Probability around the Quantum Gravity.
Part 1: Planar Pure Gravity

V.A.Malyshev *

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Abstract

In this paper we study stochastic dynamics which leaves quantum gravity equilibrium distribution invariant. We start theoretical study of this dynamics (earlier it was only used for Monte-Carlo simulation). Main new results concern the existence and properties of local correlation functions in the thermodynamic limit. The study of dynamics constitutes a third part of the series of papers where more general class of processes were studied (but it is self-contained), those processes have some universal significance in probability and they cover most concrete processes, also they have many examples in computer science and biology. At the same time the paper can serve an introduction to quantum gravity for a probabilist: we give a rigorous exposition of quantum gravity in the planar pure gravity case. Mostly we use combinatorial techniques, instead of more popular in physics random matrix models, the central point is the famous $\alpha = -\frac{7}{2}$ exponent.

*Postal address:INRIA - Domaine de Voluceau, Rocquencourt, BP105 - 78153 - Le Chesnay Cedex, France.

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1 Introduction

Some history I should say some words about the history of discrete gravity. Classical gravity deals with a smooth (not necessarily four-dimensional) manifold M, pseudo-metric tensor g_{ij} on it and the classical Einstein-Hilbert action

$$S = \int_M (\lambda R(x) + L(x) + \mu)\sqrt{\det g}dx$$

the various stationary points of which are studied. Here $R(x)$ is the intrinsic curvature at the point x, L - some functional of matter fields $\phi(x)$. In the pure (no matter) gravity case $L = 0$, we consider only this case here. Quantum gravity takes into account not only stationary points but also all other configurations with some weights, that is with a formal (but which becomes positive for Euclidean metrics) density

$$Z^{-1}\exp(-\mu S)$$

on some configuration space Ω of matter fields and metric tensors. All earlier attempts to do this brought the conclusion that Ω should also include smooth structures on M and even M itself, that is the space, its topology, should be random. Now the only reasonable way to pursue this program is to discretize everything from the beginning and then to perform some scaling limits. That is the space becomes a finite complex, smooth structure becomes a piecewise linear structure, metrics and curvature are encrypted in one-dimensional and two-dimensional skeletons of the complex, matter fields are spins which live on the cells of the complex. It appears that such quantization (discretization) is equally applicable to other physical systems: relativistic particles, strings etc., but with different interpretations. For example, the quantized (in such a way) string consists of a two-dimensional complex (representing a coordinate system and metrics on the string itself) and spins - vectors in R^d which provide a mapping of the vertices of the complex into the d-dimensional Euclidean space, thus approximating the classical string.

The discretization of the classical gravity was first considered by Regge [29] where he gave definitions of some exact mathematical objects related to the classical general relativity: finite discrete space time, its curvature and Einstein-Hilbert action. It was afterwards included in the fundamental monograph [30] but in seventies it was still considered outside of the main streamline of physics and only rare papers were devoted to it. Among them however there was a well known paper by S. Hawking [32] where the applications to quantum gravity were discussed.

In eighties there are already more than 100 papers concerning discrete quantum gravity.

In nineties the the number of papers is more than 1000 and still grows at the moment. Mainly it is due to the appearing algebraic formal techniques to deal with such problems. This formal techniques follows physical insights on relations of quantum gravity with string theory, random matrix models etc. Moreover, recent papers in theoretical physics often contain the following sententions: "Two-dimensional random geometry is now placed at the heart of many models of modern physics, from string theory and two-dimensional quantum gravity, attempting to describe fundamental interactions, to membranes and interface fluctuations in various problems of condensed matter physics", see [39].

For a probabilist the quantum gravity is a source of inspiration and also new mathematics and new philosophy of probability. The paper can serve an introduction to quantum gravity for a probabilist: it is a mathematical text on the quantum gravity for the planar pure gravity case.

Dynamics contre equilibrium Mostly we consider combinatorial techniques, instead of more popular in physics random matrix models, the central point is the famous $\alpha = -\frac{7}{2}$ exponent. Another goal of the paper is to consider stochastic dynamics which leaves quantum gravity equilibrium distribution invariant. We start theoretical study of this dynamics (earlier it was only used for Monte-Carlo simulation). The study of dynamics constitutes (but mainly it is self-contained) a third part of the series of papers (see [43, 44]) where more general class of processes was studied. These processes have also some universal character in probability: they cover most concrete processes. Also they have many examples in computer science and biology.

Here the probability is the classical probability. The quantum gravity constitutes a bunch (a lot !) of papers overfilling last 10 years well-known physical journals. Discrete quantum gravity is now considered as a promising direction towards unifying largest and smallest scales in the nowadays picture of nature. I consider one part of this field which evidently uses probabilistic intuition but it is difficult to find even formulations (I do not mention proofs !) which could be satisfactory for a mathematician: even when the "probabilities" are hopefully positive they are not normalized. And this is not because of negligence of the authors but because some deep reasons seem to be behind the curtains.

In the existing physical literature a permanently developing algebraic and geometric techniques overwhelms the subject. Thus it can be useful to step away from algebra and geometry, discussing some simple probabilistic aspects of quantum gravity: even such simple project appeared to rise many natural but still not answered questions.

There are now two variants in the discrete approaches to quantum gravity: Quantum Regge Calculus (where links (edges) have lengths as random variables) and Dynamical Triangulations (where lengths of edges are constant). The word *dynamical* in the second approach is a little bit misleading because there is no dynamics at all in this approach: main techniques uses Gibbs equilibrium distributions on large matrices. That is why I will call here these approaches equilibrium.

The dynamics appeared earlier in Monte Carlo simulations of quantum gravity. Here I try to give a probabilistic (not numerical) study of relevant Markov processes. What is new here (I do not know earlier rigorous results) is that we want to advocate not numerical but analytic and probabilistic studies of such processes. Why such processes can be useful not only in computer Monte-Carlo experiments but also as giving theoretical information ? There are many reasons - we give here a short list.

- Well known difficulty in averaging over all topologies is that, in 4 dimensions, it includes some questions which are known to be algorithmically unsolvable. Dynamics substitutes this problem with a new one: instead of averaging we are looking for a process (with arbitrary initial state) which will generate all topologies it can generate. This process should have some symmetries but also it should be a legitimate (for example non-exploding) stochastic process.

- What one would like to have (as in the stochastic quantization in quantum field theory and Glauber dynamics in statistical mechanics) is a Markov process leaving Gibbs measure invariant. This is quite natural in quantum field theory where there are Whiteman axioms and in statistical mechanics where there is a deterministic dynamics more fundamental than the Gibbs measure itself. In quantum gravity both these factors are absent and an alternative viewpoint could be advertised: that the process itself can be taken to be more fundamental than the Gibbs measure itself.

- Dynamics allows to consider the region below the critical point where equilibrium distribution has no sense. On the contrary this region is even more natural for the dynamics - like a growing universe (in the computer time, the term which I know from a paper by A. Migdal). Moreover dynamics gives also some sense to distributions in the critical point without performing scaling limits. I do not know the physical counterpart of all this but its naturalness from probability point of view is evident.

- I have absolutely no physical arguments for the choice and even relevance of the dynamical models, but that is also true for all modern approaches due to the lack of experimental confirmation. The leading thread can only be probabilistic intuition and beauty. Relevant question are: what is universality and generic situation ? It was argued recently, see [33], that computer science could play some role in future physical theories. Probabilistic aspects which we discuss here make this relation quite evident by a preliminary model of the universe growing via some grammar (more exactly a graph grammar) similarly to the random evolution of a language.

- Mathematical thermodynamic theory existing for statistical mechanics and quantum theory brought many new ideas. It is some surprise that an attempt to construct similar theory for growing complexes brought quite unexpected phenomena (see [43],[44]) (hopefully having some physical significance). One of the effects is that one cannot fix an origin in an infinite universe without Zermelo axiom, any constructive introducing of a local observer changes drastically the space time in his neighborhood.

- It cannot be easy to find critical exponents by Monte-Carlo simulation because the asymptotic is dominated by the exponential term which depends strongly on the details of the model. What is usually simulated is the uniform distribution on the set of triangulations with fixed number of cells. If we consider a growing complex then we could not find a Markov process giving the necessary exponents (the famous $\alpha = -\frac{7}{2}$) but only some random transformation of measures, that is called usually a nonlinear Markov processes, giving these exponents.

Contents of the paper One-dimensional case (section 2) is useful in particular as emphasizing links between classical probability and two-dimensional quantum gravity.

In section 3 the minimum of necessary definitions are given concerning complexes and curvature in two dimensional case.

Section 4 contains introductory definitions, problems and some known results. In section 3.2 we give a short exposition of RMT approach to pure planar gravity, the only goal of this exposition is to emphasize some points, related to the combinatorial approach.

In section 5 we study a dynamical model, where the cells are appended in random to the boundary of the disk. This model is solvable (via random walks) and we calculate some main quantities. The exponent for this model is -2 and thus it belongs to a different universality class than models accepted in physics. But continuum limit in this model is well defined, gives space with a constant curvature as in the physical model.

Section 6 is the central in the paper. We construct and study nonlinear Markov processes (where also changes are possible only on the boundary) which render the equilibrium distribution invariant. We use the Tutte functional equation method to prove that one gets $-\frac{7}{2}$ exponent. We develop new combinatorial techniques to study local correlation functions.

In section 7 we consider dynamics where changes can be done elsewhere in the complex. We study large time behavior of such Markov processes.

Acknowledgments I thank L. Pastur for elucidating to me some points of the Random Matrix Theory and S. Shlosman for reading the paper and very helpful comments.

2 One dimensional gravity

For the physical interpretation of the one-dimensional gravity and many beautiful calculations we refer to chapter 2 of Ambjorn's lectures [31]. Our goal here is to give a probabilistic viewpoint and discuss new approaches. There is no topology in one dimension: the underlying structure (cell complex) is one-dimensional - a linear graph.

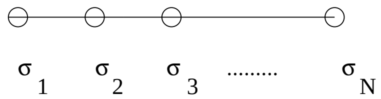

Figure 1: Linear spin graph

A chain of symbols from some alphabet can be considered as a function on vertices of such graphs (see Figure 1).

2.1 Equilibrium distribution

We give now the basic definition in more abstract terms than in [31], without prior embedding in Euclidean space: this corresponds more to Polyakov string quantization. We shall consider distributions on the set of finite linear spin graphs (sometimes we use the terminology from our previous papers but mostly it can be skipped). That is the distributions on the set Ω of strings (here string comes from computer science terminology) $\sigma = \{s_1, ..., s_N\}$, where $N = 0, 1, 2, ...$ and $s_i \in S$, where S is an alphabet (spin space). For example S can be the unit sphere in Z^d or in R^d. Case $N = 0$ corresponds to the empty string with no S-value prescribed. Define the nonnegative measure on Ω by

$$Q(\sigma) = \exp(-\mu N - \beta \sum_{i=0}^{N-1} (f_1(s_{i+1}) + f_2(s_i, s_{i+1})) \tag{1}$$

for some function f. It is convenient to assume that some element s_0 is fixed. Simplest example is when $s \in Z^d, f_2 = 0, f_1(s) = \infty$ for all s except a finite set. This measure can be normalized if

$$Z = \sum_{\sigma} Q(N, \sigma) < \infty$$

Intuitively, a sequence of arrays $s_0, s_0 + s_1, ..., s_0 + s_1 + ... + s_N$ can be considered as a "random walk". But it is quite different from the classical random walks. We shall see below its relationship with some computer science problems. We shall also see how this formal object can be tied to the Euclidean space: in physical papers one can also see similar steps - abstract object (random triangulation, internal metrics etc.) mapped finally to the physical space-time.

Simplest examples In the first example $S = Z^d, f_2 \equiv 0, f_1(\sigma) = 1$ if $|\sigma| = 1$ and ∞ otherwise, assume $s_0 = 0$. Otherwise speaking we have the non-normalized distribution $\exp(-l(s))$ on all possible finite paths $r = (0, s_1, s_1 + s_2, ..., s_1 + s_2 + ... + s_N)$ in Z^d, starting from 0 where $N = l(r)$ is the length (number of steps) of r. It does not always exists. There exists $0 < \mu_{cr} < \infty$ such that the series

$$Z = \sum_{s} \exp(-\mu l(r))$$

for the partition function converges for $\mu > \mu_{cr}$ and diverges for $\mu \leq \mu_{cr}$. In our case $\mu_{cr} = \ln 2d$.

For the second example $S = R^d, e^{-f_1(s)} = \delta(|s| - 1)$, $f_2(s, s') = \phi((s, s'))$ for some bounded function ϕ of the angle between the two vectors. For the third example $S = R^d, f_2 \equiv 0, f_1(s) = s^2 = (s^1)^2 + ... + (s^d)^2$.

These examples, highly simple and having nothing special from the probabilistic viewpoint, correspond to one-dimensional analogs of rather famous actions: free relativistic point particle in d-dimensional space-time (l - length parameter in the Euclidean space, $L = L(0, x)$ is a path from 0 to

x)

$$S(L) = \mu \int_L dl$$

Hilbert-Einstein action (κ is the curvature of the curve embedded in an Euclidean space)

$$S(L) = \mu \int_L dl + \lambda \int_L |\kappa| \, dl$$

and bosonic string action (g is some metrics on the parameter interval)

$$S(L) = \frac{1}{\alpha'}(\int_L \sqrt{g(\xi)} d\xi (g(\xi) \frac{d^2 x}{d\xi^2} + \mu)$$

One can consider the introduced distribution as a quantization of the corresponding classical action. Each link of the discrete path is assumed to have unit length and thus the length of a path is the number of links.

We shall discuss only the first example, other two are similar, see [31]. For $\mu > \mu_{cr}$ we can define the probability distribution on the set of all finite paths starting from 0

$$P(r) = Z^{-1} \exp(-l(r))$$

Green function are defined as a measure on Z^d

$$\begin{aligned} G(x) &= \sum_{s:0\to x} \exp(-\mu l(r)) = \sum_N C(N; x) \exp(-\mu N) \\ &= \sum_N P_{RW}^{(N)}(x) e^{(-\mu+\mu_{cr})N} \end{aligned}$$

where $C(N; x)$ is the number of paths from 0 to x of length N, $P_{RW}^{(N)}(x)$ - N-step transitions probabilities from 0 to x for the classical simple random walk in Z^d. The number of such paths (by the local limit theorem) is

$$C(N; x) \sim (2\pi)^{-\frac{d}{2}} (2d)^N N^{-\frac{d}{2}}$$

if $x = O(\sqrt{N})$. Green functions have their origin in physics and they also look like Green functions for a Markov process

$$G(0, x) = \sum_n p_{0x}^{(n)} = \sum_{L:0\to x} p(L)$$

where L are all paths from 0 to x. But there is no Markov process here.

The following observations are important:

- neither μ_{cr} nor the exponent $-\frac{d}{2}$ do not depend on the choice of x.
- μ_{cr} is not universal, it depends on the dimension and on the lattice. Also we will get different values of μ_{cr} if we take piecewise linear paths in R^d with sides of fixed length (see [31]);
- however the exponent $-\frac{d}{2}$ does not depend on the lattice, which follows immediately from the local limit theorem;
- for $\mu = \mu_{cr}$ the series converges iff $d > 2$.

Now we want to study the scaling limit $\mu \to \mu_{cr} + 0$. Denote $x = s_1 + ... + s_N$. In the scaling limit one studies the exponents: mass (inverse correlation length) exponent ν, susceptibility exponent γ, anomalous dimension η, Hausdorf dimension d_H. They are defined via the leading term behavior for small $\mu - \mu_{cr}$

$$m(\mu) = \lim_{|x| \to \infty} \frac{-\ln G(x)}{|x|} \approx (\mu - \mu_{cr})^{\nu}, \sum_x G(x) \approx (\mu - \mu_{cr})^{-\gamma},$$

$$G(x) \approx |x|^{-d+2-\eta}, \sum NC(N; x) \exp(-\mu N) \approx x^{d_H}$$

We give now more detailed explanations. We have immediately that

$$\chi(\mu) = \sum_x G(x) = \sum_N e^{(-\mu+\mu_{cr})N} = \frac{1}{1 - \exp(-\mu + \mu_{cr})}$$

does not depend on the dimension of s and holds also for more general spins and interactions (however not always). Thus $\gamma = 1$.

If $(\mu - \mu_{cr})x \ll 1$ then $G(x) \approx G_{RW}(x)$, the Green functions for the simple random walk. That is why $\eta = 0$.

Let $p^{(N)}(0, x)$ be the transition probabilities for the simple random walk on Z^d and $f(p, z) = \sum_N \sum_x p^{(N)}(0, x) e^{ipx} z^N, z \in C$, - their generating function. Then

$$G(p) = f(p, e^{\mu_{cr} - \mu}) = \sum_N e^{(-\mu+\mu_{cr})N} \left(\frac{1}{d} \sum \cos p_i\right)^N = \frac{1}{1 - e^{-\mu+\mu_{cr}} \frac{1}{d} \sum \cos p_i}$$

$$\approx \frac{1}{\mu - \mu_{cr} + \frac{1}{d} \sum \cos p_i}$$

which is the classical propagator for quantum relativistic free particle with mass $\sqrt{\mu - \mu_{cr}}$.

It can be also proved that

$$d_H = \nu^{-1}, \gamma = \nu(2 - \eta)$$

In the second example different exponents can be obtained for $\lambda \to \infty, \mu \to \mu_{cr}$.

2.2 Gravity as a queueing model

One can construct a reversible dynamics with respect to which the measure $Q(\sigma)$ introduced above is invariant. It is a continuous time Markov chain. The state

$$\sigma = s_1 ... s_N$$

is interpreted as a queue, where N is the length of the queue, S is the set of customer types and $s_1 + ... + s_N$ is a generalized length of the queue (taking into account customer types and signs of jobs). This is a LIFO type queue (last in first out) and transitions consist of appending and deleting links on the right hand side (like arriving and service of customers in queueing theory). More exactly, for all $\sigma = s_1 ... s_N$, with rate ν

$$s_1 ... s_N \to s_1 ... s_{N-1}$$

and with rate λ

$$s_1 ... s_N \to s_1 ... s_N s_{N+1}$$

where all values of s_{N+1} are equiprobable. Transitions from the empty queue $\emptyset$ are $\emptyset \to s_1$ with rate λ and equiprobable s_1.

Lemma 1 *If $\nu > \lambda$ then the process is ergodic and the distribution (1) is invariant with respect to this dynamics.*

To prove this, note that the restriction $N_t = N(\sigma(t))$ of the process $\sigma(t)$ is also a Markov chain - a birth-death process, its stationary probabilities are

$$Z^{-1}\exp(-\mu N - N\ln 2), \mu = \ln\frac{\nu}{\lambda}$$

For models 2 and 3 similar dynamics leaves the distributions invariant. Note that a system of two queues would correspond to two interacting particles etc.

Supercritical case Now we see that they cases $\nu \leq \lambda$ have no sense in the equilibrium approach but in the dynamical picture they are no worse than the case $\nu > \lambda$. One cannot write down equilibrium distribution for $\nu \leq \lambda$, but at any time moment there exists some distribution $Q(t)$ and its limiting properties as $t \to \infty$ could have interesting properties. Denote $\sigma(t)$ the string at time t.

Theorem 1 *For $\lambda > \nu$ we have $N(\sigma(t)) \to \infty$ a.s. Moreover there exist limiting local correlation functions (not too close to the ends of the string) which define a translation invariant Gibbs field. For example*

$$\lim_{k\to\infty}\lim_{t\to\infty} P(s_k(t) = i) \to p_i$$

In fact for the first example this Gibbs field is a Bernoulli sequence in all three regions: $\nu > \lambda, \nu = \lambda, \nu < \lambda$, see [45].

Critical case and scaling limit There are two possibilities to consider the critical case.

The first one is to consider the dynamics for critical parameter values. The properties of this dynamics define the critical exponents. There are results for sufficiently general transitions: given two positive functions $\nu(s), \lambda(s,s')$, define the transition rates as

$$\nu(s_1...s_N \to s_1...s_{N-1}) = \nu(s_N), \lambda(s_1...s_N \to s_1...s_{N+1}) = \lambda(s_N, s_{N+1})$$

thus depending on the right symbols. Assume that the functions $\nu(s), \lambda(s,s')$ are such that the Markov chain is null-recurrent, see the conditions in [47]. Let S be finite with values $a_1, ..., a_k$. Let $n(t) = (n_1(t), ..., n_k(t)), n_i(t)$ is the number of symbols a_i in the string $\sigma(t)$. Then

Theorem 2 *The central limit theorem holds for the random vector $n(t)$, that is the following limit as $t \to \infty$ exists in distribution*

$$\frac{n(t)}{\sqrt{t}} \to |w|\, c$$

where w has the standard gaussian distribution and c is a constant vector.

This gives the same canonical exponents. Note that for the reversible case the proof reduces to the reflected random walks. The proof for non-reversible dynamics is more involved: for finite S see it in [47]. For compact S it should be similar. For non-compact it would be interesting to find examples with non-Gaussian limiting distribution.

The second approach corresponds to the scaling limit in equilibrium case. In dynamics the parameters are scaled together with time t, the parameters tend to the critical line $\nu = \lambda$ and $\nu - \lambda$ is scaled as $t^{-\frac{1}{2}}$. In such dynamics the scaling limit corresponds to the diffusion approximation in queueing theory. One gets the Brownian motion with drift for the dynamics of x under the following scaling

$$t = \tau N, x = r\sqrt{N}, \nu - \lambda = N^{-\frac{1}{2}}$$

The drift defines the mass gap in the spectrum of the infinitesimal generator of the corresponding diffusion process. The proofs here can be obtained by the application of the techniques known for the critical case.

Random grammars We considered the dynamics, that is called right linear grammar (not necessarily context free) in the computer science terminology. Now we shall speak about more general dynamics when transitions can occur at any place of the string, not only in its right end.

For the first example one can construct the following reversible Markov chain, leaving invariant the distribution, that appears to be a context-free random grammar (see [43]). Each symbol of the string is deleted with rate ν and for each $i = 0, 1, ..., n$ we insert a new symbol between symbols s_i and s_{i+1} (where for $i = 0$ we put it before s_1, and for $i = n$ - after s_n) of the string $s_1 ... s_n$ with rate λ. Appended symbol with probability $\frac{1}{2d}$ will have one of $2d$ coordinate vectors e. To prove it note that this this dynamics restricted to Z_+, the set of path lengths, is also Markov. It is in fact a birth and death process on Z_+ with jump rates $q_{i,i+1} = \lambda(i-1), q_{i,i-1} = \nu i, q_{0,1} = \lambda$. Then its stationary probabilities are (if $\lambda < \nu$)

$$\pi_k = \frac{\lambda_o \lambda_1 ... \lambda_{n-1}}{\nu_1 ... \nu_n} \sim C \exp(-\mu k), \mu = \ln \frac{\nu}{\lambda}$$

For two other examples the dynamics (not context free) can also be constructed, we shall do it in another paper in more general cases.

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3 Spin Complexes

3.1 Cell structures

Here we present the minimum of basic definitions concerning cell structures.

A complex is obtained by gluing together its elementary constituents - cells, like the matter consists of molecules. One should be very careful in defining the rules of gluing and the arising probability distributions. On the other hand it seems doubtful that some type of cellular structure has some a priori advantages in front of others. There are no definite physical reasons to prefer one cell structure or gluing rule etc., over another. Thus various possibilities should be studied to see what universal laws they share. In this paper paper we shall encounter two universal classes, one of them is popular in physics now. Moreover, having some flexibility in choosing a cell structure one can gain more simplicity in the probabilistic description and even get solvable models.

3.1.1 Abstract complexes

A (labelled) complex Γ is a set of elements called cells, there is a function $\dim A$ on Γ, the dimension of the cell A, taking values $0, 1, 2, ...$ The dimension of Γ is $\dim \Gamma = \sup_A \dim A$. Let $\Gamma_d \subset \Gamma$ be the set of cells of dimension d. For each cell $A \in \Gamma_d, d > 0$, is defined a subset $\partial A \subset \cup_{i=0}^{d-1} \Gamma_i$, the boundary of A. Subcomplex Γ' of Γ is a subset of Γ such that if $A \in \Gamma'$ then $\partial A \subset \Gamma'$.

Isomorphism of two complexes is one-to-one mapping respecting dimension and boundaries. Equivalence classes of complexes with respect to these isomorphisms are called unlabelled complexes.

The star $St(A)$ of the cell A is the subcomplex containing A and all cells B such that either $A \in \partial B$ or $B \in \partial A$ or $\partial B \cap \partial A \neq \emptyset$.

Note that complexes Γ can be considered as particular cases of spin graphs $(G = G(C), s)$, see[44]. The correspondence can be constructed in different ways. For example, let the vertices i of G correspond to cells of C, the function $s(i)$ is the dimension of the corresponding simplex. Links are defined by the incidence matrix: two vertices A and A' of G are connected by a link iff $A' \in \partial A$.

Labelled spin complex is a pair (G, s) where G is a complex and $s : C(G) \to S$ is a function on the set $C(G)$ of cells of G with values in some spin space S. Isomorphism of two spin complexes is an isomorphism of the complexes respecting spins. The equivalence classes are called (unlabelled) spin complexes. Unless otherwise stated we consider only functions s defined on the cells of maximal dimension; by dualisation it is often equivalent to functions s restricted to vertices.

3.1.2 Topological complexes

There many topological incarnations of abstract complexes. In each of them a cell is represented by an open disk. A CW-complex is a topological space K which is defined by the inductive construction of its d-dimensional skeletons K_d. Let $K_0 = \Gamma_0$ be a disconnected set of points (vertices) - cells of dimension 0. In general, K_d is obtained from K_{d-1} as follows. Each cell A of dimension d is identified with an open d-dimensional disk D_A and some continuous (attaching) map $\phi_A : \partial D_A \to K_{d-1}$ is fixed. Then K_d is the factor space of the union of K_{d-1} and $\cup_{A:\dim A=d} D_A$ via identifications of $x \in D_A$ with $\phi_A(x)$.

For example, K_1 is a graph with vertices, zero-dimensional cells, and links (edges), one-dimensional cells. Link A is a loop if the boundary of A is mapped to one vertex. Often some restrictions on the attaching maps are imposed. Here we restrict ourselves to the case $\dim \Gamma = 2$ and for all $A, \dim A = 2$, the boundary ∂D_A is the union of some cells $B, \dim B = 0, 1$ (in some books, see for example [1], CW-complexes are defined as already satifying this restriction). With such CW-complex one can associate an abstract complex with $\partial A = \{B \in K_0 \cup K_1 : B \subset \phi_A(D_A)\}$.

We get the class of simplicial complexes (where the cells are called simplices) if for each 2-cell A its boundary has 3 one-dimensional cells and the set $\partial A \cap K_0$ of vertices uniquely defines A. Any graph without multiple edges, no loops is a simplicial complex.

3.1.3 Cell surfaces

In the paper we consider different classes of (two-dimensional) complexes. The class can be defined either by imposing further restrictions on the class of complexes defined above or by some constructive procedures to get all complexes in this class. Anyway such classes are a particular case of a language defined by some substitutions in a graph grammar, see [43, 44].

The following restrictions hold for all complexes in this paper: complex is a (closed compact) surface. Pseudosurface (closed compact) is a topological space isomorphic to a finite 2-dimensional simplicial complex with the following property: each link is contained in the boundary of exactly two faces (two-dimensional cells). A surface has an additional property that the neighbourhood of each vertex is homeomorphic to a disk.

This is the list of all compact closed (without holes) 2-dimensional surfaces. Orientable surfaces are just $S_\rho, \rho = 0, 1, 2, \ldots$, - sphere with ρ handles. Nonorientable surfaces are P_1 (projective plane), P_2 (Klein bottle), ..., $P_k, \ldots$, - sphere in which k holes are cut and to each of them a Moebius band (crosscup) is attached along its boundary.

In this case K_1 is a graph homeomorphically imbedded to the surface S. Such complexes are studied in the topological graph theory (see [6]) and in combinatorics, where topological complexes are called maps. Surface with holes is obtained from a closed surface by cutting out finite number of disks with non-intersecting boundaries. If the surface has a boundary then the boundary belongs to K_1.

Isomorphism of maps is an isomorphism of abstract complexes. In other words, two maps are called isomorphic if there is a homeomorphism of S such that vertices map onto vertices, edges on edges, cells on cells.

A map B is a subdivision of the map A if the graph $K_1(A)$ of A is a subgraph of the graph $K_1(B)$ of B. By Hauptvermutung if two topological complexes are homeomorphic as topological spaces there exist their subdivisions isomorphic as abstract complexes.

If the surface is closed the Euler characteristics of the complex is defined as

$$\chi = V - L + F$$

where F is the number of faces, V- number of vertices, L - number of links. It does not depend on the complex but only on the surface itself: for orientable surfaces the Euler characteristics $\chi = 2 - 2\rho$ where ρ is the genus (number of handles), for nonorientable surfaces $\chi = 2 - n$ where n is the number of crosscups.

We shall use in fact only the following 4 classes.

Arbitrary maps This is the class we have just defined. No further restrictions are imposed. Simplest examples are a vertex inside the sphere (vertex map), an edge with two vertices inside the sphere - edge map.

Smooth Cell Surfaces Smooth cell surface (see [2]) is a compact connected smooth two-dimensional manifold M with finite number of closed subsets (cells) F_i such that:

1. $\cup F_i = M$;
2. for each i there exists a one-to-one smooth mapping f_i of F_i onto a polygon with $n_i \geq 3$ faces;
3. for $i \neq j$ either $F_i \cap F_j = \emptyset$ or $f_i(F_i \cap F_j)$ is an edge or a vertex of the corresponding polygon.

Triangulations This is a smooth cell surface with all $n_i = 3$. The set V of vertices is called a cut if there are two subgraphs G_1, G_2 such that $G_1 \cup G_2 = G, G_1 \cap G_2 = V$. Disk-triangulation is a smooth cell surface, homeomorphic to the sphere, where there is one distinguished (that will be the outer face) face and for all other faces $n_i = 3$, and there is no cuts with one vertex. Then it can be considered as the triangulation of the disk (sphere with a hole). For triangulations the absence of cuts is equivalent to the absence of loops.

Simplicial complexes These are triangulations without multiple edges, where moreover every three edges define not more than one cell. Note that a triangle (cycle of length 3) having inside and outside at least one vertex, is not considered as a cell.

Convex polyhedra Quantizing smooth via piecewise linear structures is possible because the convex polyhedra have combinatorial counterparts. For example, convex polyhedra can be considered as maps with $\rho = 0$. There is a pure combinatorial characterization of maps corresponding to convex polyhedra. If a triangulation has no loops and no multiple edges, then, if $L \geq 4$, it corresponds (by Steinitz-Rademacher theorem), to a convex polyhedron.

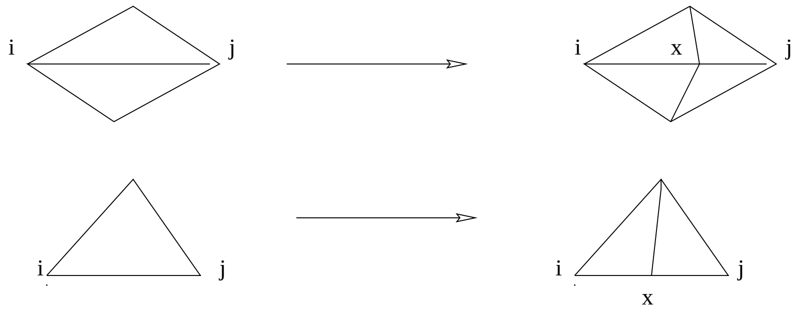

Figure 2: Alexander moves

3.1.4 Local observer (root)

Labels in complexes are not necessarily given explicitly but the complex is considered to be labelled if the set V is claimed to be fixed. Labels are useful for fixing coordinate system in the space but are superfluous for the geometry and topology. There is a very convenient way to avoid the superfluous labelling but at the same time giving some algorithmic way to get a complete coordinatization. Root (local observer) in a (labelled) complex is an array (f, l, v) where f is a two-dimensional cell, $l =$ its edge, v - vertex of l. Isomorphism of two complexes with roots is an isomorphism of complexes respecting the roots. Rooted map (rooted complex, complex with a local observer) of class **A** is an equivalence class of isomorphisms of complexes with a root in the class **A** of complexes. Assume that the rooted edge is directed from the rooted vertex. For disk triangulations we agree that one (the outer) face i is rooted, it is possible that $n_i \neq 3$.

Lemma 2 *The automorphism group of any rooted map is trivial.*

This is easily proved by induction on the number of cells by subsequent extending the automorphism from the rooted face to its neighbors.

3.1.5 Moves

Graph grammars corresponding to transformations (substitutions here are called moves) of complexes were studied very little. In the next section we shall consider Tutte moves, see Fig. 7, which consist in appending an edge between two vertices of a cell or joining together two disjoint graphs by identifying two of their vertices. In topology subdivisions played always a big role. There are two papers (see [7, 8]) where some moves are studied in detail.

Let A be the commutative associative algebra over Z_2 (simplicial chains over Z_2) generated by the symbols of some (countable) alphabet L with commutation relations $s_i^2 = 0, s_i s_j = s_j s_i$. Thus it is a linear span generated by the strings (simplices) $\alpha = s_1 ... s_n$. Define the boundary operator as a linear operator $\partial : A \to A$ such that

$$\partial\alpha = \sum \beta$$

where the sum runs over all subsets of α with the number of elements $|\alpha| - 1$. We shall consider here only two dimensional complexes.

There are other linear operators in this algebra (called Alexander moves) $S_{i,j;x}, i, j, x \in L$. They are defined as follows. Let $\alpha = ij\beta$, then

$$S_{i,j;x} ij\beta = x(i + j)\beta$$

Next example: Gross-Varsted moves. It is proved in [8] that each Alexandre move can be obtained

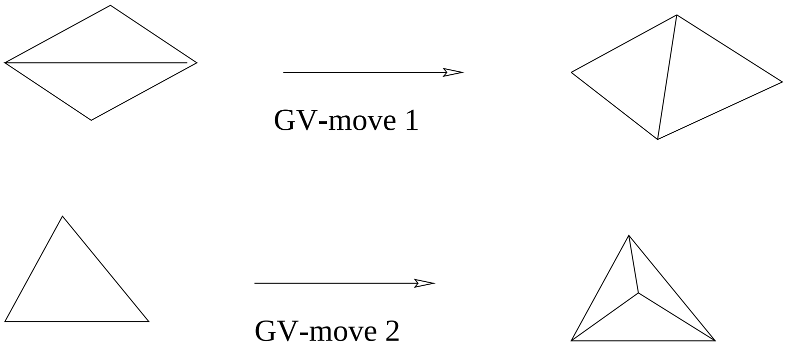

Figure 3: Gross-Varsted moves

via Gross-Varsted moves and vice versa. We say that a set of moves is irreducible in the class $\mathbf{A}$ of complexes if for each pair T_1, T_2 of complexes from $\mathbf{A}$ there is a sequence of moves giving T_2 from T_1 (in the physical literature the term ergodic is used in this cased, but we want to use the standard probabilistic terminology).

Theorem 3 *In the class* $\mathbf{A}$ *of simplicial complexes the set of Alexander moves and the set of Gross-Varsted moves as well are irreducible.*

Proof see in [7, 8].

3.1.6 Automorphism group

Let $\mathbf{A}$ be any of the five classes of complexes introduced above.

Theorem 4 *For most complexes with N two-dimensional cells from* $\mathbf{A}$ *the automorphism group is trivial, that is* $\frac{\mathbf{A}_{nontrivial}(N)}{\mathbf{A(N)}} \to 0$ *if* $N \to \infty$*, where* $\mathbf{A}(N)$ *(*$\mathbf{A}_{nontrivial}(N)$*) is the set of all complexes with N two-dimensional cells from* $\mathbf{A}$ *(the same but with nontrivial automorphism group).*

Earlier Tutte remarked that it is very intuitive that almost all triangulations have no nontrivial automorphism. Many rigorous results appeared afterwards, see [18, 21, 22]. Proof for the case of disk-triangulations see in [23].

3.2 Metrics and Curvature

The metric structure is defined once it is defined for each closed cell so that on the edges the lengths are compatible. There are two basic approaches for defining the metric structure: Dynamical Triangulations - when all edges have length one and Quantum Regge Calculus - when they are random. We shall use the first one. Then all cells with the equal number of edges are identical and on faces the metrics is standard.

One can do it differently. Let first the graph K_1 be embedded in the plane, the edges being smooth arcs. Define the metric structure on the graph K_1 so that the edge lengths are all equal to a constant. Inside a cell with n edges we define the metric structure via some smooth one-to-one mapping of an equilateral polygon Q_n with n edges onto this cell, so that the smoothness hold also in vicinity of each point on the edge. Then inside cells the curvature is zero. On edges also: this is shown on the figure in piecewise linear case.

We shall define curvature R_v at vertex v. As always the curvature is measured by parallel transport (Levi-Civita connection) of a vector (lying in the plane in piecewise linear situation) along a closed path: along the internal part of a triangle as on the Euclidean plane, through an edge - by unfolding

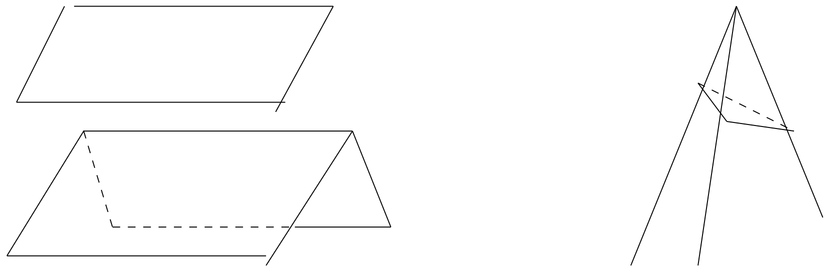

Figure 4: Curvature

the two half planes separated by this edge to a plane. One sees immediately that only paths around vertices may give nonzero difference. Around the vertex v the angle between the initial and the transported vector is $\varepsilon_v = 2\pi - \sum_f \varphi_{fv}$, where φ_{fv} is the angle of the simplex f at vertex v. Note that

$$2\pi V - \sum_v \varepsilon_v = \sum_v \sum_f \varphi_{fv} = \sum_f \sum_v \varphi_{fv} = \pi F$$

Using the Euler formula $\chi = V - L + F$ one can get from this the Gauss-Bonnet formula

$$\sum_v \varepsilon_v = 2\pi\chi$$

for triangulations where $L = \frac{3F}{2}$.

Gauss-Bonnet formula for smooth surfaces is $\int k dA = \frac{1}{2\pi} \int k(x)\sqrt{g} dx = 2\pi\chi$. Its relationship with the discrete case for a partition with n_i-gons with areas A_i of the unit sphere (the area A of the triangle is $A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$) is given by the formula

$$\sum k_i A_i = \sum_i \left[\sum_j \alpha_{ij} - (n_i - 2)\pi \right] = \sum_{ij} \alpha_{ij} - \pi \sum n_i + 2\pi F = 2\pi V - 2\pi E + 2\pi F$$

Classical examples are: positive curvature - elliptic geometry (sphere, projective plane); zero curvature, Euclidean geometry (plane, torus, Klein bottle); negative curvature - hyperbolic geometry (all others).

Now we shall show that the curvature R_i at vertex i is defined by the number q_i of edges incident with i. Einstein-Hilbert action on the smooth manifold is

$$\int (c_1 R + c_2)\sqrt{g} dx$$

where R is the Gaussian curvature, g - metrics. It is known that $\int R d\sigma = \int R\sqrt{g} dx = 4\pi\chi$. Thus the discrete action should be (up to a constant) $\lambda\rho + \mu N$, where ρ is the genus and N is the number of triangles. We want to write down a discrete analog of this action with a discrete curvature summing over vertices instead of summing over triangles. Assume all triangles to be equilateral and scale their area to 1. Thus each vertex gets area $\frac{1}{3}$ from each incident triangle, thus $\frac{q_i}{3}$ in total. Then $\int R d\sigma \simeq \sum_i \frac{q_i}{3} R_i$ and the formula

$$\sum_i \frac{q_i}{3} R_i = 4\pi\chi$$

holds if only we put the curvature at the vertex i equal $R_i = 2\pi \frac{6 - q_i}{q_i}$.

4 Equilibrium planar pure gravity

There are two kind of techniques used in the two-dimensional gravity. Historically the first one is the combinatorial approach, that was initialized by Tutte and continued (without any mention of physics) by many researchers, the papers are published in journals on combinatorics. The second one is Random Matrix Theory (RMT) approach, that was originated in physics itself. Calculations in the second approach are very persuasive but the arguments are not completely rigorous. As far as I know, no explicit connections between these approaches were established. We use the first approach and give a short review of the latter approach.

4.1 Definitions and combinatorial approach

Let $\mathbf{A}$ be some class of complexes (for example defined in the previous section) T, homeomorphic to the sphere, $|T| = F(T)$ - number of cells of dimension 2 in T, $C(N) = \natural\{T \in \mathbf{A} : |T| = N\}$. The main example is the class of all triangulations of a sphere.

The grand canonical ensemble is defined by

$$P(T) = Z^{-1}\exp(-\mu F(T)), Z = \sum_{T\in\mathbf{A}} \exp(-\mu F(T)) = \sum_N C(N)\exp(-\mu N) \tag{2}$$

In particular, the conditional distribution of $T \in \mathbf{A}$ with N fixed is uniform. Easy and general methods to estimate $C(N)$ are useful sometimes, but can provide only bounds.

Lemma 3 *(exponential a priori bounds)* $c_1\gamma_l^N < C(N) < c_2\gamma_u^N, 1 < \gamma_l < \gamma_u < \infty, c_i > 0$

Proof. Lower bound: this is quite trivial and can be proved in many ways. For example, take two following complexes homeomorphic to the ring with the same number of boundary edges from both sides. First one - alternating up and down triangles (that is standing on an edge and on the vertex correspondingly), second - two triangles up and two triangles down etc. These two kind of triangles can be glued sequentially one-after-one in all possible 2^n ways.

The following method of proof of upper bounds works even in some more general situations. One can give an algorithm to construct all possible complexes with N cells of dimension two. Start with one cell. We enumerate its edges as $1, 2, 3$. On each step we add not more than one cell to the boundary and enumerate new edges immediately after already used numbers. Now we describe the inductive construction. We take the edge with number one and make one of the 4 decisions: 1) not to add anymore triangles to this edge, 2) add to it a triangle having exactly two new edges, 3) add triangle to this edge and to the next edge on the boundary (in clockwise direction), 4) the same for counterclockwise direction. For each of 4^k decision sequences ω let $f(k,\omega)$ be the number of edges after k steps, $f(k,\omega) \leq 3 + 2k$. Moreover, if there are n triangles there cannot be more than $3n$ type 1 decisions.

One needs however exact asymptotics. All known examples exhibit the following asymptotic behavior

$$C(N) \sim c_1 N^{\alpha} c^N \tag{3}$$

From (3) it follows

Corollary 1 *There exists $0 < \mu_{cr} < \infty$ such that for $\mu > \mu_{cr}$ the series (2) converges. It diverges if $\mu < \mu_{cr}$. If $\mu = \mu_{cr}$ then $Z < \infty$ iff $\alpha < -1$.*

Thus, for the parameters $\mu < \mu_{cr}$ the distribution does not exist. However, the dynamics introduced later allows to consider such μ and for them local correlation functions make sense.

No general results are known however. None of the constants $c_1 > 0, c > 1$, is universal, but for all known examples α is. Universality of α is not at all simple intuitive fact. For example, predictions based on physical non-rigorous arguments (see, for example, [42]) failed to predict famous $\alpha = -\frac{7}{2}$ in the planar case.

Theorem 5 *The asymptotics (3) holds for all four classes, defined in the previous section. Moreover* $\alpha = -\frac{7}{2}$ *in all cases.*

Proof. We shall prove it only for triangulations; other cases see in references cited in "Enumeration of two-dimensional maps". In the similar way we shall define the distribution on the class $\mathbf{A}_0$ of rooted complexes

$$P_0(T) = Z_0^{-1}\exp(-\mu F(T)), Z_0 = \sum_{T \in A_0} \exp(-\mu F(T)) = \sum_N C_0(N)\exp(-\mu N)$$

where index zero means that we consider rooted complexes of class $\mathbf{A}$.

Lemma 4 *For triangulations*

$$C(N) \sim (3N)^{-1}C_0(N)$$

It follows from triviality of automorphism groups for most complexes (see theorem 4). Then we can take as a root any of N cells of dimension 2, choose one of its edges and orient it in 2 ways.

Denote $C(N,m)$ the number of disk-triangulations where the outer face has m edges, $C_0(N,m)$ - where the outer face is moreover rooted. The following result is similar but can be proved easier.

Lemma 5 $C(N,m) \sim m^{-1}C_0(N,m)$ *for large* N *and fixed* m.

Proof. Enumerate the edges of the boundary in a cyclic order: $1, 2, ..., m$. An automorphism ϕ is uniquely defined, if $j = \phi(1)$ is given. We shall show that almost all complexes do not have an automorphism such that $j = \phi(1)$.

To prove this we shall show that for each complex A having a nontrivial automorphism ϕ we can subdivide the complex on two parts $A_1 \cup A_2 = A$ where each cell belongs to only one part, such that $\phi A_1 = A_2$. This can be done by induction as follows. Take some boundary edge, take a triangle T with this edge and refer it to A_1, then put $\phi T \in A_2$. Each step of induction consists of taking one more triangle having common edge with already constructed part of A_1. Now we can modify A_2 inside in a number $u(N)$ of ways, bounded from below by some function $u(N) \to \infty$ as $N \to \infty$, uniformly in A. This can be done by choosing $u(N)$ triangles in A_2, not too close from each other, and modifying independently some neighborhood of each keeping the boundary of the neighborhood and the number of cells n in this neighborhood fixed. This is possible as $C(n,b) > 1$, where b is the number of edges on the boundary. Thus for given A_1 the proportion of complexes with $\phi A_1 = A_2$ is small.

We have proved that only "small" number of complexes have an automorphism ϕ such that $j = \phi(1)$. As m is fixed then multiplying this number on m gives again a "small" number.

To prove the theorem we should prove that $C_0(N) \sim c_2 N^{-\frac{5}{2}} c^N$. The universal nature of (3) is strongly supported by the fact that, for all such examples, the first positive singularity of the generating function $\sum_N C_0(N)z^N$ is an algebraic singularity, that gives the asymptotics (3) .

Remark 1 *Assume an algebraic function $y(x)$ is analytic at 0, has minimal positive singularity at point $a>0$. We say that its leading exponent is b if there exist such b_i and functions $g_i(x)$ analytic at $x=a$ such that $y(x)=\sum_{i=1}^{n} g_i(x)(a-x)^{b_i}+g_{n+1}(x)$. Then we have the following expansion*

$$y(x)=\sum_n c_n x^n, c_n \sim C n^{-b-1} a^n$$

In our case (for $C_0(N)$) $b=\frac{3}{2}$. One could also apply tauberian theorems in such situation.

We give some examples where all constants in the asymptotics are known, see the same references. First example is the class of triangulations defined above. Here $C_0(N) \sim \gamma_2 N^{-\frac{5}{2}} c^N, c=3\sqrt{\frac{3}{2}}$. For convex polyhedra we have $C_0(N) \sim \gamma_3 N^{-\frac{5}{2}} c_1^N, c_1=\frac{16}{3\sqrt{3}}$. For simplicial triangulations $C_0(N) \sim \gamma_4 N^{-\frac{5}{2}} c^N, c=\frac{3\sqrt{3}}{2}$. Many other examples can be given; it is interesting however to understand the general underlying mechanism.

Tutte [13, 14, 15] has begun to study the asymptotics for $C(N,m)$ and developed a beautiful and efficient "quadratic" method. Afterwards many authors contributed by developing the method itself and obtaining asymptotics for various classes A (see review [19] and more recent papers [23]).

The main idea of Tutte are the following recurrent equations for $C(N,m), N=0,1,...; m=2,3,...$

$$C(N,m)=C(N-1,m+1)+\sum_{N_1+N_2=N-1, m_1+m_2=m+1} C(N_1,m_1)C(N_2,m_2), m \geq 3, N \geq 1$$

$$C(0,2)=1, C(0,m)=0, m>2$$

These equations are easily derived as follows from the following picture where the orientation of the rooted edge is marked by arrow, rooted vertex is the first vertex of the arrow, rooted face is to the right of the arrow (containing the north pole of the sphere), see Figure 7 Take any rooted map with $(N-1,m+1)$ and do Tutte move 1, take any ordered pair of rooted maps $(N_1,m_1),(N_2,m_2)$ and perform Tutte move 2. Any rooted map (N,m) can be uniquely obtained in this way. $(0,2)$ corresponds to the so called edge map with one edge only which is counted twice.

If we introduce the generating function

$$U(x,y)=\sum_{N=0}^{\infty}\sum_{m=2}^{\infty} C(N,m) x^N y^{m-2}$$

the following functional equation

$$U(x,y)=U(x,y)xy^{-1}+U^2(x,y)xy+1-xy^{-1}U(x,0)$$

holds. We shall deduce from this equation that $U(x,0)$ is algebraic and compute its first singularity, below in this paper, in a bit more general setting.

Green functions Consider a class $\mathbf{A}$ of complexes. Let $\mathbf{A}(m_1,...,m_k)$ be a class of complexes, defined with the same restrictions as $\mathbf{A}$, homeomorphic to the sphere with k holes with m_i edges on the boundaries of these holes. We assume also that these boundaries do not intersect each other. The Green functions are defined as follows

$$Z(m_1,...,m_k)=\sum_{T\in A(m_1,...,m_k)} \exp(-\mu F(T))=\sum_N C(N,m_1,...,m_k)\exp(-\mu N) \tag{4}$$

Z corresponds to the case $k = 0$. Rooted Green functions are defined similarly

$$Z_0(m_1, ..., m_k) = \sum_{T \in A_0(m_1,...,m_k)} \exp(-\mu F(T)) = \sum_N C_0(N, m_1, ..., m_k) \exp(-\mu N)$$

where the index 0 everywhere means that we consider complexes with a distinguished edge on the first boundary with m_1 edges, the local observer in the terminology of [43, 44]. One would like to have an expression for the Green functions in terms of the basic probabilities (as for Markov chains).

Green functions are associated with the derivatives $\frac{d^n Z}{d\mu^n} = -\chi^{(n)}(\mu)$, that is the factorial moments of N.

Lemma 6 *The partition function and its two first derivatives are finite for $\mu = \mu_{cr}$ and for $n > 2$ we have as $\mu \to \mu_{cr} + 0$*

$$\frac{d^n Z(\mu)}{d\mu^n} \sim c(n) \left(\mu - \mu_{cr}\right)^{-\alpha-1-n}$$

Proof. We shall see later that Z is an algebraic function of $z = e^{-\mu}$ and has the principal singularity $C(1 - \frac{z}{z_0})^{-\alpha-1}$ at the point $z_0 = e^{-\mu_{cr}}$. In the vicinity of z_0 we have $1 - \frac{z}{z_0} = 1 - e^{-(\mu-\mu_{cr})} \sim \mu - \mu_{cr}$.

This is in a good agreement with the following simple intuitive counting argument.

Lemma 7 *For fixed $k, m_1, ..., m_k$ there exist constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 N^{k-1} C_0(N) < C_0(N, m_1, ..., m_k) < c_2 N^{k-1} C_0(N)$$

Proof. Take first $k = 1$ and prove the upper bound. Take some complex $A \in \mathbf{A}_0(m)$ with N faces and glue up the hole with some complex $B \in \mathbf{A}_0(m)$ with r faces where r depends only on m. We shall get some complex $C(A) \in \mathbf{A}_0$ with $N + r$ faces. For given A and $C \in \mathbf{A}_0$ with $N + r$ faces we shall get not more than p complexes $C(A) = C$ where p depends only on m. In fact, for any C the number of subcomplexes with r faces from $\mathbf{A}_0(m)$ having the same root is bounded by $C_0(r, m)$. That is why $C_0(N, m) < cC_0(N + r), r = r(m)$. The lower bound can be proved similarly. For $k > 1$ the proof is similar but one should first choose $k - 1$ faces along which paths with $m_2, ..., m_k$ edges will pass. This will give the factor N^{k-1}. This can be done by induction in k.

4.1.1 Uniform asymptotics

Two questions arise: what is the asymptotics of $C(N, m)$ if both N, m tend to infinity and what is the asymptotics of other global variables, such as the number of vertices etc. ? We shall see that these two questions are related.

V, L, F are well-defined random variables in the grand canonical ensemble and one could would like to have their joint distribution. In general only two of them are independent due to the Euler formula $V - L + F = 2$. For triangulations, where each face has 3 incident edges, we have only one independent variable as $L = \frac{3F}{2}$. For the class of all rooted maps, where two variables are independent, we have the following lemma.

Lemma 8 *Let $E(V \mid F = N)$ be the conditional mean number of vertices if the number of faces is N. Then*

$$E(V \mid F = N) \sim cN$$

for some $c > 0$.

As it follows from the formula on p. 157 of [12] the number of rooted maps with $N+1$ faces and $m+1$ vertices is

$$c(N,m) = \frac{1}{2N-1} C^m_{2N+m-2} \frac{1}{2m-1} C^n_{2m+N-2}$$

Thus

$$E(V \mid N) = \frac{\sum_m mc(N,m)}{\sum_m c(N,m)}$$

is defined by the maximum in $m = \alpha N$ of $\ln(C^m_{2N+m-2} C^n_{2m+N-2})$ by large deviation asymptotics.

Consider now one-particle Green functions.

Lemma 9 *The following series*

$$\sum_{N,m} e^{-\mu N - \nu m} C(N,m)$$

converges above some nondecreasing function $\nu(\mu)$, see Figure 5 Thus the series $\sum_{N,m} e^{-\mu N} C(N,m)$ diverges.

Proof. It is quite obvious because the series has all coefficients positive.

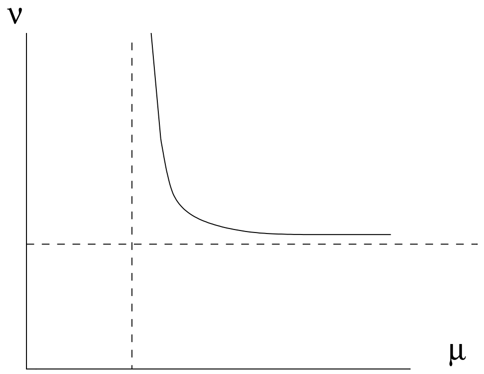

Figure 5: Critical curve

Thus we have a family of distributions $P(\mu, \nu)$. It is of interest to study the asymptotics and exponents when $\mu = a\nu$, where $a, 0 \le a \le \infty$ is fixed.

The explicit formula (see [12]) for the number $C_0(N,m)$ of triangulations with a distinguished edge on the boundary (rooted triangulations) is

$$C_0(N,m) = \frac{2^{j+2}(2m+3j-1)!(2m-3)!}{(j+1)!(2m+2j)!((m-2)!)^2} \tag{5}$$

where $N = m + 2j$ is the number of inner cells. $C_0(N,m) = 0$ if $N - m$ is odd.

Then as $j, m \to \infty$

$$C_0(N,m) \sim const \frac{\sqrt{m}}{j(2m+3j)} 2^{j+2m} C^j_{2m+3j}$$

in particular for $j \sim \beta m, \beta > 0$

$$C_o(N,m) \sim const \frac{1}{m^2} 2^{j+2m-\beta m \log \frac{\beta}{2+3\beta} - (2+2\beta) m \log \frac{2+2\beta}{2+3\beta}}$$

Thus for all $0 < \beta < \infty$ the exponent is $\alpha = -2$. For fixed m the exponent does not depend on m and is $\alpha = -\frac{5}{2}$ and moreover

$$C_0(N,m) \sim \phi(m) N^{-\frac{5}{2}} c^N$$

We have also $\phi(m) \sim C m^{\frac{1}{2}} c_1^m$ as $m \to \infty$.

4.2 RMT approach

Random Matrix Model is the following probability distribution μ on the set of selfadjoint $n \times n$-matrices $\phi = (\phi_{ij})$ with the density

$$\frac{d\mu}{d\nu} = Z^{-1} \exp(-tr(\frac{\phi^2}{2h}) - tr(V))$$

where $V = \sum a_k \phi^k$ is a polynomial of ϕ bounded from below, ν is the Lebesgue measure on real n^2-dimensional space of vectors $(\phi_{ii}, \mathrm{Re}\,\phi_{ij}, \mathrm{Im}\,\phi_{ij}, i < j)$. It can be written also as

$$\frac{d\mu}{d\mu_0} = Z_0^{-1} \exp(-tr(V))$$

where μ_0 is the Gaussian measure. It is easy to see that μ_0 has covariances $\langle \phi_{ij}, \phi_{kl}^* \rangle = \langle \phi_{ij}, \phi_{lk} \rangle = h\delta_{ik}\delta_{jl}$. Note that for mere existence of the probability measure μ one needs that the senior coefficient a_p of V were positive and p were even. In this case there exists a well-developed probability theory of such models, which we shall not review here, see [37].

The fundamental connection (originated from t'Hooft) between RM model and two-dimensional complexes is provided by the formal series

$$\log Z_0 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} < tr(V), ..., tr(V) >= \sum \frac{(-1)^k}{k!} \sum_{D_k} I(D_k)$$

where D_k is the sum of all connected diagrams with k vertices. Take for example $V = a_4 \phi^4$. Then each diagram has labelled vertices $1, ..., k$, each vertex has labelled thick legs $1, 2, 3, 4$, corresponding to the product $\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}$. Each thick leg can be seen as a narrow strip with two sides, each side is marked with a matrix index. Dividing by 4^k we eliminate the numbering of the four legs leaving them however cyclically ordered. After coupling legs and their sides (note that coupled sides have the same index and, as each vertex have two sides with the same index, we get index loops) and summing over indices we get a factor n^N where N is the number of index loops. After this we are left with

$$\sum \frac{(-a_4)^k}{k!} \sum_{D_k} h^{L(D_k)} n^{N(D_k)}$$

For each graph D choose the minimal cell embedding $f(D)$ of D in a compact orientable surface of genus ρ (topological graph theory [6]). Assume clockwise order of legs. It has k vertices, $2k$ edges and N faces. Putting $a = -a_4$ and using Euler formula $k = N + 2\rho - 2$ we have

$$\sum \frac{(-a_4)^k}{k!} \sum_{D_k} h^{L(D_k)} n^{N(D_k)} = \sum_{N,\rho} C(N,\rho) a^k h^{2k} n^N =$$

$$= \sum_{N,\rho} C(N,\rho)(ah^2 n)^N (ah^2)^{2\rho-2} = a^2 \exp(-\mu N - \nu\rho)$$

with $\mu = -\ln(ah^2 n), \nu = -2\ln(ah^2)$.

The calculations in RMM can be done only for $n \to \infty$, thus to get finite μ one should scale as $ah^2 = \frac{b}{n}, b = e^{-\mu}$. In the limit $n \to \infty$ we have $\nu \to \infty$ and only $\rho = 0$ survives giving thus only plane imbeddings. The limit $\lim_{\mu \to \mu_{cr}} \lim_{n \to \infty}$ is called the simple scaling limit. It was proved (see [34]) that $\alpha = -\frac{7}{2}$ in this case showing again the stability of this exponent.

There are important points in this approach which should be mentioned:

- In case $V = a_4\phi^4$ the order of all vertices equals 4, this is some restriction on the class of maps;
- Automorphism group of our labelled diagram factores in two factors. The first one $C(D)$ related to the permutation of vertices, and second one $C_l(D)$ related to the permutation of legs in each vertex. Almost all diagrams have the first factor trivial, but for some of them $C_v(D) > 1$. We can sum over nonlabelled diagrams then but each unlabelled diagram will have a factor

$$\sum \frac{C_l(D)}{C_v D)}$$

  This means that the counting does not coincide with the natural counting used in the combinatorial approach;
- We should fix also f somehow: normally one chooses embedding $f(D)$ to the minimal possible ρ. But anyway not all possible triangulations are taken into account because a given graph can be embedded to surfaces with different ρ. This gives one more reason that the counting rule does NOT coincide with natural counting where all maps from some fixed class are counted exactly once. But this should not be taken seriously: anyway this counting is no worse and no better than others.
- There appears a contradiction if one wants to get probability distributions simultaneously for the matrix model itself and for graph embeddings. We have probability distribution for the matrix model if $a > 0$, but the probability distribution on the diagrams is achieved only if $a < 0$. Thus one should always perform analytic continuation from $a > 0$ to $a < 0$. The free energy for the scaling mentioned above can be rigorously calculated but the complete argument leading to the graph counting is still lacking.
- There are other pure gravity models treated with this approach: more general pure gravity model counts the number $n(q, T)$ of vertices v with $q_v = q$:

$$Z = \sum_{T \in A} \prod_{q > 2} t_q^{n(q,T)}$$

where t_q are the parameters, see [39].

5 Linear boundary dynamics

The probability distribution

$$P(T) = Z^{-1} \exp(-\mu F(T))$$

on some set $\mathbf{A}$ of complexes is invariant with respect to the following simple Markov process. Let at time t the triangulation be $T(t)$. The process is defined by the following infinitesimal transition rates. With rate $\lambda_+(N), N = |T|$, at time t we destroy T, add one more cell and glue anew all cells randomly together, that is if $N = F(T(t))$ then we choose uniformly $T(t + 0)$ among complexes of the class A with $N + 1$ cells. With rate $\lambda_-(N)$ we do random choice of a complex with $N - 1$ cells.

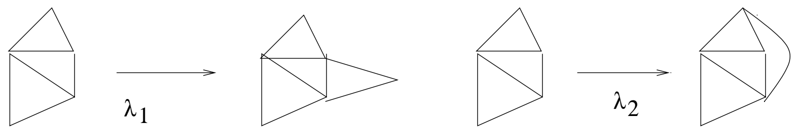

Figure 6: Dynamics on the boundary

What dependence on N can be ? If $\lambda_+(T) = bf(N), \lambda_-(T) = df(N-1)$ for some positive function $f(N)$ then the probability distribution $P(T)$ is an invariant distribution with respect to this process. Proof consists of the remark that the induced process on N is a reversible Markov chain: a birth and death process on Z_+ with jump rates $q_{i,i+1} = bf(i), q_{i,i-1} = df(i-1), q_{0,1} = 1$.

The simplest way of Monte-Carlo simulation is to take sufficiently large N and simulate uniform distribution, but it is impossible to find the exponent in this way. One should compare different N and this can be done via such a process. Apart from this such dynamics is of no interest, it is not constructive, especially in higher dimensions. In the rest of this paper we shall study local dynamics. We start with a simplest local dynamics of two-dimensional planar complexes. The distribution appears not to be invariant with respect to the first model dynamics. Thus, there could be two possibilities: either it will nevertheless give the same exponents for the invariant measure or its invariant measure belongs to another universality class (being however irreducible and ergodic). We shall show that the second one holds.

5.1 Local Pure Growth

We consider smooth cell surfaces and assume the cells be triangles. One starts with one triangle and each step consists in attaching a new triangle on the boundary. There are two kinds of attachment (see Figure 6): to one or to two edges with the same vertex: for any edge on the boundary we attach to it a triangle with rate λ_1. For any pair of neighboring edges on the boundary we attach to them a triangle with rate λ_2. At any time the complex is homeomorphic to a closed two dimensional disk and its boundary - to a circle. We assume that the initial state is the only triangle and that if the number of edges m on the boundary is equal to 3 then only λ_1-transitions are possible. We can consider the states with $m = 3$ as giving a triangulation of the sphere itself (all other states as disk-triangulations), the outside of the triangle being the cell containing the north pole on the sphere. One can interpret it as the closing up of the hole in the sphere (the external part of the complex).

Remark 2 *It is important to note that one could consider two other variants of this dynamics. First one is when we consider equivalence classes of cell surfaces. Then transition rates would be $\lambda_i m', i = 1, 2$, instead of $\lambda_i m$, where m' is m divided by the number of automorphisms of the disk triangulation. Second, we shall use its analog later in more complicated situations, is that there is a distinguished (rooted) edge on the boundary and transitions can occur only if they touch this edge.*

There are 3 cases with quite different behavior of this Markov process: sub-critical or ergodic, critical or null recurrent, supercritical or non-recurrent. For all these cases we shall study the behavior of local correlation functions and of the following global variables at time t:

$$F(t) = F(T(t)), V(t) = V(T(t)), L(t) = L(T(t)), m(t)$$

where L is the total number of edges and m is the number of edges on the boundary.

Subcritical case By definition it is the case when $\lambda_2 > \lambda_1$.

Theorem 6 *If $\lambda_2 > \lambda_1$ then*

$$\lim_{t\to\infty} P(m(t) = N) = \pi(m = N) \sim_{N\to\infty} CN^{-1}\exp(-\nu N), \nu = \ln\frac{\lambda_2}{\lambda_1}$$

Let τ_3 be the random number of jumps until first return to the state 3 and put $P(T) = P(T(\tau_3) = T)$ for any triangulation of the sphere with a distinguished face (outer face). Then

$$P(F(T) = N) \sim C\left(\frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}\right)^N$$

Proof. Note first that the length $m(t)$ of the boundary is itself a Markov process: the evolution of the boundary can be seen as the simplest (context free) random grammar with the alphabet consisting of one symbol 1 (representing one edge) and with the substitutions

$$\lambda_1 : 1 \to 11, \lambda_2 : 11 \to 1$$

This process is obviously reduced to the branching process with one particle type where λ_1 is the birth rate, λ_2 is the death rate. Denote this process $m(t)$ - it is a continuous time Markov process states of which are the points of the lattice interval $[3, \infty)$. It has jumps ± 1 and the corresponding rates $\lambda_1 k$ and $\lambda_2 k$ from the point $k \in [3, \infty]$. Its initial state is 3. The process starts anew with the point 3. The stationary measure for the process $m(t)$ is

$$\pi(m = k) = \frac{\prod_{i=3}^{k-1}\lambda_1 i}{\prod_{i=4}^{k}\lambda_2 i} \sim Ck^{-1}\exp(-\nu k), \nu = -\ln\frac{\lambda_1}{\lambda_2}$$

Remark 3 *Note that for dynamics with a local observer the same considerations show that*

$$\pi(m = k) = \frac{\prod_{i=3}^{k-1}\lambda_1}{\prod_{i=4}^{k}\lambda_2} \sim C\exp(-\nu k)$$

The exponents differ, as normal, by 1.

Consider the (discrete time) jump process $z_n = m(t_n)$ for $m(t)$, where $t_1 < t_2 < ...$ are the moments of jumps. Let $f(s) = \frac{1}{\lambda_1+\lambda_2}(\lambda_2 + \lambda_1 s^2)$ be the generating function for the transitions of the jump process and $f_n(s)$ be its iterates, let $n(\omega)$ be the first time when $m(t_n) = 3$. It is known that, see [46],

$$P(n(\omega) = N) = (f_N(0) - f_{N-1}(0)) \sim Cm^N, m = \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}$$

This gives the proof.

Supercritical case If $\lambda_1 > \lambda_2$ then the boundary has exponential growth. The array (V, L, F, m) behaves like a degenerate branching process with four particle types and the following rates

$$\lambda_1 m : V \to V + 1, L \to L + 2, F \to F + 1, m \to m + 1$$

$$\lambda_2 m : V \to V, L \to L + 2, F \to F + 1, m \to m - 1$$

Then.for some random variable ξ

$$m(t) \sim \xi\exp(\lambda_1 - \lambda_2)t, F(t) \sim c_1\xi\exp(\lambda_1 - \lambda_2)t, V(t) \sim c_2\xi\exp(\lambda_1 - \lambda_2)t,$$

$$L(t) \sim c_3\xi\exp(\lambda_1 - \lambda_2)t$$

taking into account the Euler formula $V - L + F = 1 + m$ and $L = \frac{3F}{2}$, the proof being the same as for similar statements in [44].

Critical case and the exponents If $\lambda_2 = \lambda_1$ then the process $m(t)$ is null-recurrent. One cannot speak about its stationary probabilities but there exists an infinite stationary measure. It has however the same exponent $\alpha = -1$.

We could consider a different analog of the stationary probabilities: we consider $P(T(\tau) = N)$, where $T(\tau)$ is the number of cells at the final moment τ (when the hole closes up).

Lemma 10 *Let $\lambda_1 = \lambda_2 = \lambda$. Then*

$$P(T(\tau_3) = N) \sim \frac{C}{N^2}$$

It is well-known, [46].

Remark 4 *The exponent is slightly different from the equilibrium case: $\alpha = -2 \neq -\frac{5}{2}$. But also the following phenomenon occurs. Exponents in ergodic and null-recurrent case are different: $\alpha = 0$ and $\alpha = -2$ correspondingly. That is the partition function for critical case converges and the limit of the partition functions as $\mu \to \mu_{cr}$ is infinite.*

Local correlation functions Even if the exponents are different from the physical theory it would be interesting to study local correlation functions, they define fluctuations of the curvature.

Denote by i the i-th appeared vertex. Put $q_i(t) = 0$ if i did not appear before time t and otherwise put $q_i(t)$ equal to the number of triangles (or edges) incident to i. Let $\tau(i)$ be the first time when the vertex i appeared.

Theorem 7 *There exist $\chi_k, \sum \chi_k = 1$, such that for $\lambda_1 > \lambda_2$*

$$\lim_{s\to\infty} \lim_{i\to\infty} P(q_i(\tau(i) + s) = k) \to \chi_k$$

For some constants $a, b > 0$ and any two vertices $i < j$ such that the initial distance $\rho(i, j)$ (that is the distance at time $\tau(j)$) between them equals d

$$|P(q_i(t) = k, q_j(t) = l) - \chi_k\chi_l| < b\exp(-ad)$$

for all i and t sufficiently large. If $\lambda_1 \geq \lambda_2$ then

$$\lim_{s\to\infty} \lim_{m\to\infty} P(q_i(\tau(i) + s|m(\tau(i)) = m) = k) \to \chi_k$$

Also the same exponential decay property holds. Moreover, all local correlation functions χ_k are analytic functions of λ_1, λ_2 for all values of $\lambda_1, \lambda_2 > 0$.

Proof. For the vertex i put $\xi_i(s) = q_i(\tau(i)+s)$. Note that $\xi_i(0) = 2$ and the number q_i will increase until both adjacent links to this vertex will not enter one new triangle, denote this random time σ_i, it has exponential distribution with the parameter λ_2. Thus χ_k is equal to the probability that the Poisson process with rate $2\lambda_1 + 2\lambda_2$ will have exactly $k - 3$ jumps, that is appending triangles to one or two vertices from the left (or from the right) of i. In fact, take vertex i and its two edges. Until this moment from both sides new triangles in i appear. The crucial argument is that this process is independent of m as far as during this time interval $m > 3$ and independent on all events which do not touch vertex i.

The second assertion of the theorem is proven quite similarly if we remark that the processes $q_i(t)$ and $q_j(t)$ become dependent only when the distance between i, j becomes less than 4. For this to occur there should be many λ_2-events in-between i and j, which has exponentially small probability.

Define the mean curvature

$$k = ER = \sum 2\pi \frac{6-q}{q} \chi_q$$

From the exponential decay it follows the central limit theorem for the scaled curvature

Theorem 8 *For any sequence of sets I of vertices*

$$\frac{\sum_{i \in I} R_i - k|I|}{\sqrt{|I|}}$$

converges to Gaussian distribution as $|I| \to \infty$.

5.2 Reversible boundary processes

The reasons why for this dynamics we did not get the desirable invariant measure on the complexes are rather delicate. We shall give now an intuitive explanation. One could expect a simple invariant measure for a process which is reversible. To get a reversible process we should add the possibility of deletion of faces. For example, let $\lambda_1 = \lambda_2 = \lambda$ and assume that each boundary triangle can be deleted with rate μ unless the resulting state does not belong to the class $\mathbf{A}$ of complexes.

Lemma 11 *This Markov process is reversible.*

Proof. Let α_i be complexes. We shall consider closed paths $\alpha_1, \alpha_2, ..., \alpha_n, \alpha_1$ where each α_{i+1} is obtained from α_i by appending or deleting a triangle on the boundary. The number of appending in such closed path $\alpha_1, \alpha_2, ..., \alpha_n, \alpha_1$ should be equal to the number of deletions, thus it is $\frac{n}{2}$. Denote λ_{ab} the rate of transitions from complex α to complex β. Thus

$$\lambda_{\alpha_1 \alpha_2} ... \lambda_{\alpha_n \alpha_1} = \lambda_{\alpha_1 \alpha_n} ... \lambda_{\alpha_2 \alpha_1} = (\lambda\mu)^{\frac{n}{2}}$$

There are however complexes with arbitrary N where no triangle can be deleted, because otherwise we get states, where two graphs intersect only in one vertex, which are not allowed. The set of such states has a sufficiently complicated nature and it is difficult to use standard procedure to get stationary probabilities via balance equations. However, we could split these graphs into two parts belonging to our class, then we can get splitting the complex into connected components. It is exactly the latter operation which brings us to nonlinear Markov processes.

6 Nonlinear boundary dynamics

In the preceding section we considered a Markov dynamics, the states were the complexes themselves. This dynamics was a local dynamics: the changes could occur at any point of the boundary. We saw that this did not give us exponents accepted in the physical literature. Here we will construct dynamics giving exactly the exponent $\alpha = -\frac{7}{2}$. This dynamics appears not a Markov process. One can think about infinite number of universes interacting with each other, but not like an infinite particle system. Simplest pairwise interaction (gluing) of two universes will give us quadratic functional equations and, as a result, the necessary exponents.

6.1 Quadratic quasi processes

Let Δ be the set of probability measures on some space S. We shall consider a class of transformations $M : \Delta \to \Delta$, generalization of Markov chains. These transformations are not generated by random maps $S \to S$, they are nonlinear on Δ.

Let a (denumerable) set S be given and let a Markov chain be defined on the state space S with transition probabilities $p_{\alpha\beta}, \sum_{\beta:\beta\neq\alpha} p_{\alpha\beta} = 1$, from α to β. It defines a linear transformation L

$$q = \{q_\beta\} \to Lq = \{\sum_\alpha q_\alpha p_{\alpha\beta}\}$$

on Δ. Let also a probability kernel $P((\alpha,\gamma) \to \beta) : S \times S \to S$ be given, $\sum_\beta P((\alpha,\gamma) \to \beta) = 1$. It is symmetric $P((\alpha,\gamma) \to \beta) = P((\gamma,\alpha) \to \beta)$ and $P((\alpha,\gamma) \to \gamma) = 0$. It can be deterministic. It defines the quadratic transformation Q on Δ

$$q = \{q_\alpha\} \to Qq = \{\sum_{\beta,\gamma} q_\beta q_\gamma P((\beta,\gamma) \to \alpha)\}$$

Then taking a convex combination we have the transformations (formally $c_0 + c_1 L + c_2 Q$) on the class of measures on S

$$q(\alpha, t+1) = r_1 \sum_\beta q(\beta,t) p_{\beta\alpha} + r_2 \sum_{\beta,\gamma} q(\beta,t) q(\gamma,t) P((\beta,\gamma) \to \alpha) + (1 - r_1 - r_2) c_0(\alpha) \tag{6}$$

for some probability measure $c_0(\alpha)$ and nonnegative numbers r_1, r_2 such that $0 \leq r_1 + r_2 \leq 1$.We see that the total mass $\sum_\alpha q(\alpha) = 1$ is conserved.

This can be interpreted as follows. Consider a denumerable number of particles on S, q_α is the mean number of particles at point α. With probability r_1 each particle (independently of the others) makes a jump according to the probabilities $p_{\alpha\beta}$. This gives the linear transformation. With probability r_2 particles form pairs so that mean number of pairs (α, β) is the product of means, then each pair (α,γ), independently of each other, gives birth to one particle at β with probability $P(\alpha,\gamma \to \beta)$. Also with probability $1 - r_1 - r_2$ one has an immigration with mean $c_0(\alpha)$ of particles to α. We want to emphasize that there is no stochastic process here in its standard sense but only the transformation of measures. This seems to be related also to field theory of strings (second quantization of strings) but this physical theory does not have a mathematical status.

Some probabilistic theory There is no theory of such quadratic quasi-processes and we have to discuss it here. We can rewrite it in more general terms

$$T = p_1 T_1 + p_2 T_2 + (1 - p_1 - p_2) T_3$$

Let k_1 be the contraction coefficient for T_1, that is for any probability measures μ_1, μ_2 we have $\|T_1(\mu_1 - \mu_2)\| \leq k_1 \|\mu_1 - \mu_2\|$. Let $k_2(x)$ be the contraction coefficient for the stochastic matrix $P_{yz}(x) = P((x,y) \to z)$ and $k_2 = \sup_x k_2(x)$.

Theorem 9 *Assume $p_1 k_1 + 2 p_2 k_2 < 1$. If the number of states is finite then there is exactly one fixed point of T and the convergence to it is exponentially fast. If the number of states is countable the same assertion holds under the condition that there exists a (Lyapounov) function $f(x)$ such that $\sum_x f(x) < \infty$ and if $\mu(x) \leq f(x)$ then also $(T\mu)(x) \leq f(x)$.*

Proof. Take two probability measures ν and $\mu = \nu + \varepsilon$. Then we have the following contraction property for $\rho(\nu,\mu) = \|\varepsilon\|$

$$\rho(T\mu, T\nu) = \left\| p_1 T_1 \varepsilon + p_2 \sum \mu(x)\varepsilon(y) P((x,y) \to z) + p_2 \sum \nu(x)\varepsilon(y) P((x,y) \to z) \right\|$$

$$\leq (p_1 k_1 + 2 p_2 k_2) \|\varepsilon\|$$

Then the first assertion of the theorem follows. To prove the second assertion note that by compactness there is a fixed point ν in $A = \{\mu : \mu(x) \leq f(x)\}$and for each μ_0 the sequence $T^n \mu_0$ converges to ν.

Continuous time quasi-processes are defined similarly. Instead of probabilities p we introduce rates λ: with rate λ_1 we do the linear transformation, with rate λ_2 the quadratic transformation, and immigration arrives with rate λ_0. The equations for the stationary measure are the following

$$(\lambda_1 + \lambda_2 + \lambda_0)\pi(\alpha) = \lambda_1 \sum_{\beta} \pi(\beta,t) p_{\beta\alpha} + \lambda_2 \sum_{\beta,\gamma} \pi(\beta,t)\pi(\gamma,t) P((\beta,\gamma) \to \alpha) + \lambda_0 c_0(\alpha)$$

and can be reduced to the previous case. The time evolution is governed by the following equation

$$\frac{dq(\alpha,t)}{dt} = \lambda_1 \sum_{\beta} (q(\beta,t) - q(\alpha,t)) p_{\beta\alpha} +$$

$$+\lambda_2 \sum_{\beta,\gamma} (q(\beta,t)q(\gamma,t) - q(\alpha,t)) P((\beta,\gamma) \to \alpha) + \lambda_0 (c_0(\alpha) - q(\alpha,t))$$

6.2 Generating functions

Now we consider the dynamics with the set S of all plane disk-triangulations with the root on the boundary and the projection of this dynamics onto Z_+^2, where the points of Z_+^2 are denoted $\alpha = (N,m)$, N is the number of faces (not outer) and m is the number of boundary edges. That is the measure of the point (N,m) is the sum of measures of the corresponding complexes. This projection will appear to be also a quadratic quasi-process. Introduce the generating function

$$U(x,y) = \sum_{N,m=0}^{\infty} q(N,m) x^N y^m$$

We assume homogeneity: for all $\alpha, \beta, \alpha_1, \beta_1, \gamma$ such that $\alpha, \beta, \alpha+\gamma, \beta+\gamma, \alpha_1, \beta_1, \alpha_1+\beta_1+\gamma$ all belong to the quarter plane

$$p_{\alpha,\alpha+\gamma} = a_\gamma, P((\alpha,\beta) \to \alpha+\beta+\gamma) = P((\alpha_1,\beta_1) \to \alpha_1+\beta_1+\gamma) = b_\gamma$$

together with the following bounded jumps assumptions: $p_{\alpha\beta} = 0$ if $|\alpha - \beta| > d$ for some fixed integer d, $P((\alpha,\beta) \to \gamma) = 0$ if $|\alpha+\beta-\gamma| > d$, $c_0(\alpha) \neq 0$ only for finite number of α. Introduce the generating functions

$$A(x,y) = \sum_{n,m} a_{(n,m)} x^n y^m, B(x,y) = \sum_{n,m} b_{(n,m)} x^n y^m, C(x,y) = \sum_{n,m} c_0(n,m) x^n y^m$$

Now we get a functional equation for the generating function of the stationary measure

$$U = r_1 U A(x,y) + r_2 U^2 B(x,y) + (1 - r_1 - r_2) c_0(x,y) + b.c.$$

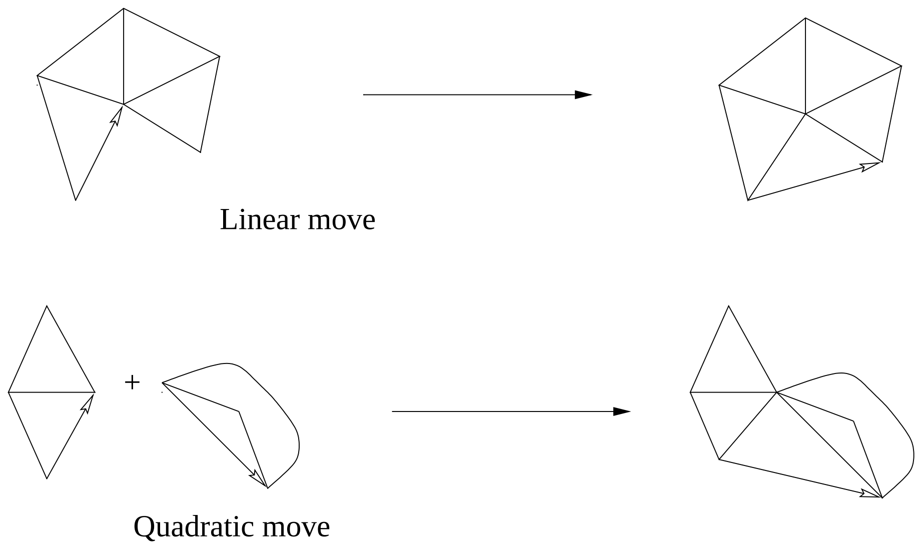

Figure 7: Linear and quadratic moves

where boundary terms can appear because there is no homogeneity in the vicinity of axes of Z_+^2. If $d=1$ then the boundary terms are linear combinations of functions $U(x,0)$ and $U(0,y)$ and finite number of $\pi(N,m)$.

Consider now the case where the jumps correspond to the moves shown on the Figure 7

Here the jumps are

$$p_{(N,m),(N+1,m-1)} = 1$$

$$P((N_1,m_1),(N_2,m_2)) \to (N_1+N_2+1, m_1+m_2-1) = 1, c_0((0,2)) = 1$$

To calculate boundary terms it is convenient to introduce a complex consisting of one edge with two vertices, it corresponds to $N=0, m=2$. Then c_0 is the unit measure on this complex. Gluing of two such complexes gives a triangle with $N=1, m=3$. From the line $m=2$ linear jumps are not possible, this is the only line where the homogeneity is destroyed. Thus the equation is

$$U = r_1 U x y^{-1} + r_2 U^2 x y^{-1} + (1-r_1-r_2) y^2 - x y^{-1} r_1 \sum_N q(N,2) x^N y^2$$

One can do the following scaling in the main equation

$$x \to x r_1^{-1}, q(N,m) \to a q(N,m)$$

If we put $a r_2 = 1, \beta = \frac{(1-r_1-r_2) r_2}{r_1}$ then it becomes

$$U = U x y^{-1} + U^2 x y^{-1} + \beta y^2 - x y^{-1} \sum_N q(N,2) x^N y^2$$

We call this equation canonical. Note that the case $\beta = 1$ or $r_1 = \frac{r_2(1-r_2)}{1+r_2}$ corresponds to the counting problem and was completely solved by Tutte. We only reproduce his analysis in a more general setting.

If there exists a solution U with nonnegative coefficients $q(N,m)$ of the canonical equation then the invariant measure $\widetilde{q}(N,m)$ for the dynamics will be $\widetilde{q}(N,m) = a^{-1} r_1^N q(N,m)$.

We shall be interested only in the class E of invariant measures satisfying the following assumptions:

1. (exponential bounds) $q(N,m) < C^{N+m}$ for some $C > 0$;

2. $q(N,m)$ are nonnegative

The main result is the following theorem.

Theorem 10 *A unique positive invariant measure exists for all r_1, r_2. It is finite iff $\frac{32}{27}\frac{2(1-r_1-r_2)r_1r_2}{27} < 1$. The measure of the sets $\{(N,m)\}$ for fixed m is finite iff $\frac{2(1-r_1-r_2)r_1r_2}{27} \leq 1$.*

In fact we have to prove the following

Lemma 12 *For any β there exists a unique solution of the canonical equation with positive coefficients in the considered class of measures. The series $\sum_N q(N,m) < \infty$ converges for each m iff $\beta \leq \frac{2}{27}$. The series $\sum_{N,m} q(N,m)$ converges iff $\frac{27}{32}\sqrt{\frac{2}{27\beta}} > 1$.*

Proof of the Theorem. One can easily prove that for any initial measure the difference of measures for any two complexes with the same (N,m) tends to zero. Thus, any invariant measure has the property that all complexes with the same (N,m) have the same invariant measure. Then we can use the above scaling argument. The theorem follows.

6.3 Analysis of the functional equation

Here we prove the Lemma. Putting $U = y^2 W$ and defining $S_m(x)$ by

$$W(x,y) = \sum_{m=2}^{\infty} S_{m-2}(x) y^{m-2}$$

we can rewrite the functional equation

$$W = \beta + xyW^2 + xy^{-1}(W - S), S(x) = S_0(x) = W(x,0) \tag{7}$$

In this case there is only one boundary term S (this simplifies strongly). This is the generating function for the number of disk-triangulations with rooted face, having exactly 2 boundary edges. In the general case the equations would be more complicated. It demands joining together the quadratic method of Tutte and the methods developed by the author for random walks in a quarter plane.

If we know $S_0(x)$ as a formal series then all S_m are defined recursively by

$$yS_0 = \beta y + yxS_1, y^2 S_1 = y^2 x S_0^2 + y^2 x S_2, ...$$

We partially follow the derivation in [12]. We rewrite the functional equation (7) in the form

$$(2xU + x - y)^2 = 4x^2y^2S + (x-y)^2 - 4\beta xy^3 = D \tag{8}$$

Consider the analytic set $\{(x,y) : 2xU + x - y = 0\}$ in a small neighborhood of $x = y = 0$. Note that it is not empty, $(0,0)$ belongs to this set and it defines a function $y(x) = y(x) = x + O(x^2)$ in the neighborhood of $x = 0$. We shall prove that $y(x)$ and $S(x)$ are algebraic functions. We have two equations valid at the points of this set

$$D = 0, \frac{\partial D}{\partial y} = 0$$

or

$$4x^2y^2S(x) + (x-y)^2 - 4\beta xy^3 = 0 \tag{9}$$

$$8x^2yS(x) - (x-y) - 12\beta xy^2 = 0$$

from where we shall get both $y(x)$ and $S(x)$

$$x = y(1-2\beta y^2), S = \frac{\beta(1-3\beta^2y^2)}{(1-2\beta y^2)^2}$$

The algebraic function $y(x)$ satisfies the equation $y^3 + py + q = 0$ with

$$p = -\frac{1}{2\beta}, q = \frac{x}{2\beta}$$

Its discriminant

$$\Delta = -4p^3 - 27q^2 = p^2(\frac{2}{\beta} - 27x^2)$$

is not a square in the field of rational functions. Then (see [49]) the Galois group is S_3 and the ramification points are $x = \pm\sqrt{\frac{2}{27\beta}}$. Cardano (formal) solution of the cubic equation is

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Note that $\left|\frac{q}{2}\right| \neq \left|\sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right|$ and thus there is no singularities inside the circle of radius $x_1 = \sqrt{\frac{2}{27\beta}}$. Two terms with opposite signs give the cancellation of the lowest order singularity $(x-x_1)^{\frac{1}{2}}$ and we have thus the leading singularity $(x-x_1)^{\frac{3}{2}}$. We need the branch where $y(0) = 0$, as $y(x_1) > 0$ then it is known that if the discriminant is zero then $y(x_1) = \sqrt[3]{\frac{x_1}{2\beta}}$. Then iterating the equation

$$y = \frac{x}{1-2\beta y^2}$$

we get that the expansion of $y(x)$ at $x = 0$ has all coefficients positive. S is an algebraic function, analytic for $|x| < \sqrt{\frac{2}{27\beta}}$. In fact, S could have a pole for $|x| < \sqrt{\frac{2}{27\beta}}$ only if $1 - 2\beta y^2(x) = 0$ but it would imply $x = 0$ which is impossible. To visualize the expansion of U denote now $y_0(x) = y(x)$ and substitute

$$S = \frac{\beta(1-3\beta^2y_0^2(x))}{(1-2\beta y_0^2(x))^2}, x = y_0(x)(1-2\beta y_0^2(x))$$

into (8). We get

$$\begin{aligned} D &= 4y^2y_0^2(1-3\beta^2y_0^2) + (y_0(x)(1-2\beta y_0^2(x)) - y)^2 - 4\beta y^3 y_0(x)(1-2\beta y_0^2(x)) \\ &= (y-y_0)^2(a(x) + b(x)y) \end{aligned}$$

as $y = y_0(x)$ is a double root of the main equation, and

$$a(x) = (1-2\beta y_0^2(x))^2, b(x) = -4\beta y_0(x)(1-2\beta y_0^2(x))$$

Choosing minus sign we have then

$$U(x,y) = \frac{y-x}{2x} - \frac{\sqrt{D}}{2x} == \frac{-x + y_0(x)}{2x} - (y - y_0(x))\frac{\sqrt{a(x)+b(x)y} - 1}{2x}$$

that gives a legitimate expansion.

For given x the convergence radius of U as the function of y is defined by zeros of $\sqrt{a(x)+b(x)y}$ or $\sqrt{1 - \frac{4\beta y_0^2}{x}y}$. As $\frac{y_0^2}{x}$ increases on the interval $[0, x_1]$ then the convergence radius for $x = x_1$ is $R = \frac{x_1}{4\beta y_0^2(x_1)}$. We have $R = \frac{27}{32}x_1$.

6.4 Correlation functions

Using the tree representation introduced below we prove that for most vertices the conditional distributions of the random variables q_v converge to the unique limit as $N \to \infty$. One should know how to specify a vertex v if they are not labelled. Below we give some way to do it, we discussed this problem in [43, 44] in a wider extent. The proof is combinatorial but without use of analytic methods, it is sufficiently involved and we present it not in a completely formal way.

6.4.1 Tree representation

The generation process of maps, given by recurrent application of Tutte moves, will be represented as a planar tree. Moreover, this will give a one-to-one correspondence between maps and some class of planar trees. A planar tree has a root vertex and grows upwards (it is shown on Figure 8 by arrows). Denote $\mathbf{A}_0$ the class of rooted disk-triangulations. We shall denote vertices of trees by v and vertices of maps by w.

Denote $\mathbf{T}_0$ the class of all planar trees characterized as follows. There can be 3 types of vertices: 0, 1, 2 according to how many edges go upwards from this vertex. Vertices of type 0 are also called end vertices. Denote n_i the number of vertices of type i. Among 2-vertices there are vertices which are incident to one 0-vertex, let their number be n_{20} and which have two incident 0-vertices, their number is denoted by n_{00}. The only further restriction on this class of trees is the following. For any vertex v denote T_v the tree consisting of the vertex v and all vertices above v. Denote $n_i(v)$ etc. - the corresponding numbers for the tree T_v. Note that $n_0(v) = n_2(v) + 1$. Then the class $\mathbf{T}_0$ is characterized by the following restriction: $n_0(v) - n_1(v) - 1 \geq 0$ for all vertices v of type 1.

Lemma 13 *There is a one-to-one correspondence between $\mathbf{A}_0$ and $\mathbf{T}_0$.*

Show first that each map generates a planar tree in $\mathbf{T}_0$, that is there is a function f from maps to planar trees. We prove this by induction on the number of faces. The map itself is represented by the root vertex of the tree. From the root vertex we draw upwards one edge in case of move 1 of Tutte, and two edges, in case of move 2, corresponding to splitting the map on two maps. We can distinguish the latter maps corresponding to the right or left vertices of the tree accordingly to the orientation of the rooted edge of the map. In fact, the rooted edges of the two maps, on which the map is split, have the same orientation. Thus one of then precedes the other one. The map with the preceding rooted edge we consider as corresponding to the left vertex (with smaller numbers) and the map with subsequent rooted edge - to the right vertex. Each step reduces the number of faces by 1, thus we come to induction hypothesis. If the map is the edge map (consisting of one edge), then the corresponding vertex is the end vertex of the tree.

Let us note that $N = n_1 + n_2, L = n_0 + n_1 + n_2, V = n_0 + 1, m = n_0 - n_1 + 1$. Here N, L, m, V refer to the map, and numbers n_i, n_{00} etc. - to the tree. To prove the latter equality note that each 1-vertex and each 2-vertex (except those which are incident to 0-vertices), when passed downwards, diminish m on one, each of n_{20} vertices gives one more edge to m and each of n_{00} vertices gives three more edges to m. Thus $m = n_{20} + 3n_{00} - n_1 - (n_2 - n_{00} - n_{20}) = n_0 - n_1 + 1$. As $m \geq 2$ for each map, we have the restriction.

Now show that each planar tree of class $\mathbf{T}_0$ generates a map - we shall show that f is one-to-one. Take a tree and proceed by induction from upper to lower vertices. All end vertices we declare to be edge maps. Take V end edges, enumerate them from left to right as the end vertices of the tree $v = 1, 2, ..., V$. These edges will give $V + 1$ vertices in the complex. We mark all end vertices. Each induction step we take a vertex v such that there are only marked vertices above it. If v is of type i then the induction step consists of Tutte move i. We mark v after this step. Then we proceed by

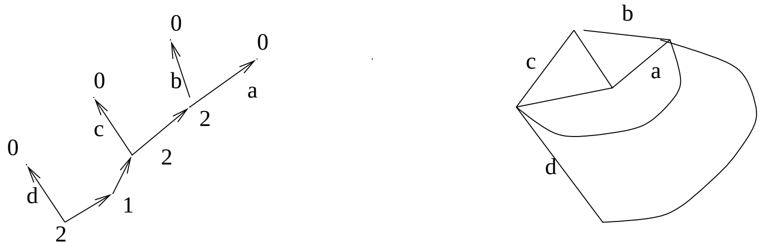

Figure 8: Planar trees and maps

induction. Inversely, the map constructed in this way generates the tree from which we started. All maps are legitimate because $m \geq 2$. Lemma is proved.

Contribution of the tree $G(T)$ is defined as the product $r_0^{n_0} r_1^{n_1} r_2^{n_2}$ where $n_i = n_i(T)$ is the number of i-type vertices. This is equal of course to the probability of the corresponding rooted map.

Remark 5 *Planar trees are in one-to-one correspondence with the parenthesis systems, which can be put the product $a_1 ... a_n$ in non associative non commutative algebra, see Figure 9. However, the restrictions posed on $\mathbf{T}_0$ make this one-dimensional grammar more involved.*

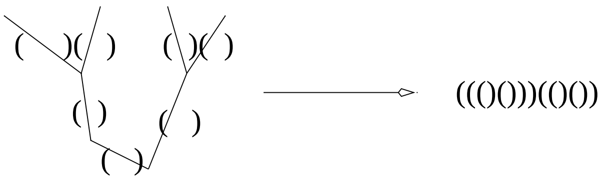

Figure 9: Planar trees and parenthesis

6.4.2 Local curvature

Take some end vertex (or end edge) v and put $l(v) = \max(v, V - v)$.

Theorem 11 *If $l(v) \to \infty$ (necessarily $V \to \infty$ and $N \to \infty$) then for the equilibrium distribution P there exists the limit*

$$\lim P(q_v = k) = p(k), \sum_{k=2}^{\infty} p(k) = 1$$

Proof. One can imagine V edge maps, corresponding to the end vertices, put horizontally along the line and directed from left to right. Take some v and the left (that is the rooted) vertex $w = w(v)$ of the corresponding edge map. We fix orientation of the boundary to be counterclockwise. Define the history of each vertex $w(v)$. This history can be described in terms of maps and in terms of the tree, it will be convenient to use both descriptions. By definition it consists of several parts, that we call history parts, of the unique path from v to the root of the tree in the tree T in the downward direction. In the sequence of maps the history ends either when the vertex $w(v)$ disappears from the boundary of the map (this means that q_w will not change anymore) or in the root of the tree, if w is on the boundary of the final map. The first part of the history lasts until the vertex will be covered by an edge. q_w may change only on such history parts. From the tree point of view the first part of the path starts with the end vertex v, goes downwards vertex by vertex, and lasts until the first right 2-vertex.We say that the 2-vertex is a right vertex if our vertex $w(v)$ is in the right map. Next part of the history starts when $w(v)$ becomes to belong to the rooted edge of the map and ends exactly as in the first part.

If the vertex a of the tree is above $v = w(v)$ then w belongs to the corresponding map $f^{-1}(a)$. Denote $a(w)$ the infimum of a such that $a < v$ and w is on the boundary of $f^{-1}(a)$.

Denote $b(w)$ the infimum of b such that $b < v$ and w is still a rooted vertex in the corresponding complex for each b_i on the interval $v > b_1 > b_2 > ... > b$ in the tree. The length $L(v(w))$ of the latter path in the tree is exactly the difference $q_w(j) - q_w(k)$ where j is the first complex of the path (edge map), k - the last one. We shall prove that $P_V(L(v(w)) = n) < C\exp(-\gamma n)$ for some $\gamma, C > 0$. Note that we can use for this any invariant measure as this conditional probability is the same for fixed N, m (in fact we are interested only in $m = 3$).

If from the tree one deletes all 1-vertices then the resulting tree without 1-vertices will be called a bare tree, It defines an equivalence class of dressed trees, each of them can be obtained by appending some number of 1-vertices to the bare tree. Each nonnegative measure on trees induces a measure on equivalence classes - bare trees. We start with bare trees.

Case of bare trees It is the case when there are no 1-vertices at all, that is $m = n_0 + 1 = N$. In this case all vertices are on the boundary, and once the vertex was covered it does not participate in the process anymore. It can be covered only when a left join occurs and until it was covered there can be k right joins which give k extra edges to q_w. Any right join joins to w a complex corresponding to some tree covering an interval to the right of v.

For example, let us estimate the probability $P(k; j_1, ..., j_k)$ that there are exactly k joins to the intervals of lengths $j_1, ..., j_k$ covering the interval $I_r = \{v+1, ..., v+R\}$ before the left join covering the interval $I_l = \{v-L, ..., v-1\}$. We shall use the following Markov property. Call a vertex $v' = v(I)$ I-separating if it covers exactly the interval I. Then the number of trees with the separating vertex v' is equal to $t(I)t(V - I + v')$ where $t(I)$ is the number of trees on the interval I, because $t(V - I + v')$ is also the number of factor trees with respect to the set of trees on I. Thus the probability that v' is I-separating can be estimated as

$$\frac{t(I)t(V - I + v')}{t(V)} \sim \min(b, cI^{-\frac{3}{2}})$$

as $t(n+1) = \frac{1}{n+1} C_{2n}^{n} \sim cn^{-\frac{3}{2}} 2^n$ are well-known Catalan numbers and $b < 1$. It follows that the probability P_v that the vertex v will have a right join earlier than a left join. It is clear that $P_v < d < 1$. Then using the Markov property we shall get by induction for large r and some $a < 1$

$$P(q_v \geq r) < a^r$$

Thus we got the exponential estimates. The existence of the thermodynamic limit follows from the fact that the influence of the boundaries takes place also with probabilities less than $a^{l(v)}$.

Case $n_1 > 0$. We fix a bare tree and consider one auxiliary problem (urn problem) concerning the distribution of 1-vertices on the bare tree. The estimates are uniform in bare trees (equivalence classes).

Consider first the probability that the vertex w will get large value of q_w due to 1-vertices until it will covered at the first time. We shall do such estimates separately for each history part and consider in detail only the first history part. Let v be the vertex where the first parts ends. Let T_v be the tree over this vertex (with the root v) and $f^{-1}(v)$ be the corresponding complex.

First consider the case when T_v has only one 00-vertex, then 0-vertices join sequentially to already existing trees. It means that different ways to put 1-vertices to the bare tree can be identified with all possible ordered arrays of nonnegative integers $a_1, ..., a_n$ such that for all $k = 1, 2, ..., n$ we have $\sum_{i=1}^{k} a_i \leq k - 2$.

We can formulate the following abstract urn model. Let we have n urns and m balls in these urns, a_i - the number of balls in the urn i. Let $c(n, m)$ be the number of arrays $a_1, ..., a_n$ such that $\sum_{i=1}^{k} a_i \leq k, \sum_{i=1}^{m} a_i = m \leq n$. Then we have the following recurrence

$$c(n, m) = \sum_{k=0}^{m} c(n-1, m-k)$$

or

$$c(n, m) = c(n, m-1) + c(n-1, m-1)$$

from where it is not difficult to get asymptotics for the number $c(n, m; i) = c(n-1, m-i)$ of arrays among $c(n, m)$ such that there are exactly i balls in the last urn. Then we have an explicit formula for the generating function

$$f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} c(n, m) x^n y^m = \frac{y}{1 - y(1+x)} \sum_{k=1}^{\infty} k x^k$$

the coefficients coincide with ours for $n \geq m$. We want to prove that $\frac{c(n,m)}{c(n,m-1)} < b$ for some $b < 1$, and to find a method (not using generating functions) which could work in the general situation. For this we rewrite $c(n, m)$ in terms of the number of paths starting at the line $m = 1$ and ending at the point (n, m). Steps of the paths are either $(0, 1)$ or $(1, 1)$. We have

$$c(n, m) = \sum_{k=m+1}^{n} k L(k; n, m)$$

where $L(k; n, m)$ - the number of paths from the point $(k, 1)$ to the point (n, m), as $c(k, 1) = k$. As $L(k; n, m-1) = L(k-1; n-1, m-1)$, then $\frac{c(n,m-1)}{c(n-1,m-1)} = 1 + O(\frac{1}{m})$. The result follows.

Consider now the general case. Instead of the urn problem on the interval $[1, n]$ we have ah urn problem on an arbitrary planar tree T under the conditions

$$\sum_{i \in T_v} a_v \leq V(T_v), \sum_{v \in T} a_v = m$$

where a_v is the number of balls in the urn (vertex) v of the tree, $V(T_v)$ is the number of vertices of the tree T_v. Let $c(T_v, m)$ be the number of such arrays on the tree T_v with m balls. If for example from the vertex v only two edges go upwards to the vertices $v(1), v(2)$, then

$$c(T_v, m) = \sum_{i=0}^{m} c(T_v, m; i), c(T_v, m; i) = \sum_{m_1 + m_2 = m - i} c(T_{v(1)}, m_1) c(T_{v(2)}, m_2)$$

Then the argument is similar to the previous one. Note that $c(T_v, 1) = V(T_v)$. We want to compare $c(T_v, m; i)$ and $c(T_v, m; i+1)$, for this we iterate the latter recurrent equation for $c(T_v, m; i+1)$ to the very end, that is we get the sum of terms B_s^{i+1}, in each of them all factors equal $c(T_{v'}, 1)$ for some v'. The iteration process for $c(T_v, m; i+1)$ there corresponds the similar process for $c(T_v, m; i)$, that is why to each term B_s^{i+1} there corresponds the term B_s^i in the expansion for $c(T_v, m; i)$. In that term one of the factors is $c(T_{v'}, 2)$ instead of the factor $c(T_{v'}, 1)$ in the term B_s^{i+1}. Thus as before $\frac{c(T_v,m)}{c(T_v,m-1)} \gtrsim 2$. From this bounds uniform in bare trees follow. The influence of the boundary is exponentially small.

Similarly one can estimate other correlation functions, for example, the decay of correlations.

Theorem 12 *Let $V \to \infty$ and take two vertices v_1, v_2 with $l(v_i) \to \infty$. Then*

$$|\langle q_{v_1} q_{v_2} \rangle - \langle q_{v_1} \rangle \langle q_{v_2} \rangle| < c \exp(-\alpha |v_1 - v_2|)$$

7 Internal dynamics

We considered above only a growth of the boundary, that was quite natural: many modern technologies follow this principle. But also another dynamics is possible where all cells (even inside the building) can evolve. We shall consider here some questions related to such dynamics.

Note that Gross-Varsted moves can be used not only for simplicial complexes but for other classes as well, as it is seen from the picture. Consider GV-moves 1 and 2 and the inverse one to 2, consider the Markov chain with rates $\lambda_i, i = 1, 2, \mu$ for these moves correspondingly.

Thermodynamic limit of local processes If $\lambda_2 = \mu = 0$ then V, L, N are invariants. Let $A' \subset A(N, L)$ be an irreducible component of the set of (nonequivalent) complexes with given N and L and $C(A') = |A'|$. We make an assumption that a move can only be done if it gives non-equivalent complex. We formulate the following lemma without proof.

Lemma 14 *If $\lambda_2 = \mu = 0$ then the Markov chain on each A' is reversible with respect to the uniform measure. On the class of simplicial complexes this component coincides with the whole class.*

Proof. Reversibility is verified via the condition $\pi_\alpha \lambda_{\alpha\beta} = \frac{\lambda}{C(A')} = \pi_\beta \lambda_{\beta\alpha}$ if $\lambda_{\beta\alpha} = \lambda_{\alpha\beta} = \lambda$.

The following example shows that large time and large N limits are not interchangeable, that is

$$\lim_{t\to\infty} \lim_{N\to\infty} \neq \lim_{N\to\infty} \lim_{t\to\infty}$$

for local quantities. This the simulation is slow and dangerous in this case. Consider the sequence of such chains $\xi^{(N)}(t)$ having the embedded state spaces

$$... \subset A^N \subset A^{N+1} \subset ...$$

Take a vertex v at time 0 and consider random variables $q_v^{(N)}(t)$ - number of edges at v at time t. We have $L = \frac{3N}{2}, V = \frac{N+4}{2}$ and it could be natural to think that $q_v \approx \frac{L}{V} \to 3$. But the following argument shows more complicated situation.

Lemma 15 *Consider the class of simplicial complexes. As $N \to \infty$ the limiting process exists and is the random walk on $[3, \infty)$ with transition rates $\lambda_{i,i+1} = \lambda_{i,i-1} = \lambda i$. Thus $P(q_v^{(N)}(t) = k) \to_{N\to\infty} 0$.*

Proof. For fixed N the process $q_v^{(N)}(t)$ is Markov with state space $3, ..., N$ with rates $\lambda_{i,i+1} = \lambda_{i,i-1} = \lambda i$. In fact, each edge incident to v can be changed to a transversal and, for each triangle containing v, its edge not containing v can be erased by GV-move, this will give one more incident edge. The limiting random walk is null recurrent and thus big fluctuations in it occur until it reaches equilibrium for fixed N.

Similar proof does not hold for other classes of complexes.

Now consider Markov chains where the only transitions are A-moves. To get ergodic chains we change the generator which produces jumps. Now the jumps are produced by any vertex i with rates λ or μ. For fixed i with rate λ take randomly (that is with probability q_i^{-1}) one of the edges on the boundary of $St(i)$ and do the A-move corresponding to this edge. Let μ be the rate of the inverse A-move at vertex i, also for each possible vertex v of degree 4 on $\partial St(i)$ with equal probability we take one pair of triangles (on the right hand side of the A-move) and do the inverse A-move. Once the vertex i appeared it can disappear afterwards. Let $t(i)$ the time when vertex i appeared.

Theorem 13 *If $\lambda > \mu$ then $q_i(t) \to \infty$ with positive probability. If $\lambda < \mu$ then the vertex disappears a.s. and $Eq_i(t)$ is uniformly bounded.*

Proof. Let for each vertex v $a(t) = a_v(t)$ be number of vertices j on $\partial St(v)$ with $q_j = 4$, let $b(t) = q_v(t) - a(t)$. Fix vertex i. If $v \in \partial St(i)$ then $v+1$ is the next vertex on $\partial St(i)$ in the clockwise direction.

Consider the process $(a_i(t), b_i(t))$ and for fixed configuration outside $St(i)$ write down its infinitesimal jumps in Z_+^2. For a direct and inverse A-move there can be only three possibilities:

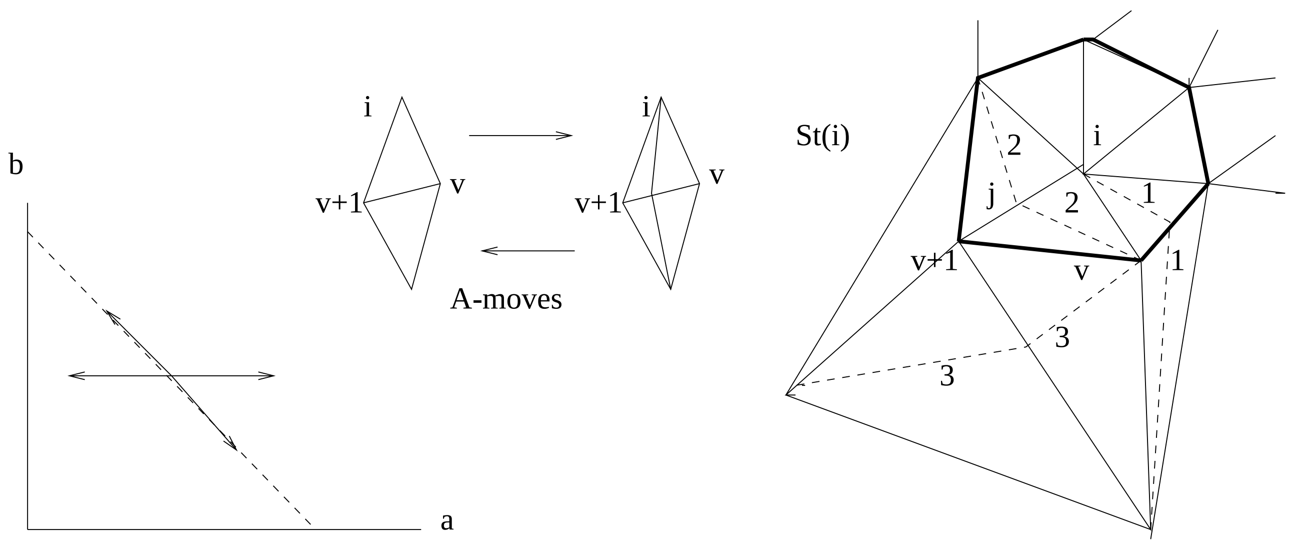

Figure 10: Proof of the theorem

1. Two edges (marked 1 on the figure) appear on some link ($(v-1, v)$ on the figure). Thus here the transition is $a, b \to a+1, b$ with rate $2\lambda \frac{q}{q} = 2\lambda$. Here and further factor 2 because the same move can be produced also by the opposite vertex. Inverse move $a, b \to a, b+1$ with rate $2\mu \frac{a}{a} = 2\mu$;

2. This move is produced by vertex $v \in \partial St(i)$ (dotted edges 2 on the figure), the new vertex appears on the edge $(i, v+1)$. It produces a change in the vector (a, b) only if $q_{v+1} \neq 4$. Thus here $a, b \to a+1, b-1$ with rate $2\lambda a \frac{b}{q_v}$. Inverse move gives the jump $a, b \to a-1, b$ with rate $2\alpha\mu \frac{a}{a_v}$.

3. Next move is also produced by vertex v (edges 3 on the figure). q_v can be transformed $4 \to 3, 5 \to 4$.

In fact we do not need rates for 2 and 3: note only that these jumps conserve $q_i = a + b$. Assume first $\mu > \lambda$. Then the embedded process $f(n) = q_i(t_n) = a(t_n) + b(t_n)$, where t_n are the jump moments, satisfies the following inequality

$$E(f(n+1) \mid f(n)) < f(n) - \varepsilon$$

for some fixed $\varepsilon > 0$. By the submartingale techniques (see, for example, [48]) we have the proof. In the opposite case we have

$$E(f(n+1) \mid f(n)) > f(n) + \varepsilon$$

and again the techniques of [48] works.

It seems plausible that if $\lambda < \mu$ then for all sequences $t(i) \to \infty$ the process $q_i(t)$ tends to some proper distribution if $s = t - t(i)$ is fixed. If $\lambda \ll \mu$ it can be proved. On the contrary for the critical case $\mu = \lambda$ random variables $q_i(t)$ fluctuates as for the Brownian motion. Compared with the results in the previous section this gives argument that we do not get the physical invariant measure here.

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