The Mean Field Theory for Percolation Models of the Ising Type

L. Chayes

Department of Mathematics,

University of California, Los Angeles, CA 90095-1555, USA

A. Coniglio

Dipartimento di Scienze Fisiche,

Università "Federico II" Mostra d'Oltremare Pad.19 I–80125 Napoli, Italy

J. Machta

Department of Physics and Astronomy,

University of Massachusetts, Amherst, MA 01003-3720

K. Shtengel

Department of Physics,

University of California, Los Angeles, CA 90095-1547, USA

Abstract

The $q = 2$ random cluster model is studied in the context of two mean–field models: The Bethe lattice and the complete graph. For these systems, the critical exponents that are defined in terms of finite clusters have some anomalous values as the critical point is approached from the high–density side which vindicates the results of earlier studies. In particular, the exponent $\tilde{\gamma}'$ which characterizes the divergence of the average size of finite clusters is $1/2$ and $\tilde{\nu}'$, the exponent associated with the length scale of finite clusters is $1/4$. The full collection of exponents indicates an upper critical dimension of 6. The standard mean–field exponents of the Ising system are also present in this model ($\nu' = 1/2$, $\gamma' = 1$) which implies, in particular, the presence of two diverging length–scales. Furthermore, the finite cluster exponents are stable to the addition of disorder which, near the upper critical dimension, may have interesting implications concerning the generality of the disordered system/correlation length bounds.

A. Introduction

The close connection between spin–systems and percolation models has lead to many important developments in statistical physics. For a broad class of spin models there is a mapping to an equivalent *graphical representation* having a percolation transition corresponding to the phase transition of the spin model (see e.g. [1]). In particular, the graphical representations of the q-state Potts models are the "q-state" random cluster models [2,3]. For $q = 2$ the spin representation is the Ising model while the $q = 1$ random cluster model is ordinary bond percolation. The equivalence between spin models and graphical models has lead to a new class of highly efficient *cluster* Monte Carlo methods [4]. Cluster methods simulate both the spin model and the graphical representation and it is often more efficient to measure thermodynamic quantities via their graphical analogs [5].

As an example of the correspondence between spin models and graphical models, consider the order parameter exponent β [6] that is defined for spin systems by, $m \sim (T_c - T)^\beta$ where T is the temperature, T_c the critical temperature and m the order parameter. For graphical models, the order parameter is the fraction of sites in the percolating cluster, P_∞ and $\tilde{\beta}$ is defined as, $P_\infty \sim (p - p_c)^{\tilde{\beta}}$ where p is the bond occupation probability and p_c the percolation threshold. For a given q-state Potts model and the corresponding random cluster model. It turns out that the order parameter, namely the spontaneous magnetization, is exactly equal to the percolation density: $m = P_\infty$. Thus $\tilde{\beta} = \beta$. Analogs of other thermodynamic exponents may be defined for percolation models; $\tilde{\gamma}$ ($\tilde{\gamma}'$) characterize the divergence of the average size of the *finite* cluster containing the origin below (above) the percolation threshold. Similarly, $\tilde{\nu}$ ($\tilde{\nu}'$) characterizes the divergence of average scale of the *finite* clusters below (above) threshold. The connection between the thermal and geometric exponents follows from the fact that the spin-spin correlation function between sites

i and j is equal to (a constant times) the probability that i and j are in the same cluster in this graphical representation. Thus, in the high temperature phase the correlation lengths for the two models are equal and the magnetic susceptibility is equal to (or proportional to) the average size of the connected clusters. Thus, in particular, $\gamma = \tilde{\gamma}$ and $\nu = \tilde{\nu}$.

On the other hand, the corresponding relation for γ' and $\tilde{\gamma}'$ or (ν' and $\tilde{\nu}'$) is *not* a straightforward consequence of the mapping to the graphical representation. (We remind the reader that in general, primed exponents denote quantities pertaining to the approach to the critical point from the low temperature or percolating phase.) In particular, for the Potts models, the graphical expression for the truncated two point function between the sites i and j consists of two terms: (i) The probability that the two sites are in the same finite cluster and (ii) the correlation of the infinite cluster density at the sites i and j. For the random cluster models (with $q \geq 1$) the second term is positive so the geometric quantities are smaller. This leads to the inequalities $\tilde{\nu} \geq \tilde{\nu}'$ and $\tilde{\gamma} \geq \tilde{\gamma}'$. On general grounds, one would expect that items (i) and (ii) are comparable and for independent percolation ($q = 1$) this is the subject of a rigorous theorem [7]. It is therefore something of a surprise that for percolation models of the Ising–type, the equality of $\tilde{\gamma}$ and $\tilde{\gamma}'$ breaks down in mean–field. This was first discovered in a series of papers co–authored by one of us [9–11] both in the context of a polymer/solvent model and in a mean–field calculation for the Potts/percolation droplet model.

The principal focus of this paper is to underscore (and bolster) the conclusions inherent in the result $\gamma' \neq \tilde{\gamma}'$ and to clarify the relationship between the Ising magnet and its graphical representation. In particular, the result $\gamma' \neq \tilde{\gamma}'$ in these sorts of systems will be shown in some generality: A full fledged Bethe lattice calculation and for the $q = 2$ random cluster model defined on the complete graph. Furthermore (at least in the context of the Bethe lattice) there is the additional mean–field exponent $\tilde{\nu}' = \frac{1}{4}$. Taken along with $\tilde{\beta} = \beta = \frac{1}{2}$ and $\eta = 0$ (and an effective $\delta = 2$) these exponents satisfy the standard scaling relations and lead to the tentative prediction $d_c = 6$ for the upper critical dimension. Of particular interest is therefore the behavior of the $q = 2$ random cluster model itself: a different set of exponents above and below threshold and the appearance of two distinct diverging length scales as the critical point is approached from the percolating phase. Since these models are the starting point of the $\infty > d \gg 1$ expansions, it is our belief that the behavior uncovered here also holds in sufficiently high dimensions.

It must be emphasized that the discrepancies between random cluster and Ising exponents are due to a difference in definitions not content. The same information is inherent in both a spin model and its graphical representation and it is always possible to define appropriate graphical quantities to represent any spin quantity. On the other hand, the fact that two natural sets of definitions lead to different exponents indicates the presence of real and interesting features of the Ising–type random cluster model in high dimensions. Some of these are discussed at the conclusion of this paper.

B. The Bethe Lattice

Consider a half-space (rooted) Bethe lattice of coordination number 3 (a binary Cayley tree). For a k-level tree define the interacting bond percolation problem with the weights of configurations ω given by $W(\omega) = B_{p_e}(\omega)\, q^{l(\omega)}$. Here $B_{p_e}(\omega)$ is the Bernoulli probability of ω at the parameter value $p_e \in [0, 1]$. The quantity $l(\omega)$ counts the number of loops. In the present context, the only mechanism for the formation of loops is via connections to the boundary; all points on the boundary are regarded as "pre-connected" [8]. As is well known (for appropriate q) this is the random cluster representation the q-state Potts model given by the Hamiltonian $-\beta H = J\sum_{<i,j>}\delta_{\sigma_i,\sigma_j}$ subject to the boundary condition that the boundary sites are all locked in the same state. The parameter p_e is given by $p_e = [1-e^{-J}]/[1+(q-1)e^{-J}]$ (which for the Ising case $q=2$ becomes $p_e = \tanh(J/2)$).

The partition function on the k-level binary tree is given by $Z_k \equiv \sum_\omega W_k(\omega)$. There are two types of bond configurations: those in which the root site is connected to the boundary, and those in which the root site is not. Let I_k denote the total weight of configurations connected to the boundary and F_k denote the total weight of bond configurations not connected to the boundary; $Z_k = I_k + F_k$. Let us now take two such k-level trees and join them by attaching their root sites to a new root site, thus creating a $(k+1)$-level tree. The recursion relations for F_{k+1}, I_{k+1}, and Z_{k+1} are:

$$F_{k+1} = [F_k + (1-p_e)I_k]^2 = (Z_k - p_e I_k)^2\ ; \tag{1a}$$

$$I_{k+1} = 2p_e I_k Z_k + (q-2)p_e^2 I_k^2\ ; \tag{1b}$$

$$Z_{k+1} = Z_k^2 + (q-1)p_e^2 I_k^2\ . \tag{1c}$$

We define $P_\infty^{(k)} \equiv I_k/Z_k$ to be the probability that a root site of a k-level tree is connected to the boundary. Using the above recursion relations (1) and taking the limit of $k\to\infty$, we arrive at the well-known expression for the percolation density:

$$P_\infty \equiv \lim_{k\to\infty} P_\infty^{(k)} = \frac{2p_e P_\infty + (q-2)p_e^2 P_\infty^2}{1+(q-1)p_e^2 P_\infty^2}\ . \tag{2}$$

Analyzing the non-trivial solution of this equation one can see that there is a continuous phase transition at $p_e = 1/2$ if $0 < q \le 2$ with the critical exponent $\tilde\beta = 1$ for $0 < q < 2$ and $\tilde\beta = 1/2$ for $q = 2$. For $q > 2$ the transition becomes discontinuous.

We now look at the average *finite* cluster size. Let us define a random variable c_k to be the size of the connected component of the root site, whenever this cluster is disconnected from the boundary, and $c_k = 0$ if the root site is connected to the boundary. Let also $X_k \equiv \sum_\omega c_k W_k(\omega)/Z_k$ be its average. Since only the bond configurations that contribute to F_k enter into this average, we can write $X_k = f_k F_k/Z_k$ where f_k is the average cluster size *given* that it is disconnected from the boundary. Again, merging two k-level trees into a new $(k+1)$-level tree one finds the recursion relation for f_k:

$$f_{k+1}F_{k+1} = (1-p_e)^2 Z_k^2 + (1+f_k)p_e F_k(1-p_e)Z_k + (1+2f_k)p_e^2 F_k^2 \,. \tag{3}$$

Dividing both sides of (3) by Z_{k+1} and recalling the definitions of X_k and $P_\infty^{(k)}$ we obtain

$$X_{k+1} = 1 - P_\infty^{(k+1)} + X_k \frac{2p_e^2(1-P_\infty^{(k)}) + 2p_e(1-p_e)}{1+(q-1)[p_e P_\infty^{(k)}]^2}. \tag{4}$$

Taking the limit $k \to \infty$ (the existence of which is guaranteed by a straightforward argument; i.e. subtracting the purported answer from both sides) we obtain:

$$X(p_e) \equiv \lim_{k\to\infty} X_k = \frac{(1-p_e P_\infty)^2}{1-2p_e+2p_e^2 P_\infty + (q-1)p_e^2 P_\infty^2} \,. \tag{5}$$

This expression is valid on both the ordered and disordered sides of the transition. Indeed, for $p_e < 1/2$, $P_\infty = 0$ and Eq.(5) reduces to the well known expression $X = 1/(1-2p_e)$ (which gives $\tilde{\gamma} = 1$ for all $0 < q \leq 2$).

The situation on the ordered side ($p_e > 1/2$) is changed by the presence of the term proportional to P_∞ in the denominator. For $0 < q < 2$, $P_\infty(p_e)$ is asymptotically linear in $(p_e - 1/2)$ resulting in the anticipated $\tilde{\gamma}' = 1$. However, for $q = 2$,

$$P_\infty(p_e) \equiv m(p_e) \sim (p_e - \frac{1}{2})^{1/2} \gg (2p_e - 1) \tag{6}$$

for $(p_e - 1/2)$ small, so that the terms involving P_∞ are dominant. Indeed, substituting this expression for m directly into Eq. (5) we arrive at

$$X = \frac{1-m}{p_e m} \qquad \text{for } q=2 \text{ and } p_e > 1/2. \tag{7}$$

The result for $q = 2$ is $\tilde{\gamma}' = 1/2 \neq \gamma$.

Now we look at the finite cluster distribution. The probability that the root site of a k-level tree belongs to a cluster of size n that does not touch the boundary is defined as

$$P_n^{(k)} = \frac{F_k(n)}{Z_k} = \frac{F_k(n)}{F_k + I_k}\,, \tag{8}$$

where $F_k(n)$ is simply the total weight corresponding to such event (clearly, $F_k = \sum_n F_k(n)$). Once again, merging two k-level trees into a new one we find the following relations between these weights:

$$F_{k+1}(1) = (1-p_e)^2 Z_k^2\,; \tag{9a}$$

$$\begin{aligned} F_{k+1}(n) &= 2p_e(1-p_e) Z_k F_k(n-1) \\ &\quad + p_e^2 \sum_{i=1}^{n-2} F_k(i) F_k(n-i-1)\,, \quad (n \neq 1)\,. \end{aligned} \tag{9b}$$

The next step is to divide both parts of these equations by Z_{k+1} and to arrive (with the help of (1c)) at the expression for $P_n^{(k+1)}$ in terms of $P_i^{(k)}$ and $P_\infty^{(k)}$. Letting $k \to \infty$ (which is again easily justified) we obtain:

$$P_1 = \frac{(1-p_e)^2}{1 + (q-1)p_e^2 P_\infty^2}\,; \tag{10a}$$

$$\begin{aligned} P_n &= \frac{1}{1 + (q-1)p_e^2 P_\infty^2} \Big\{ 2p_e(1-p_e) P_{n-1} \\ &\quad + p_e^2 \sum_{i=1}^{n-2} P_i P_{n-i-1} \Big\}\,, \qquad (n \neq 1)\,. \end{aligned} \tag{10b}$$

It should be noted that (10a) and (10b) are independent of q when $P_\infty = 0$. Thus, below *and at* the percolation threshold ($p_e \leq 1/2$) the finite cluster distribution is identical to that for the case of independent percolation! Since the critical exponents τ (and η) are defined at the critical point they must take on their mean field percolation values $\tau = 5/2$ (and $\eta = 0$) for any $q \in (0, 2]$. Since $\delta = 3$ for the mean field Ising model, the exponent relation $\tau = 1/\delta - 2$ is violated.

The last critical exponent of our interest here is the correlation length exponent $\tilde{\nu}$. In order to find it we adopt the standard definition of the correlation length on the Bethe lattice (see, e.g., Ref. [12]):

$$\xi(p_e) = \sqrt{\frac{1}{X(p_e)} \sum_x |x|^2 \, \tau^f_{0,x}(p_e)} \tag{11}$$

where $\tau^f_{0,x}(p_e)$ is the probability of the origin being connected to the site x while *not* being connected to the boundary. The probability $\tau^f_{0,x}$ depends only on the level n of the Cayley tree that the site x belongs to. Thus, using the metrics relation $|x|^2 = n$, we can rewrite the sum in (11) as

$$\sum_x |x|^2 \, \tau^f_{0,x} = \sum_{n=0}^{\infty} n 2^n \tau^f_n = 2 \sum_{n=0}^{\infty} (n+1) 2^n \tau^f_{n+1} \; . \tag{12}$$

Performing the same procedure of merging two k-level trees together to form a $k+1$-level tree and taking a limit of $k \to \infty$ we generate a recursion relation for the probability τ^f_n:

$$\tau^f_{n+1} = \tau^f_n \frac{p_e(1 - p_e P_\infty)}{1 + (q-1) p_e^2 P_\infty^2} \; . \tag{13}$$

Combining this result with (12) we obtain the desired expression for the correlation length:

$$\xi(p_e) = \sqrt{\frac{2p_e(1 - p_e P_\infty)}{1 - 2p_e + 2p_e^2 P_\infty + (q-1) p_e^2 P_\infty^2}} = \sqrt{\frac{2 p_e X(p_e)}{1 - p_e P_\infty}} \tag{14}$$

with $X(p_e)$ given by Eq. (5). This brings us to the following relation between the critical exponents: $\tilde{\nu} = \tilde{\gamma}/2$ on both sides of the transition (in agreement with the scaling relation $\gamma = 2(\nu - \eta)$ since $\eta = 0$), which in turn leads to a surprising result: in the Ising case ($q = 2$) $\nu' = 1/2$ while $\tilde{\nu}' = 1/4$!

So far we have dealt with the half-space Bethe lattice where the root site has only two nearest neighbors as opposed to three for any inner site. We claim, however, that all full-space quantities experience the same critical behavior as the corresponding half-space quantities above. In fact, they can be explicitly calculated if we attach the root sites of two identical trees together thus forming a complete full-space Bethe lattice (cf. [8]):

$$\mathcal{P}_\infty = P_\infty \frac{1 + p_e + (q-2)p_e P_\infty}{1 + (q-1)p_e P_\infty^2}\,; \tag{15a}$$

$$\mathcal{X}(p_e) = X(p_e) \frac{1 + p_e - 2p_e P_\infty}{1 + (q-1)p_e P_\infty^2}\,; \tag{15b}$$

$$\zeta(p_e) = \xi(p_e) \sqrt{\frac{3(1 - p_e P_\infty)}{2(1 + p_e - 2p_e P_\infty)}}\,. \tag{15c}$$

Here $\mathcal{P}_\infty$, $\mathcal{X}$, and ζ refer to the isotropic, full-space quantities, which are just non-singular modifications of the corresponding half-space quantities given by Eqs. (2), (5) and (14).

The disordered case. The addition of disorder has almost no effect on the previous set of results. This fact leads to some interesting consequences that will be discussed in the final section. Here we will demonstrate this stability to disorder confining attention to the exponent $\tilde{\gamma}'$ for the case $q = 2$. The setup is as follows: The couplings are be given by $(J_{i,j})$ which are identical and independent non–negative random variables. The quantity $P_\infty^{(k)}$ is now a random function of these couplings that obeys the distributional equation

$$P_\infty^{(k+1)} =_d \frac{p_{e,L} P_{\infty,L}^{(k)} + p_{e,R} P_{\infty,R}^{(k)}}{1 + p_{e,L} P_{\infty,L}^{(k)} p_{e,R} P_{\infty,R}^{(k)}} \tag{16}$$

with $P_{\infty,L}^{(k)}$ and $P_{\infty,R}^{(k)}$ identical and independent representing the percolation probabilities for a k–level tree and with $p_{e,L}$ and $p_{e,R}$, distributed according to $\tanh(J_{i,j}/2)$, representing the effective strength of the bonds connecting the root site to the two k–level trees situated above the root site. Let $\overline{P}_{\infty}^{(k)}$ denote the (quenched) average of $P_{\infty}^{(k)}$ and similarly let $\overline{p}_e$ denote the average of $p_{e,L}$ or $p_{e,R}$. It is assumed that the distribution for the $J_{i,j}$ depends on a parameter (e.g. width, temperature) that can be changed continuously. The phase transition occurs at $\overline{p}_e = 1/2$ and the exponent β is the same as in the non–random case.

Our analysis begins with the random analog of Eq. (4). After a certain amount of work, the relevant generalization is seen to be

$$X_{k+1} =_d 1 - P_{\infty}^{(k+1)} + \frac{p_{e,L} X_k (1 - p_{e,R} P_{\infty,R}^{(k)}) + p_{e,R} X_k' (1 - p_{e,L} P_{\infty,L}^{(k)})}{1 + p_{e,L} P_{\infty,L}^{(k)} p_{e,R} P_{\infty,R}^{(k)}} \tag{17}$$

Since the $(J_{i,j})$'s are all non–negative, all the terms in the denominator are non–negative and we may write

$$X_{k+1} \leq_d 1 - P_{\infty}^{(k+1)} + p_{e,L} X_k (1 - p_{e,R} P_{\infty,R}^{(k)}) + p_{e,R} X_k' (1 - p_{e,L} P_{\infty,L}^{(k)}) \tag{18}$$

Noting that all the relevant quantities in Eq. (18) are independent we perform the disorder average to obtain

$$\overline{X}_{k+1} \leq 1 - \overline{P}_{\infty}^{(k+1)} + 2\overline{p}_e \overline{X}_k (1 - \overline{p}_e \overline{P}_{\infty}^{(k)}) \tag{19}$$

In the setup with wired boundary conditions and all the $(J_{i,j})$ non–negative, it is not difficult to show that these average quantities tend to a definite limit as $k \to \infty$. Hence

$$\overline{X}_{\infty} \leq \frac{1 - \overline{P}_{\infty}}{1 - 2\overline{p}_e + \overline{p}_e^2 \overline{P}_{\infty}}. \tag{20}$$

Using $\overline{P}_{\infty} \sim (\overline{p}_e - 1/2)^{1/2}$ – which is not hard to prove – we obtain the (significant) first half: $\tilde{\gamma}' \leq 1/2$.

Opposite bounds are obtained by expanding the denominator

$$X_{k+1} \geq_d 1 - P_\infty^{(k+1)} + [p_{e,L} X_k (1 - p_{e,R} P_{\infty,R}^{(k)}) + p_{e,R} X_k' (1 - p_{e,L} P_{\infty,L}^{(k)})] \times$$

$$\times [1 - p_{e,L} p_{e,R} P_{\infty,L}^{(k)} P_{\infty,R}^{(k)}] \tag{21}$$

Neglecting positive terms and using $p_{e,L} P_{\infty,L}^{(k)} \leq 1$, $p_{e,R} P_{\infty,R}^{(k)} \leq 1$ we arrive at

$$\overline{X}_{k+1} \geq_d 1 - \overline{P}_\infty^{(k+1)} + 2\overline{p}_e \overline{X}_k - 4\overline{p}_e^2 \overline{P}_\infty^{(k)} \overline{X}_k \tag{22}$$

which leads to a bound similar to Eq. (20) but in the opposite direction. Hence we conclude $\tilde{\gamma}' = 1/2$.

C. The Complete Graph

Similar results can be established for the random cluster model on the complete graph. Here, the calculations are as straightforward as the Bethe lattice, however, the rigorous justification of these calculations requires some unpleasant analysis. We will again be content with the discussion of the exponent $\tilde{\gamma}'$.

The underlying lattice consists of N sites with bonds of uniform strength between all pairs. The weight for a bond configuration ω is given by $W(\omega) = B_{p_e(N)}(\omega) q^{l(\omega)} \propto B_{p_N}(\omega) q^{c(\omega)}$ with c the number of connected components and p_N defined to be $1 - e^{-J/N}$. C.f. [13] for a more detailed description of the random cluster model in this context. Let G denote the size of the largest (giant) cluster. As is not hard to show, the probability of belonging to this cluster, G/N, converges to $m(J)$ where $m(J)$ satisfies the mean field equation $m = [1 - e^{-Jm}]/[1 + (q-1)e^{-Jm}]$ [13]. We assume throughout that $q = 2$ and $J > 2$ so we are in the low–temperature phase.

If i is a site in the graph, let $C_i(N, p_N)$ denote the analog of the quantity c_k for the Bethe lattice; that is $C_i(N, p_N)$ is zero if i is connected to the giant cluster and otherwise is the size of the cluster at i. The strategy will be to fix G and obtain estimates on (the distribution of) C_i. These estimates are, more or less, the desired result if the random G is replaced by mN. The large deviation result of [13] in essence allows this replacement.

Thus suppose that there are G sites in the giant cluster of an N site graph with parameter p (not necessarily equal to p_N) and consider the cluster of the origin. For fixed ϵ, let us assume that $|G/N - m| < \epsilon$ – otherwise for the upper bounds we will assign $C_0 = N$ and for the lower bounds, $C_0 = 0$. We start with the upper bounds. As before, let F_0 denote the size of the cluster at 0 given that it is not attached to the giant cluster. Given the condition of detachment, the origin and the other $N - G - 1$ sites act like an autonomous random cluster model subject to the condition that no cluster has size exceeding G – and for the upper bounds, we may neglect this condition. We will use the methods introduced in [11,14] known as the Edwards–Sokal coupling which for complete graphs is particularly easy: Divide the remaining $N - G$ sites into two groups of N_1 and N_2 sites. One of these is identified as plus and the other as minus. There can be no bonds between spins of opposite type and bonds between spins of the same type occur with probability p_N. Taking into account the relevant energetics and combinatorial factor, the result is seen to be a complete graph Ising problem with $N - G$ sites and temperature parameter $p = p_N$. Since $G \geq (m(J) - \epsilon)N$ (with ϵ small) it is not hard to show that the Ising system is above the critical temperature. Thus, with large probability, N_1 is close to half of $N - G$ – say $|2N_1 - (N - G)| < \epsilon N$. Hence, when all is said and done, we are reduced to the problem of the cluster distribution for (subcritical) percolation on the complete graph with $N_1 \approx (N/2)(1 - m)$ sites and bond probability $\approx J/N$. Let us temporarily denote these parameters by n and α/n – with $\alpha < 1$. As is not hard to show, the distribution for the cluster size at any one of these sites is bounded by a Bernoulli branching process with a mean of α and a maximum of n offspring. Thus $F_0 \leq_d I$ where I satisfies the distributional equality

$$I = 1 + \frac{\alpha}{n} \sum_{j=1}^{n} I^{(j)} \tag{23}$$

with the $I^{(j)}$ independent and identical in distribution to I.

Let $g_N(\epsilon)$ denote the probability that $|G - mN| > \epsilon N$. Let $\phi_N(\epsilon)$ denote the probability that $|N_1 - 1/2(N-G)| > \epsilon N$ optimized over all G's such that $|G - mN| < \epsilon N$. Then, solving Eq. (23) (after expectation) we find

$$(1 - g_N)(1 - \phi_N)\langle F_0 \rangle_{N,J} \leq N g_N + (1 - g_N) \frac{N\phi_N + (1 - \phi_N)}{1 - \frac{1}{2}J(1-m) - 2J\epsilon} \tag{24}$$

Using the large deviation estimate of [13] for g_N and a similar (easily derived) estimate for ϕ_N we get, letting $N \to \infty$ and then $\epsilon \to 0$

$$\langle F_0 \rangle_J \leq \frac{1}{1 - \frac{1}{2}J(1-m)} \tag{25a}$$

hence

$$\langle C_0 \rangle_J \leq \frac{1-m}{1 - \frac{1}{2}J(1-m)}. \tag{25b}$$

The derivation of the lower bound involves a few more details. We still condition on $G \approx mN$ and $N_1 \approx N_2$; now we must pay lip service to the possibility of another large cluster. Let n and α be as before. We will imagine that there are already $n-1$ sites present and that we add the n^{th} at the origin. Let c satisfy $c^2 n = G$. Then, in order to get a cluster of size G, either (i) the pre-existing collection of $n-1$ sites must contain a cluster of size larger than $c\sqrt{n}$ or (ii) the new site must give rise to at least $c\sqrt{n}$ bonds. For ease of future exposition, we will replace (i) by the weaker condition that some cluster contains at least $c\sqrt{n}$ bonds (rather than sites). Denoting by $\mathbf{X}_i$ and $\mathbf{X}_{ii}$ the indicators of these events, it is not hard to see, by comparison with the aforementioned branching process, that for the described values of α and n, both probabilities tend to zero at least as fast as $\exp\{-b(\alpha)\sqrt{n}\}$ for some $b > 0$.

Now consider the clusters of the first $n-1$ sites which we denote by $K_1, K_2, \ldots K_s$. Let π_j denote the probability that the new site connects to the j^{th} cluster. Clearly $p|K_j| \geq \pi_j \geq p|K_j| - p^2|K_j|^2$. Hence

$$F_0 \geq (1 - \mathbf{X}_i)(1 - \mathbf{X}_{ii})[1 + \sum_{j=1}^{s}(p|K_j|^2 - p^2|K_j|^3)] \tag{26}$$

The first term in the sum is obviously $p\sum_j F_j$ – essentially our upper bound. But given that $\mathbf{X}_i \neq 1$, each $|K_j| \leq c\sqrt{n}$ and thus

$$F_0(N, p_N) \geq (1 - \mathbf{X}_i)(1 - \mathbf{X}_{ii})[1 + (1 - \frac{\alpha}{\sqrt{n}})\frac{\alpha}{n}\sum_j F_j(N-1, p_N). \tag{27}$$

The only remaining difficulty is that the $F's$ on the right hand side of Eq. (27) are slightly out of balance with regards to their arguments. However, the density p_N may be obtained from the density p_{N-1} by independently removing occupied bonds with probability $1/N$. Thus writing $F_j(N-1, p_N) = F_j(N-1, p_{N-1}) + F_j(N-1, p_N) - F_j(N-1, p_{N-1})$, the difference term is (distributionally) negative and may be bounded below by $-F_j(N-1, p_{N-1})$ if even a *single* bond in the cluster of j gets removed. However since we may operate under the stipulation that there are never more than $c\sqrt{n}$ bonds in any of these clusters, the probability of such a loss is of the order $N^{-1/2}$. Putting all these ingredients together, we arrive at the recursive inequality

$$\langle F_0 \rangle_{J,N} \geq [1 + \frac{1}{2}J(1 - m(J))\langle F_0 \rangle_{J,N}]e(N, \epsilon) \tag{28}$$

with $e(N, \epsilon) < 1$ satisfying $\lim_{\epsilon \to 0} \lim_{N \to \infty} e(N, \epsilon) = 0$ After a straightforward limiting argument, the desired result

$$\langle C_0 \rangle_J = \frac{1-m}{1-\frac{1}{2}J(1-m(J))} \tag{29}$$

now follows from the upper and lower bounds. This gives us $\tilde{\gamma}' = 1/2$ for the complete graph.

D. Conclusions/Speculations

As indicated by our notation the exponents $\tilde{\gamma}'$, $\tilde{\beta}$ and $\tilde{\nu}'$ have direct counterparts in spin–systems. A standard hyper-scaling relation, $d\gamma'/(2\beta+\gamma') = 2-\eta$, would therefore predict the upper critical dimension $d_c = 6$ (see also Ref. [15]) This, once again, is surprising since it differs from the usual value ($d_c = 4$) associated with Ising systems.

What has so far been demonstrated (in a tautological sense) is a breakdown in some of the anticipated relations between thermodynamic and geometric exponents. Let us illustrate this further. For magnetic systems, the critical state may be perturbed by a magnetic field h which serves to define the exponent δ. This exponent is related to the geometric exponent τ by the following argument: If $h \ll 1$, only clusters of size on the order of or exceeding $1/h$ will be aligned with the field. Hence, $m(h) \sim \sum_{n \gtrsim 1/h} P_n \sim h^{\tau-2} \equiv h^{1/\delta}$, i.e. $\tau - 2 = 1/\delta$. This relationship has broken down on the Bethe lattice and thus it may be presumed not to hold in sufficiently high dimension. It therefore must be reinterpreted as a hyperscaling relation. But there is a further point to be considered, namely that the above argument must also break down.

Although this argument is far from rigorous, it appears to be irrefutable *provided that one assumes that the distribution of clusters with* $h \gtrsim 0$ *is not significantly disturbed from the* $h = 0$ *distribution and that the contribution from the infinite cluster scales as the contribution from finite clusters.* We expect that the breakdown of $\tau - 2 = 1/\delta$ comes from the fact that the contribution from finite clusters which scale with an exponent $\tau - 2 = 1/2$ is different from the contribution of the infinite cluster which scales with the Ising exponent $1/\delta = 1/3$.

Let us now present some speculations/conjectures concerning the behavior of these systems in finite dimensions. Under the assumption that $d_c = 6$, the exponents $\tilde{\nu}'$, $\tilde{\gamma}'$ etc. calculated here would be valid for $d \geq 6$ (perhaps with logarithmic modifications at $d = 6$). This would give $\tau_{d=4} = 7/3$ while $\tau_{d=6} = 5/2$. But what about 5 dimensions? The simplest scenario is as follows: Noting that δ is not a geometric exponent, let us eliminate this object in favor of η (which is both geometric and thermodynamic and coincides with the Ising exponent) via the usual hyperscaling relation. This gives us the hyperscaling relation

$$\tau = \frac{3d + 2 - \eta}{d + 2 - \eta} \tag{30}$$

For $d \geq 4$ we may set $\eta = 0$. We note that this relation gives the right value for $d = 4$ and $d = 6$ and breaks down for $d > 6$ which is consistent with the upper critical dimensionality $d_c = 6$. We therefore may suppose the relation to hold for $d \leq 6$. In particular for $d = 5$, it gives us the prediction $\tau_{d=5} = 17/7$. Similar reasoning leads to the predictions for $4 \leq d \leq 6$ of $\tilde{\gamma}' = 2/(d-2)$ and $\tilde{\nu}' = 1/(d-2)$, i.e. $\tilde{\gamma}'_{d=5} = 2/3$ and $\tilde{\nu}'_{d=5} = 1/3$. Both hyperscaling and scaling relations predict $\tilde{\alpha}' = \frac{1}{2}$. On the other hand any *thermodynamic* definition concerning specific heat leads us to $\alpha' = 0$. This dichotomy may be due to the fact that the singularity $\tilde{\alpha}' = \frac{1}{2}$ has a vanishing amplitude. [15]

Perhaps the most significant feature uncovered by these calculations is the appearance of two divergent length scales, ξ' and $\tilde{\xi}'$ corresponding to the infinite and the finite clusters. Let us pause to reinterpret the former. Note that the (truncated) correlation function for the probability that two sites belong to the infinite cluster is also the correlation function for the probability that the pair does *not* belong to the infinite cluster. Thus, ξ' can be related to the typical scale of cavities in the infinite cluster. In this light, the equality of ξ' and $\tilde{\xi}'$ – as is the case for ordinary percolation

– is eminently reasonable: Cavities in the infinite cluster of sizes up to some scale (ξ') that are populated with finite cluster that range up to a comparable scale. However our situation is quite different. For $0 < T_c - T \ll 1$ (and $d > 4$) we expect the cavities to be filled with relatively small scale finite clusters.

Finally, let us emphasize the conclusions of the final calculations in Section B: The mean field exponents obtained all appear to be stable to the presence of disorder. Taken in conjunction with the previous discussion, – which includes the assumption/prediction that $d_c = 6$ – this would imply a violation of the Harris criterion–type bound "$\nu \geq 2/d$" in dimensions 5–8. Since the above bound holds in all systems where it is possible to define an equivalent *finite–size scaling* correlation length, [16] the implication here is that it is not possible to define such a length–scale.

We caution the reader that the discussions in this section are highly speculative; the various scenarios might all be wrong. We are not definitive in any of these "predictions" – they have all been made without the benefit of derivations or supplementary calculations (let alone rigorous proofs). However in $d = 5$ & 6 the system under consideration is certainly within reach of currently available numerical methods which would shed some light on these issues.

One of us (JM) was supported by NSF 96-32898.

REFERENCES

[1] L. Chayes and J. Machta, Physica A **239**, 542 (1997), L. Chayes and J. Machta, *Graphical Representations and Cluster Algorithms Part II* (Preprint), L. Chayes, *Discontinuity of the Spin-Wave Stiffness in the Two–Dimensional XY Model* (Preprint).

[2] C. M. Fortuin and P. W. Kasteleyn, Physica (Amsterdam) **57**, 536 (1972).

[3] A. Coniglio and W. Klein, J. Phys. A**13**, 2775 (1980).

[4] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. **58**, 86 (1987).

[5] M. Sweeny, Phys. Rev. B**27**, 4445 (1983).

[6] Throughout this work, we will simply *assume* the existence of critical exponents.

[7] J.T. Chayes, L. Chayes, G.R. Grimmett, H. Kesten and R. Schonmann, Ann. Probab. **17** 1277 (1989).

[8] J. T. Chayes, L. Chayes, J. P. Sethna, and D. J. Thouless, Commun. Math. Phys. **106**, 41 (1986).

[9] A. Coniglio, H. E. Stanley and W. Klein, Phys. Rev. B**25**, 6805 (1982).

[10] J. Roussenq, A. Coniglio and D. Stauffer, J. Physique Lett. **43**, L703 (1982).

[11] A. Coniglio, F. de Liberto, G. Monroy and F. Peruggi, J. Phys. A (Math. Gen.) **22**, L837 (1989).

[12] G. R. Grimmett, *Percolation* (Springer, Berlin, 1989).

[13] B. Ballobás, G. R. Grimmett, S. Janson, Probab. Theory Relat. Fields **104**, 283 (1996)

[14] R. G. Edwards and A. D. Sokal Phys. Rev. D **38**, 2009 (1988)

[15] A. Coniglio and T. C. Lubensky, J. Phys. A **13**, 1783 (1980).

[16] J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).