi\nu}{2}}}{2\vartheta(\lambda_0)\Gamma(-\mathrm{i}\nu)} \tag{56}$$

$$= \tfrac{1}{2}|\sin(\eta/2)|\sqrt{\nu}\exp((\lambda_0^2-\beta)/2+\mathrm{i}\Psi_0) \quad , \tag{57}$$

where

$$\Psi_0 = -\frac{3\pi}{4}+\arg\Gamma(\mathrm{i}\nu)+\Psi_1 \quad , \tag{58}$$

$$\Psi_1 = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda\, \mathrm{sign}(\lambda-\lambda_0)\ln|\lambda-\lambda_0|\partial_\lambda \ln(\varphi(\lambda,\beta)) \quad . \tag{59}$$

An alternative rigorous derivation of these formulae can be performed via the Deift-Zhou nonlinear steepest descent method [14]. It is also worth comparing equations (57)-(59) with the asymptotic formula for the solution of the Cauchy problem of the NLS equation obtained by Zakharov and Manakov [15] (see also [12, 13] and [16]).

Let us summarize our results. Inserting (50), (51), (55) and (57) into (9) and (10) we obtain the following expressions for the leading asymptotics of the correlation function,

$$\begin{aligned} g^+(x,t) &= e^{\mathrm{i}x^2/2t+2\mathrm{i}t\beta}e^{-2\mathrm{i}t(h+\ln(2\mathrm{ch}(h)))} \\ &\qquad \int_{-\pi}^{\pi} d\eta\,\frac{F(\gamma,\eta)}{1-\cos(\eta)}\,C^+(\lambda_0,\beta,\eta)\,(4t)^{\frac{1}{2}(\nu-\mathrm{i})^2} \\ &\qquad \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda\,|x+2\lambda t|\ln(\varphi(\lambda,\beta))\right\} \quad , \end{aligned} \tag{60}$$

$$g^-(x,t) = e^{-\mathrm{i}x^2/2t-2\mathrm{i}t\beta}e^{2\mathrm{i}t(h+\ln(2\mathrm{ch}(h)))}$$

$$\int_{-\pi}^{\pi} d\eta \, \frac{F(\gamma,\eta)}{2\gamma} \, C^-(\lambda_0,\beta,\eta)\,(4t)^{\frac{1}{2}(\nu+\mathrm{i})^2}$$
$$\exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \, |x+2\lambda t| \ln(\varphi(\lambda,\beta))\right\} \quad , \tag{61}$$

where

$$\begin{aligned} C^+(\lambda_0,\beta,\eta) &= -|\sin(\eta/2)|\frac{\sqrt{\nu}}{2\pi}\exp\left\{\frac{1}{2}(\lambda_0^2-\beta)+\mathrm{i}\Psi_0+\frac{\nu^2}{2}\right. \\ &\quad -\int_{-\infty}^{\beta} d\beta\,(\mathrm{i}\nu/2+\nu\partial_\beta\Psi_0) \qquad (62) \\ &\quad \left.+\frac{1}{2\pi^2}\int_{-\infty}^{\beta} d\beta\,\left(\partial_\beta\int_{-\infty}^{\infty} d\lambda\,\mathrm{sign}(\lambda-\lambda_0)\ln(\varphi(\lambda,\beta))\right)^2\right\}, \\ C^-(\lambda_0,\beta,\eta) &= C^+(\lambda_0,\beta,\eta)\,\exp(-(\lambda_0^2-\beta)-2\mathrm{i}\Psi_0)/\sin^2(\eta/2) \quad . \qquad (63) \end{aligned}$$

These equations are valid for large t and fixed finite ratio $\lambda_0 = -x/2t$. Correlations in the pure space direction $t = 0$ will be discussed in a separate publication. We would like to emphasize that (60) and (61) hold for arbitrary temperatures. Note that there is no pole of the integrand at $\eta = 0$, since $\sqrt{\nu} \sim |\eta|$ for small η and thus $C^+(\lambda_0,\beta,\eta) \sim \eta^2$.

Asymptotics in the low temperature limit

For the following steepest descent calculation we transform the η-integrals in (60), (61) into complex contour integrals over the the unit circle, setting $z = e^{\mathrm{i}\eta}$. Since we would like to consider low temperatures, we have to restore the explicit temperature dependence by scaling back to the physical space and time variables x and t and to the physical correlation functions $G^\pm_{\uparrow\uparrow}$. Recall that in the previous sections we have suppressed an index 'r' referring to 'rescaled'. Let us restore this index in order to define $k_0 = \lambda_0\sqrt{T} = x/2t$, $\vartheta(k) = \vartheta_r(k/\sqrt{T})$, $\varphi(k,\beta) = \varphi_r(k/\sqrt{T},\beta)$, $C^\pm(k_0,\beta,z) = C^\pm_r(\lambda_0,\beta,\eta)$, $F(\gamma,z) = F_r(\gamma,\eta)$. Then

$$\begin{aligned} G^+_{\uparrow\uparrow}(x,t) &= 2\mathrm{i}\sqrt{T}\,e^{\mathrm{i}x^2/4t+\mathrm{i}t(\mu-B)} \\ &\quad \oint dz\,\frac{F(\gamma,z)}{(z-1)^2}C^+(k_0,\beta,z)(2Tt)^{\frac{1}{2}(\nu(z)-\mathrm{i})^2}e^{tS(z)} \quad , \qquad (64) \end{aligned}$$

$$G^-_{\uparrow\uparrow}(x,t) \;=\; -\mathrm{i}\sqrt{T}\, e^{-\mathrm{i}x^2/4t-\mathrm{i}t(\mu-B)}$$
$$\oint dz\, \frac{F(\gamma,z)}{2\gamma z} C^-(k_0,\beta,z)(2Tt)^{\frac{1}{2}(\nu(z)+\mathrm{i})^2} e^{tS(z)} \quad , \tag{65}$$

where

$$S(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, |k-k_0|\ln(\varphi(k,\beta)) \quad . \tag{66}$$

We would like to calculate the contour integrals (64), (65) by the method of steepest descent. For this purpose we have to consider the analytic properties of the integrands. Let us assume that $k_0 \geq 0$, and let us cut the complex plane along the real axis from $-\infty$ to $-e^{-\beta}$ and from $-e^{\beta-k_0^2/T}$ to 0. The integrands in (64) and (65) can be analytically continued as functions of z into the cut plane with the only exception of the two simple poles of $F(\gamma,z)$ at $z=\gamma^{\pm 1}$. We may therefore deform the contour of integration as long as we never cross the cuts and take into account the pole contributions, if we cross $z=\gamma$ or $z=\gamma^{-1}$.

The saddle point equation $\partial S/\partial z = 0$ can be represented in the form

$$\int_0^\infty \frac{dk\, k}{1+z^{-1}e^{-\beta}e^{(k-k_0)^2/T}} = \int_0^\infty \frac{dk\, k}{1+ze^{-\beta}e^{(k+k_0)^2/T}} \quad . \tag{67}$$

This equation was discussed in the appendix of [6]. In [6] it was shown that (67) has exactly one real positive solution which is located in the interval $[0,1]$. It was argued that this solution gives the leading saddle point contribution to (64) and (65). At small temperatures (67) can be solved explicitly. There are two solutions $z_\pm = \pm z_c$, where

$$z_c = \frac{T^{3/4}}{2\pi^{1/4}k_0^{3/2}}\, e^{-k_0^2/2T} \quad . \tag{68}$$

In the derivation of (68) we assumed that $k_0 \neq 0$. The case $k_0 = 0$ has to be treated separately (see below).

The phase $tS(z)$ has the low temperature approximation

$$tS(z) = -2k_0 Dt\left\{\left(1-\frac{1}{z}\right)z_c^2 + 1 - z\right\} \quad . \tag{69}$$

Here $D = \partial P/\partial\mu$ is the density of the electron gas. The low temperature expansion (69) is valid in an annulus $e^{\beta-k_0^2/T} \ll |z| \ll e^{-\beta}$, which lies in our cut plane. The unit circle and the circle $|z| = z_c$ are inside this annulus. We may thus first apply (69) and then deform the contour of integration from the unit circle to the small circle $|z| = z_c$. Let us parameterize the small circle as $z = z_c e^{\mathrm{i}\alpha}$, $\alpha \in [-\pi, \pi]$. Then $S(z(\alpha)) = -2k_0 D((z_c - 1)^2 + 2z_c(1 - \cos(\alpha)))$, which implies that the small circle is a steepest descent contour and that on this contour $S(z_-) \le S(z) \le S(z_+)$. The maximum of $S(z)$ on the steepest descent contour at $z = z_+$ is unique and therefore provides the leading saddle point contribution to (64), (65) as $t \to \infty$. The saddle point approximation becomes good when $tS(z(\alpha)) = -2k_0 Dt((z_c - 1)^2 + z_c\alpha^2 + \mathcal{O}(\alpha^4))$ becomes sharply peaked around $\alpha = 0$. Hence, the relevant parameter for the calculation of the asymptotics of $G^{\pm}_{\uparrow\uparrow}$ is $2k_0 Dt = xD$ rather than t. xD has to be large compared to z_c^{-1}. The parameter xD has a simple interpretation. It is the average number of particles in the interval $[0, x]$. Let us consider two different limiting cases.

(i) $xD \to 0$, the number of electrons in the interval $[0, x]$ vanishes. In this regime the interaction of the electrons is negligible. An electron propagates freely from 0 to x. $G^{\pm}_{\uparrow\uparrow}$ cannot be calculated by the method of steepest descent. We have to use the integral representation (60), (61) instead. Since $tS(z)$ and $\nu(z)$ tend to zero on the contour of integration, the integrals in (60) and (61) are easily calculated. We find

$$G^{+}_{\uparrow\uparrow}(x,t) = \frac{e^{-\mathrm{i}\frac{\pi}{4}}}{2\sqrt{\pi}} t^{-\frac{1}{2}} e^{\mathrm{i}t(\mu - B)} e^{\mathrm{i}x^2/4t} \quad , \tag{70}$$

$$G^{-}_{\uparrow\uparrow}(x,t) = \frac{e^{\mathrm{i}\frac{\pi}{4}}}{2\sqrt{\pi}} e^{(\mu - B - k_0^2)/T} t^{-\frac{1}{2}} e^{-\mathrm{i}t(\mu - B)} e^{-\mathrm{i}x^2/4t} \quad . \tag{71}$$

As expected, equations (70) and (71) are the well known result for free fermions.

(ii) $xD \gg z_c^{-1}$, the average number of electrons in the interval $[0, x]$ is large. This is the *true asymptotic region*, $x \to \infty$. In this region the interaction becomes important. At the same time the method of steepest descent can be used to calculate $G^{\pm}_{\uparrow\uparrow}$. This case will be studied below.

In the process of deformation of the contour from the unit circle to the small circle of radius z_c we may cross the pole of the function $F(\gamma, z)$ at $z = \gamma^{-1}$. Then we obtain a contribution of the pole to the asymptotics of $G^{\pm}_{\uparrow\uparrow}$. It turns out that the pole contributes to $G^{\pm}_{\uparrow\uparrow}$, when the magnetic field is below a critical positive value, $B_c = k_0^2/4$. Below this value the pole contribution always dominates the contribution of the saddle point. Hence, we have to distinguish two different asymptotic regions, $B > B_c$ and $B < B_c$. On the other hand, if we consider the asymptotics for fixed magnetic field, we have to treat the cases $B > 0$ and $B \leq 0$ separately. For $B > 0$ we have to distinguish between a time like regime ($k_0^2 < 4B$) and a space like regime ($k_0^2 > 4B$). In these respective regimes we obtain the asymptotics:

Time like regime ($x < 4t\sqrt{B}$):

$$G^+_{\uparrow\uparrow}(x,t) = \frac{e^{-\mathrm{i}\frac{\pi}{4}}}{2\sqrt{\pi}} \frac{1}{\sqrt{4\pi z_c x D_\downarrow}} t^{-\frac{1}{2}-\mathrm{i}\nu(z_c)} e^{\mathrm{i}t(\mu-B)} e^{\mathrm{i}x^2/4t} e^{-xD_\downarrow}\,, \tag{72}$$

$$G^-_{\uparrow\uparrow}(x,t) = \frac{e^{\mathrm{i}\frac{\pi}{4}}}{2\sqrt{\pi}} \frac{e^{(\mu-B-k_0^2)/T}}{\sqrt{4\pi z_c x D_\downarrow}} t^{-\frac{1}{2}+\mathrm{i}\nu(z_c)} e^{-\mathrm{i}t(\mu-B)} e^{-\mathrm{i}x^2/4t} e^{-xD_\downarrow}\,, \tag{73}$$

where

$$\nu(z_c) = -\frac{2D_\downarrow k_0^{3/2} e^{-k_0^2/2T}}{\pi^{1/4} T^{5/4}}\,, \tag{74}$$

$$D_\downarrow = \tfrac{1}{2}\sqrt{\tfrac{T}{\pi}} e^{(\mu+B)/T}\,. \tag{75}$$

$D_\downarrow = \partial P/\partial(\mu+B)$ is the low temperature expression for the density of down-spin electrons.

Space like regime ($x > 4t\sqrt{B}$):

$$G^+_{\uparrow\uparrow}(x,t) = \frac{e^{-\mathrm{i}\frac{\pi}{4}}}{2\sqrt{\pi}} t^{-\frac{1}{2}-\mathrm{i}\nu(\gamma^{-1})} e^{\mathrm{i}t(\mu-B)} e^{\mathrm{i}x^2/4t} e^{-xD_\downarrow}\,, \tag{76}$$

$$G^-_{\uparrow\uparrow}(x,t) = \frac{e^{\mathrm{i}\frac{\pi}{4}}}{2\sqrt{\pi}} e^{(\mu-B-k_0^2)/T} t^{-\frac{1}{2}+\mathrm{i}\nu(\gamma^{-1})} e^{-\mathrm{i}t(\mu-B)} e^{-\mathrm{i}x^2/4t} e^{-xD_\downarrow}\,, \tag{77}$$

where

$$\nu(\gamma^{-1}) = -\frac{e^{(3B+\mu-k_0^2)/T}}{2\pi}\,. \tag{78}$$

For $B \leq 0$ there is no distinction between time and space like regimes. The asymptotics is always given by the equations (76) and (77).

It is instructive to compare the asymptotic expressions (72)-(78) for the correlation functions with the corresponding asymptotics for free fermions, (70), (71). We can interpret the factors $t^{-\frac{1}{2}\pm\mathrm{i}\nu(z_+)} e^{-xD_\downarrow}/\sqrt{4\pi z_c x D_\downarrow}$ in (72) and

(73) and $t^{\pm i\nu(\gamma^{-1})} e^{-xD_\downarrow}$ in (76) and (77) as low temperature corrections to the free fermionic correlation functions. The occurrence of the density of down-spin electrons in the exponential factors in equations (72)-(78) has a natural interpretation. Correlations decay due to interaction. Because of the Pauli principle and the locality of the interaction in our specific model, equation (1), up-spin electrons only interact with down-spin electrons. Therefore the correlation length is expected to be a decreasing function of the density of down-spin electrons, which diverges as the density of down spin electrons goes to zero. We see from (72)-(78) that the low temperature expression for the correlation length is just $1/D_\downarrow$ and thus meets these expectations.

In the limit $B \to -\infty$, $\mu \to -\infty$, $\mu - B$ fixed there are no $\downarrow$-spin electrons left in the system, $D_\downarrow \to 0$. This is the free fermion limit. In the free fermion limit $B < B_c$, and the asymptotics of $G^+_{\uparrow\uparrow}(x,t)$ and $G^-_{\uparrow\uparrow}(x,t)$ are given by equations (76), (77), which turn into the expressions (70), (71) for free fermions.

The pure time direction $k_0 = 0$ requires a separate calculation. For $k_0 = 0$ the saddle point equation (67) has the solutions $z = \pm 1$ for all temperatures. The unit circle is a steepest descent contour with unique maximum of $S(z)$ at $z = 1$, which gives the leading asymptotic contribution to the integrals in (64) and (65). We find algebraically decaying correlations,

$$G^+_{\uparrow\uparrow}(0,t) = C_0^+ t^{-1} e^{it(\mu-B)} \quad , \quad G^-_{\uparrow\uparrow}(0,t) = C_0^- t^{-1} e^{-it(\mu-B)} \quad , \tag{79}$$

where

$$\begin{aligned} C_0^+ &= \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{2\pi T}}(1 + 2e^{-2B/T}) \\ &\quad \left[(e^{(\mu+B)/T} + e^{(\mu-B)/T})(1 + e^{(\mu+B)/T} + e^{(\mu-B)/T})\right]^{-\frac{1}{2}} \quad , \end{aligned} \tag{80}$$

$$C_0^- = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi T}} \frac{1 + 2e^{-2B/T}}{1 + e^{2B/T}} \left[\frac{e^{(\mu+B)/T} + e^{(\mu-B)/T}}{1 + e^{(\mu+B)/T} + e^{(\mu-B)/T}}\right]^{\frac{1}{2}} \quad . \tag{81}$$

These formulae are valid at any temperature.

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