Stability of Propagating Fronts in Damped Hyperbolic Equations

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Abstract. – *We consider the damped hyperbolic equation*

$$\varepsilon u_{tt} + u_t = u_{xx} + F(u) , \quad x \in \mathbf{R} , \quad t \geq 0 , \quad u \in \mathbf{R} ,$$

where ε is a positive, not necessarily small parameter. We assume that $F(0) = F(1) = 0$ and that F is concave on the interval $[0,1]$. Under these assumptions, our equation has a continuous family of monotone propagating fronts (or travelling waves) indexed by the speed parameter $c \geq c_*$. Using energy estimates, we first show that the travelling waves are locally stable with respect to perturbations in a weighted Sobolev space. Then, under additional assumptions on the non-linearity, we obtain global stability results using a suitable version of the hyperbolic Maximum Principle. Finally, in the critical case $c = c_*$, we use self-similar variables to compute the exact asymptotic behavior of the perturbations as $t \to +\infty$. In particular, setting $\varepsilon = 0$, we recover several stability results for the travelling waves of the corresponding parabolic equation.

Keywords : damped hyperbolic equations, travelling waves, stability, asymptotic behavior, self-similar variables

AMS classification codes (1991) : 35B40, 35B35, 35B30, 35L05, 35C20

1. Introduction

Mathematical models for spreading and interacting particles or individuals are very common in chemistry and biology, especially in genetics and population dynamics. If the spatial spread of the particles is described by Brownian motion, these models usually take the form of reaction-diffusion equations or systems for the population densities [10,27]. Depending on the precise form of the interaction, such systems exhibit interesting solutions like propagating fronts or travelling waves, whose existence and stability properties have attracted a lot of attention in recent years [32].

It can be argued, however, that diffusion is not a realistic model of spatial spread for short times, because particles performing Brownian motion can move with arbitrarily high speed, and the directions of motion at successive times are uncorrelated. This drawback can be eliminated by replacing the diffusion process with a velocity jump process which is more satisfactory for short times and has equivalent long-time properties [17,22,31]. In one space dimension, this procedure leads to damped wave equations instead of reaction-diffusion systems [6,19,20,33]. Under the same assumptions as in the parabolic case, the damped hyperbolic equations also have travelling wave solutions [18] with analogous stability properties [14,16]. The aim of this paper is to review some of these stability results in the simplest case of a scalar equation with a non-linearity of "monostable" type.

We thus consider the damped hyperbolic equation

$$\varepsilon u_{tt} + u_t \;=\; u_{yy} + F(u) \;, \tag{1.1}$$

where $y \in \mathbf{R}$, $t \in \mathbf{R}_+$ and ε is a positive, *not necessarily small* parameter. We assume that the non-linearity $F \in \mathcal{C}^2(\mathbf{R},\mathbf{R})$ has the following properties:

(H1) $\quad F(0) = F(1) = 0 \;, \quad F'(0) > 0 \;, \quad F'(1) < 0 \;, \quad F''(u) < 0 \;$ for $u \in (0,1]$.

In particular, $u \equiv 1$ is a stable equilibrium point of Eq.(1.1), and $u \equiv 0$ is unstable. For simplicity, we assume F being concave on $[0,1]$, but this condition is more restrictive than what we really need and could be relaxed in several ways. A typical non-linearity satisfying (H1) is $F(u) = u - u^m$, with $m \in \mathbf{N}$, $m \geq 2$.

Under the assumptions (H1), Eq.(1.1) has monotone travelling wave solutions (also called propagating fronts) connecting the equilibrium states $u = 1$ and $u = 0$. Indeed, setting

$$u(y,t) \;=\; h(\sqrt{1+\varepsilon c^2}\,y - ct) \;\equiv\; h(x) \;, \tag{1.2}$$

where $c > 0$, we obtain for the function h the ordinary differential equation

$$h''(x) + ch'(x) + F(h(x)) \;=\; 0 \;, \qquad x \in \mathbf{R} \;. \tag{1.3}$$

As is well-known [1,26], Eq.(1.3) has a solution satisfying $h'(x) < 0$ for all $x \in \mathbf{R}$, $h(-\infty) = 1$, $h(+\infty) = 0$ if and only if $c \geq c_* = 2\sqrt{F'(0)}$. This solution is unique up to a translation in the variable x. Therefore, for all $\varepsilon > 0$, Eq.(1.1) has a continuous family of monotone travelling waves indexed by the speed parameter $c \geq c_*$. Note that the actual speed of the wave is not c, but $c/\sqrt{1+\varepsilon c^2}$, a quantity which is bounded by $1/\sqrt{\varepsilon}$ for all $c \geq c_*$.

In the limit $\varepsilon \to 0$, Eq.(1.1) reduces to the semilinear parabolic equation $u_t = u_{yy} + F(u)$, which has been intensively studied since the pioneering work of Fisher [11] and Kolmogorov, Petrovskii and Piskunov [26]. Using the parabolic Maximum Principle and probabilistic techniques, the long-time behavior of a large class of solutions has been explicitly determined [1,2]. In a more general context, a local stability analysis of the travelling waves using functional-analytic methods has been initiated by Sattinger [30] and extended by many authors [7,23,25]. In particular, the decay rate in time of the perturbations in the critical case $c = c_*$ has been computed using renormalization techniques [3,13]. Since the equilibrium state $u = 0$ ahead of the front is linearly unstable, all these stability properties are restricted to perturbations which decay to zero at least as fast as h itself as $x \to +\infty$. Following [14,16], the aim of this paper is to show how these results can be extended to the hyperbolic case $\varepsilon > 0$.

To investigate the stability of the travelling wave (1.2) as a solution of (1.1), we go to a moving frame using the change of variables

$$u(y,t) \;=\; v(\sqrt{1+\varepsilon c^2}y - ct, t) \equiv v(x,t) \;,$$

where $x = \sqrt{1+\varepsilon c^2}y - ct$. The equation for v is

$$\varepsilon v_{tt} + v_t - 2\varepsilon c v_{xt} \;=\; v_{xx} + cv_x + F(v) \;, \tag{1.4}$$

and, by construction, $h(x)$ is a stationary solution of (1.4). Setting $v(x,t) = h(x) + w(x,t)$, we obtain for the perturbation w the equation

$$\varepsilon w_{tt} + w_t - 2\varepsilon c w_{xt} \;=\; w_{xx} + cw_x + F'(h)w + \mathcal{N}(h,w)w^2 \;, \tag{1.5}$$

where

$$\mathcal{N}(h,w) \;=\; \int_0^1 (1-\sigma)F''(h+\sigma w)\,\mathrm{d}\sigma\ .$$

Rewriting in the usual way the second order equation (1.5) as a first order system for the pair (w, w_t), we shall study the stability of the origin $(w, w_t) = (0,0)$ in an weighted Sobolev space Z which we now describe.

For $k \in \mathbf{N}$, we denote by $\mathrm{H}^k = \mathrm{H}^k(\mathbf{R})$ the usual (real) Sobolev space of order k over $\mathbf{R}$, with $\mathrm{H}^0(\mathbf{R}) = \mathrm{L}^2(\mathbf{R})$. Following [30], we introduce for $s > 0$ the weight function $p_s(x) = 1 + \mathrm{e}^{sx}$ and, for $k \in \mathbf{N}$, we denote by H^k_s the Hilbert space $\mathrm{H}^k(\mathbf{R}, p_s^2\,\mathrm{d}x)$ defined by the norm

$$\|u\|^2_{\mathrm{H}^k_s} \;=\; \int_{\mathbf{R}} \left(\sum_{i=0}^{k} |\partial_x^i u(x)|^2\right) p_s(x)^2\,\mathrm{d}x\ . \tag{1.6}$$

Setting $\mathrm{L}^2_s = \mathrm{H}^0_s$, we define the product space $Z = \mathrm{H}^1_s \times \mathrm{L}^2_s$ equipped with the norm

$$\|(w_1,w_2)\|^2_Z \;=\; \|w_1\|^2_{\mathrm{H}^1_s} + \|w_2\|^2_{\mathrm{L}^2_s}\ . \tag{1.7}$$

Finally, it will be convenient to denote by Z_ε the space Z equipped with the ε-dependent norm

$$\|(w_1,w_2)\|^2_{Z_\varepsilon} \;=\; \|w_1\|^2_{\mathrm{H}^1_s} + \varepsilon\|w_2\|^2_{\mathrm{L}^2_s}\ . \tag{1.8}$$

The perturbation space Z clearly depends on the choice of $s > 0$ and becomes smaller when s increases. It is thus natural to look for the the smallest value of $s > 0$ for which the origin in (1.5) is linearly stable in Z. This is most conveniently done by setting $w(x,t) = \mathrm{e}^{-sx}\omega(x,t)$ and studying the equation for ω, namely:

$$\begin{aligned} &\varepsilon\omega_{tt} + (1+2\varepsilon cs)\omega_t - 2\varepsilon c\omega_{xt} \;= \\ &\qquad \omega_{xx} + (c-2s)\omega_x + (F'(h) - cs + s^2)\omega + \mathcal{N}(h, \mathrm{e}^{-sx}\omega)\mathrm{e}^{-sx}\omega^2\ . \end{aligned} \tag{1.9}$$

A straightforward computation in the Fourier variables shows that the origin in (1.9) is linearly stable in $\mathrm{H}^1 \times \mathrm{L}^2$ only if $F'(0) - cs + s^2 \le 0$. In fact, this condition can easily be inferred from the coefficient of ω in (1.9). Therefore, the largest perturbation space Z for which we can expect stability of the front h is obtained by choosing

$$s \;=\; \frac{1}{2}(c - \sqrt{c^2 - c_*^2})\ . \tag{1.10}$$

Note that this value corresponds to the decay rate of h as $x \to +\infty$, since $h(x) \sim \mathrm{e}^{-sx}$ if $c > c_*$ and $h(x) \sim x\mathrm{e}^{-sx}$ if $c = c_*$ [1]. Thus, if (1.10) holds, the translations $h(x+x_0)-h(x)$ do not belong to the perturbation space H^1_s. In view of the translation invariance of Eq.(1.1), this is of course necessary to obtain an asymptotic stability result. In the sequel, we always assume that the condition (1.10) holds.

In this paper, we present three different results which show that the travelling wave h is stable with respect to perturbations in Z_ε or in a subspace of it. In Section 2, we give a local stability result valid for all $c \geq c_*$ and all $\varepsilon > 0$. Using appropriate energy functionals, we show that, if $(w(0), w_t(0))$ is sufficiently small in Z_ε, then the solution $(w(t), w_t(t))$ of (1.5) stays in a neighborhood of the origin in Z_ε and converges to zero as $t \to +\infty$ in a slightly weaker norm. In the critical case $c = c_*$, we also give an estimate of the convergence rate. These results have been proved in [14] in the particular case where $F(u) = u - u^2$. Their proofs can be adapted, with minor changes, to cover the general case of a non-linearity F satisfying (H1).

Section 3 is devoted to global stability results. We first recall the Maximum Principle for hyperbolic equations [29] in a version adapted to our problem. Then, under the additional assumption that $F'(u)$ be strictly negative for $u \geq 1$, we show that the travelling wave h is stable with respect to "large" perturbations in Z_ε, provided some positivity conditions are satisfied. Furthermore, if $1 + 4\varepsilon F'(1) \geq 0$ and $F''(u) \leq 0$ for $u \geq 0$, we obtain linear upper and lower bounds for the solutions of Eq.(1.5), as well as a decay rate in time for the quantity $\|p_s w(t)\|_{\mathrm{L}^\infty}$. The proofs of these results can also be found in [14] if $F(u) = u - u^2$.

In Section 4, we restrict our analysis to the critical case $c = c_*$, and we study the long-time behavior of the solutions (w, w_t) of (1.5) in a slightly smaller function space. In particular, we show that

$$w(x,t) \;=\; \frac{\alpha}{t^{3/2}}\, h'(x)\varphi^*\left(\frac{x\sqrt{1+\varepsilon c_*^2}}{\sqrt{t}}\right) + o(t^{-3/2})\;, \quad t \to +\infty\;,$$

where $\alpha \in \mathbf{R}$ and $\varphi^* : \mathbf{R} \to \mathbf{R}$ is a *universal* profile. In the parabolic case $\varepsilon = 0$, this asymptotic expansion has been obtained by Gallay [13] using the renormalization group method [5] combined with resolvent estimates. We follow here a simpler and fairly different approach based on self-similar variables and energy estimates only. We refer to [16] for the detailed proof.

To conclude this introduction, we would like to point out the striking similarity between the stability results presented here and the corresponding statements in the parabolic case. This is an illustration of the more general fact that the long-time behavior of solutions to damped hyperbolic equations such as (1.1) is essentially parabolic [15]. A similar phenomenon can be observed in the context of hyperbolic conservation laws with damping [21,28].

2. Local Stability of the Travelling Waves

In this section, we show that the travelling wave h is stable with respect to sufficiently small perturbations in the space Z_ε. Our main result is:

Theorem 2.1. *Assume that (H1) holds, and let $\varepsilon_0 > 0$, $c \ge c_*$. Then there exist constants $\delta_0 > 0$ and $K_0 \ge 1$ such that, for all $0 < \varepsilon \le \varepsilon_0$, the following result holds: for all $(\varphi_0, \varphi_1) \in Z_\varepsilon$ such that $\|(\varphi_0,\varphi_1)\|_{Z_\varepsilon} \le \delta_0$, there exists a unique solution $(w, w_t) \in \mathcal{C}^0([0,\infty), Z_\varepsilon)$ of (1.5) with initial data $(w(0), w_t(0)) = (\varphi_0, \varphi_1)$. Moreover, one has*

$$\|(w(t), w_t(t))\|_{Z_\varepsilon} \;\le\; K_0 \|(\varphi_0,\varphi_1)\|_{Z_\varepsilon}\ , \quad t \ge 0\ , \tag{2.1}$$

and

$$\lim_{t\to+\infty} \left(\|w(t)\|_{\mathrm{H}^1} + \|(p_s w(t))_x\|_{\mathrm{L}^2} + \|p_s w_t(t)\|_{\mathrm{L}^2}\right) \;=\; 0\ , \tag{2.2}$$

where $p_s(x) = 1 + \mathrm{e}^{sx}$ and s is given by (1.10).

Sketch of the proof. We follow the lines of the proof of Theorem 1.1 in [14]. Let

$$\mathcal{N}_1(h) \;=\; \int_0^1 F''(\sigma h)\,\mathrm{d}\sigma\ , \quad \mathcal{N}_2(h,w) \;=\; \frac{1}{2}\int_0^1 (1-\sigma)^2 F''(h+\sigma w)\,\mathrm{d}\sigma\ .$$

Given $0 < \varepsilon \le \varepsilon_0$ and $c \ge c_*$, we assume that $(w, w_t) \in \mathcal{C}^0([0,T], Z_\varepsilon)$ is a solution of (1.5) satisfying

$$\text{(A1)} \qquad \|w(t)\|_{\mathrm{H}^1_s} \;\le\; \delta\ , \quad t \in [0,T]\ ,$$

for some (sufficiently small) $\delta > 0$. As in (1.9), we set $w(x,t) = \mathrm{e}^{-sx}\omega(x,t)$. To control the behavior of w on $[0,T]$, we introduce two families of energy functionals:

$$\begin{aligned}
E_0(t) \;&=\; \frac{1}{2}\int_{\mathbf{R}} \left(\varepsilon\omega_t^2 + \omega_x^2 - \omega^2(h\mathcal{N}_1(h) + 2w\mathcal{N}_2(h,w))\right)\,\mathrm{d}x\ ,\\
E_1(t) \;&=\; \int_{\mathbf{R}} \left((\tfrac{1}{2} + \varepsilon cs)\omega^2 + \varepsilon\omega\omega_t\right)\,\mathrm{d}x\ ,\\
E_2(t) \;&=\; \alpha_0 E_0(t) + E_1(t)\ ,
\end{aligned}$$

where $\alpha_0 = \max(2\varepsilon, 1/(2c^2))$, and

$$\begin{aligned}
\mathcal{E}_0(t) &= \frac{1}{2}\int_{\mathbf{R}} \left(\varepsilon w_t^2 + w_x^2 + s^2 w^2 - w^2(h\mathcal{N}_1(h) + 2w\mathcal{N}_2(h,w))\right) \,\mathrm{d}x \ , \\
\mathcal{E}_1(t) &= \int_{\mathbf{R}} \left((\tfrac{1}{2} - \varepsilon cs)w^2 + \varepsilon w w_t\right) \,\mathrm{d}x \ , \\
\mathcal{E}_2(t) &= \alpha_1 \mathcal{E}_0(t) + \mathcal{E}_1(t) + \alpha_2 (E_0(t)E_2(t))^{1/2} \ ,
\end{aligned}$$

where $\alpha_1 = \max(2\varepsilon, \theta/(2c^2))$ and θ, α_2 are positive constants. Note that the functionals E_i, $\mathcal{E}_i$ control the behavior of the perturbation w "ahead" and "behind" the front respectively.

Arguing as in [14, Section 2], one can show that, if δ, θ and α_2^{-1} are sufficiently small, then the functions E_0, E_2, $\mathcal{E}_2$ are non-negative and satisfy the differential inequalities

$$\begin{aligned}
&\frac{\mathrm{d}E_0}{\mathrm{d}t}(t) \le \frac{1}{2}(c^2 - c_*^2)\, E_0(t) \ , \quad \frac{\mathrm{d}E_2}{\mathrm{d}t}(t) + E_0(t) \le 0 \ , \\
&\frac{\mathrm{d}\mathcal{E}_2}{\mathrm{d}t}(t) + \alpha_3 \mathcal{E}_2(t) \le C_1 (E_0(t)E_2(t))^{1/2} \ , \quad t \in [0,T] \ ,
\end{aligned} \tag{2.3}$$

for some $\alpha_3, C_1 > 0$. In addition, there exists a constant $C_2 \ge 1$ such that

$$C_2^{-1} \|(w,w_t)\|_{Z_\varepsilon}^2 \le E_2(t) + \mathcal{E}_2(t) \le C_2 \|(w,w_t)\|_{Z_\varepsilon}^2 (1 + \Psi(\|w(t)\|_{\mathrm{L}^\infty})) \ , \tag{2.4}$$

where the function $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$ is defined by

$$\Psi(K) = \sup_{0 \le u \le 1+K} |F'(u)| \ . \tag{2.5}$$

Combining the estimates Eqs.(2.3), (2.4), we obtain the bound

$$\|(w(t), w_t(t))\|_{Z_\varepsilon} \le C_0 \|(w(0), w_t(0))\|_{Z_\varepsilon} (1 + \Psi(\|w(0)\|_{\mathrm{L}^\infty}))^{1/2} \ , \quad t \in [0,T] \ , \tag{2.6}$$

where C_0 is a positive constant depending only on ε_0, c and F. Since the Cauchy problem for Eq.(1.5) in Z_ε is locally well-posed [14], this proves global existence of the solution (w, w_t) provided the right-hand side of (2.6) is smaller than the quantity δ appearing in (A1). Then, the differential inequalities (2.3) imply that $E_0(t)$ and $\mathcal{E}_2(t)$ converge to zero as $t \to +\infty$, thus proving (2.2). $\square$

Remarks.

1) In the critical case $c = c_*$, it follows from (2.3) that E_0 is non-increasing in time, and that $tE_0(t) + t^{1/2}\mathcal{E}_2(t) \to 0$ as $t \to +\infty$. In particular, we have

$$\lim_{t\to+\infty} \Big(t^{1/4}(\|w(t)\|_{\mathrm{H}^1} + \|w_t(t)\|_{\mathrm{L}^2}) + t^{1/2}(\|\omega_x(t)\|_{\mathrm{L}^2} + \|\omega_t(t)\|_{\mathrm{L}^2}) \Big) \;=\; 0 \;. \qquad (2.7)$$

This decay rate is not optimal, see Section 4 below.

2) Combining (2.1), (2.2), (2.7), we obtain

$$\lim_{t\to+\infty} \|p_s w(t)\|_{\mathrm{L}^\infty} \;=\; 0 \;\text{ if } c > c_* \;, \quad \text{and} \quad \lim_{t\to+\infty} t^{1/4}\|p_s w(t)\|_{\mathrm{L}^\infty} \;=\; 0 \;\text{ if } c = c_* \;. \qquad (2.8)$$

3) All the estimates in the proof of Theorem 2.1 are uniform in ε for $\varepsilon \in (0, \varepsilon_0]$. In particular, taking the limit $\varepsilon \to 0$ everywhere, we obtain a proof of the local stability of the travelling wave h for the corresponding parabolic equation.

4) If $c = c_*$, the stability condition $F'(0) - cs + s^2 = (s - c_*/2)^2 \le 0$ implies $s = c_*/2$, hence (1.10) is the only possibility. On the other hand, if $c > c_*$ and s is chosen so that $F'(0) - cs + s^2 < 0$ (in contrast to (1.10)), one can show using the same spectral estimates as in the parabolic case that the origin in (1.5) is exponentially stable in Z_ε. The fastest decay rate in time for the perturbations is obtained if we set $s = \hat{s}(\varepsilon)$, where

$$\hat{s}(\varepsilon) \;=\; \frac{c}{2}\sqrt{\frac{1 + 4\varepsilon F'(0)}{1 + \varepsilon c^2}} \;.$$

3. Global Stability of the Travelling Waves

Throughout this section, we assume that the non-linearity F satisfies (H1) and

$$\text{(H2)} \qquad\qquad F'(u) \;\le\; -\mu < 0 \;, \quad u \ge 1 \;,$$

for some $\mu > 0$. Under this additional assumption, it is known in the parabolic case that the front h is stable with respect to large perturbations in H^1_s satisfying a positivity condition. This property is a consequence of the classical Maximum Principle for parabolic equations. In this section, we show a similar global stability result for $\varepsilon > 0$ using a hyperbolic Maximum Principle.

Our starting point is the following crucial observation. Let $0 < \varepsilon \le \varepsilon_0$, $c \ge c_*$, $d \in (0,1]$, and assume that $(w, w_t) \in \mathcal{C}^0([0,T], Z_\varepsilon)$ is a solution of (1.5) satisfying, instead of (A1),

$$\text{(A2)} \qquad w(x,t) \;\ge\; -(1-d)h(x)\ , \quad x \in \mathbf{R}\ , \quad t \in [0,T]\ .$$

In other words, we assume that $v(x,t) \equiv h(x) + w(x,t) \ge dh(x)$. Then it can be shown (see [14] in the case $F(u) = u - u^2$) that the a priori estimates (2.3), (2.4), (2.6) still hold for the solution (w, w_t), with constants C_0, C_1, C_2 depending only on ε_0, c, d and F. Thus, we can remove the smallness condition (A1) in the proof of Theorem 2.1 provided we are able to show that the positivity condition (A2) is satisfied for all times. This in turn can be obtained from the Maximum Principle under appropriate assumptions on the initial data.

3.1. The Hyperbolic Maximum Principle

Motivated by (1.4) or (1.5), we consider the hyperbolic operator L with constant coefficients

$$Lu \;=\; u_{xx} + 2\varepsilon c u_{xt} - \varepsilon u_{tt} + c u_x - u_t\ . \tag{3.1}$$

Assume that $\ell : \mathbf{R} \times [0,T] \to \mathbf{R}$ is a continuous function such that

$$\ell(x,t) \;\ge\; \underline{\ell}\ , \quad x \in \mathbf{R}\ , \quad t \in [0,T]\ , \tag{3.2}$$

where T is a positive number and $\underline{\ell} \in \mathbf{R}$ satisfies

$$1 + 4\varepsilon\underline{\ell} \;\ge\; 0\ . \tag{3.3}$$

The following result is a consequence of the Maximum Principle given by Protter and Weinberger (see [29, Chapter 4, Theorem 1] or [14, Appendix A, Theorem A.1]).

Theorem 3.1. *Let $\varepsilon > 0$, $c > 0$, and assume that the conditions (3.2) and (3.3) are satisfied. If $(u(x,t), u_t(x,t))$ belongs to $\mathcal{C}^0([0,T], H^1_{\rm loc}(\mathbf{R}) \times L^2_{\rm loc}(\mathbf{R}))$, with $u_{xx} + 2\varepsilon c u_{xt} - \varepsilon u_{tt}$ in $L^2_{\rm loc}(\mathbf{R} \times (0,T))$, and if*

$$\big(L + \ell(x,t)\big)u(x,t) \;\ge\; 0\ , \quad \text{a.e.}(x,t) \in \mathbf{R} \times [0,T]\ , \tag{3.4}$$

$$u(x,0) \;\le\; 0\ , \quad \forall\, x \in \mathbf{R}\ , \tag{3.5}$$

$$\varepsilon u_t(x,0) - \varepsilon c u_x(x,0) + \frac{1}{2}u(x,0) \;\le\; 0 \;, \quad \text{a.e. } in \; \mathbf{R} \;, \tag{3.6}$$

then $u(x,t) \le 0$ *for all* $(x,t) \in \mathbf{R} \times [0,T]$.

As an application, we define for $d \in [0,1]$, $K \ge 0$ the function

$$\Lambda_d(K) \;=\; \inf \left\{ \frac{F(v)-F(u)}{v-u} \,\middle|\, 0 \le u \le d \,,\; u < v \le 1{+}K \right\} \;\le\; 0 \;. \tag{3.7}$$

We have the following result:

Proposition 3.2. *Assume that (H1), (H2) hold. Let* $\varepsilon > 0$, $c \ge c_*$, $d \in [0,1]$ *and let* K *be a non-negative constant such that*

$$1 + 4\varepsilon\Lambda_d(K) \;\ge\; 0 \;. \tag{3.8}$$

For some $T > 0$, *assume that* $(w, w_t) \in \mathcal{C}^0([0,T], Z_\varepsilon)$ *is a solution of (1.5) with initial data* (φ_0, φ_1) *satisfying*

$$\varphi_0(x) \;\ge\; -(1-d)h(x) \;, \quad x \in \mathbf{R} \;, \tag{3.9}$$

$$\varepsilon\varphi_1(x) \;\ge\; \varepsilon c(\varphi_0'(x) + (1{-}d)h'(x)) - \frac{1}{2}(\varphi_0(x) + (1{-}d)h(x)) \;, \quad \text{a.e. } in \; \mathbf{R} \;. \tag{3.10}$$

Suppose moreover that

$$w(x,t) \le K \;, \quad (x,t) \in \mathbf{R} \times [0,T] \;. \tag{3.11}$$

Then

$$w(x,t) \ge -(1-d)h(x) \;, \quad (x,t) \in \mathbf{R} \times [0,T] \;. \tag{3.12}$$

Proof. Without loss of generality, we may assume that $F'(u) \ge 0$ for all $u \le 0$. Let $u(x,t) = dh(x) - v(x,t)$, where $v(x,t) = h(x) + w(x,t)$. Since $v(x,t)$ is a solution of (1.4), it is straightforward to verify that $(L+\ell)u(x,t) = F(dh(x)) - dF(h(x))$, where L is defined in (3.1) and

$$\ell(x,t) \;=\; \frac{F(v(x,t)) - F(dh(x))}{v(x,t) - dh(x)} \;.$$

In view of (H1), one has $F(dh(x)) - dF(h(x)) \ge 0$ for all $x \in \mathbf{R}$. Furthermore, since $v(x,t) \;\le\; 1 + K$ by (3.11), we have $\ell(x,t) \;\ge\; \underline{\ell} \;=\; \Lambda_d(K)$. Indeed, this inequality follows immediately from (3.7) if $v(x,t) \geq dh(x)$; in the converse case, we observe that $\ell(x,t) \geq \min(F'(dh(x)),0) \geq \Lambda_d(K)$. Thus (3.8) implies (3.3), and the conditions (3.9), (3.10) are nothing else as the hypotheses (3.5) and (3.6). Therefore Theorem 3.1 shows that $u(x,t) = dh(x) - v(x,t) \leq 0$ for all $(x,t) \in \mathbf{R} \times [0,T]$, which is (3.12). $\square$

Remark. Theorem 3.1 suggests the definition of a partial order in $H^1_{\text{loc}}(\mathbf{R}) \times L^2_{\text{loc}}(\mathbf{R})$ as follows. We say that $(\varphi_0, \varphi_1) \leq (\psi_0, \psi_1)$ if

$$
\begin{aligned}
\varphi_0(x) \;&\leq\; \psi_0(x)\ , \quad x \in \mathbf{R}\ , \\
\varepsilon\varphi_1(x) - \varepsilon c\varphi_0'(x) + \frac{1}{2}\varphi_0(x) \;&\leq\; \varepsilon\psi_1(x) - \varepsilon c\psi_0'(x) + \frac{1}{2}\psi_0(x) \quad \text{a.e. in } \mathbf{R}\ ,
\end{aligned}
$$

see (3.5), (3.6). Then, if $(\varphi_0, \varphi_1) \leq (\psi_0, \psi_1)$, the solution of the linear hyperbolic equation $(L + \ell)u(x,t) = 0$ satisfying $u(x,0) = \varphi_0(x)$, $u_t(x,0) = \varphi_1(x)$ stays for all $t \in \mathbf{R}_+$ below the solution of the same equation with initial data (ψ_0, ψ_1). This order has the property that we can write any $(\varphi_0, \varphi_1) \in H^1_{\text{loc}} \times L^2_{\text{loc}}$ as the sum of a "positive" part $(\varphi_0^+, \varphi_1^+) \geq 0$ and a "negative" part $(\varphi_0^-, \varphi_1^-) \leq 0$. This decomposition is unique if we impose that $(\varphi_0^+, \varphi_1^+) = 0$ whenever $(\varphi_0, \varphi_1) \leq 0$ and $(\varphi_0^-, \varphi_1^-) = 0$ whenever $(\varphi_0, \varphi_1) \geq 0$. The formulas for $(\varphi_0^\pm, \varphi_1^\pm)$ are given by

$$
\begin{aligned}
&\varphi_0^+(x) = \sup(0, \varphi_0(x))\,, \\
&\varphi_1^+(x) = c(\varphi_0^+)'(x) - \frac{1}{2\varepsilon}\varphi_0^+(x) + \sup(0, (\varphi_1 - c\varphi_0' + \frac{1}{2\varepsilon}\varphi_0)(x))\,,
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
&\varphi_0^-(x) = \inf(0, \varphi_0(x))\,, \\
&\varphi_1^-(x) = c(\varphi_0^-)'(x) - \frac{1}{2\varepsilon}\varphi_0^-(x) + \inf(0, (\varphi_1 - c\varphi_0' + \frac{1}{2\varepsilon}\varphi_0)(x))\,.
\end{aligned}
\tag{3.14}
$$

Remark that, if $(\varphi_0, \varphi_1) \in Z_\varepsilon$, then $(\varphi_0^\pm, \varphi_1^\pm) \in Z_\varepsilon$. Moreover, it can be verified that $|\varphi_1^\pm(x)| \leq |\varphi_1(x)| + c|\varphi_0'(x)|$ a.e. in $\mathbf{R}$.

3.2. A General Global Stability Result

Combining the a priori estimates of Section 2 and the Maximum Principle, we are now able to state and prove our main stability result:

Theorem 3.3. *Assume that (H1), (H2) hold, and let* $\varepsilon_0 > 0$, $c \geq c_*$, $d \in (0,1]$. *There exists a constant* $C_0 \geq 1$ *such that, for all* $0 < \varepsilon \leq \varepsilon_0$ *and all* $K > 0$ *satisfying*

$$1 + 4\varepsilon\Lambda_d(K) \geq 0 , \tag{3.15}$$

the following result holds: If $K_* > 0$ *is such that*

$$C_0 K_*(1 + \Psi(K_*))^{1/2} < K , \tag{3.16}$$

where Ψ *is defined in (2.5), then for any* $(\varphi_0, \varphi_1) \in Z_\varepsilon$ *satisfying the inequalities (3.9), (3.10) and the bound* $\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \leq K_*$, *there exists a unique solution* $(w, w_t) \in \mathcal{C}^0([0,\infty), Z_\varepsilon)$ *of (1.5) with initial data* (φ_0, φ_1). *Moreover, one has*

$$\|(w(t), w_t(t))\|_{Z_\varepsilon} \leq K , \quad w(x,t) \geq -(1-d)h(x) , \tag{3.17}$$

for all $x \in \mathbf{R}$, $t \in \mathbf{R}_+$, *and (2.2), (2.7) hold.*

Remark. The constant C_0 in (3.16) is the same as in (2.6).

Sketch of the proof. By Proposition 3.2, the solution (w, w_t) with initial data (φ_0, φ_1) satisfies $w(x,t) \geq -(1-d)h(x)$ as long as $w(x,t) \leq K$. On the other hand, in view of (3.16), the a priori estimate (2.6) implies that $w(x,t) \leq \|(w(t), w_t(t))\|_{Z_\varepsilon} < K$ as long as $w(x,t) \geq -(1-d)h(x)$. Combining these facts and using a contradiction argument, we show that the solution is globally defined and satisfies (3.17) for all times. The differential inequalities (2.3) then imply (2.2), (2.7). $\square$

Remarks.

1) The relations (3.15), (3.16) imply that K (hence K_*) can be chosen very big if ε is sufficiently small. In this case, Theorem 3.3 shows that the travelling wave h is stable with respect to large perturbations, provided the positivity conditions (3.9), (3.10) are satisfied. Conversely, if ε is large, then K (hence K_*) has to be very small, and Theorem 3.3 reduces to a local stability result similar to Theorem 2.1.

2) As a simple example, consider the non-linearity $F(u) = u - u^2$ [14] which satisfies (H1), (H2). Then $\Lambda_d(K) = -(d+K)$, and the condition (3.15) reads $1 - 4\varepsilon(d+K) \geq 0$. Also $\Psi(K) = 1 + 2K$, hence the hypothesis $\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \leq K_*$ can be replaced by

$$\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \leq CK(1+K)^{-1/3} \ ,$$

where C is a sufficiently large positive constant.

3.3. Further Stability Results if $1 + 4\varepsilon F'(1) \geq 0$

The previous results are still incomplete for at least two reasons. First, if $c > c_*$, they fail to give a decay rate of the perturbations as $t \to +\infty$, see (2.8). Next, they do not provide any global existence result if $d = 0$, that is if $v(x,0) \geq 0$. Assuming that the non-linearity F satisfies the stronger assumption

$$\text{(H3)} \qquad F''(u) \leq 0 \ , \quad u \geq 0 \ ,$$

we can give a partial answer to both questions when $1 + 4\varepsilon F'(1) \geq 0$. Indeed, in this case, the Maximum Principle allows us to compare the solution $w(x,t)$ of (1.5) with solutions of the linear equations

$$\varepsilon \tilde{w}_{tt} + \tilde{w}_t - 2\varepsilon c \tilde{w}_{xt} = \tilde{w}_{xx} + c\tilde{w}_x + F'(h)\tilde{w} \ , \tag{3.18}$$

and

$$\varepsilon \tilde{w}_{tt} + \tilde{w}_t - 2\varepsilon c \tilde{w}_{xt} = \tilde{w}_{xx} + c\tilde{w}_x + F'(0)\tilde{w} \ . \tag{3.19}$$

We denote by $(\tilde{w}(t), \tilde{w}_t(t))$ the solution of (3.18) with initial data (φ_0, φ_1), and by $(\tilde{w}^{\pm}(t), \tilde{w}_t^{\pm}(t))$ the solution of (3.19) with initial data $(\varphi_0^{\pm}, \varphi_1^{\pm})$, the "positive" and "negative" parts of (φ_0, φ_1) defined in (3.13), (3.14). Following the arguments in [14, Section 4], we obtain our last stability result:

Theorem 3.4. *Assume that (H1), (H3) hold and that* $1+4\varepsilon F'(1) \geq 0$*. Given* $c \geq c_*$*,* $d \in [0,1]$*, there exists a constant* $C_3(c) \geq 1$ *such that, for all* $K \geq 0$ *satisfying*

$$1 + 4\varepsilon\Lambda_1(K) \geq 0 \ ,$$

the following result holds: for any $(\varphi_0, \varphi_1) \in Z_\varepsilon$ *satisfying (3.9), (3.10) and*

$$\inf\left(\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \ , \ \|(\varphi_0^+, \varphi_1^+)\|_{Z_\varepsilon}\right) \leq \frac{K}{C_3(c)},$$

there exists a unique solution $(w, w_t) \in \mathcal{C}^0([0,\infty), Z_\varepsilon)$ *of (1.5) with initial data* (φ_0, φ_1). *Moreover, we have*

$$-(1-d)h(x) \le w(x,t) \le K \ , \tag{3.20}$$

and

$$\tilde{w}^-(x,t) \le w(x,t) \le \tilde{w}(x,t) \le \tilde{w}^+(x,t) \ ,$$

for all $x \in \mathbf{R}$, $t \in \mathbf{R}_+$. *Finally, if* $d > 0$ *and* $1 - 4\varepsilon F'(0) > 0$, *one has*

$$\lim_{t\to+\infty} t^{1/4} \left(\|p_s w(t)\|_{\mathrm{L}^\infty} + \|w(t)\|_{\mathrm{H}^1} + \|w_t(t)\|_{\mathrm{L}^2}\right) = 0 \ .$$

Remark. If $(\varphi_0^+, \varphi_1^+) = 0$, *i.e.* if the initial data are non-positive, then (3.20) with $K = 0$ shows that the solution $w(x,t)$ remains non-positive for all times.

4. Asymptotic Expansions in the Critical Case $c = c_*$

In this section, we restrict ourselves to the critical case $c = c_*$, and we consider perturbations of the travelling wave h in a strict subspace of Z_ε. Using self-similar variables and energy estimates, we are able to compute explicitly the long-time asymptotics of the perturbations as $t \to +\infty$. In particular, we recover the results obtained by Gallay [13] in the parabolic case $\varepsilon = 0$.

Following Kirchgässner [25], we consider solutions of (1.4) of the following form

$$v(x,t) = h(x) + h'(x) W\left(x, \frac{t}{1+\varepsilon c_*^2}\right) \ ,$$

i.e. we set $w(x,t) = h'(x)W(x, t/(1+\varepsilon c_*^2))$. Then W satisfies the equation

$$\eta W_{tt} + (1-\nu\gamma(x))W_t - 2\nu W_{xt} = W_{xx} + \gamma(x)W_x + h'(x)W^2\mathcal{N}(h(x), h'(x)W) \ , \tag{4.1}$$

where

$$\eta = \frac{\varepsilon}{(1+\varepsilon c_*^2)^2} \ , \quad \nu = \frac{\varepsilon c_*}{1+\varepsilon c_*^2} \ , \quad \gamma(x) = c_* + 2\frac{h''(x)}{h'(x)} \ , \quad x \in \mathbf{R} \ .$$

In (4.1) and in the sequel, the second argument of the function W is simply denoted by t, instead of $t/(1+\varepsilon c_*^2)$.

From [1] we know that the travelling wave h (with $c = c_*$) satisfies

$$h(x) \;=\; \begin{cases} 1 + \mathcal{O}(\mathrm{e}^{\beta x}) & \text{as } x \to -\infty \;, \\ (a_1 x + a_2)\mathrm{e}^{-c_* x/2} + \mathcal{O}(x^2 \mathrm{e}^{-c_* x}) & \text{as } x \to +\infty \;, \end{cases}$$

where $a_1 > 0$, $a_2 \in \mathbf{R}$, and $\beta = \frac{1}{2}(-c_* + \sqrt{c_*^2 - 4F'(1)}) > 0$. Similar asymptotic expansions hold for the derivatives h', h'', hence

$$\gamma(x) \;=\; \begin{cases} \gamma_- + \mathcal{O}(\mathrm{e}^{\beta x}) & \text{as } x \to -\infty \;, \\ 2/(x{+}x_0) + \mathcal{O}(x\mathrm{e}^{-c_* x/2}) & \text{as } x \to +\infty \;, \end{cases} \tag{4.2}$$

where $\gamma_- = c_* + 2\beta = 2\sqrt{F'(0) - F'(1)}$ and $x_0 = (a_2/a_1 - 2/c_*)$. The hypothesis (H1) on F also implies that $\gamma'(x) < 0$ for all $x \in \mathbf{R}$.

To study the long-time behavior of the solution W of (4.1), we use the *scaling variables* or *self-similar variables* defined by

$$\xi \;=\; \frac{x}{\sqrt{t{+}t_0}} \;, \qquad \tau = \log(t{+}t_0) \;,$$

for some $t_0 > 0$. These variables have been widely used to study the long time behavior of solutions to parabolic equations, in particular to prove convergence to self-similar solutions [4,8,9,12,24]. In [15], it has been shown by the authors that these variables are also a powerful tool in the framework of damped hyperbolic equations. Following [15], we define the rescaled functions U and V by

$$U(\xi,\tau) \;=\; \mathrm{e}^{3\tau/2} W(\xi \mathrm{e}^{\tau/2}, \mathrm{e}^{\tau} - t_0) \;, \quad V(\xi,\tau) \;=\; \mathrm{e}^{5\tau/2} W_t(\xi \mathrm{e}^{\tau/2}, \mathrm{e}^{\tau} - t_0) \;, \tag{4.3}$$

or equivalently

$$\begin{aligned} W(x,t) \;&=\; \frac{1}{(t{+}t_0)^{3/2}}\, U\left(\frac{x}{\sqrt{t{+}t_0}}, \log(t{+}t_0)\right) \;, \\ W_t(x,t) \;&=\; \frac{1}{(t{+}t_0)^{5/2}}\, V\left(\frac{x}{\sqrt{t{+}t_0}}, \log(t{+}t_0)\right) \;. \end{aligned} \tag{4.4}$$

Then $U(\xi,\tau), V(\xi,\tau)$ satisfy the system

$$\begin{aligned} &U_\tau - \frac{\xi}{2} U_\xi - \frac{3}{2} U \;=\; V \;, \\ &\eta \mathrm{e}^{-\tau}\big(V_\tau - \frac{\xi}{2} V_\xi - \frac{5}{2} V\big) + (1 - \nu\gamma(\xi \mathrm{e}^{\tau/2}))V - 2\nu \mathrm{e}^{-\tau/2} V_\xi \;=\; \\ &\qquad\qquad U_{\xi\xi} + \mathrm{e}^{\tau/2}\gamma(\xi \mathrm{e}^{\tau/2}) U_\xi + \mathrm{e}^{-\tau/2} h'(\xi \mathrm{e}^{\tau/2}) U^2 N(\xi,\tau) \;, \end{aligned} \tag{4.5}$$

where $N(\xi,\tau) = \mathcal{N}(h(\xi e^{\tau/2}), e^{-3\tau/2}h'(\xi e^{\tau/2})U)$.

We now introduce function spaces for the rescaled perturbations U, V. For $\tau \ge 0$, we denote by $\mathcal{X}_\tau, \mathcal{Y}_\tau$ the Hilbert spaces of measurable functions on $\mathbf{R}$ defined by the norms

$$\begin{aligned} \|V\|_{\mathcal{Y}_\tau}^2 &= \int_{-\infty}^0 e^{2\beta\xi e^{\tau/2}} |V(\xi)|^2 \,d\xi + \int_0^{+\infty} (1+\xi^6)|V(\xi)|^2 \,d\xi \;, \\ \|U\|_{\mathcal{X}_\tau}^2 &= \|U\|_{\mathcal{Y}_\tau}^2 + \|U_\xi\|_{\mathcal{Y}_\tau}^2 \;. \end{aligned}$$

We denote by $\mathcal{Z}_\tau$ the product space $\mathcal{Z}_\tau = \mathcal{X}_\tau \times \mathcal{Y}_\tau$ equipped with the standard norm

$$\|(U,V)\|_{\mathcal{Z}_\tau}^2 = \|U\|_{\mathcal{X}_\tau}^2 + \|V\|_{\mathcal{Y}_\tau}^2 \;.$$

As is easily verified, if the functions (U,V) and (W,W_t) are related through (4.3) or (4.4), then $(U(\cdot,\tau),V(\cdot,\tau)) \in \mathcal{Z}_\tau$ if and only if the actual perturbation $w(x,t) = h'(x)W(x,t/(1+\varepsilon c_*^2))$ satisfies

$$\int_{-\infty}^0 (w^2+w_x^2+w_t^2)(x,t)\,dx + \int_0^\infty (1+x^4)e^{c_* x}(w^2+w_x^2+w_t^2)(x,t)\,dx \;<\; \infty \;. \quad (4.6)$$

Therefore, the perturbation space considered in this section is slightly smaller than the space Z defined in (1.7), due to the factor $(1+x^4)$ in (4.6).

Before stating the main result of this section, we explain its content in a heuristic way. Taking formally the limit $\tau \to +\infty$ in (4.5) and using (4.2), we see that U satisfies the linear parabolic equation

$$U_\tau \;=\; \mathcal{L}_\infty U \;\stackrel{\text{def}}{=}\; U_{\xi\xi} + \Bigl(\frac{\xi}{2}+\frac{2}{\xi}\Bigr)U_\xi + \frac{3}{2}U \quad \text{if } \xi > 0 \;, \quad U_\xi \;=\; 0 \quad \text{if } \xi \le 0 \;. \quad (4.7)$$

Therefore, it is reasonable to expect that the long-time behavior of the solutions to (4.5) will be determined by the spectral properties of the operator $\mathcal{L}_\infty$ on $\mathbf{R}_+$, with Neumann boundary condition at $\xi = 0$. Now, as is easily verified, this limiting operator is just the image under the scaling (4.4) of the radially symmetric Laplacian operator in three dimensions. Indeed, if U and W are related through (4.4), the equation $U_\tau = \mathcal{L}_\infty U$ is equivalent to $W_t = W_{xx} + (2/x)W_x$, $x > 0$. This crucial observation allows to compute exactly the spectrum of $\mathcal{L}_\infty$ in various function spaces, see [15]. For instance, in the space $L^2(\mathbf{R}_+, (1+\xi^6)d\xi)$, the spectrum of $\mathcal{L}_\infty$ consists of a simple, isolated eigenvalue at $\lambda = 0$, and of "continuous" spectrum filling the half-plane $\{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\,\lambda \leq -1/4\}$. The eigenfunction corresponding to $\lambda = 0$ is the gaussian $\mathrm{e}^{-\xi^2/4}$. Therefore, we expect that the solution $U(\xi,\tau)$ of (4.5) converges as $\tau \to +\infty$ to $\alpha\varphi^*(\xi)$ for some $\alpha \in \mathbf{R}$, where

$$\varphi^*(\xi) \;=\; \frac{1}{\sqrt{4\pi}} \begin{cases} 1 & \text{if } \xi < 0 \;, \\ \mathrm{e}^{-\xi^2/4} & \text{if } \xi \geq 0 \;. \end{cases}$$

This function is normalized so that $\int_0^\infty \xi^2 \varphi^*(\xi)\,\mathrm{d}\xi = 1$. Since $V = U_\tau - \frac{\xi}{2}U_\xi - \frac{3}{2}U$, we also expect that $V(\xi,\tau) \to \alpha\psi^*(\xi)$, where $\psi^* = -\frac{\xi}{2}\varphi^*_\xi - \frac{3}{2}\varphi^*$. It is crucial to note that Eq.(4.7) is independent of ε: this explains why the solutions of (4.1), hence of (1.1), behave for large times in a similar way to those of the corresponding parabolic equations.

Our last result shows that the heuristic arguments above are indeed correct:

Theorem 4.1. *Assume that (H1) holds, and let $\varepsilon > 0$, $c = c_*$. There exist $\tau_0 > 0$ and $\delta_0 > 0$ such that, for all $(U_0, V_0) \in \mathcal{Z}_{\tau_0}$ satisfying $\|(U_0,V_0)\|_{\mathcal{Z}_{\tau_0}} \leq \delta_0$, the system (4.5) has a unique solution $(U,V) \in \mathcal{C}^0([\tau_0,+\infty),\mathcal{Z}_\tau)$ with $(U(\tau_0),V(\tau_0)) = (U_0,V_0)$. In addition, there exists $\alpha^* \in \mathbf{R}$ such that*

$$\|U(\tau) - \alpha^*\varphi^*\|^2_{\mathcal{X}_\tau} + \int_{\tau_0}^{\tau} \mathrm{e}^{-(\tau-\sigma)/2} \|V(\sigma) - \alpha^*\psi^*\|^2_{\mathcal{Y}_\sigma}\,\mathrm{d}\sigma \;=\; \mathcal{O}(\tau^2 \mathrm{e}^{-\tau/2}) \;,$$

as $\tau \to +\infty$.

Remark. We say that $(U,V) \in \mathcal{C}^0([\tau_0,+\infty),\mathcal{Z}_\tau)$ is a solution of the system (4.5) if there exists a solution $(W,W_t) \in \mathcal{C}^0([0,+\infty),\mathcal{Z}_0)$ of (4.1) such that (4.3), (4.4) hold, with $t_0 = \mathrm{e}^{\tau_0}$.

In terms of the original variables, Theorem 4.1 implies that [16]

$$\sup_{x\in\mathbf{R}} \left(1 + \frac{\mathrm{e}^{c_* x/2}}{1+|x|}\right) \left| w(x,t) - \frac{\alpha}{t^{3/2}}\, h'(x)\varphi^*\Big(\frac{x\sqrt{1+\varepsilon c_*^2}}{\sqrt{t}}\Big)\right| \;=\; \mathcal{O}(t^{-7/4}\log t) \;,$$

as $t \to +\infty$, where $\alpha = \alpha^*(1+\varepsilon c_*^2)^{3/2}$, $w(x,t) = h'(x)W(x,t/(1+\varepsilon c_*^2))$ and W is given by (4.4). In the parabolic case $\varepsilon = 0$, this result has been obtained in [13] using slightly different function spaces. Remarkably enough, the asymptotic profile φ^* is *universal*: it does not depend on the initial data, nor on the parameter $\varepsilon \geq 0$, nor on the precise form of the non-linearity F.

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