A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbb{T}^2$

Amadeu Delshams[1*] Rafael de la Llave[2†]
Tere M. Seara[1‡]

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[1] Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Diagonal 647, 08028 Barcelona, Spain

[2] Department of Mathematics
University of Texas at Austin
Austin, TX, 78712, USA

Abstract

We give a proof based in geometric perturbation theory of a result proved by J.N. Mather using variational methods. Namely, the existence of orbits with unbounded energy in perturbations of a generic geodesic flows in $\mathbb{T}^2$ by a generic periodic potential.

*amadeu@ma1.upc.es
†llave@math.utexas.edu
‡tere@ma1.upc.es

1 Introduction

The goal of this paper is to give a proof, using geometric perturbation methods, of a result proved by J.N. Mather using variational methods. We will prove:

Theorem 1.1 *Let g be a $\mathcal{C}^r$ generic metric on $\mathbb{T}^2$, $U : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{R}$ a generic $\mathcal{C}^r$ function (on $\mathbb{T}^2$, and periodic in time), r sufficiently large.*

Consider the time dependent Lagrangian

$$L(q, \dot{q}, t) = \frac{1}{2} g_q(\dot{q}, \dot{q}) - U(q, t).$$

Then, the Euler-Lagrange equation of L has a solution $q(t)$ whose energy

$$E(t) = H(\dot{q}(t), q(t), t) = \frac{1}{2} g_q(\dot{q}(t), \dot{q}(t)) + U(q(t), t),$$

goes to infinity as $t \to \infty$.

Remark 1.2 Note that, in fact, the only unbounded part in $H(\dot{q}(t), q(t), t)$ is $\dot{q}(t)$, so that the theorem could be expressed as unbounded growth in the velocity. $\square$

Remark 1.3 As it is usually the case in these problems of diffusion, one not only constructs orbits whose energy grows unbounded, but also orbits whose energy makes more or less arbitrary excursions. We formulate this precisely in Theorem 4.24, and deduce Theorem 1.1 from it. $\square$

Remark 1.4 The argument presented here shows that $r \geq 15$ is large enough for Theorem 1.1. (See the proof of Lemma 4.18.) We do not claim that this is optimal for the geometric method to go through. $\square$

Actually, the results of Mather contain this as a particular case, as well as ours. This theorem as stated seems to be just a common ground that allows some comparison of the methods. Notably, Mather can deal with situations involving much less regularity.

Remark 1.5 The assumptions of genericity will be made quite explicit in Theorem 4.24, a more general result than Theorem 1.1. They amount to the existence of a closed hyperbolic geodesic with a homoclinic connecting orbit for the metric g, and that a certain functional, called Melnikov potential or Poincaré function, computed from the potential on the homoclinic orbit, is not constant. □

The work of Mather [Mat96] also requires similar assumptions. As far as we can see, the main difference in the hypothesis of [Mat96] and this paper is that [Mat96] also uses that the periodic orbits and the connecting ones are minimizing and class A. On the other hand, the differentiability hypotheses of this note are much more restrictive than those in [Mat96]. The orbits with growing energy produced in this work and those produced in [Mat96] are not necessarily the same: the orbits we produce here shadow smooth invariant curves, whereas those in [Mat96] shadow minimizing Aubry-Mather sets (which could be Cantor sets). We think that it is remarkable that the functional that needs not to be constant is the same in both approaches. We hope that this could lead to a more geometric understanding of [Mat96], which could perhaps lead to some new results.

We note that it is possible to choose g and U as arbitrarily close to the flat metric and zero as desired in an analytic topology. Hence, this could be considered as an analogue of Arnol'd diffusion. Depending on what one defines precisely as Arnol'd diffusion it may not be appropriate to call the phenomenon described in [Mat96] and here by this name. Since a universally accepted precise definition of Arnol'd diffusion seems to be lacking we just point out that the phenomena described here has a similar flavor and indeed the methods that we use here are very similar to the methods traditionally used in the field.

The analogy with the traditional approaches of Arnol'd diffusion is much closer when we consider what happens for a bounded range of (rather high) energies. We note that in this case, there are two smallness parameters. One is the distance of the metric to the flat metric and another one is the size of the potential. For high energy, the potential is a very small perturbation of the geodesic flow (we will make all this more precise later). If we chose g close to flat, for the theorem to go through we need to choose the energy for which the potential can be considered as a sufficiently small perturbation. The same feature of two smallness parameters was present in the original example [Arn64].

Note that the geodesic flow, which in our case plays the role of the unperturbed system, is assumed to have some hyperbolicity properties. In the notation of [CG94] this is an *a priori* unstable system.

An important feature of this problem is that, besides two smallness parameters, it has two time scales. For high energy, the frequency of the unperturbed problem is high while the frequency of the perturbation is small. Hence, one can bring to bear methods of adiabatic theory to obtain small gaps between KAM tori. (This phenomenon also happens in the models considered in [CG94], who emphasized the important role played by this fact in the conclusions and also identified several physical models where this is a natural assumption.)

The main difference of the methods presented here with more traditional approaches to Arnold diffusion is the reliance in hyperbolic perturbation theory and center manifold reduction rather than the exclusive reliance in KAM perturbation theories. (An sketch of the method proposed, was known in [dlL96].)

We think that the locally invariant normally hyperbolic manifold is an interesting structure since one can study the dynamics on it using the powerful methods of two dimensional dynamics, notably Aubry-Mather theory. We hope to come back to these issues exploiting the many structures present in the invariant center manifold in a near future. Similar ideas were used in [dlLW89]. We note that the use of methods based in normal hyperbolicity to deal with systems with two scales of time in a geometric way has been successfully used for a long time (See e.g. [Fen79]).

We want to draw attention to [BT98], which presents another method to obtain similar results without using the theory of normally hyperbolic manifolds. They construct the transition chain relying in standard KAM theory and Poincaré-Melnikov theory.

Of course, the methods used in [Mat96] are completely different from all the methods based on perturbation theory.

We have hopes that a blending of the traditional methods, with hyperbolic perturbation theory, a more geometric understanding and variational methods could lead to progress in the problem of Arnol'd diffusion.

1.1 Summary of the method

The proof we present here can be conveniently divided into different stages.

In a first stage, we use classical Riemannian geometry to establish the existence of a family of periodic orbits. The whole family is a two dimensional normally hyperbolic manifold which carries an exact symplectic form (restriction of the symplectic form in the phase space). Its stable and unstable manifolds intersect transversally and the motion on it is a twist map with an unbounded frequency. This step is due to Morse and Mather, and is covered is sections 2 and 3.2.

In a second stage (section 4.2), we show that, for high enough energy, the perturbation introduced by the potential can be considered small. This is just an elementary scaling argument. We give full details mainly to set the notation.

In a third stage, we use perturbation theory of normally hyperbolic manifolds to show that this normally hyperbolic manifold persists into a locally invariant normally hyperbolic manifold, and its stable and unstable manifolds keep on intersecting transversally. Also, we note that the perturbed invariant manifold inherits a symplectic structure from the ambient space and that, therefore, the rich methods of Hamiltonian perturbation theory can be brought to bear on the motion restricted to it. A brief summary of hyperbolic perturbation theory is presented in appendix A, and the application to our problem is presented in sections 3.3 and 4.3. The motion on this invariant manifold is faster than the perturbation introduced by the potential.

In a fourth stage (section 4.4.1), we use averaging theory to eliminate the fast angles from the Hamiltonian to obtain that the motion on the normally hyperbolic invariant manifold can be reduced to integrable up to an error which is of very high order in the perturbation parameter. Hence, the error decreases as an inverse power of the energy.

In a fifth stage (section 4.4.2), we use quantitative versions of KAM theory to show that the smallness of the perturbation in the invariant manifold leads to the fact that this invariant manifold is filled very densely with KAM tori, and we obtain explicit expressions for the main terms of these tori.

In a sixth stage we use Poincaré-Melnikov theory to compute the change of energy in a homoclinic excursion and show that, under appropriate nondegeneracy assumptions, the—very close—KAM tori have to have homoclinic and heteroclinic intersections.

These calculations are not completely standard due to the presence of two time scales. We also note that the literature about Melnikov functions for quasiperiodic objects is somewhat confusing. Notably, some of the terms that make the naive Melnikov integrals not absolutely converging are incorrectly omitted in many papers. Hence, we decided to present rather full details in section 4.6.

In a seventh stage (section 4.7) we use the results which show that given transition chains, one can find orbits that shadow them.

We emphasize that all these stages use only readily available theorems and techniques. (Perhaps the less standard part is the part on the calculation of Poincaré-Melnikov functions, so it appears fully expanded.) Moreover, these stages are significantly independent, so that if we assume—or arrive by other methods—to the conclusions of one, all the subsequent results apply.

In particular, the above strategy was designed to be compatible with variational methods. The invariant manifolds produced using the theory of normally hyperbolic manifolds carry Aubry-Mather sets, as pointed by J.N. Mather. Moreover, variational methods can be used to provide powerful shadowing lemmas that can be used in the last stage.

2 Classical geometry of the geodesic flow

The following geometric facts were proved by Morse and Mather and their relevance for the problem we are considering was discovered and emphasized in [Mat96].

Theorem 2.1 *For a $\mathcal{C}^r$ open and dense set of metrics in $\mathbb{T}^2$, $r = 1, \ldots, \infty, \omega$, there exists a closed geodesic "Λ" which is hyperbolic in the dynamical systems sense as a periodic orbit of the unit geodesic flow.*

Moreover, there exists another geodesic "γ" and real numbers a_+, a_-, such that

$$\mathrm{dist}(\text{“}\Lambda\text{”}(t + a_\pm), \text{“}\gamma\text{”}(t)) \to 0 \quad \textit{as} \quad t \to \pm\infty \tag{2.1}$$

Here we will take the standard definition that a geodesic "Λ" is a curve "Λ" $: \mathbb{R} \to \mathbb{T}^2$, parameterized by arc length which is a critical point for the length among any two of its points. Later, we will consider curves in the cotangent bundle that are orbits of the geodesic flow. Clearly, these orbits are closely related to the geometric geodesics in the manifold. We will use for the orbits in the cotangent bundle the same letter as for the geodesics but suppress the quotation marks. When we want to speak about the orbits of the geodesic flow as manifolds in phase space (more properly, the range of the mapping Λ), we will use a ˆ (i.e. $\hat{\Lambda} = \mathrm{Range}\Lambda$). Note that by changing the origin of time, we obtain another geodesic, so that the geodesics satisfying geometric properties are always one parameter families. This consideration will be important when we consider time dependent perturbations.

Actually Morse showed much more. He showed that there exists one "Λ" in each free homotopy class. Moreover, he showed that "Λ" can be taken to be minimizing and "γ" satisfies other minimizing properties (class A). Such properties play an important role in the work [Mat96], but we will not need them here. For the approach presented here, the hyperbolicity and the connection property are what is important. In particular, we note that dynamical systems theory shows that the homoclinic connections γ imply the existence of other periodic orbits and other homoclinic connections. Even if these are not minimizing, the proof developed here works without any change.

3 The unperturbed problem

3.1 Hamiltonian formalism and notation

The present problem admits natural Lagrangian and Hamiltonian formulations. From our point of view neither of them plays a large role, but it seems that the Hamiltonian point of view is somewhat more convenient. Hence, this is the formalism that we will consider.

The phase space of the geodesic flow is $\mathbf{T}^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$. We will denote the coordinates in $\mathbb{T}^2$ by q and the cotangent directions by p. Note that we are taking some advantage—but mainly in the notation—of the fact that the cotangent bundle of $\mathbb{T}^2$ is trivial.

We point out that, as it is well known, the phase space, being a cotangent bundle admits a canonical symplectic form, which moreover is exact.

It is well known that for a cotangent manifold $\mathbf{T}^*\mathbb{T}^2$ there is a unique 1-form θ such that $\alpha^*\theta = \alpha$ for any one form α on $\mathbb{T}^2$. (Here we think of forms as maps from $\mathbb{T}^2$ to $\mathbf{T}^*\mathbb{T}^2$.)

Then, $\Omega = d\theta$ is a symplectic form. In local coordinates, $\theta = \sum_i p_i dq_i$,

$\Omega = \sum_i dp_i \wedge dq_i$.

With respect to the form Ω, the geodesic flow is Hamiltonian and the Hamiltonian function is $H_0(p,q) = \frac{1}{2}g_q(p,p)$. We will denote by Φ_t this geodesic flow.

For each E, we will denote $\Sigma_E = \{(p,q) \mid H_0(p,q) = E\}$, and observe that, for any $E_0 > 0$ (later we will use this for large E_0), $\tilde{\Sigma}_{E_0} = \cup_{E \geq E_0} \Sigma_E \simeq [E_0, \infty) \times \mathbb{T}^1 \times \mathbb{T}^2$, that is, we can take the energy as a part of a coordinate system. Note that the energy is one half the square of $|p|$ so that the energy can be used as a radial coordinate in p. This is quite convenient. We will also need an angle coordinate, to complete the polar coordinate system.

We also note that Σ_E—a three dimensional manifold diffeomorphic to $\mathbb{T}^1 \times \mathbb{T}^2$—is invariant under the geodesic flow.

Given an arbitrary geodesic "λ" $: \mathbb{R} \to \mathbb{T}^2$ we will denote by $\lambda_E(t) = (\lambda_E^q(t), \lambda_E^p(t))$ the orbit of the geodesic flow that lies in the energy surface Σ_E, and whose projection over q turns along the range of "λ". Moreover, we fix the origin of time in λ_E so that it corresponds to the origin of the parameterization in "λ". (Formally $H_0(\lambda_E(t)) = E$, and Range("λ") $=$ Range(λ_E^q), "λ"$(0) = \lambda_E^q(0)$.)

It is easy to check that the above conditions determine uniquely the orbit of the geodesic flow, in particular determine $\lambda_E^p(t)$. Note that

$$(\lambda_E^p(t), \lambda_E^q(t)) = \left(\sqrt{E}\lambda_1^p(\sqrt{E}t), \lambda_1^q(\sqrt{E}t)\right), \tag{3.1}$$

so that, for the geodesic, the role of E is just a rescaling of time.

3.2 Hyperbolicity properties

Extending the methods of Morse for Theorem 2.1, J.N. Mather showed:

Theorem 3.1 *For a $\mathcal{C}^r$ generic metric, $r = 2, \ldots, \infty, \omega$, and for any value of the Hamiltonian $H_0(p,q) = E > 0$, there exists a periodic orbit $\Lambda_E(t)$, as in (3.1), of the geodesic flow whose range $\hat{\Lambda}_E$ is a normally hyperbolic invariant manifold in the energy surface. Its stable and unstable manifolds $W^{s,u}_{\hat{\Lambda}_E}$ are two dimensional, and there exists a homoclinic orbit $\gamma_E(t)$, that is, its range $\hat{\gamma}_E$ satisfies*

$$\hat{\gamma}_E \subset \left(W^{\mathrm{s}}_{\hat{\Lambda}_E} \setminus \hat{\Lambda}_E\right) \cap \left(W^{\mathrm{u}}_{\hat{\Lambda}_E} \setminus \hat{\Lambda}_E\right).$$

Moreover, this intersection is transversal as intersection of invariant manifolds in the energy surface along $\hat{\gamma}_E$.

For $E = 1$, we have that

$$\operatorname{dist}(\Lambda_1(t + a_\pm), \gamma_1(t)) \to 0 \quad as \quad t \to \pm\infty. \tag{3.2}$$

We also note that since $\hat{\gamma}_E$ is one dimensional, $W^{\mathrm{s}}_{\hat{\Lambda}_E}$, $W^{\mathrm{u}}_{\hat{\Lambda}_E}$ are two dimensional, and the ambient manifold Σ_E is three dimensional, we have $T_x\hat{\gamma}_E = T_x W^{\mathrm{s}}_{\hat{\Lambda}_E} \cap T_x W^{\mathrm{u}}_{\hat{\Lambda}_E}$ for all the points $x \in \hat{\gamma}_E$. Hence, by the implicit function theorem, $\hat{\gamma}_E$ is the locally unique intersection. Since we are considering manifolds invariant under flows, their intersection has to contain orbits and γ_E is locally the only possible—up to change in the origin of the parameter—orbit in the intersection of $W^{\mathrm{s}}_{\hat{\Lambda}_E}$ and $W^{\mathrm{u}}_{\hat{\Lambda}_E}$.

In this situation the energy is preserved and then the dynamics can be analyzed on the energy surface. This, however, will not be useful when we consider the external periodic potential which changes the energy. Hence, it will be useful to discuss what happens for all energy surfaces. The following is a description of the behavior of $\Lambda = \bigcup_{E \ge E_0} \hat{\Lambda}_E$ for all values of the energy.

Lemma 3.2 *Define $\Lambda = \bigcup_{E \ge E_0} \hat{\Lambda}_E$. This is a manifold with boundary which is diffeomorphic to $[E_0, \infty) \times \mathbb{T}^1$, and the canonical symplectic form Ω on $\mathbf{T}^*\mathbb{T}^2$ Ω restricted to Λ is non-degenerate. The form $\Omega|_\Lambda$ is invariant under the geodesic flow Φ_t.*

We have for some $C, \alpha > 0$ and for all $x \in \hat{\Lambda}_E$,

$$T_x\Sigma_E = E^s_x \oplus E^u_x \oplus T_x\hat{\Lambda}_E$$

with $||D\Phi_t(x)|_{E^s_x}|| \le Ce^{-\alpha t}$ for $t \ge 0$, $||D\Phi_t(x)|_{E^u_x}|| \le Ce^{\alpha t}$ for $t \le 0$ and $||D\Phi_t(x)|_{T_x\hat{\Lambda}_E}|| \le C$ for all $t \in \mathbb{R}$.

The stable and unstable manifolds to Λ: W^{s}_Λ, W^{u}_Λ, are three dimensional manifolds diffeomorphic to $[E_0, \infty) \times \mathbb{T}^1 \times \mathbb{R}$, and

$$\gamma = \bigcup_{E \ge E_0} \hat{\gamma}_E \subset (W^{\mathrm{s}}_\Lambda \setminus \Lambda) \cap (W^{\mathrm{u}}_\Lambda \setminus \Lambda)$$

is diffeomorphic to $[E_0, \infty) \times \mathbb{R}$.

We also note that, since the definition of transversal intersection of manifolds only requires that the tangent spaces span the ambient space, when we add an extra dimension (in this case the energy, but later we will consider other parameters) the intersection of the extended manifolds is still transversal. The intersection of the extended manifolds will not be just one orbit but we will have

$$T_x\gamma = T_x W^{\mathrm{s}}_\Lambda \cap T_x W^{\mathrm{u}}_\Lambda.$$

Hence, γ will still be a locally unique intersection.

We note that the only properties of the geodesic flow that we will use are the conclusions of Theorem 3.1 and Lemma 3.2.

3.3 Extended phase spaces

Since we are going to consider periodic perturbations, it will be convenient to introduce an extra angle variable, which we will denote by s, which moves at a constant rate 1. Then, the phase space will be $\mathbf{T}^*\mathbb{T}^2 \times \mathbb{T}^1$.

We will introduce the notation $\tilde{\Lambda} = \Lambda \times \mathbb{T}^1$, and analogously for γ, etc. to denote the corresponding objects in the extended phase space.

In the case that we do not have any external potential, the dynamics in this extended phase space is just the product of the unperturbed dynamics in $\mathbf{T}^*\mathbb{T}^2$ and the motion with constant speed in the circle.

In this extended phase space the results of section 3.2 simply translate into:

- $\hat{\Lambda}_E \times \mathbb{T}^1$ is a two dimensional invariant manifold. Its (un)stable manifold is a three dimensional manifold. They intersect transversally in $\Sigma_E \times \mathbb{T}^1$. (Of course, they are not transversal in the whole extended space since they lie on the energy surface.)

- When we consider the results for all the energies, we obtain normal hyperbolicity.

  $\Lambda \times \mathbb{T}^1$ is a 3-dimensional manifold, and it is normally hyperbolic for the extended flow $\tilde{\Phi}_t$ (see Definition A.1 in appendix A). The (un)stable manifolds of $\Lambda \times \mathbb{T}^1$ are $W^{\mathrm{u,s}}_\Lambda \times \mathbb{T}^1$, and are 4-dimensional.

- Moreover, $\gamma \times \mathbb{T}^1$ lies in the intersection of $W^s_\Lambda \times \mathbb{T}^1 = W^s_{\Lambda\times\mathbb{T}^1}$ and of $W^u_\Lambda \times \mathbb{T}^1 = W^u_{\Lambda\times\mathbb{T}^1}$, and the intersection is transversal.

- The extended flow $\tilde{\Phi}_t$ restricted to the invariant manifold $\tilde{\Lambda}$ is neither contracting nor expanding:

$$||D\tilde{\Phi}_t(x)|_{T_x\tilde{\Lambda}}|| \leq C \quad \forall t \in \mathbb{R}, \quad x \in \tilde{\Lambda}. \tag{3.3}$$

These observations will be important because they will allow us to use the rich theory of hyperbolic invariant manifolds summarized in appendix A when we consider the problem with the external potential.

This extended phase space is obviously not symplectic (it has odd dimension). In order to perform some other calculations, we will find it convenient to perform a symplectic extension. This is accomplished by adding another real variable a symplectically conjugate to s, which does not change with time.

Then, the symplectically extended phase space is $\mathbf{T}^*\mathbb{T}^2 \times \mathbb{R} \times \mathbb{T}^1$. The symplectic form in this space is $\tilde{\Omega} = \Omega + da \wedge ds$. The flow is Hamiltonian and its Hamiltonian function is $h(a, s, p, q) = a + H_0(p, q)$.

Since a is conserved, the restriction of the flow of h to each of the manifolds $a =$ cte is identical to the flow of H_0 in the extended phase space. In this case, the neutral direction given by a spoils all the hyperbolicity properties. This situation is very common in Hamiltonian systems since the neutrality along a manifold as in (3.3) implies similar bounds for the symplectic conjugate space.

3.4 The inner map

We will consider F the time 1 map of the geodesic flow restricted to Λ i. e. $F = \Phi_1|_\Lambda$. (This will make it easier to analyze the time periodic external forcing.) As we are dealing with the autonomous case, we note:

1. It is still true that Λ is a normally hyperbolic surface for Φ_1.

2. The stable and the unstable manifolds for Φ_1 are the same as for the flow Φ_t. In particular, they are still transversal.

3. $\Omega|_\Lambda$ is a symplectic form on Λ.

4. ${\Phi_1}^*\Omega = \Omega$. Hence $F^*\Omega|_\Lambda = \Omega|_\Lambda$.

5. We have the canonical 1-form θ, called the symplectic potential, such that $d\theta = \Omega$. We note that $\Omega|_\Lambda = d\theta|_\Lambda$.

6. ${\Phi_1}^*\theta = \theta + dS$. Hence, $F^*\theta|_\Lambda = \theta|_\Lambda + dS|_\Lambda$. Therefore, the map F restricted to Λ is an exact symplectic map.

Remark 3.3 Note that the rescaling properties (3.1) of the geodesic flow imply scaling properties for the variational equations. As a consequence of them, the angle $\langle T_{\hat\Lambda_E} W^{\rm s}_{\hat\Lambda_E}, T_{\hat\Lambda_E} W^{\rm u}_{\hat\Lambda_E}\rangle$ between the stable and the unstable bundles in $\hat\Lambda_E$, remains bounded independently of E. On the other hand, the Lyapunov exponents scale with $\sqrt{E}$. Therefore,

$$\begin{aligned} \left\| D\Phi_1|_{T_{\hat\Lambda_E} W^{\rm s}_{\hat\Lambda_E}} \right\| &\leq \alpha^{\sqrt{E}} \\ \left\| D\Phi_{-1}|_{T_{\hat\Lambda_E} W^{\rm u}_{\hat\Lambda_E}} \right\| &\leq \alpha^{\sqrt{E}} \end{aligned}$$

where $\alpha < 1$ is independent of E, even if it depends on the metric. $\square$

3.5 A coordinate system on Λ

Now we want to describe a coordinate system in Λ that can be used to compute the motions on it as well as their perturbations. We want coordinate functions that are not only defined on Λ but also on a neighborhood of it. This will be particularly important for us mainly in the calculation of the Poincaré function. Since the manifolds we are going to consider are cylinders, we will take one real coordinate and one angle coordinate.

The real coordinate will be the unperturbed energy H_0, which is natural, given the fundamental role played by it in the problem. For the conjugate angle, we will choose an angle coordinate (in the q-space) of the geodesic "Λ" and extend it to $\mathbb{T}^2$, and then, lift it to $\mathbf{T}^*\mathbb{T}^2$. We can identify a point in Λ with the values (E, φ) taken by these coordinates. The coordinate representation of the periodic orbit $\Lambda_E(t)$ in these coordinates (see (3.1)) is given by $(E, \sqrt{E}t)$. Note that for any $\varphi_0 \in \mathbb{R}$, $\Lambda_E(t + \varphi_0/\sqrt{E})$ is another periodic orbit that in these coordinates is given by $(E, \varphi_0 + \sqrt{E}t)$.

For emphasis, the unperturbed map will be denoted also by F_0. Then:

$$F_0(E, \varphi) = (E, \varphi + \sqrt{E}) \tag{3.4}$$

We call attention to the fact that F_0 is a twist map. The restriction of the symplectic form to Λ can be expressed in these coordinates as

$$\Omega|_\Lambda = \frac{1}{\sqrt{E}} dE \wedge d\varphi = d(2\sqrt{E}) \wedge d\varphi$$

and is clearly preserved by F_0. Moreover, we have that

$$\begin{aligned} \theta|_\Lambda &= 2\sqrt{E} d\varphi \\ F_0^* \theta|_\Lambda &= \theta|_\Lambda + dE. \end{aligned}$$

3.6 The outer map

Another important ingredient in our approach is the map $S : \Lambda \to \Lambda$ that we will call the "scattering map" or the "outer map" associated to γ. This map S will transform the asymptotic point at $-\infty$ of a homoclinic orbit to Λ into the asymptotic point at $+\infty$. We define the scattering map as $x_+ = S(x_-)$ if $W^s(x_+) \cap W^u(x_-)$ lies in a curve which is in a neighborhood of γ.

We note that this map is well defined locally due to the smooth dependence on the base point of the (un)stable manifold and to the fact that the intersection is locally unique.

The following construction will make this definition more precise and will make it possible to discuss perturbations.

If we fix an small section $\mathcal{S}$ of the flow near a portion of the homoclinic orbit γ, then $\mathcal{V} = W^s_\Lambda \cap W^u_\Lambda \cap \mathcal{S}$ is a differentiable manifold because of the explicit description in section 3.2. (Later, for perturbed mappings, we will use another argument from appendix A.)

We can define two mappings $x_\pm : \mathcal{V} \to \Lambda$ by $p \in W^s_{x_+(p)}$, $p \in W^u_{x_-(p)}$. That is, we associate to a point $p \in \mathcal{V}$, the point in Λ to whose strong (un)stable manifold the point belongs.

If we consider $x_+(\mathcal{V}) \times x_-(\mathcal{V}) \subset \Lambda \times \Lambda$, this manifold is the graph of the mapping S in an small neighborhood in Λ.

Clearly, the mapping S associates to an asymptotic orbit in $-\infty$ another asymptotic orbit at $+\infty$.

We note that this definition of the mapping S is independent of the choice of the transversal section $\mathcal{S}$ because the points $x_\pm$ are a property of the orbits in $\mathcal{V}$ and choosing another transversal section does not change the pairs (x_-, x_+) that appear. (More geometrically, we only use the transversal section to be able to parameterize the space of orbits.)

We call attention to the fact that all the elements in the above construction of the map S are based in intersecting manifolds constructed using hyperbolicity theory and that, therefore, using the results of appendix A, it will be easy to show that they persist under perturbation and that they depend somewhat smoothly on parameters. See section 4.5.

We could worry that, as x_- moves around in a loop, $S(x_-)$ does not come back to the same point. This, however cannot happen because in such a case we would have two points in the manifold Λ, x_+^1, x_+^2 such that their stable manifolds intersect. Hence, we conclude that S is a well defined map, in the universal cover of Λ and that a deck transformation in the domain induces a deck transformation in the range: $S \circ g = g' \circ S$.

A direct calculation shows that in our case $S_\# : \pi(\Lambda) \to \pi(\Lambda)$ verifies $S_\# = \mathrm{Id}$. In any case, we note for future references that even if S gets deformed in a continuous way, $S_\#$ cannot depend on parameters since it is an element of a discrete group.

Remark 3.4 We note that even if the map S is well defined as a map on the manifold Λ, the mapping that to each point x_- associates the orbit (with parameterization) effecting the connection, even if defined locally, is not well defined globally. If we move x_- along a non-trivial loop in Λ, the connecting orbit gets transformed into another orbit with a different parameterization. (This amounts to incrementing or decrementing $a_\pm$ by a unit). This remark will play a role when we discuss the Poincaré function since it is expressed as an integral over the connecting orbit. □

We now proceed to compute the outer map for the unperturbed problem, which will be denoted by $S_0 : \Lambda \to \Lambda$. For x_+, x_- being two points of Λ, $x_+ = S_0(x_-)$ means that $\exists z \in \gamma \subset \mathbf{T}^*\mathbb{T}^2$, such that

$$\mathrm{dist}\,(\Phi_t(x_\pm), \Phi_t(z)) \to 0\ , \quad \text{as} \quad t \to \pm\infty$$

From the results of Morse of Theorem 3.1:

$$\mathrm{dist}(\Lambda_1(t + a_\pm), \gamma_1(t)) \to 0, \quad \text{as} \quad t \to \pm\infty \tag{3.5}$$

and the rescaling properties give

$$\mathrm{dist}(\Lambda_E(t/\sqrt{E} + a_\pm/\sqrt{E}), \gamma_E(t/\sqrt{E})) \to 0 \quad \text{as} \quad t \to \pm\infty \tag{3.6}$$

and for any $\varphi_0 \in \mathbb{R}$,

$$\operatorname{dist}(\Lambda_E(t/\sqrt{E} + \varphi_0/\sqrt{E} + a_\pm/\sqrt{E}), \gamma_E(t/\sqrt{E} + \varphi_0/\sqrt{E})) \to 0 \quad \text{as} \quad t \to \pm\infty \tag{3.7}$$

Hence, expressing the map S_0 in the coordinates (E, φ), we have $S_0(\Lambda_E) = \Lambda_E$, and

$$S_0(E, a_- + \varphi) = (E, a_+ + \varphi)$$

or more simply, calling $\Delta = a_+ - a_-$ the phase shift:

$$S_0(E, \varphi) = (E, \varphi + \Delta)\,. \tag{3.8}$$

4 The problem with external potential

4.1 Summary

The main idea is that, for high energy, the external potential is an small (and slow) perturbation of the geodesic flow.

Therefore, all the geometric structures that we constructed based on normal hyperbolicity and transversality persist for high energy. In particular, the manifold Λ will persist as well as the transversality of the intersection of its stable and unstable manifolds. This will allow us to define F, S analogues of the maps F_0, S_0, and to compute them perturbatively.

Using the information that we have of these maps, we will construct a sequence $n_1, \ldots, n_k, \ldots$, such that there is some point x

$$p_k = F^{n_k} \circ S \circ \cdots \circ F^{n_1} \circ S(x) \to \infty. \tag{4.1}$$

This sequence of points p_k will be used as the skeleton for orbits of the perturbed geodesic flow whose energy grows to infinity. Heuristically, these orbits can be described as follows: the orbits make excursions roughly along the homoclinic orbit when the external potential has a phase that helps to gain energy, but they bid their time between jumps staying close to the unperturbed periodic orbits till the phase becomes favorable again. As we will see later, this can also be interpreted in the usual language of whiskered tori and their heteroclinic intersections. By choosing the time when to perform the jumps, it will be possible for the orbits to keep on gaining energy.

Therefore, the main technical goal will be to compute perturbatively, for high energy, the inner and the outer maps F and S, show that applying them alternatively we can construct sequences p_k as in (4.1) and then, show that these orbits can be shadowed by real orbits.

The existence of p_k will require some non-degeneracy assumptions on the external potential (namely, that there are times at which jumping produces a gain in energy). It turns out that the gain in energy is expressed by an integral—commonly termed the Poincaré function—which depends on the phase at which the jump takes place (relative to the phase of the potential). If this function, as a function of the jumping time, is not constant, it is indeed possible to make jumps that gain energy. Rather remarkably, the same integral and the same condition appears in J.N. Mather's approach even if with a very different motivation.

Remark 4.1 It is interesting to note that the variational construction in [Mat96] also involves jumps separated by orbits that stay close to Λ. Nevertheless, his construction emphasizes a great deal that the orbits satisfy minimizing properties. In the construction here, it seems that minimality does not play any role. On the other hand, we need regularity conditions for the invariant sets (they are smooth tori) and we could not consider jumps to Cantor sets, as the variational method can. □

Remark 4.2 We recall attention to the fact that the problem has two different smallness parameters. One is how close is the metric to the integrable one. Another one is the inverse of the energy. For large values of the energy, the potential can be considered as a perturbation of the geodesic flow. Since the last perturbation is done on an unstable system, the problem can be considered as *a priori* unstable in the notation of [CG94].

We also note that there are two different time scales involved. One is the time scale of the period of the perturbation ($O(1)$) and the second one is that of the period of the geodesic ($1/\sqrt{E}$), which is also a characteristic time of the homoclinic trajectory. □

4.2 The scaled problem

In order to make the perturbative structure of the problem more apparent we will scale the variables and the time. Thus, we pick a (large) number E_* and introduce $\varepsilon = 1/\sqrt{E_*}$.

Recall that the original Hamiltonian is $H(p,q,t) = \frac{1}{2}g_q(p,p) + U(q,t)$, hence $\varepsilon^2 H(p,q,t) = \frac{1}{2}g_q(\varepsilon p, \varepsilon p) + \varepsilon^2 U(q,t)$. If we denote $\varepsilon p = \bar{p}$ and consider the symplectic form $\bar{\Omega} = d\bar{p} \wedge dq = \varepsilon \Omega$, we note that $q, \bar{p}$ are conjugate variables in $\bar{\Omega}$. We also introduce a new time $\bar{t} = t/\varepsilon$. We see that the equations

$$\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial q} = -\frac{1}{2}\frac{\partial g_q}{\partial q}(p,p) - \frac{\partial U}{\partial q}(q,t), \\ \frac{dq}{dt} &= \frac{\partial H}{\partial p} = g_q(p,\cdot), \end{aligned}$$

are equivalent to

$$\begin{aligned} \frac{d\bar{p}}{d\bar{t}} &= -\frac{1}{2}\frac{\partial g_q}{\partial q}(\bar{p},\bar{p}) - \varepsilon^2 \frac{\partial U}{\partial q}(q,\bar{t}\varepsilon), \\ \frac{dq}{d\bar{t}} &= g_q(\bar{p},\cdot), \end{aligned}$$

which are Hamiltonian equations in $\bar{\Omega}$, for the time $\bar{t}$, with respect to the Hamiltonian

$$\bar{H}_\varepsilon(\bar{p},q,\bar{t}\varepsilon) = \frac{1}{2}g_q(\bar{p},\bar{p}) + \varepsilon^2 U(q,\bar{t}\varepsilon). \tag{4.2}$$

We also introduce $\bar{E} = E/E_*$. For our purposes, it suffices to analyze a fixed range in scaled energies (which we will fix arbitrarily to be $[1/2, 2]$) and establish that for large enough E_*, we can find pseudo-orbits which are often close to Λ and whose energy increases from $\approx 1/2$ to ≈ 2. Then, using that the result is valid for all the large enough energies, we can construct a pseudo-orbit whose energy grows unboundedly.

From now on and until further notice, we will drop the bar from the problem. We will refer to the bar variables as the rescaled variables and the original ones as the physical variables. Then the Hamiltonian H_ε and all the function derived from it will be $1/\varepsilon$ periodic in time. In order to do this more apparent we will use the notation given in (4.2). Now, we want to compute rescaled versions of the maps F_0 and S_0.

In order to compute F_0, we note that H_0 is the Hamiltonian for the geodesic flow and that what we are doing is just considering the time $1/\varepsilon$ in the region $\cup_{1/2 \le E \le 2}\Sigma_E$.

We note that in the calculation of the inner map, ε enters in two different ways. One is as a parameter in the Hamiltonian (4.2) and another one is that

$1/\varepsilon$ is the time that we have to follow the flow. We introduce the notation f_ε^δ as the time $1/\delta$ map of the Hamiltonian (4.2). We will find it convenient to compute $f_\varepsilon^\varepsilon$ as a perturbation of f_0^ε. We will show that f_0^ε can be understood in terms of the geodesic flow and, then, the perturbation theory required to pass from f_0^ε to $f_\varepsilon^\varepsilon$ is only regular perturbation theory for (4.2).

Note also that when we consider extended phase spaces as introduced in section 3.3, the coordinate s, being an angle of period $1/\varepsilon$ moving at unit speed, is invariant under the time $1/\varepsilon$ map. Hence, when we consider $1/\varepsilon$ time maps, it suffices to consider only the coordinates in $T^*\mathbb{T}^2$. Of course, when the flow is non-autonomous, the map could depend on s, the phase at which the forcing happens.

We note that f_0^ε leaves invariant the manifold $\hat{\Lambda}_E$. In the coordinates (E,φ), this map restricted to Λ is (compare to (3.4)):

$$\begin{aligned} f_0^\varepsilon : \Lambda &\rightarrow \Lambda \\ (E,\varphi) &\mapsto (E, \varphi + \sqrt{E}/\varepsilon) \end{aligned}$$

Moreover, Λ is normally hyperbolic, and the hyperbolicity properties of the manifold $\hat{\Lambda}_E$ under this map can be easily ascertained:

1. The manifold $\hat{\Lambda}_E$ is invariant under f_0^ε.

2. The stable and unstable bundles do not depend on ε. (As stated in Remark 3.3, they do not depend on E when we identify the energy surfaces using the rescaling of the flow.)

3. The contraction and expansion factors can be bounded

$$\begin{aligned} \left\| Df_0^\varepsilon|_{T_{\hat{\Lambda}_E} W^{\mathrm{s}}_{\hat{\Lambda}_E}} \right\| &\leq \alpha^{\sqrt{E}/\varepsilon} \\ \left\| D(f_0^\varepsilon)^{-1}|_{T_{\hat{\Lambda}_E} W^{\mathrm{u}}_{\hat{\Lambda}_E}} \right\| &\leq \alpha^{\sqrt{E}/\varepsilon} \end{aligned}$$

   where α does not depend on ε.

In the next pages, we will study the perturbation theory for the manifold Λ and the time $1/\varepsilon$ map restricted to it.

4.3 The perturbed invariant manifold

Using the hyperbolicity properties of the manifold Λ for the geodesic flow (see section 3.2), we will apply the results of hyperbolic perturbation theory summarized in the appendix A.

In order to do perturbation theory for the manifold Λ, it will be more convenient to use the flow rather than the time $1/\varepsilon$ map. Notice that the Lyapunov exponents of the unperturbed map are $\pm\infty$. Even if this does not interfere with stability (roughly, the larger the Lyapunov exponents are, the more stable the system is), it is cumbersome to write the arguments. Later we will see that the computation of the inner map F_ε can also be done more easily in this framework.

As indicated in section 3.3, we find it more convenient to introduce the extra variable s and consider the autonomous flow

$$\begin{aligned} \dot{p} &= -\frac{\partial H_0}{\partial q}(p,q) - \delta\frac{\partial H_1}{\partial q}(p,q,s/\mathcal{T}) \\ \dot{q} &= \frac{\partial H_0}{\partial p}(p,q) + \delta\frac{\partial H_1}{\partial p}(p,q,s/\mathcal{T}) \\ \dot{s} &= 1 \end{aligned} \tag{4.3}$$

defined on the extended phase space $\mathbf{T}^*\mathbb{T}^2 \times \mathcal{T}\mathbb{T}^1$. This problem is equivalent to our original one if we set $\delta = \varepsilon^2$, $\mathcal{T} = 1/\varepsilon$, and $H_1(p,q,s/\mathcal{T}) = U(q,\varepsilon s)$.

We will denote the flow of (4.3) by $\tilde{\Phi}_{t,\mathcal{T},\delta}(p,q,s) = (\Gamma^{s,s+t}_{\mathcal{T},\delta}(p,q), s+t)$, where $\Gamma^{t,t'}_{\mathcal{T},\delta}(p,q)$ is the non-autonomous flow. Note that as usual $\Gamma^{t',t''}_{\mathcal{T},\delta} \circ \Gamma^{t,t'}_{\mathcal{T},\delta} = \Gamma^{t,t''}_{\mathcal{T},\delta}$ in the domains where these compositions make sense.

We note that setting $\delta = 0$ in (4.3) we have that

$$\tilde{\Lambda} := \Lambda \times \mathcal{T}\mathbb{T}^1 \simeq [E_0,\infty) \times \mathbb{T}^1 \times \mathcal{T}\mathbb{T}^1$$

is a manifold locally invariant for the flow. This manifold is also normally hyperbolic in the sense of Definition A.1.

Using Theorem A.13 and observation 1 after it, we have:

Theorem 4.3 *Assume that we have a system of equations as in* (4.3)*, where the Hamiltonian* $H = H_0 + \delta H_1$ *is* $\mathcal{C}^r$, $2 \le r < \infty$. *Then, there exists a* $\delta^* > 0$ *such that for* $|\delta| < \delta^*$, *there is a* $\mathcal{C}^{r-1}$ *function*

$$\mathcal{F} : [E_0 + K\delta, \infty) \times \mathbb{T}^1 \times \mathcal{T}\mathbb{T}^1 \times (-\delta^*, \delta^*) \longrightarrow \mathbf{T}^*\mathbb{T}^2 \times \mathcal{T}\mathbb{T}^1$$

such that

$$\tilde{\Lambda}_{\mathcal{T},\delta} = \mathcal{F}\left([E_0 + K\delta, \infty) \times \mathbb{T}^1 \times \mathcal{T}\mathbb{T}^1 \times \{\delta\}\right) \tag{4.4}$$

is locally invariant for the flow of (4.3).

Therefore, $\tilde{\Lambda}_{\mathcal{T},\delta}$ *is* δ $\mathcal{C}^{r-2}$*-close to* $\tilde{\Lambda}_{\mathcal{T},0} = \tilde{\Lambda}$.

From now on, we are going to fix our attention to the case $\delta = \varepsilon^2$ and $\mathcal{T} = 1/\varepsilon$, and we will call $\tilde{\Lambda}_\varepsilon = \tilde{\Lambda}_{1/\varepsilon,\varepsilon^2}$, $\tilde{\Phi}_{t,\varepsilon} = \tilde{\Phi}_{t,1/\varepsilon,\varepsilon^2}$ and $\Gamma_\varepsilon^{t,t'} = \Gamma_{1/\varepsilon,\varepsilon^2}^{t,t'}$.

Remark 4.4 Even if the Theorem 4.3 only guarantees local invariance for $\tilde{\Lambda}_\varepsilon$, we will show later that KAM theory will provide with invariant boundaries consisting of KAM tori. Therefore, it is possible to take $\tilde{\Lambda}_\varepsilon$ invariant. Since the results in hyperbolic theory for locally invariant manifolds are somewhat sharper for invariant manifolds (they include uniqueness statements and a geometric definition of stable and unstable manifolds), this will allow us later to state slightly sharper results. The main results in this paper can be obtained without this improvement, hence we will just develop it in remarks. □

4.4 The perturbed inner map

4.4.1 Averaging theory

First of all, let us note that in $\tilde{\Lambda}$ the unperturbed inner map is given by:

$$\begin{aligned} f_0^\varepsilon : \tilde{\Lambda} &\to \tilde{\Lambda} \\ (E, \varphi, s) &\mapsto \left(E, \varphi + \frac{\sqrt{E}}{\varepsilon}, s + \frac{1}{\varepsilon}\right) = \left(E, \varphi + \frac{\sqrt{E}}{\varepsilon}, s\right). \end{aligned}$$

In order to compute the inner map $F := f_\varepsilon^\varepsilon : \tilde{\Lambda}_\varepsilon \to \tilde{\Lambda}_\varepsilon$, we note that the external perturbation not only is small but is also slow. In this situation, we can apply the methods of the adiabatic perturbation theory (see, for example [AKN88, LM88]).

For a fixed s, we denote $\Lambda_\varepsilon^s \subset \mathbf{T}^*\mathbb{T}^2$ the manifold obtained by fixing s in $\tilde{\Lambda}_\varepsilon$ given by (4.4):

$$(\Lambda_\varepsilon^s, s) = \mathcal{F}\left([E_0 + K\varepsilon^2, \infty) \times \mathbb{T}^1 \times \{s\} \times \{\varepsilon^2\}\right)$$

By Theorem 4.3, Λ^s_ε is ε^2-close to the unperturbed manifold Λ in the $\mathcal{C}^{r-2}$ sense. In particular, if we denote by Ω^s_ε the restriction of the symplectic form Ω to these manifolds, it is a symplectic form. We also have $\Omega^s_\varepsilon = d\theta^s_\varepsilon$, where θ^s_ε is the restriction of the symplectic potential form to Λ^s_ε.

The classical results of adiabatic perturbation theory we want to quote refer to time dependent Hamiltonian flows on a fixed manifold with a fixed symplectic structure, whereas we have a time dependent manifold. Thus, we introduce changes of variables that keep the manifold fixed and study the flow induced in the fixed manifold. Since the Hamiltonian character is important in adiabatic perturbation theory, we pay attention to the Hamiltonian structure of the changes of variables.

Since $\tilde{\Lambda}_\varepsilon$ is invariant by the flow $\tilde{\Phi}_{t,\varepsilon}(p,q,s) = (\Gamma^{s,s+t}_\varepsilon(p,q), s+t)$ of (4.3), we have that $\Gamma^{t,t'}_\varepsilon : \Upsilon^t_\varepsilon \subset \Lambda^t_\varepsilon \to \Lambda^{t'}_\varepsilon$ (where Υ^t_ε excludes a neighborhood of order ε^2 outside the boundary of Λ^t_ε). Moreover, this flow transforms the symplectic structure in one manifold to the one of the image ${\Gamma^{t,t'}_\varepsilon}^* \Omega^t_\varepsilon = \Omega^{t'}_\varepsilon$. Furthermore, it is an exact transformation, that is, ${\Gamma^{t,t'}_\varepsilon}^* \theta^t_\varepsilon = \theta^{t'}_\varepsilon + dS^{t,t'}_\varepsilon$ where $S^{t,t'}_\varepsilon$ is a real valued function in $\Lambda^{t'}_\varepsilon$ and the d refers to the exterior differential in that manifold.

Now, since the manifolds Λ^s_ε are close to the standard one Λ we can find coordinate maps $C^s_\varepsilon : \Lambda^s_\varepsilon \to \Lambda$. We claim that it is possible to choose these C^s_ε in such a way that they transform the symplectic form into the standard one. In effect, if we push forward the symplectic forms Ω^s_ε, we obtain a family of symplectic forms in Λ which are close to Ω. These symplectic forms are also exact. Applying Moser's method [Wei77], we can find maps from Λ to Λ that transform these symplectic forms into the standard one. We will just redefine the C^s_ε to include the composition with these mappings in Λ. A proof that these maps can be chosen to be $\mathcal{C}^{r-2}$ jointly with the parameters can be found in complete detail in [BdlLW96].

If we now consider $C^{t'}_\varepsilon \circ \Gamma^{t,t'}_\varepsilon \circ (C^t_\varepsilon)^{-1}$ we see that it is a flow of exact symplectic mappings in Λ. The Hamiltonian $k_\varepsilon(E, \varphi, \varepsilon s)$ generating this flow is the push-forward by C^s_ε of the Hamiltonian $H_\varepsilon(p,q,s/\mathcal{T}) = H_\varepsilon(p,q,\varepsilon s)$ generating the flow of (4.3) ($\mathcal{T} = 1/\varepsilon$). In particular, it is a $\mathcal{C}^{r-2}$ flow, $1/\varepsilon$ periodic and it is an small perturbation of the constant flow $\dot{E} = 0, \dot{\varphi} = \sqrt{E}$ of Hamiltonian $2/3E^{3/2}$.

In this situation, we can apply the Adiabatic Invariance Theorem.

Theorem 4.5 *Let $k_\varepsilon(E,\varphi,\varepsilon s)$ be a $\mathcal{C}^n$ Hamiltonian, 1-periodic in φ and εs, such that $k_\varepsilon(E,\varphi,\varepsilon s) = \frac{2}{3}E^{3/2} + \varepsilon^2 k_1(E,\varphi,\varepsilon s;\varepsilon)$.*

Then, for any $0 < m < n$, and $E > 0$, there exists a canonical change of variables $(E,\varphi,s) \mapsto (I,\psi,s)$, 1-periodic in φ and εs, which is ε^2-close to the identity in the $\mathcal{C}^{n-m}$ topology, such that transforms the Hamiltonian system of Hamiltonian $k_\varepsilon(E,\varphi,\varepsilon s)$ into a Hamiltonian system of Hamiltonian $K_\varepsilon(I,\psi,\varepsilon s)$. This new Hamiltonian is a $\mathcal{C}^{n-m}$ function of the form:

$$K_\varepsilon(I,\psi,\varepsilon s) = K^0_\varepsilon(I,\varepsilon s) + \varepsilon^{m+1}K^1_\varepsilon(I,\psi,\varepsilon s)$$

where $K^0_\varepsilon(I,\varepsilon s) = \frac{2}{3}I^{3/2} + \mathrm{O}_{\mathcal{C}^1}(\varepsilon^2)$, and the notation $\mathrm{O}_{\mathcal{C}^1}(\varepsilon)$ means a function whose $\mathcal{C}^1$ norm is $\mathrm{O}(\varepsilon)$.

Proof.

The proof of this Theorem is standard. For more details and applications of the analytic case, one can see [AKN88]. We will just go over the proof to show that it works for finite differentiable Hamilttonians.

Calling a the action conjugate of time s, we have the 2-degrees of freedom Hamiltonian $a + k_\varepsilon(E,\varphi,\varepsilon s)$, which has a fast angle φ and a slow one εs.

We look for a canonical change of variables which eliminates the fast angle φ. The change will be obtained through a composition of changes of variables. Each of these changes will be generated through a generating function of the form:

$$Ps + I\varphi + \varepsilon^{q+2}S_q(I,\varphi,\varepsilon s;\varepsilon) \tag{4.5}$$

where S_q is 1-periodic on φ and εs.

In this way, through the implicit equations

$$\begin{aligned} E &= I + \varepsilon^{q+2}\frac{\partial S_q}{\partial\varphi}(I,\varphi,\varepsilon s;\varepsilon) \\ a &= P + \varepsilon^{q+3}\frac{\partial S_q}{\partial \varepsilon s}(I,\varphi,\varepsilon s;\varepsilon) \\ \psi &= \varphi + \varepsilon^{q+2}\frac{\partial S_q}{\partial I}(I,\varphi,\varepsilon s;\varepsilon) \end{aligned}$$

we obtain a canonical change of variables $(E,\varphi,a,\varepsilon s) \to (I,\psi,P,\varepsilon s)$, where

$$(E,\varphi,a) = (I,\psi,P) + \varepsilon^{q+2}\psi_q(I,\psi,\varepsilon s;\varepsilon) \tag{4.6}$$

which, by the implicit function theorem, has one degree less of differentiability than its generating function (4.5).

We will apply the following inductive Lemma:

Lemma 4.6 *Let us consider a Hamiltonian of the form*

$$a + K_q(E, \varphi, \varepsilon s; \varepsilon) = a + K_q^0(E, \varepsilon s; \varepsilon) + \varepsilon^{q+2} K_q^1(E, \varphi, \varepsilon s; \varepsilon)$$

where $K_q^0 = \frac{2}{3}E^{3/2} + \mathrm{O}_{\mathcal{C}^1}(\varepsilon^2)$ *is* $\mathcal{C}^{n-q+1}$ *and* K_q^1 *is* $\mathcal{C}^{n-q}$.

Let us consider $S_q(I, \varphi, \varepsilon s; \varepsilon)$ *the function verifying*

$$\frac{\partial}{\partial I} K_q^0(I, \varepsilon s; \varepsilon) \frac{\partial S_q}{\partial \varphi}(I, \varphi, \varepsilon s; \varepsilon) + K_q^1(I, \varphi, \varepsilon s; \varepsilon) = \overline{K}_q^1(I, \varepsilon s; \varepsilon) \tag{4.7}$$

where

$$\overline{K}_q^1(I, \varepsilon s; \varepsilon) = \int_0^1 K_q^1(I, \varphi, \varepsilon s; \varepsilon) d\varphi.$$

Then, the change (4.6) generated by (4.5) transforms the Hamiltonian $a + K_q(E, \varphi, \varepsilon s; \varepsilon)$*, into a Hamiltonian*

$$a + K_{q+1}(I, \psi, \varepsilon s; \varepsilon) = a + K_{q+1}^0(I, \varepsilon s; \varepsilon) + \varepsilon^{q+3} K_{q+1}^1(I, \psi, \varepsilon s; \varepsilon)$$

where

$$K_{q+1}^0(I, \varepsilon s; \varepsilon) = K_q^0(I, \varepsilon s; \varepsilon) + \varepsilon^{q+2} \overline{K}_q^1(I, \varepsilon s; \varepsilon) = \frac{2}{3} I^{3/2} + \mathrm{O}_{\mathcal{C}^{n-q}}(\varepsilon^2)$$

is $\mathcal{C}^{n-q}$ *and* K_q^1 *is* $\mathcal{C}^{n-q-1}$.

Proof.

Note that $S_q = \int \mathrm{d}\varphi \left(K_q^1 - \overline{K}_q^1 \right) / \partial_I K_q^0$, as well as $\partial S_q / \partial q$ are $\mathcal{C}^{n-q}$. The new Hamiltonian is given by

$$\begin{aligned}
&P + \varepsilon^{q+3} \frac{\partial S_q}{\partial \varepsilon s}(I, \varphi, \varepsilon s; \varepsilon) \\
&+ K_q^0 \left(I + \varepsilon^{q+2} \frac{\partial S_q}{\partial \varphi}(I, \varphi, \varepsilon s; \varepsilon), \varepsilon s; \varepsilon \right) \\
&+ \varepsilon^{q+2} K_q^1 \left(I + \varepsilon^{q+2} \frac{\partial S_q}{\partial \varphi}(I, \varphi, \varepsilon s; \varepsilon), \varphi, \varepsilon s; \varepsilon \right) \\
&= P + K_{q+1}^0(I, \varepsilon s; \varepsilon) + \varepsilon^{q+3} K_{q+1}^1(I, \psi, \varepsilon s; \varepsilon)
\end{aligned}$$

where, Taylor expanding K_q^0 and K_q^1 and using the Definition (4.7) of the generating function, we get:

$$K_{q+1}^0(I, \varepsilon s; \varepsilon) = K_q^0(I, \varepsilon s; \varepsilon) + \varepsilon^{q+2}\overline{K}_q^1(I, \varepsilon s; \varepsilon)$$

and

$$\begin{aligned}
\varepsilon^{q+3}K_{q+1}^1(I, \psi, \varepsilon s; \varepsilon) &= K_q^0\left(I + \varepsilon^{q+2}\frac{\partial S_q}{\partial \varphi}(I, \varphi, \varepsilon s; \varepsilon), \varepsilon s; \varepsilon\right) \\
&-K_q^0(I, \varepsilon s; \varepsilon) - \frac{\partial}{\partial I}K_q^0(I, \varepsilon s; \varepsilon)\varepsilon^{q+2}\frac{\partial S_q}{\partial \varphi}(I, \varphi, \varepsilon s; \varepsilon) \\
&+\varepsilon^{q+2}\left(K_q^1\left(I + \varepsilon^{q+2}\frac{\partial S_q}{\partial \varphi}(I, \varphi, \varepsilon s; \varepsilon), \varphi, \varepsilon s; \varepsilon\right) - K_q^1(I, \varphi, \varepsilon s; \varepsilon)\right) \\
&+\varepsilon^{q+3}\frac{\partial S_q}{\partial \varepsilon s}(I, \varphi, \varepsilon s; \varepsilon)
\end{aligned}$$

where, in these formulas, φ has to be expressed in terms of the variables $(I, \psi, \varepsilon s; \varepsilon)$ using the change of variables (4.6).

Since K_q^0 is $\mathcal{C}^{n-q+1}$ and K_q^1 is $\mathcal{C}^{n-q}$, it is clear that S_q, $\partial_\varphi S_q$ are $\mathcal{C}^{n-q}$, and the change of variables (4.6) is $\mathcal{C}^{n-q-1}$. (Note that in the equation above, only the term $\varepsilon^{q+3}\partial_{\varepsilon s}S_q$ is $\mathcal{C}^{n-q-1}$.) Then one has that K_{q+1}^0 is $\mathcal{C}^{n-q}$ and K_{q+1}^1 is $\mathcal{C}^{n-q-1}$. $\square$

To finish the proof of Theorem 4.5, we only need to apply the inductive Lemma 4.6 for $q = 0, 1, \ldots m-1$, and we obtain the desired result. For $q = 0$, it is important to note that $K_0^0(E, \varepsilon s; \varepsilon) = \frac{2}{3}E^{3/2}$ is $\mathcal{C}^\infty$ if $E > 0$, and $K_0^1(E, \varphi, \varepsilon s; \varepsilon) = k^1(E, \varphi, \varepsilon s; \varepsilon)$ is $\mathcal{C}^n$. Then the last Hamiltonian will be of class $\mathcal{C}^{n-m}$. $\square$

Lemma 4.7 *In the conditions of Theorem 4.5 with $n = r-2$, the map $f_\varepsilon^\varepsilon$ restricted to Λ_ε^0, which is exact symplectic, can be written in the coordinates (I, ψ) introduced in Theorem 4.5 as*

$$f_\varepsilon^\varepsilon(I, \psi) = \left(I, \psi + \frac{1}{\varepsilon}A(I, \varepsilon)\right) + \varepsilon^m R(I, \psi; \varepsilon), \tag{4.8}$$

where $A(I, \varepsilon) = \varepsilon\int_0^{1/\varepsilon} D_1K_\varepsilon^0(I, \varepsilon s)ds = \sqrt{I} + \mathrm{O}_{\mathcal{C}^0}(\varepsilon^2)$, and R is a $\mathcal{C}^{r-m-4}$ function.

Proof.

Recall that $f_\varepsilon^\varepsilon$ in the (I, ψ) coordinates is the time $1/\varepsilon$ map of the $\mathcal{C}^{r-2-m}$ Hamiltonian K_ε whose flow is $\mathcal{C}^{r-3-m}$. The flow, in these coordinates, is the flow of an integrable Hamiltonian K_ε^0 plus some Hamiltonian of order $\mathrm{O}(\varepsilon^{m+1})$. Hence, using variational equations, we obtain that the time $1/\varepsilon$ map differs from that of the integrable part by an amount not larger than ε^m in the $\mathcal{C}^{n-4-m}$ topology. $\square$

4.4.2 K.A.M. theory

We now recall a quantitative version of KAM Theorem. The version below is somewhat weaker than that of [Her83] (we do not use fractional regularities so we loose whole integer number of derivatives in the conclusion while an arbitrary real number would suffice), but is enough for our purposes. We recall that a real number ω is called a Diophantine number of exponent θ if there exists a constant $C > 0$ such that $|\omega - p/q| \geq C/q^{\theta+1}$ for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

Theorem 4.8 *Let $f : [0,1] \times \mathbb{T}^1 \mapsto [0,1] \times \mathbb{T}^1$ be an exact symplectic $\mathcal{C}^l$ map, with $l \geq 4$.*

Assume that $f = f_0 + \delta f_1$, where $f_0(I, \psi) = (I, \psi + A(I))$, A is $\mathcal{C}^l$, $\left|\dfrac{dA}{dI}\right| \geq M$, and $\|f_1\|_{\mathcal{C}^l} \leq 1$.

Then, if $\delta^{1/2} M^{-1} = \rho$ is sufficiently small, for a set of σ of Diophantine numbers of exponent $\theta = 5/4$, we can find invariant tori which are the graph of $\mathcal{C}^{l-3}$ functions u_σ, the motion on them is $\mathcal{C}^{l-3}$ conjugate to the rotation by σ, and $\|u_\sigma\|_{\mathcal{C}^{l-3}} \leq \mathrm{cte}\, \delta^{1/2}$, and the tori cover the whole annulus except a set of measure smaller than $\mathrm{cte}\, M^{-1} \delta^{1/2}$.

Moreover, if $l \geq 6$ we can find expansions $u_\sigma = u_\sigma^0 + \delta u_\sigma^1 + r_\sigma$, with $\|r\|_{\mathcal{C}^{l-4}} \leq \mathrm{cte}\, \delta^2$, and $\|u^1\|_{\mathcal{C}^{l-4}} \leq \mathrm{cte}$.

Applying Theorem 4.8 to the map $f_\varepsilon^\varepsilon$ restricted to Λ_ε^0, in the coordinates given in Lemma 4.7, we obtain KAM invariant tori of system (4.3), as long as this map is $\mathcal{C}^l$ with $l := r - m - 4 \geq 6$. Note that according to (4.8), the frequencies of f_0^ε are roughly $(1/\varepsilon) A(I, \varepsilon)$ with $A(I, 0)$ the frequencies of the unperturbed Hamiltonian flow. Hence, the $\mathcal{C}^{l-4}$ distance between invariant tori is not bigger than $\varepsilon^{m/2+1}$. We note that these invariant circles for $f_\varepsilon^\varepsilon$ correspond to invariant two dimensional tori for the extended flow. An invariant circle for $f_\varepsilon^\varepsilon$ with frequency ω corresponds to a two dimensional invariant torus for $\tilde{\Phi}_{t,\varepsilon}$ with frequency (ω, ε).

Remark 4.9 Note that these KAM tori that we have produced for the map $f_\varepsilon^\varepsilon$ are really whiskered tori for the extended flow $\tilde{\Phi}_{t,\varepsilon}$. They could have been produced also by appealing to the Graff-Zehnder Theorem.

In particular, proceeding as in Zehnder [Zeh75, Zeh76] we can obtain a normal form for the Hamiltonian $H_\varepsilon(p, q, \varepsilon s)$ in a neighborhood of these KAM tori:

$$\begin{aligned} G(I, a, \varphi, s, z^s, z^u) &= \omega I + a + \frac{\Gamma}{2} I^2 + < z^s, \Omega(\varphi, s) z^u > \\ &+ g(I, \varphi, s, z^s, z^u) \end{aligned} \tag{4.9}$$

Such normal forms are commonly used in the study of inclination lemmas for whiskered tori. However, we will perform our study of inclination lemmas in the normal form for whiskered tori with one dimensional whiskers introduced in [FM98] (Section (4.1)). This normal form does not require that the motion on the tori satisfies Diophantine conditions—only that it is an irrational rotation—, and requires much less regularity. □

Remark 4.10 When the metric and the potential are $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$, even if the argument using the hyperbolic invariant manifold only allows to construct finitely differentiable tori, appealing to the results in [Zeh75, Zeh76], we can conclude that these tori we constructed are indeed $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$. □

Remark 4.11 Note that KAM tori produced by Theorem 4.8 are of codimension 1 inside $\tilde{\Lambda}_\varepsilon$. If we choose a submanifold whose boundary consists of two KAM tori, this submanifold will be an invariant manifold for the extended flow. The results of hyperbolic perturbation theory of appendix A can be extended to include uniqueness as is explained in observation 4 after Theorem A.13. □

Once we have the existence of the invariant tori of system (4.3), it is worthwhile to obtain some explicit approximations for them in the coordinate system given by the phases (φ, s) and the value of the Hamiltonian H_ε.

(Note that since the hamiltonian H_ε is close to the coordinate E, $(H_\varepsilon, \varphi, s)$ constitute a good system of coordinates.)

We will find it convenient to write

$$U(q,\tau) = \bar{U}(\tau) + \tilde{U}(q,\tau)$$

where the functions $\bar{U}(\tau)$, and $\tilde{U}(q,\tau)$ are given by

$$\bar{U}(\tau) = \int_0^1 U(\Lambda_1(\varphi),\tau)d\varphi, \qquad \tilde{U}(q,\tau) = U(q,\tau) - \bar{U}(\tau). \tag{4.10}$$

This decomposition is natural because of the different scales involving the problem. We are separating explicitly the average on the fast variables. We call attention to the fact that $\bar{U}(\tau)$, being independent of q, does not affect the dynamics.

Lemma 4.12 *Let ω be one of the frequencies σ allowed in Theorem 4.8. Then, in the coordinate system $(H_\varepsilon, \varphi, s)$, we can write the torus of frequency (ω, ε) as the graph of a function $G(\varphi, s; \varepsilon)$. Moreover, we can write*

$$G(\varphi, s; \varepsilon) = E + \varepsilon^2 \bar{U}(\varepsilon s) + \varepsilon^3 \tilde{g}(\varphi, s; \varepsilon) + \mathrm{O}_{\mathcal{C}^{l-4}}(\varepsilon^4) \tag{4.11}$$

where $E = \omega^2$, $\tilde{g}(\varphi, \tau; \varepsilon)$ is a 1-periodic in (φ, τ) function which verifies

$$\omega D_1 \tilde{g}(\varphi, \tau; \varepsilon) + \varepsilon D_2 \tilde{g}(\varphi, \tau; \varepsilon) = D_2 \tilde{U}(\Lambda_1^q(\varphi), \tau) + \mathrm{O}_{\mathcal{C}^{l-4}}(\varepsilon^3) \tag{4.12}$$

and $||\tilde{g}(\cdot, \cdot; \varepsilon)||_{\mathcal{C}^{l-4}}$ is bounded uniformly in ε.

Furthermore, we can choose $\tilde{g}$ in such a way that $\tilde{g} = D_2 \tilde{h}$. This $\tilde{h}$ satisfies (obviously)

$$\omega D_1 \tilde{h}(\varphi, \tau; \varepsilon) + \varepsilon D_2 \tilde{h}(\varphi, \tau) = \tilde{U}(\Lambda_1^q(\varphi), \tau) + \mathrm{O}_{\mathcal{C}^{l-4}}(\varepsilon^3) \tag{4.13}$$

and $||\tilde{h}(\cdot, \cdot; \varepsilon)||_{\mathcal{C}^{l-4}}$ is bounded uniformly in ε.

We call attention to the fact that the functions as we have defined them are not unique. On the other hand, as we will see later, the ambiguities only arise in subdominant terms.

Proof. We will first present a formal proof and then we will work out the relation with perturbative methods such as Lindstedt-Poincaré, which are somewhat subtle since the problem involves singular perturbations. (One frequency is much larger than the other.)

The KAM theorem 4.8 provides us with parameterizations

$$(p(\psi, \varepsilon s), q(\psi, \varepsilon s), s)$$

of the torus in terms of variables ψ, s which satisfy $\dot{\psi} = \omega$, $\dot{s} = 1$.

These parameterizations are $O_{\mathcal{C}^{l-4}}(\varepsilon^3)$ close to constant when expressed in terms of the averaged variables.

We denote by

$$G(\psi, \varepsilon s; \varepsilon) = H_\varepsilon(p(\psi, \varepsilon s), q(\psi, \varepsilon s), \varepsilon s) \tag{4.14}$$

and note that the derivative with respect to the flow of this equation is

$$\begin{aligned} \frac{d}{dt} G \circ \Phi_{t,\varepsilon}|_{t=0} &= \omega D_1 G(\psi, \varepsilon s; \varepsilon) + \varepsilon D_2 G(\psi, \varepsilon s; \varepsilon) \\ &= \varepsilon^3 D_2 U(q(\psi, \varepsilon s; \varepsilon), \varepsilon s). \end{aligned} \tag{4.15}$$

We note that the first two terms of the averaging transformations are $E+\varepsilon^2 \bar{U}(\varepsilon s)$ and that, as a consequence of the hyperbolic perturbation theory, the averaging method and the KAM theory, the KAM tori are close to an orbit Λ of the unperturbed system. If we perform this substitution in (4.15), we obtain the desired result. □

Remark 4.13 The previous calculation can be also understood as a modification of Lindstedt-Poincaré method. Since the Lindstedt-Poincaré method is a commonly used tool in singularly perturbed systems, we thought it could be interesting to some readers to develop a comparison. We refer to [Gal94] for a survey of Lindstedt methods for analytic systems that includes a treatment of singularly perturbed systems through the use of tree-like diagrams.

Since we are considering a system with two time scales, the most standard method, which fixes the frequency and, then, seeks parameterizations of tori with the prescribed frequency as expansions in powers of ε, cannot work since the frequency dependence in ε will cause the composed frequency to go through resonances on which we do not expect tori to exist.

Nevertheless, we will see that it is possible to compute systematically parameterizations $p(\psi, \varepsilon s; \varepsilon)$, $q(\psi, \varepsilon s; \varepsilon)$ that satisfy the equations of motion to a very high accuracy and whose coefficients are, furthermore, of moderate size. Once we have that, the Newton method started on them will lead to a true solution which is close to these approximate solutions. (See [Zeh75, Zeh76])

If we seek a parameterization of the torus with frequency vector (ω, ε), as above, we obtain a system of equations

$$\begin{aligned} [\omega D_1 + \varepsilon D_2]\; p(\psi, \varepsilon s; \varepsilon) &= -D_q H_\varepsilon(p(\psi, \varepsilon s; \varepsilon), q(\psi, \varepsilon s; \varepsilon), s) \\ [\omega D_1 + \varepsilon D_2]\; q(\psi, \varepsilon s; \varepsilon) &= D_p H_\varepsilon(p(\psi, \varepsilon s; \varepsilon), q(\psi, \varepsilon s; \varepsilon), s) \end{aligned} \tag{4.16}$$

Even if, as we will soon see, it is a bad idea to try to obtain solutions that are just powers of ε with coefficients that are functions only of the other variables, it is quite feasible to obtain expansions in powers of ε with coefficients that are functions of all the variables—including ε—which solve (4.16) up to a high order power in ε and such that all the coefficients are of order 1. These coefficients are not unique since the term of a certain order is only defined up to terms of higher order.

The main observation is that, given Γ with $\int_{\mathbb{T}} \Gamma(\psi, \varepsilon s; \varepsilon)\, d\psi = 0$ and smooth, the equation for η

$$[\omega D_1 + \varepsilon D_2]\eta(\psi, \varepsilon s; \varepsilon) = \Gamma(\psi, \varepsilon s; \varepsilon) \tag{4.17}$$

can be satisfied up to high order error in ε by functions whose size is comparable to Γ. As it is well known, this is the homology equation and the Lindstedt series can be computed by recursively solving this equation on expressions that involve only previously computer quantities.

If we try to solve (4.17) using Fourier analysis, we find that it is equivalent to

$$\hat{\eta}_{k_1,k_2} = (2\pi\imath(\omega k_1 + \varepsilon k_2))^{-1}\, \hat{\Gamma}_{k_1,k_2} \tag{4.18}$$

If we choose η in such a way that its Fourier coefficients with $|k| \le \varepsilon^{-1/2}$ are obtained according to (4.18) and the other ones are zero, we note that:

a) If Γ is $\mathcal{C}^m$ then

$$|\hat{\Gamma}_{k_1,k_2}| \le C|k|^{-m} ||\Gamma||_{\mathcal{C}^m}.$$

Hence, η solves the equation (4.17) up to an error whose $\mathcal{C}^l$ norm can be bounded by $C||\Gamma||_{\mathcal{C}^m} \sum_{|k|\ge\varepsilon^{1/2}} |k|^{l-m}$, which can, in turn, be bounded by $C||\Gamma||_{\mathcal{C}^m}\varepsilon^{(-l+m-2)/2}$ when $l - m + 1 < -1$.

b) Since Γ has no Fourier coefficients with $k_1 = 0$, then the denominators of (4.18) are uniformly bounded from below and we have, using the same estimates as above, $||\eta||_{\mathcal{C}^l} \leq C||\Gamma||_{\mathcal{C}^m}$ when $l - m + 1 < -1$.

By repeating this construction in all the steps that we have to solve (4.17) in the calculation of the Lindstedt series, we obtain functions of size bounded uniformly in ε which satisfy (4.16) up to an error which can be bounded by a power of ε. This power can be made arbitrarily high if we are considering systems that are differentiable enough.

Note that these approximate solutions—in contrast to those of the standard Lindstedt method—are not unique since they include choices such as the level of truncation (we took $|k| \leq \varepsilon^{-1/2}$ but could have made other choices).

The above procedure makes it clear that it is a bad idea solving the equations (4.17) exactly. If we considered in (4.18) the coefficients with $|k| \approx \varepsilon^{-1}$ or bigger we would indeed have to consider small divisors. This is a reflection of the fact that there is no number ω such that (ω, ε) is a nonresonant vector for an interval of ε around zero.

Once we have parameterizations that solve (4.16) with a very small error, we can apply an appropriate version of KAM theorem to produce an exact solution. Indeed, this Lindstedt method is an alternative to the averaging method that we used in the main text.

We emphasize that for the applications that we have in mind here, there suffices to compute only a finite number of terms to obtain approximations to $O(\varepsilon^n)$. Hence, there is no need to discuss convergence and we only need that the functions involved are finitely differentiable. □

4.5 The perturbed outer map. Theoretical results

Now, we turn to the problem of computing the map S. We will be mainly interested in computing the image under S of one of the KAM circles invariant under F. Moreover it will suffice to study the change in energy, and, more importantly, characterize when this change of energy produces a jump from a torus to another.

The theory developed in appendix A will show that the construction of the map S can be carried out for small perturbations, giving rise to a differentiable map that also depends differentiably on parameters. Note that once we now that the map is differentiable with respect to parameters, we will be able to compute the derivatives just matching powers.

In effect, if we fix a transversal section $\mathcal{S}$ to the flow in a neighborhood of γ, we can consider $\mathcal{V}_\varepsilon = W^u_{\tilde\Lambda,\varepsilon} \cap W^u_{\tilde\Lambda,\varepsilon} \cap \mathcal{S}$, where $W^s_{\tilde\Lambda,\varepsilon}$ is the stable manifold under the perturbed flow $\tilde\Phi_{t,\varepsilon}$ of $\tilde\Lambda_\epsilon$—the manifold invariant under $\tilde\Phi_{t,\varepsilon}$ close to Λ. (See section 3.2 and appendix A.)

Because of the smooth dependence on parameters of the stable and unstable manifolds, for small enough ε, this $\mathcal{V}_\varepsilon$ is a smooth manifold that depends smoothly on the parameter ε. (See Theorem A.13 for details on how smooth the flow may be chosen to be.)

On $\mathcal{V}_\varepsilon$ we can define functions $x_{\pm,\varepsilon} : \mathcal{V}_\varepsilon \to \tilde\Lambda_\varepsilon$. These functions are jointly smooth in the variables and in the parameters. For some precisions on the rather standard method to define smooth functions on smooth families of manifolds and for the degree of differentiability that one can expect in this situation, we refer to Theorem A.13. We note that, for small ε, we can assume that, when r, the differentiability of the flow satisfies $r < \infty$, all the differentiability properties of the manifolds and of the dependence with respect to parameter are just r since the other limiting regularities in the general theory of hyperbolic systems are higher than r.

We argued already that $x_{+,0}(\mathcal{V}_0) \times x_{-,0}(\mathcal{V}_0) \subset \Lambda_0 \times \Lambda_0$ is the graph of a function. Hence, for small ε, $x_{+,\varepsilon}(\mathcal{V}_\varepsilon) \times x_{-,\varepsilon}(\mathcal{V}_\varepsilon) \subset \tilde\Lambda_\varepsilon \times \tilde\Lambda_\varepsilon$ will also be the graph of a smooth function, and it will depend smoothly on parameters.

We call attention to the fact that, as discussed in appendix A, the stable and unstable manifolds for $\tilde\Lambda_\varepsilon$ are foliated by the stable and unstable manifolds of the points in $\tilde\Lambda_\varepsilon$. For the tori produced in Theorem 4.8, these (un)stable manifolds are called the whiskers.

As we know already from Lemma 4.12 an approximate characterization of the KAM tori in $\tilde\Lambda_\varepsilon$ in terms of H_ε, we will be interested in computing the change of H_ε in the outer map.

Due to the fact that $\tilde\Lambda_\varepsilon$ is a hyperbolic manifold and that its stable and unstable manifolds intersect, we know that any point on $\tilde\Lambda_\varepsilon$ is connected to another point through some "homoclinic" (to $\tilde\Lambda_\varepsilon$) orbit.

The fact that $x_+ = S(x_-)$ for two points $x_\pm \in \tilde\Lambda_\varepsilon$ is, by definition, equivalent to the existence of a point $z \in \mathbf{T}^*\mathbb{T}^2 \times \mathcal{T}\mathbb{T}^1$, for $\mathcal{T} = 1/\varepsilon$, such that the trajectory $\tilde\gamma(t)$ starting on it converges for $t \to \pm\infty$ exponentially fast to the trajectories $\tilde{\Lambda}^{\pm}$ started in $x_{\pm}$ respectively. i.e. $\tilde{\Lambda}^{\pm}(0) = x_{\pm}$ and

$$\operatorname{dist}(\tilde{\Lambda}^{\pm}(t), \tilde{\gamma}(t)) \leq Ce^{-\beta|t|} \quad \text{as} \quad t \to \pm\infty. \tag{4.19}$$

We want to give criteria to decide whether x_+ and x_- belong to the same torus (and $\tilde{\gamma}$ would be a homoclinic orbit to this torus), or if they are in different tori (and $\tilde{\gamma}$ is heteroclinic).

4.6 The perturbed outer map. The Poincaré function.

The characterization of orbits homoclinic to a torus will be done in two steps. First we will compute $x_+ = S(x_-)$. From (3.8), and using the fact that the perturbation is $\mathrm{O}(\varepsilon^2)$, see (4.2), we already know that $\varphi_+ = \varphi_- + \Delta + O(\varepsilon^2)$, where $\Delta = a_+ - a_-$ is the phase shift. Moreover, the asymptotic condition (4.19) and the fact that $\dot{s} = 1$ even for the perturbation, implies $s_+ = s_- = s$, where s is the time coordinate of the homoclinic point z.

The first goal of this section will be to compute the leading term of the difference $H_\varepsilon(x_+) - H_\varepsilon(x_-)$ (which is zero in the unperturbed case). This is done in Lemma 4.14 using the classical Poincaré-Melnikov theory. In a second step, we need to decide whether $S(x_-) = x_+$ and x_- lie in the same torus of those produced by Theorem 4.8. Since the energy H_ε is not constant on the tori, it is not enough to compare $H_\varepsilon(x_+)$ with $H_\varepsilon(x_-)$. Instead, we will consider the point y belonging to the torus of x_- but with the same phases as x_+ and compute $H_\varepsilon(x_+) - H_\varepsilon(y)$, which will be our Melnikov function. See figure 1.

Therefore, the main goal of this section will be to give an explicit expression of the difference $H_\varepsilon(x_+) - H_\varepsilon(y)$. To compute $H_\varepsilon(y)$ we will follow a general method, which consists in using the equations (4.11) that express the torus containing x_- as a graph of the energy H_ε over the phases. If we evaluate this function at the phases of x_+, we obtain a Poincaré function (4.27) which gives the leading term of $H_\varepsilon(x_+) - H_\varepsilon(y)$ [Tre94]. This will be done in Lemma 4.15.

For the system at hand, we can take advantage of the slow dynamics and we can use the fact that the point $\tilde{\Phi}_{\Delta/\omega,\varepsilon}(x_-) \equiv u$ has the same phases $(\varphi, \varepsilon s)$ as y up to order ε. Using this fact, in Lemma 4.18 we will give an explicit formula for the leading term of the Poincaré function in terms of the potential U and the unperturbed geodesics, with no need to solve any small divisors

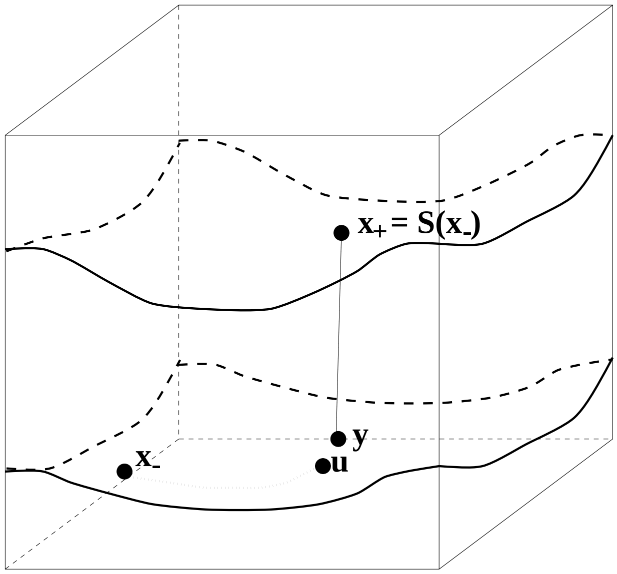

Figure 1: Illustration of the perturbed tori and the outer map.

equation to obtain $H_\varepsilon(y)$. This explicit expression will be quite important to establish that, for high enough energies—in the scaled variables for small enough ε—the KAM tori have homoclinic and heteroclinic intersections.

Lemma 4.14 *Let x_- and x_+ be two points on $\tilde{\Lambda}_\varepsilon$ such that $x_+ = S(x_-)$, then*

$$
\begin{aligned}
H_\varepsilon(x_+) - H_\varepsilon(x_-) \;=\; & \varepsilon^3 \lim_{(T_1,T_2)\to\infty} \Bigg[\int_{-T_1}^{T_2} dt\, D_2\tilde{U}(\gamma_E^q(t + \frac{\varphi_0}{\sqrt{E}}), \varepsilon s + \varepsilon t) \\
& - \int_{-T_1}^{0} dt\, D_2\tilde{U}(\Lambda_E^q(t + \frac{\varphi_0 + a_-}{\sqrt{E}}), \varepsilon s + \varepsilon t) \\
& - \int_{0}^{T_2} dt\, D_2\tilde{U}(\Lambda_E^q(t + \frac{\varphi_0 + a_+}{\sqrt{E}}), \varepsilon s + \varepsilon t) \Bigg] + \mathrm{O}(\varepsilon^5)
\end{aligned}
\tag{4.20}
$$

where

$$
x_\pm = \mathcal{F}(E, \varphi_\pm, s, \varepsilon^2) = \left(\Lambda_E \left(\frac{\varphi_0 + a_\pm}{\sqrt{E}} \right) + \mathrm{O}(\varepsilon^2), s \right)
$$

for some $\varphi_0 \in \mathbb{R}$, where $\mathcal{F}$ is introduced in Theorem 4.3, as a parameterization of the invariant manifold $\tilde{\Lambda}_\varepsilon$ and $\tilde{U}$, introduced in (4.10), is the forcing potential minus its average on the periodic orbit Λ.

Proof.

Recall that if a trajectory $\tilde{\lambda}(t) = (\lambda^p(t), \lambda^q(t), s+t)$ satisfies (4.3) then:

$$\frac{d}{dt} H_\varepsilon \circ \tilde{\lambda}(t) = \varepsilon^3 D_2 U(\lambda^q(t), \varepsilon s + \varepsilon t).$$

Therefore, for any two trajectories $\tilde{\lambda} = (\lambda^p, \lambda^q, s+t))$, $\tilde{\mu} = (\mu^p, \mu^q, r+t)$ of (4.3), we have, by the fundamental theorem of Calculus,

$$\begin{aligned} H_\varepsilon(\tilde{\lambda}(T)) - H_\varepsilon(\tilde{\mu}(T)) &= H_\varepsilon(\tilde{\lambda}(0)) - H_\varepsilon(\tilde{\mu}(0)) \\ + \quad \varepsilon^3 \int_0^T dt\, D_2 U(\lambda^q(t), \varepsilon s + \varepsilon t) &- \varepsilon^3 \int_0^T dt\, D_2 U(\mu^q(t), \varepsilon r + \varepsilon t). \end{aligned} \tag{4.21}$$

As $x_+ = S(x_-)$, we know that there exists $z \in \mathbf{T}^*\mathbb{T}^2 \times \mathcal{T}\mathbb{T}^1$, $\mathcal{T} = 1/\varepsilon$, such that the trajectory $\tilde{\gamma}_{(\varepsilon)}$ starting on it and $\tilde{\Lambda}_{+,(\varepsilon)}$, $\tilde{\Lambda}_{-,(\varepsilon)}$, the trajectories starting on x_+, x_-, verify (4.19). Therefore the three points have the same time coordinate s.

Now we can use (4.21) and, by (4.19), taking limits at $\pm\infty$ as appropriate,

$$\begin{aligned} 0 &= H_\varepsilon(x_+) - H_\varepsilon(z) \\ &\quad + \lim_{T_2 \to \infty} \varepsilon^3 \int_0^{T_2} dt\, \Big(D_2 U(\Lambda^q_{+,(\varepsilon)}(t), \varepsilon s + \varepsilon t) - D_2 U(\gamma^q_{(\varepsilon)}(t), \varepsilon s + \varepsilon t) \Big) \\ 0 &= H_\varepsilon(x_-) - H_\varepsilon(z) \\ &\quad + \lim_{T_1 \to \infty} \varepsilon^3 \int_0^{-T_1} dt\, \Big(D_2 U(\Lambda^q_{-,(\varepsilon)}(t), \varepsilon s + \varepsilon t) - D_2 U(\gamma^q_{(\varepsilon)}(t), \varepsilon s + \varepsilon t) \Big). \end{aligned}$$

Subtracting these two equations we obtain

$$\begin{aligned} & H_\varepsilon(x_+) - H_\varepsilon(x_-) = \\ & - \ \varepsilon^3 \lim_{(T_1, T_2) \to \infty} \Bigg[\int_0^{T_2} dt\, \Big(D_2 U(\Lambda^q_{+,(\varepsilon)}(t), \varepsilon s + \varepsilon t) - D_2 U(\gamma^q_{(\varepsilon)}(t), \varepsilon s + \varepsilon t) \Big) \\ & - \int_{-T_1}^{0} dt\, \Big(D_2 U(\Lambda^q_{-,(\varepsilon)}(t), \varepsilon s + \varepsilon t) - D_2 U(\gamma^q_{(\varepsilon)}(t), \varepsilon s + \varepsilon t) \Big) \Bigg]. \end{aligned} \tag{4.22}$$

By (4.19), these limits are reached uniformly in ε. We also note that the dependence of the trajectories on ε is uniform on compact intervals of time.

Hence, at the expense only of introducing an error of higher order in ε, we can substitute in (4.22) for $\Lambda_{\pm,(\varepsilon)}$ and $\gamma_{(\varepsilon)}$ the unperturbed orbits:

$$\begin{aligned}
\Lambda_{\pm,(\varepsilon)}(t) &= \Lambda_E\left(t+\frac{\varphi_0+a_\pm}{\sqrt{E}}\right)+\mathrm{O}(\varepsilon^2)\\
\gamma_{(\varepsilon)}(t) &= \gamma_E\left(t+\frac{\varphi_0}{\sqrt{E}}\right)+\mathrm{O}(\varepsilon^2)
\end{aligned}$$

with Λ_E and γ_E the orbits constructed in Theorem 3.1.

We note that the right hand side of (4.22) is linear in U. Hence if we use the decomposition $U(q,\tau)=\bar U(\tau)+\tilde U(q,\tau)$ given in (4.10), and observe that computing the right hand side of (4.22) in $\bar U$ gives zero, we obtain (4.20). □

Lemma 4.15 *Let y be a point with the phases of x_+ and which lies on the invariant torus for the perturbed flow which contains x_-, where*

$$x_+=\mathcal{F}(E,\varphi_+,\varepsilon s,\varepsilon^2)=\left(\Lambda_E\left(\frac{\varphi_0+a_+}{\sqrt{E}}\right)+\mathrm{O}(\varepsilon^2),\varepsilon s\right).$$

Then:

$$\begin{aligned}
H_\varepsilon(x_+)-H_\varepsilon(y) &= \varepsilon^3\lim_{(T_1,T_2)\to\infty}\Bigg[\int_{-T_1}^{T_2} dt D_2\tilde U(\gamma_E^q\left(t+\frac{\varphi_0}{\sqrt{E}}\right),\varepsilon s+\varepsilon t)\\
&\qquad -\tilde g(\varphi_0+a_++\sqrt{E}\,T_2,\varepsilon s+\varepsilon T_2;\varepsilon)\\
&\qquad -\tilde g(\varphi_0+a_--\sqrt{E}\,T_1,\varepsilon s-\varepsilon T_1;\varepsilon)\Bigg]\\
&\quad +\mathrm{O}(\varepsilon^5)\,.
\end{aligned}\tag{4.23}$$

where $\tilde g$ is the function given in Lemma 4.12 verifying (4.12), associated to the invariant torus of the perturbed flow which contains x_-.

Proof.

We use Lemma 4.14 for $H_\varepsilon(x_+)-H_\varepsilon(x_-)$ and Lemma 4.12 for $H_\varepsilon(x_-)-H_\varepsilon(y)$:

$$\begin{aligned}
&H_\varepsilon(x_+)-H_\varepsilon(y)=H_\varepsilon(x_+)-H_\varepsilon(x_-)+H_\varepsilon(x_-)-H_\varepsilon(y)\\
&\quad=\varepsilon^3\lim_{(T_1,T_2)\to\infty}\Bigg[\int_{-T_1}^{T_2} dt\, D_2\tilde U\left(\gamma_E^q\left(t+\frac{\varphi_0}{\sqrt{E}}\right),\varepsilon s+\varepsilon t\right)
\end{aligned}$$

$$
\begin{aligned}
& - \int_{-T_1}^{0} dt\, D_2\tilde{U}\left(\Lambda_E^q\left(t+\frac{\varphi_0+a_-}{\sqrt{E}}\right), \varepsilon s+\varepsilon t\right) \\
& - \int_{0}^{T_2} dt\, D_2\tilde{U}\left(\Lambda_E^q\left(t+\frac{\varphi_0+a_+}{\sqrt{E}}\right), \varepsilon s+\varepsilon t\right) \\
& + \tilde{g}(\varphi_0+a_-, \varepsilon s;\varepsilon) - \tilde{g}(\varphi_0+a_+, \varepsilon s;\varepsilon)\Big] \\
+ \mathrm{O}(\varepsilon^5). &
\end{aligned} \tag{4.24}
$$

Now, calling $A_-(t) = \tilde{g}(\varphi_0+a_-+\sqrt{E}t, \varepsilon s+\varepsilon t;\varepsilon)$, we have, using the functional equation (4.12) verified by $\tilde{g}$:

$$
\begin{aligned}
\dot{A}_-(t) &= \sqrt{E}D_1\tilde{g}(\varphi_0+a_-+\sqrt{E}t, \varepsilon s+\varepsilon t;\varepsilon) \\
&\quad + \varepsilon D_2\tilde{g}(\varphi_0+a_-+\sqrt{E}t, \varepsilon s+\varepsilon t;\varepsilon) \\
&= D_2\tilde{U}(\Lambda_1(\sqrt{E}t+\varphi_0+a_-), \varepsilon s+\varepsilon t) + \mathrm{O}(\varepsilon^3) \\
&= D_2\tilde{U}\left(\Lambda_E\left(t+\frac{\varphi_0+a_-}{\sqrt{E}}\right), \varepsilon s+\varepsilon t\right) + \mathrm{O}(\varepsilon^3)
\end{aligned}
$$

and a similar identity holds for $A_+(t) = \tilde{g}(\varphi_0+a_++\sqrt{E}t, \varepsilon s+\varepsilon t;\varepsilon)$, which verifies:

$$
\dot{A}_+(t) = D_2\tilde{U}\left(\Lambda_E\left(t+\frac{\varphi_0+a_+}{\sqrt{E}}\right), \varepsilon s+\varepsilon t\right) + \mathrm{O}(\varepsilon^3)
$$

Then, using the fundamental theorem of Calculus, we have for any T:

$$
A_\pm(T) - A_\pm(0) = \int_0^T dt\, D_2\tilde{U}\left(\Lambda_E\left(t+\frac{\varphi_0+a_\pm}{\sqrt{E}}\right), \varepsilon s+\varepsilon t\right) + \mathrm{O}(\varepsilon^3)
$$

and using these identities to express the second and third integrals in (4.24) with T_1 and T_2 we obtain formula (4.23). □

Remark 4.16 The function provided by Lemma 4.15:

$$
\begin{aligned}
M(\varphi_0, \varepsilon s, E;\varepsilon) = \lim_{(T_1,T_2)\to\infty} \Bigg[& \int_{-T_1}^{T_2} dt D_2\tilde{U}\left(\gamma_E^q\left(t+\frac{\varphi_0}{\sqrt{E}}\right), \varepsilon s+\varepsilon t\right) \\
& - \tilde{g}(\varphi_0+a_++\sqrt{E}T_2, \varepsilon s+\varepsilon T_2;\varepsilon) \\
& + \tilde{g}(\varphi_0+a_--\sqrt{E}T_1, \varepsilon s-\varepsilon T_1;\varepsilon)\Bigg]
\end{aligned} \tag{4.25}
$$

is usually called the Melnikov function associated to the perturbed torus. As

$$H_\varepsilon(x_+) - H_\varepsilon(y) = \varepsilon^3 M(\varphi_0, \varepsilon s, E; \varepsilon) + \mathrm{O}(\varepsilon^5) \tag{4.26}$$

the non-degenerate zeros of this function correspond to points of the torus which have a homoclinic orbit. □

Remark 4.17 Note that in (4.25) in general, neither the integral nor the other terms reach a limit as T_1, T_2, but rather oscillate quasiperiodically. Only their combination converges.

The meaning of this phenomenon can be clearly ascertained when we realize that $\tilde{g}$ measures the fact that the invariant torus is displaced. If we are interested in the intersections of the manifolds of perturbed torus, we need to consider the changes induced in the stable manifolds of the perturbed torus, not on the unperturbed one.

We warn the reader that in many places in the literature, this term is omitted. This omission is incorrect, unless special circumstances (e.g. symmetries, that the perturbation vanishes on the torus, etc.) justify it. □

As a matter of fact, the zeros of the Melnikov function are critical points of the Melnikov potential, see [DR97], also called Poincaré function, defined by:

$$\begin{aligned} L(\varphi_0, \varepsilon s, E; \varepsilon) \quad = \quad & \lim_{(T_1,T_2)\to\infty} \left[\int_{-T_1}^{T_2} dt\, \tilde{U}\left(\gamma_E^q\left(t + \frac{\varphi_0}{\sqrt{E}}\right), \varepsilon s + \varepsilon t\right) \right. \\ & - \tilde{h}(\varphi_0 + a_+ + \sqrt{E}T_2, \varepsilon s + \varepsilon T_2; \varepsilon) \quad (4.27) \\ & \left. + \tilde{h}(\varphi_0 + a_- - \sqrt{E}T_1, \varepsilon s - \varepsilon T_1; \varepsilon) \right] \end{aligned}$$

where $D_2\tilde{h} = \tilde{g}$ and $\tilde{h}$ verifies (4.13).

The Poincaré function satisfies the following properties:

1. $M(\varphi_0, \varepsilon s, E; \varepsilon) = D_2 L(\varphi_0, \varepsilon s, E; \varepsilon)$

   Note that the uniform convergence of the difference of two integrals by (4.19) readily justifies the taking of derivatives term by term.

2. $L(\varphi_0, \varepsilon s, E; \varepsilon)$ is $1/\varepsilon$-periodic in s.

3. For any $u \in \mathbb{R}$ one has:

$$L(\varphi_0 + \sqrt{E}u, \varepsilon s + \varepsilon u, E; \varepsilon) = L(\varphi_0, \varepsilon s, E; \varepsilon)$$

and, taking $u = -\varphi_0/\sqrt{E}$:

$$L(\varphi_0, \varepsilon s, E; \varepsilon) = L(0, \varepsilon(s - \varphi_0/\sqrt{E}), E; \varepsilon),$$

that is, L is a $\sqrt{E}/\varepsilon$-periodic function of φ_0.

In the following Lemma we are going to give an approximation of the Poincaré function $L(\varphi_0, \varepsilon s, E; \varepsilon)$.

Lemma 4.18

$$L(\varphi_0, \varepsilon s, E; \varepsilon) = \frac{1}{\sqrt{E}}\mathcal{L}\left(\varepsilon\left(s - \frac{\varphi_0}{\sqrt{E}}\right)\right) + \mathrm{O}_{\mathcal{C}^2}(\varepsilon) \tag{4.28}$$

where

$$\mathcal{L}(\tau) = \lim_{(T_1,T_2)\to\infty}\left[\int_{-T_1}^{+T_2} dt\ \tilde{U}(\gamma_1(t),\tau) - \int_{-T_1+a_-}^{+T_2+a_+} dt\ \tilde{U}(\Lambda_1(t),\tau)\right] \tag{4.29}$$

Proof. In order to obtain the first order terms in the Poincaré function L we write (4.27) as

$$\begin{aligned} L(0,\tau,E;\varepsilon) \ = \ \lim_{(T_1,T_2)\to\infty} \Bigg[& \int_{-T_1}^{T_2} dt\, \tilde{U}(\gamma_E^q(t), \tau + \varepsilon t) \\ & - \tilde{h}(a_+ + \sqrt{E}\,T_2, \tau + \varepsilon T_2; \varepsilon) + \tilde{h}(a_+, \tau; \varepsilon) \\ & + \tilde{h}(a_- - \sqrt{E}\,T_1, \tau - \varepsilon T_1; \varepsilon) - \tilde{h}(a_-, \tau; \varepsilon) \\ & - \tilde{h}(a_+, \tau; \varepsilon) + \tilde{h}(a_- + \Delta, \tau + \varepsilon\Delta/\sqrt{E}; \varepsilon) \\ & - \tilde{h}(a_- + \Delta, \tau + \varepsilon\Delta/\sqrt{E}; \varepsilon) + \tilde{h}(a_-, \tau; \varepsilon)\Bigg]. \end{aligned}$$

The fourth line in this expression is of order ε in the $\mathcal{C}^1$ norm due to the fact that $\tilde{h}(\cdot,\cdot;\varepsilon)$ is a bounded function with bounded derivatives, (see Lemma 4.12) and $a_- + \Delta = a_+$. In order to obtain integral expressions for the other three, we only need to use the fundamental Theorem of Calculus and the functional equation (4.13) verified by $\tilde{h}$. Then we obtain:

$$\begin{aligned}
&L(0,\tau,E;\varepsilon)\\
&= \lim_{(T_1,T_2)\to\infty}\left[\int_{-T_1}^{0} dt\,\tilde{U}(\gamma_E^q(t),\tau+\varepsilon t)-\tilde{U}\left(\Lambda_E^q\left(t+\frac{a_-}{\sqrt{E}}\right),\tau+\varepsilon t\right)\right.\\
&\qquad+\int_0^{T_2} dt\,\tilde{U}(\gamma_E^q(t),\tau+\varepsilon t)-\tilde{U}\left(\Lambda_E^q\left(t+\frac{a_+}{\sqrt{E}}\right),\tau+\varepsilon t\right)\\
&\qquad\left.-\int_0^{\frac{\Delta}{\sqrt{E}}} dt\,\tilde{U}\left(\Lambda_E^q\left(t+\frac{a_-}{\sqrt{E}}\right),\tau+\varepsilon t\right)\right]+\mathrm{O}(\varepsilon),
\end{aligned}$$

now, using the rescaling properties (3.1), and changing the variable $u=\sqrt{E}t$:

$$\begin{aligned}
&L(0,\tau,E;\varepsilon)=\\
&\frac{1}{\sqrt{E}}\lim_{(T_1,T_2)\to\infty}\left[\int_{-T_1}^{0} du\,\tilde{U}\left(\gamma_1^q(u),\tau+\frac{\varepsilon u}{\sqrt{E}}\right)-\tilde{U}\left(\Lambda_1^q(u+a_-),\tau+\frac{\varepsilon u}{\sqrt{E}}\right)\right.\\
&\qquad+\int_0^{T_2} du\,\tilde{U}\left(\gamma_1^q(u),\tau+\frac{\varepsilon u}{\sqrt{E}}\right)-\tilde{U}\left(\Lambda_1^q(u+a_+),\tau+\frac{\varepsilon u}{\sqrt{E}}\right)\\
&\qquad\left.-\int_0^{\Delta} du\,\tilde{U}\left(\Lambda_1^q(u+a_-),\tau+\frac{\varepsilon u}{\sqrt{E}}\right)\right]+\mathrm{O}(\varepsilon),
\end{aligned}$$

and taking the dominant terms in ε:

$$\begin{aligned}
&L(0,\tau,E;\varepsilon)\\
&= \frac{1}{\sqrt{E}}\lim_{(T_1,T_2)\to\infty}\left[\int_{-T_1\sqrt{E}}^{0} du\,\tilde{U}(\gamma_1^q(u),\tau)-\tilde{U}(\Lambda_1^q(u+a_-),\tau)\right.\\
&\qquad+\int_0^{T_2\sqrt{E}} du\,\tilde{U}(\gamma_1^q(u),\tau)-\tilde{U}(\Lambda_1^q(u+a_+),\tau)\\
&\qquad\left.-\int_0^{\Delta} du\,\tilde{U}(\Lambda_1^q(u+a_-),\tau)\right]+\frac{1}{\sqrt{E}}R(\tau,\varepsilon)+\mathrm{O}(\varepsilon)\\
&= \frac{1}{\sqrt{E}}\mathcal{L}(\tau)+\frac{1}{\sqrt{E}}R(\tau,\varepsilon)+\mathrm{O}(\varepsilon).
\end{aligned}$$

Where $R(\tau,\varepsilon)$ is defined so that the above is an identity. Note that it only involves the difference of integrals whose integrands have second arguments that are slightly different.

One can bound $R(\tau,\varepsilon)$, using the properties (4.19) and the fact that $\tilde{U}(q,r)$ is a periodic function with respect to its second variable, as

$$|R(\tau,\varepsilon)| \leq K\varepsilon \left(\int_{-\infty}^{+\infty} du\; e^{-\beta_0 |u|} + \int_0^{\Delta} du \right) \leq C\varepsilon.$$

Similarly, one can bound the first and second derivatives because one can take derivatives under the integral sign (the convergence of the integrand is exponentially fast) and then, similar cancelations than those used above, establish the result.

Then taking $\tau = \varepsilon(s - \varphi_0/\sqrt{E})$, we have the Lemma. □

Proposition 4.19 *Given a metric that satisfies the genericity conditions of Theorem* 3.1*, the set of periodic potentials for which the function $\mathcal{L}(\tau)$ in Lemma 4.18 is identically constant is a $\mathcal{C}^l$ closed subspace of infinite codimension for $l \geq 0$.*

Proof.

We note that, for every τ τ', the mapping $U \mapsto \mathcal{L}(\tau) - \mathcal{L}(\tau')$ is a continuous linear functional map if we give U the $\mathcal{C}^l$ topology, $l \geq 0$. This functional is non trivial as can be observed by noting that, since Λ and γ do not coincide, it is possible to choose potentials U with support near γ so that the functional does not vanish. □

4.7 Transition chains and transition lemmas

We recall that according to [Arn64, AA68], a transition chain for a Hamiltonian flow is a sequence of transition tori such that the unstable manifold of one intersects transversally the stable manifold of the next.

The definition of transition tori in [Arn64] is topological, but for our purposes we note that it has been shown in [FM98] (we will discuss it in Lemma 4.22) that all whiskered tori with one dimensional whiskers and with irrational motion are transition tori. This includes the tori produced applying Theorem 4.8 to $f_\varepsilon^\varepsilon$ in our problem.

The importance of transition chains is that there are orbits that follow them closely.

So, we will verify that we can find a sequence of tori obtained by applying Theorem 4.8 to $f_\varepsilon^\varepsilon$ and such that the stable manifold of one crosses transversally the unstable manifold of the previous one. Then, we will discuss some small modifications needed to the standard arguments (they only apply to finite sequences) that show that indeed there are orbits that follow them.

We note that, in the notation that we have introduced in this paper, the assertion that the unstable manifold of a torus contained in Λ intersects the unstable manifold of another one is equivalent to the assertion that the image of the first torus under the outer map S intersects the second.

We will refer to the tori obtained applying Theorem 4.8 to $f_\varepsilon^\varepsilon$ simply as KAM tori.

Lemma 4.20 *Assume that r is sufficiently big. If the function $\mathcal{L}(\tau)$ is not constant, we can find $K > 0$ such that for ε sufficiently small, given a KAM torus $\mathcal{T}$, we can find other KAM tori $\mathcal{T}^+$, $\mathcal{T}^-$ such that*

$$\begin{array}{rcl} W^u_{\mathcal{T}} \pitchfork W^s_{\mathcal{T}^+}, & & W^u_{\mathcal{T}} \pitchfork W^s_{\mathcal{T}^-} \\ \min H_\varepsilon(\mathcal{T}) & \geq & \max H_\varepsilon(\mathcal{T}^-) + K\varepsilon^3 \\ \max H_\varepsilon(\mathcal{T}) & \leq & \min H_\varepsilon(\mathcal{T}^+) - K\varepsilon^3 \end{array}$$

Proof.

Observe that, since $\mathcal{L}$ is periodic and C^2, if it is not constant, we can find two numbers $\tau_\pm$ such that $\mathcal{L}'(\tau_+) > 0$, $\mathcal{L}'(\tau_-) < 0$, $\mathcal{L}''(\tau_\pm) \neq 0$. Since $\mathcal{L}$ is $\mathcal{C}^2$ the same inequalities are true for small intervals around $\tau_\pm$.

We study the dynamics of the manifold Λ using the coordinates H_ε, φ.

Since $\mathcal{L}$ approximates in the $\mathcal{C}^2$ sense the Poincaré function, and the derivative of this function measures the increase in H_ε under the map S, it follows that for small enough ε, given any KAM torus $\mathcal{T}$, its image under S has to include segments $\rho_\pm$ (corresponding to the intervals around $\tau_\pm$ above) such that $\max H_\varepsilon(\rho_\pm) - \min H_\varepsilon(\rho_\pm) \geq K_1\varepsilon^3$. On the other hand, the projection of these intervals over the φ variable has a length not more that $K_2\varepsilon$.

We note that in the averaged coordinates, the KAM tori are not more that $\varepsilon^{m/2+1}$ apart and that they correspond very approximately to surfaces of constant action. Hence, in the original coordinates, they will be graphs of functions in the φ, H_ε coordinates which are not more than $\varepsilon^{m/2+1}$ apart in the $\mathcal{C}^{l-4}$ sense.

Since the interpretation of the function M (See figure 1) was the increment in energy over a torus of the map S, we see that the image of one torus has to cross two KAM tori, one of higher energy and another one of lower energy.

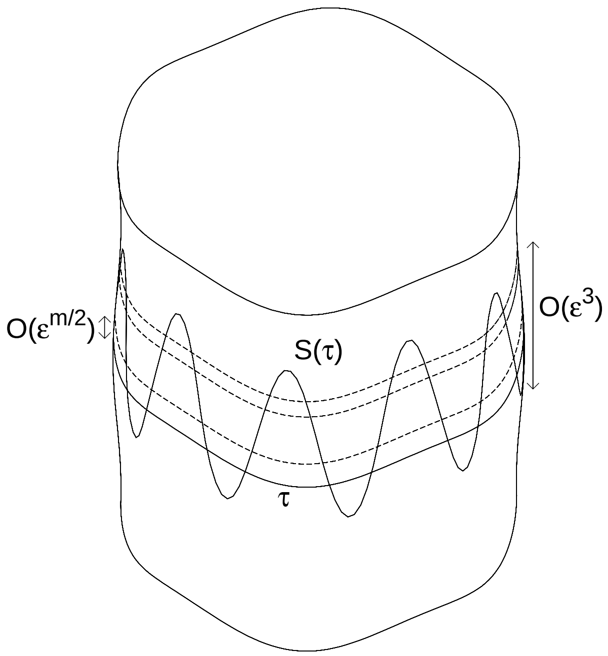

Figure 2: Illustration of the action of the map S on a torus τ.

Moreover, this intersection has to be transversal. The fact that $\mathcal{L}''(\tau_\pm) \neq 0$ implies that the derivative of the gain in energy with respect to the angle is bounded from below by a constant times ε^3. That is, if we express the torus $\mathcal{T}$, $S(\mathcal{T})$ and $\mathcal{T}^+, \mathcal{T}^-$ as graphs of functions $\Psi, \Psi_S, \Psi_\pm$ respectively, we have $|\Psi'_S - \Psi'_\pm| \geq K\varepsilon^3$ in a neighborhood of the intersection. This implies that the intersection of the tori is transversal.

As an immediate consequence, we have:

Lemma 4.21 *Assume that the metric g satisfies the assumptions of Theorem 3.1 and that V is such that $\mathcal{L}$ is not constant. Assume moreover that both g and V are $\mathcal{C}^{15}$. Then, there exist $M > 0$, $\alpha > 0$, such that if*

$$I_i = [E^i_-, E^i_+] \quad i = 1, \dots$$

is any sequence of intervals such that

$$\begin{aligned} E^i_- &\geq M \\ (E^i_+ - E^i_-) &\geq M(E^i_+)^{-\alpha} \end{aligned}$$

Then, we can find a sequence $\{\mathcal{T}_i\}$ of KAM tori such that

$$H_\varepsilon(\mathcal{T}_i) \cap I_i \neq \emptyset$$
$$W^s_{\mathcal{T}_{i+1}} \pitchfork W^u_{\mathcal{T}_i}.$$

Our next goal is to show that the pseudo orbits obtained by interspeding the KAM homoclinic jumps with the motion along the torus can be shadowed by true orbits of the system. As it is usual in the literature for Arnol'd diffusion, the key step is to find an appropriate inclination lemma (also called sometimes λ-lemma).

In the literature, one can find very sharp inclination lemmas—including even some estimates of the times needed to do the shadowing—for analytic maps, when the rotation is Diophantine, in [Mar96, Cre97, Val98]. (Related results appear in [CG94]). The result that we have found best adapted to our purposes is that of [FM98] for whiskered tori with one dimensional strong (un)stable directions—as is the case in the problem we are considering—, which works for $\mathcal{C}^1$ maps and only requires that the torus has an irrational rotation.

A particular case of the results of [FM98] is:

Lemma 4.22 *Let f be a $\mathcal{C}^1$ symplectic mapping in a $2(d+1)$ symplectic manifold. Assume that the map leaves invariant a $\mathcal{C}^1$ d-dimensional torus $\mathcal{T}$ and that the motion on the torus is an irrational rotation. Let Γ be a $d+1$ manifold intersecting $W^s_{\mathcal{T}}$ transversally.*

Then

$$W^u_{\mathcal{T}} \subset \overline{\bigcup_{i>0} f^i(\Gamma)}$$

An immediate consequence of this is that any finite transition chain can be shadowed by a true orbit. The argument for infinite chains requires some elementary point set topology.

Lemma 4.23 *Let $\{\mathcal{T}_i\}_{i=1}^\infty$ be a sequence of transition tori. Given $\{\varepsilon_i\}_{i=1}^\infty$ a sequence of strictly positive numbers, we can find a point P and a increasing sequence of numbers T_i such that*

$$\Phi_{T_i}(P) \in N_{\varepsilon_i}(\mathcal{T}_i)$$

where $N_{\varepsilon_i}(\mathcal{T}_i)$ is a neighborhood of size ε_i of the torus $\mathcal{T}_i$.

Proof.
Let $x \in W^s_{\mathcal{T}_1}$. We can find a closed ball B_1, centered on x, and such that

$$\Phi_{T_1}(B_1) \subset N_{\varepsilon_1}(\mathcal{T}_1). \tag{4.30}$$

By the Inclination Lemma 4.22

$$W^s_{\mathcal{T}_2} \cap B_1 \neq \emptyset.$$

Hence, we can find a closed ball $B_2 \subset B_1$, centered in a point in $W^s_{\mathcal{T}_2}$ such that, besides satisfying (4.30):

$$\Phi_{T_2}(B_2) \subset N_{\varepsilon_2}(\mathcal{T}_2).$$

Proceeding by induction, we can find a sequence of closed balls

$$\begin{array}{l} B_i \subset B_{i-1} \subset \cdots \subset B_1 \\ \Phi_{T_j}(B_i) \subset N_{\varepsilon_j}(\mathcal{T}_j), \quad i \leq j. \end{array}$$

Since the balls are compact, $\cap B_i \neq \emptyset$. A point P in the intersection satisfies the required property. □

Putting together Lemma 4.21 and Lemma 4.23, we obtain the following result, which clearly implies Theorem 1.1.

Theorem 4.24 *Assume that the metric g satisfies the assumptions of Theorem 3.1 and that V is such that $\mathcal{L}$ is not constant. Assume moreover that both g and V are $\mathcal{C}^{15}$. Then, there exist $M > 0$, $\alpha > 0$, such that if*

$$I_i = [E^i_-, E^i_+] \quad i = 1, \ldots$$

is any sequence of intervals such that

$$\begin{array}{rcl} E^i_- & \geq & M \\ (E^i_+ - E^i_-) & \geq & M(E^i_+)^{-\alpha} \end{array}$$

Then, we can find an orbit $p(t), q(t)$ of the Hamiltonian flow and an increasing sequence of times $t_1 < t_2 < \cdots < t_n < \cdots$, such that

$$H_\varepsilon(p(t_i), q(t_i), t_i) \in I_i.$$

and $(p(t_i), q(t_i))$ is in a neighborhood of size $M(E^i_-)^{-2}$ of the periodic orbit Λ.

Note: By assuming more differentiability in the hypothesis of the theorem, we can get α to be arbitrarily large.

Remark 4.25 A question that has often been asked us and, which is indeed quite relevant for physical applications, is what is the measure of the diffusing orbits and what is the speed of diffusion predicted by this mechanism.

We do not know at the moment of this writing how to produce a set of positive measure of diffusing orbits. Of course, the mechanism described here is presumably not the only mechanism that contributes to diffusion.

A heuristic argument—which at the moment we cannot even raise as a conjecture—suggests that the orbit following the mechanism studied in this paper can perform $\approx E^{1/2}$ homoclinic excursions in a unit of time and in each of them it can gain $\approx E^{-3/2}$ energy. Hence, the energy can increment in a unit of time a quantity proportional to the inverse if the energy. Hence, we expect that our orbits satisfy $E(t) \approx t^{1/2}$.

Note that this argument implicitly assumes that the proportion of times that are favorable for the jump and indeed the average gain in energy per jump reach a limit as the energy grows and that the time that one needs to bid preparing for the next jump is a fixed proportion of the total time.

The rigorous justification (and indeed a non-rigorous but reliable assessment) of this assumptions seems like a daunting task, but we hope some reader may be motivated to investigate this question. $\square$

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A Appendix: Brief summary of hyperbolicity theory

In this section we collect some of the results from the rich theory of hyperbolic (or normally hyperbolic or further qualifications) invariant manifolds.

The results we present are quite standard and can be found in many places, (indeed the theory seems to have been developed several times) so we will just highlight some of the more subtle points of the conclusions which affect some of the statements of the theorems we will prove. We just recommend [Fen74, HP70, Wig94] as readable and complete references. Another version—somewhat more demanding in notation and style—is in [HPS77]. Yet another point of view can be found in [SS74] and following papers. We refer to [Wig94] for a discussion of original references.

We discuss only three aspects: the regularity properties of invariant manifolds and foliations, their persistence properties and the smooth dependence on parameters.

We will present here the theory for flows. Using the standard suspension trick, all the general results for flows imply corresponding results for invertible maps. There are aspects of the theory of hyperbolic non-invertible maps without flow counterparts, but the aspects of the theory we are discussing are identical for flows and for maps. The theory of non-invertible maps is still somewhat incomplete even in the aspects we discuss here.

Definition A.1 *Let M be a manifold and Φ_t a $\mathcal{C}^r$, $r \geq 1$, flow on it. We say that a manifold $\Lambda \subset M$—possibly with boundary—invariant under Φ_t is hyperbolic when there is a bundle decomposition*

$$TM = T\Lambda \oplus E^s \oplus E^u$$

invariant under the flow, and numbers $C > 0$, $0 < \beta < \alpha$, such that for $x \in \Lambda$

$$\begin{aligned} v \in E^s_x \quad &\Longleftrightarrow \quad |D\Phi_t(x)v| \leq Ce^{-\alpha t}|v| \quad \forall\ t > 0 \\ v \in E^u_x \quad &\Longleftrightarrow \quad |D\Phi_t(x)v| \leq Ce^{\alpha t}|v| \quad \forall\ t < 0 \\ v \in T_x\Lambda \quad &\Longleftrightarrow \quad |D\Phi_t(x)v| \leq Ce^{\beta|t|}|v| \quad \forall\ t \end{aligned} \tag{A.1}$$

Remark A.2 In this paper, we will refer to (A.1) as saying that the manifold is "hyperbolic". In some references where more precision is needed, names as $\alpha - \beta$ hyperbolic or normally hyperbolic are used.

The hypothesis (A.1) are often referred as saying that the bundle decomposition satisfies exponential dichotomies. □

Remark A.3 There are two different ways of developing hyperbolicity theory. One is, as we stated, to assume that the constants in (A.1) are uniform in the bundle. Another one is to assume bounds such as those in (A.1) along an orbit and that the ratios along several constants along the orbit are bounded. The first method is the basis of [HP70] and [HPS77]. The second one was used in [Fen74, Fen77].

Clearly, the hypothesis of the bundle approach imply those of the orbit method. The difference in the bounds can be particularly significative in systems in which a geometric structure implies relations between expansion and contraction rates along an orbit but not on a bundle. One example of this situation is the study of the horospheric foliation in geodesic flows in manifolds of negative curvature ([HK90]). Moreover, the study of individual orbits leads naturally to the non-uniform hyperbolic theory [Pes76, Pes77].

For the applications we have in mind, we do not need the sharper results, so that we will state the results in the somewhat simpler language of bundles. □

Remark A.4 Note that if the inequalities (A.1) are established for $|t| \leq T$ with T sufficiently large to overcome the constant C (i.e. $Ce^{\beta - \alpha T} < 1$), then we can recover the definition we have given because we can bound $||D\Phi_{nT+s}|| \leq ||D\Phi_T||^n \cdot ||D\Phi_s||$.

This observation is useful when we want to study the persistence of these structures for sufficiently small perturbations. □

Remark A.5 Similarly, we note that, by redefining the metric in M, one can get rid of the constant C in the definition. A metric satisfying this is called the adapted metric or sometimes, specially in the East, Lyapunov metric. We refer to the general references above. □

Remark A.6 If we construct an splitting between bundles in such a way that the bundles are not assumed to be invariant but that they satisfy the inequalities in (A.1) for $|t| \leq T$, with T large enough with respect to C, α and β, then one can construct invariant bundles that satisfy similar inequalities with slightly worse constants. □

Intuitively, Definition A.1 means that the normal infinitesimal perturbations grow faster (either in the future or in the past) than the infinitesimal perturbations along the manifold.

The first result is about the existence of invariant stable and unstable manifolds for hyperbolic manifolds.

Theorem A.7 *Let Λ be a compact hyperbolic manifold (possibly with boundary) for the $\mathcal{C}^r$ flow Φ_t, satisfying Definition A.1. Then, there exists a sufficiently small neighborhood U, and a sufficiently small $\delta > 0$, such that:*

1. *The manifold Λ is $\mathcal{C}^{\min(r, r_1 - \delta)}$, where $r_1 = \alpha/\beta$.*
2. *For any x in Λ, the set*
$$\begin{aligned} W_x^s &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Phi_t(x)) \le Ce^{(-\alpha+\delta)t} \;\; \text{for} \;\; t > 0\} \\ &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Phi_t(x)) \le Ce^{(-\beta-\delta)t} \;\; \text{for} \;\; t > 0\} \end{aligned}$$
is a $\mathcal{C}^r$ manifold and $T_x W_x^s = E_x^s$.
3. *The bundles E_x^s are $\mathcal{C}^{\min(r, r_0 - \delta)}$ in x, where $r_0 = (\alpha - \beta)/\beta$, and*
$$\begin{aligned} W_\Lambda^s &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Lambda) \le Ce^{(-\alpha+\delta)t} \;\; \text{for} \;\; t > 0\} \\ &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Lambda)) \le Ce^{(-\beta-\delta)t} \;\; \text{for} \;\; t > 0\} \end{aligned}$$
is a $\mathcal{C}^{\min(r, r_0 - \delta)}$ manifold. Moreover $T_x W_\Lambda^s = E_x^s$. Finally
$$W_\Lambda^s = \cup_{x \in \Lambda} W_x^s.$$
Moreover, we can find a $\rho > 0$ sufficiently small and a $\mathcal{C}^{\min(r, r_0 - \delta)}$ diffeomorphism from the bundle of balls of radius ρ in E_Λ^s to $W_\Lambda^s \cap U$.

An analogous theorem can be stated for W_Λ^u considering the flow generated by $-X$.

The last part of the conclusions state, roughly, that all the orbits that approach fast enough the manifold Λ, approach an orbit in Λ. Moreover, for the points approaching Λ fast enough and in a sufficiently small neighborhood of Λ, the point whose orbit is approached is a well defined function in W_Λ^s and is $\mathcal{C}^{\min(r, r_0 - \delta)}$.

We point that compactness enters only mildly in the assumptions. We only need that the flow is uniformly $\mathcal{C}^r$ in a neighborhood of Λ.

Remark A.8 When $\beta = 0$, r_0 and r_1 have zero denominator. This cannot be interpreted as ∞ without care. Even if $r = \infty, \omega$, we cannot conclude that $\min(r, r_0) = \infty$ and that the manifolds are $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$. The best that can be said is that there are $\mathcal{C}^k$ manifolds for every k. There are examples where the $\mathcal{C}^\infty$ conclusions are false even for polynomial perturbations. $\square$

Remark A.9 We emphasize that, even if the manifolds W^s_x are as smooth as the flow, the dependence on x is not claimed to be smoother than r_0, which depends on the contraction factors in the tangent and (un)stable bundles. Indeed, it is sometimes the case that these bounds are sharp in $\mathcal{C}^{r_0}$ open sets. Similarly, the regularity of the manifold Λ and that of W^s_Λ can be sharp even if the flow is assumed to be analytic. An example for maps can be obtained setting

$$f : \mathbb{T}^2 \times \mathbb{R} \mapsto \mathbb{T}^2 \times \mathbb{R}$$

given by

$$f(x, y) = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x, h(x) + \frac{1}{50} y \right).$$

with $h : \mathbb{T}^2 \mapsto \mathbb{R}$ a conveniently chosen trigonometric polynomial. Using the standard suspension trick, similar examples can be obtained for flows. More examples in this line and a more detailed analysis can be found in [dlL92]. $\square$

Remark A.10 Even if the above examples show that the regularity numbers r_0, r_1 cannot be improved in general, it is possible to obtain sharper results if we introduce more parameters to characterize the exponential rates of the different bundles. Here we have used only α and β, but one obtains sharper results if one introduces different parameters for the contraction rate of the unstable bundle in the past and the stable bundle in the future. $\square$

The above theorem has as a corollary the smooth dependence on parameters of the (un)stable manifolds. The trick is completely elementary and will be used later several times.

Corollary A.11 *Assume that $\Phi_{t,\varepsilon}$ is a family of flows which is jointly $\mathcal{C}^r$ in all its variables (the base point x, the time t and the parameter ε) and that for all the values of the parameter in a ball $\Phi_{t,\varepsilon}$ leaves invariant the manifold Λ. Then, for sufficiently small $|\varepsilon|$, it is possible to apply the Theorem A.7. Moreover, the manifolds $W^s_{\Lambda,\varepsilon}$, $W^s_{x,\varepsilon}$ are $\mathcal{C}^{\min(r, r_0 - \delta)}$ jointly in x and ε.*

The idea of the proof is very simple. We just consider the extended flow $\tilde{\Phi}_t(x,\varepsilon) = (\Phi_{t,\varepsilon}(x),\varepsilon)$, on $M \times B$ with B a sufficiently small ball in ε. It is easy to check that the manifold $\Lambda \times B$ is invariant for the flow and that, for a finite time the flow satisfies the exponential dichotomy bounds in the stable and unstable subspaces. Using Remark A.6 we conclude that there are invariant bundles with very close constants. A moment's reflection shows that the dependence of manifolds of the extended system on the base point gives the dependence on parameters and the base point in the original system. □

Remark A.12 We note that the dependence on parameters cannot be more differentiable than the dependence on the base point and, indeed, the examples alluded to in Remark A.9 can be easily made into examples in which the dependence with respect to parameters is optimal. Take, for example $\Phi_{t,\varepsilon}(x) = \Phi_t(x - \varepsilon v) + \varepsilon v$ so that the invariant objects of the $\Phi_{t,\varepsilon}(x)$ are just translates by εv of the invariant objects for Φ_t and, therefore, the dependence on parameters is the same as the space dependence in the original problem.

This is in sharp contrast with the results of the usual implicit function theorem so that the formulations of these problems in terms of implicit function theorems need to involve specialized implicit function theorems that do not have the same properties as the usual one. □

Now, we continue to discuss persistence. Roughly, we state that any perturbation of a system admitting a hyperbolic manifold has to carry another hyperbolic invariant manifold which is a perturbation of that of the original system.

Theorem A.13 *Let $\Lambda \subset M$—not necessarily compact—be hyperbolic for the flow Φ_t generated by the vector field X, which is uniformly $\mathcal{C}^r$ in a neighborhood U of Λ such that* $\mathrm{dist}(M \setminus U, \Lambda) > 0$*. Let Ψ_t be the flow generated by another vector field Y which is $\mathcal{C}^r$ and sufficiently $\mathcal{C}^1$ close to X. Then, we can find a manifold Γ which is hyperbolic for Y and $\mathcal{C}^{\min(r,r_1-\delta)}$ close to Λ.*

The constants in Definition A.1 for Γ are arbitrarily close to those of Λ if Y is sufficiently $\mathcal{C}^1$ close to X.

The manifold Γ is the only $\mathcal{C}^1$ manifold, $\mathcal{C}^0$ close to Λ and invariant under the flow of Y.

There are several extensions of this result that can be readily obtained. We will just sketch the method of proof and refer to the sources mentioned above.

1. Similarly to Corollary A.11, one can obtain smooth dependence on parameters in Theorem A.13 by extending the system by another one with trivial dynamics. Again, we obtain only $\min(r, r_1 - \delta)$ regularity and this is optimal in examples.

   A convenient way of formulating this smooth dependence on parameters is using the implicit function theorem and finding a $\mathcal{C}^{\min(r,r_1-\delta)}$ mapping $\mathcal{F} : \Lambda \times B \to M$ in such a way that $\mathcal{F}(\Lambda, \varepsilon) = \Lambda_\varepsilon$, and $\mathcal{F}(\cdot, 0)$ is the identity.

2. Using the remark above, given a family of flows, we can use the map $\mathcal{F}$ to identify all the invariant manifolds of all the flows. Extending the mapping $\mathcal{F}$ to a neighborhood and changing coordinates to it, we obtain that we can reduce the study of a family of flows to the problem of a family of flows which preserve a common manifold. This is precisely the case considered in Corollary A.11.

   Hence, we can obtain that there is a $\mathcal{C}^{\min(r,r_1-\delta)}$ mapping $R : W^s_\Lambda \times B \to M$ in such a way that $R(W^s_\Lambda, \varepsilon) = W^s_{\Lambda_\varepsilon, \varepsilon}$, $R(\cdot, \varepsilon)|_\Lambda = \mathcal{F}(., \varepsilon)$, $R(W^s_x, \varepsilon) = W^s_{\mathcal{F}(x,\varepsilon),\varepsilon}$.

3. It is also possible to discuss persistence of manifolds with boundary and locally invariant manifolds.

   The idea is that we can extend the flow to a globally defined one, with $\mathcal{C}^r$ bounds which are close to the ones of the original problem and with bounds on the bundles which are also close to the ones we assumed and which agrees with our original flow in the points of the original manifold which are sufficiently far from the boundary.

   Then, we can apply Theorem A.13 to the extended system. The invariant manifold for the extended system will be locally invariant for the original one.

4. Even if Theorem A.13 includes uniqueness in its conclusions and, therefore the manifold produced is unique (under appropriate conditions) for

the extended system, the extension process is not unique and the manifold produced does depend on the extension used. Hence, one cannot claim uniqueness for the locally invariant manifold produced for the original system.

On the other hand, it follows from the uniqueness conclusions of Theorem A.7, that all the orbits that remain in a sufficiently small neighborhood of Λ and away from the boundary should be present in all the extensions that do not modify the vector field away from this neighborhood of the boundary.

Similarly, note that the definition of stable manifold of a point x or of a manifold Λ involves discussing what happens for arbitrarily large times of the time in the evolution. Such long time orbits depend on the extension used if the orbit of x is not contained in the manifold Λ away from the boundary.

On the other hand, for the orbits that indeed remain inside of Λ, the definition identifies the points of the stable manifold. Hence, the germs of these stable manifolds have to agree in all the extensions.

5. The above extension process can be combined with the dependence on parameters. We just remark that given a family of perturbations, one can perform the extension in such a way that it depends smoothly on parameters. (The extension only involves elements such as cut-off functions, mappings to identify spaces etc., that can be used for all the values of the parameter.)

   Again, the extension process is not unique and the smooth dependence on parameters should be interpreted as the possibility of finding a map $\mathcal{F}$ or R so that its range produces the invariant manifolds.

   As before, we note that the orbits that are contained in a small neighborhood of Λ away from the boundaries and the germs of their stable and unstable manifolds should be present in all the extended systems.

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