Two-Parameter Spectral Averaging and Localization for Non-Monotoneous Random Schrödinger Operators

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Abstract

We prove exponential localization at all energies for two types of one-dimensional random Schrödinger operators: the Poisson model and the random displacement model. As opposed to Anderson-type models, these operators are not monotoneous in the random parameters. Therefore the classical one-parameter version of spectral averaging, as used in localization proofs for Anderson models, breaks down. We use the new method of two-parameter spectral averaging and apply it to the Poisson as well as the displacement case. In addition, we apply results from inverse spectral theory, which show that two-parameter spectral averaging works for sufficiently many energies (all but a discrete set) to conclude localization at all energies.

1 Introduction

While the first rigorous proofs of localization for onedimensional random Schrödinger operators and their discrete counterparts were given earlier (e.g. [11, 22, 3]), it have been the results of Kotani on general ergodic potentials [18, 20] which have provided the deepest insights and become the basis of most subsequent work on onedimensional random operators. Kotani's results on Lyapunov exponents and their relation to spectral properties combined with the method of *spectral averaging* or the use of *Wegner estimates* have provided an elegant and widely applicable proof of onedimensional localization, see for example [29, 21, 4, 32].

Spectral averaging was originally introduced in the context of rank one perturbations and thus applied to situations involving random boundary conditions [19] in dimension $d = 1$ and the discrete Anderson model [29, 30], both in $d = 1$ and $d > 1$. Subsequently, it was understood that spectral averaging can be used on more general perturbations: 'rank one' can be replaced by 'monotonicity' – in form sense – and an unrestrictive compactness assumption. This lead to proofs of localization for continuum Anderson and related models, both in the onedimensional and multidimensional case, e.g. [21, 7, 8]. Other approaches to localization, in particular in higher dimension, rely on Wegner estimates and avoid spectral averaging, e.g. [9, 13, 15]. This also usually requires some sort of monotonicity in the random parameters.

There are, however, several physically important models where the monotonicity condition breaks down. Specifically, this is the case for the *Poisson model* as well as for the *random*

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displacement model to be introduced below. The main results of this paper are extensions of the method of spectral averaging which allow to prove localization at all energies for these models in dimension one. A partial result of this type, still based on a monotonicity property, was obtained for the Poisson model in [32].

Previous results on localization under lack of monotonicity were given in two papers by Klopp [17, 16]. In [17] a Wegner estimate for a multidimensional continuum Anderson model without sign-restriction on the single site potential is derived, while [16] provides a Wegner estimate for a semi-classical version of the multidimensional displacement model.

Here are the models together with the detailed assumptions which will be used hereafter in this text:

In both models we assume that the "single site potential" $f \in L^2_{loc}(\mathbb{R})$ is real-valued, $f \not\equiv 0$, with $\operatorname{supp} f \subset [-s, s]$ for some $s > 0$. Also, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with events $\mathcal{F}$ and probability measure $\mathbb{P}$.

The Displacement Model

Let $s < \frac{1}{2}$ and $d_n : \Omega \to [-d_{max}, d_{max}]$, $n \in \mathbb{Z}$, be independent and identically distributed random variables such that $d_{max} + s < \frac{1}{2}$. Assume also that the distribution measure of d_n has a non-trivial absolutely continuous component. For $\omega \in \Omega$ let

$$\begin{aligned} H_\omega^D &= -\frac{d^2}{dx^2} + V_\omega^D(x) \\ &= -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} f(x - n - d_n(\omega)), \end{aligned} \tag{1.1}$$

which is selfadjoint in $L^2(\mathbb{R})$ if defined as the closure of the corresponding operator on $C_0^\infty(\mathbb{R})$.

The d_n describe random displacements of particles in a crystal, modeling the disorder introduced by temperature effects. The existence of an absolutely continuous component in their distribution (or even pure absolute continuity) makes sense from the viewpoint of physics.

The Poisson Model

This is the random Schrödinger operator

$$\begin{aligned} H_\omega^P &= -\frac{d^2}{dx^2} + V_\omega^P \\ &= -\frac{d^2}{dx^2} + \sum_i f(x - x_i(\omega)), \end{aligned} \tag{1.2}$$

where the $x_i(\omega)$ for $\omega \in \Omega$ are the points of a Poisson process on $\mathbb{R}$ with constant density $\alpha > 0$. For almost every ω they can be labeled in a measurable way by $i \in \mathbb{Z} \setminus \{0\}$ such that

$$\ldots < x_{-1}(\omega) < 0 < x_1(\omega) < x_2(\omega) < \ldots$$

They have the property that for every Borel set $B \subset \mathbb{R}$

$$\mathbb{P}(\#\{i : x_i(\omega) \in B\} = n) = \frac{(\alpha|B|)^n}{n!} e^{-\alpha|B|}, \tag{1.3}$$

where $|\cdot|$ is Lebesgue measure. Also, if $B_1 \cap B_2 = \emptyset$, then the random variables $\#\{i : x_i(\omega) \in B_1\}$ and $\#\{i : x_i(\omega) \in B_2\}$ are independent. Setting for a moment also $x_0 := 0$ (*not* a Poisson point) we define the random distances

$$p_n = x_{n+1} - x_n, \quad n \in \mathbb{Z}. \tag{1.4}$$

The p_n are i.i.d. with distribution density $\alpha e^{-\alpha t}$ (for more details and references see [32]).

Since f is locally L^2 and compactly supported we have $\mathbb{E}(V_\omega^P(0)^2) < \infty$, e.g. [32] and thus H_ω^P is almost surely essentially selfadjoint on $C_0^\infty(\mathbb{R})$ [14].

The Poisson model is used to describe amorphous media, i.e. solids with extreme structural disorder.

H_ω^X is ergodic for both models $X = P$ and $X = D$, i.e. there is an ergodic dynamical system $\{T_y\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, where $y \in \mathbb{R}$ for $X = P$ and $y \in \mathbb{Z}$ for $X = D$, such that

$$U_y V_\omega^X U_{-y} = V_{T_y\omega}^X \qquad \text{for all } y \text{ and } \omega.$$

Here $(U_y f)(x) = f(x-y)$. This, in particular, implies that there are deterministic sets Σ_{ac}^X, Σ_{sc}^X and Σ_{pp}^X such that almost surely

$$\sigma_{ac}(H_\omega^X) = \Sigma_{ac}^X, \quad \sigma_{sc}(H_\omega^X) = \Sigma_{sc}^X, \quad \sigma_{pp}(H_\omega^X) = \Sigma_{pp}^X,$$

where σ_{ac}, σ_{sc} and σ_{pp} denote the absolutely continuous, singular continuous and point spectrum (closure of the set of eigenvalues), respectively. For the general theory of ergodic operators, see the monographs [5] and [27].

The assumption $f \in L^2_{loc}$ guarantees that the fundamental results of Kotani [18, 20] extend to the models considered here, see [14, Appendix B]. The generalizations of Kotani theory provided by Minami [23, 24] cover the case $f \in L^1_{loc}$, $f \geq 0$. While all our methods presented below work for general $f \in L^1_{loc}$, this does not seem to be covered by any of the known versions of Kotani theory (but probably could).

Since f is compactly supported, it follows from the independence properties of the Poisson respectively displacement model that H_ω^X is non-deterministic in the sense of [23] (generalizing [18]). Thus Kotani theory implies that the Lyapunov exponent $\gamma^X(E)$ is positive for almost every $E \in \mathbb{R}$. This in turn has the consequence that $\Sigma_{ac}^X = \emptyset$.

Our main result here is that in addition $\Sigma_{sc}^X = \emptyset$, and therefore the spectrum for both the Poisson model and the displacement model is pure point with probability one:

Theorem 1.1 $\Sigma_{ac}^P = \Sigma_{sc}^P = \emptyset$, *i.e.* H_ω^P *has pure point spectrum for almost every* ω. *The eigenfunctions decay exponentially at the rate of the Lyapunov exponent.*

Theorem 1.2 $\Sigma_{ac}^D = \Sigma_{sc}^D = \emptyset$, *i.e.* H_ω^D *has pure point spectrum for almost every* ω. *The eigenfunctions decay exponentially at the rate of the Lyapunov exponent.*

To our knowledge, Theorem 1.2 provides the first proof of localization at all energies for the "pure" onedimensional displacement model. In [16], Klopp studies the semi-classical model $-h^2\Delta + \sum_n f(x - n - d_n(\omega))$ in arbitrary dimension. For a certain class of single site potentials f, in particular $f \leq 0$, $f \in C_0^\infty$ with sufficiently small support, and under suitable assumptions on the distribution of the d_n, in particular the existence of a density, localization is established in certain energy regions near the bottom of the spectrum if h is sufficiently small. Some of the known proofs of localization for Anderson models extend to models with additional disorder in the displacement, but depend crucially on the existence of Anderson-type disorder, see for example [7, 17].

Theorem 1.1 extends the result of [32], where localization for the onedimensional Poisson model was proven at positive energies.

The problems arising from non-monotonicity can be illustrated by the breakdown of the methods in [32] for energies $E < 0$: One of the basic approaches to localization is to use positivity of the Lyapunov exponent in combination with the method of spectral averaging. The result for the Poisson model in [32] is based on spectral averaging for the one-parameter family of operators

$$H_a = -\frac{d^2}{dx^2} + V_a, \quad a \in (0, \infty),$$

where $V_a(x) = V(x+a)$ for $x < -a$, $V_a(x) = 0$ for $-a \le x < a$, and $V_a(x) = V(x-a)$ for $a \le x$. Here V is a fixed potential. The family H_a essentially arises by fixing all but one of the p_n in H_ω^P. For the Weyl-Titchmarsh spectral measures ρ_a of H_a it was shown in [32] that the "average measure"

$$\int_{a_1}^{a_2} \rho_a \, da \tag{1.5}$$

is absolutely continuous in $(0, \infty)$ for arbitrary $0 < a_1 < a_2 < \infty$. The proof of this fact uses monotoneous a-dependence of eigenvalues of certain finite box operators. This is established in [32] by using Prüfer transform methods going back to Carmona's work, but can most easily be understood in terms of phase space volumes: Let H_a^N be the restriction of H_a to $L^2(-N - a, N+a)$ with, say, Dirichlet boundary conditions and fixed N, then the number of eigenvalues of H_a^N below E is approximately (e.g. [28])

$$\int_{-N-a}^{N+a} \sqrt{(E - V_a)_+} \, dx,$$

which grows in a for $E > 0$, but remains constant for $E \le 0$. In fact, the average measure (1.5) is not necessarily absolutely continuous in $(-\infty, 0)$: To see this, fix an $E < 0$ and choose V on $(0, \infty)$ such that $-u'' + Vu = Eu$ has a square-integrable solution u_+ on $[0, \infty)$ with $u_+'(0)/u_+(0) = \sqrt{|E|}$ (a suitable rectangular well will do). In a related way, we can choose V on $(-\infty, 0)$ such that $-u'' + Vu = Eu$ has a solution u_- on $(-\infty, 0]$ with $u_-'(0)/u_-(0) = \sqrt{|E|}$. This choice of V in the definition of V_a provides an example in which E is an eigenvalue of H_a for every $a > 0$, the corresponding eigenfunction u_a given by

$$u_a(x) = \begin{cases} u_-(x+a), & x \le -a \\ c_1 \exp(\sqrt{|E|}x), & -a < x < a \\ c_2 u_+(x-a), & x \ge a \end{cases}$$

with suitable constants c_1 and c_2. It follows for this example that $\rho_a(\{E\}) > 0$ for all a and therefore $\int_{a_1}^{a_2} \rho_a(\{E\}) \, da > 0$. Thus the average measure has a non-trivial pure point component in $(-\infty, 0)$.

Similarly, fixing all but one d_n in the displacement model leads to a family of the form $\tilde{H}_a = -d^2/dx^2 + W(x) + f(x-a)$, where W is supported in $\mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and a sufficiently small such that $f(\cdot - a)$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$. It is again possible to make non-trivial choices of W and f which give rise to a negative eigenvalue of $\tilde{H}_a$ which is stationary in a.

Thus the traditional method of *one parameter* spectral averaging does not work in either model. Here are our basic ideas to overcome the problem of non-monotonicity:

(i) H^P_ω and H^D_ω depend on many parameters, which vary independently (p_n resp. d_n). If an eigenvalue E is stationary under varying one parameter (as in the above examples), then it should be less likely (respectively happen for fewer E's) that it is stationary under independent variation of two or more parameters. This leads to the idea of using *multiparameter spectral averaging* (here: two parameters). This will allow to exclude continuous spectrum from all but a discrete set of energies, which itself can not carry continuous spectrum.

(ii) "Monotonicity" of the model enters the proofs in [21] and [32] in the form of monotoneous dependence of Prüfer angles on one of the random parameters (equivalent to monotonicity of eigenvalues of finite box operators). Lacking monotonicity, we will use that it is sufficient to have analytic and non-constant dependence on the parameters. This shows *local monotonicity* away from a discrete set of parameters. To show non-constancy for the various models requires results from inverse spectral theory. Basically, these results say that if $f \neq 0$ in one of our models, then at least some solutions will satisfy different data than in the case $f = 0$. We will collect these results in Section 3.

(iii) It is sufficient to do spectral averaging locally in parameter intervals where the considerations from (ii) yield monotonicity. As long as these parameter values appear with positive probability (guaranteed by requiring absolutely continuous distributions), this leads to a proof of localization with positive probability. Non-randomness of the spectral type implies almost sure localization.

In a separate publication we will use similar ideas to generalize known results for continuum onedimensional Anderson type models of the form $H^A_\omega = -d^2/dx^2 + V_{per}(x) + \sum_n q_n(\omega) f(x-n)$, where V_{per} is a periodic background potential and the q_n are i.i.d. coupling constants. Almost sure localization at all energies can be shown for H^A_ω without sign restrictions on the single site potential f, i.e. without the monotonicity properties exploited in most of the existing work on Anderson models.

Our use of results from inverse spectral theory as given in Section 3 is closely related to ideas from scattering theory as was pointed out to us by S. Molchanov and B. Simon. The use of scattering theoretic concepts in proofs of localization was discussed in [14] as well as in [26]. We plan to investigate these connections further in future work. In particular, this should allow to extend our results to models with non-compactly supported single site potential f.

Some of the new ideas presented here, in particular the strategy of replacing monotonicity by analyticity and non-constancy, should also provide insights into related questions in higher dimensions. In particular, they may be useful in proving Wegner estimates for the Poisson model and for the 'non-semi-classical' version of the displacement model (i.e. without a small parameter h as required in [16].

The proof of our main results is organized as follows: In Section 2 we use known facts from Kotani theory and the method of subordinate solutions to reduce the proofs of localization to results on two-parameter spectral averaging. Results from inverse theory, which will enter the proofs of spectral averaging, are provided in Section 3. Sections 4 and 5 make up the technical core of the paper, the proofs of spectral averaging for the Poisson model and displacement model, respectively. The details of this are somewhat easier for the Poisson model, while the method used for the displacement model in Section 5 has the advantage of supplying a second independent proof of localization for the Poisson case. We outline this briefly in Section 6.

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2 General Concepts

Let $Q \in L^1_{loc}$ be real-valued and such that $L = -d^2/dx^2 + Q$ is limit point at $\pm\infty$, and let ρ be the corresponding Weyl-Titchmarsh spectral measure (cf. [6]), $\rho = \rho_{ac} + \rho_{sc} + \rho_{pp}$ the Lebesgue decomposition of ρ. Subordinacy theory [10] shows that

$$\rho_{ac}(\mathbb{R} \setminus M_{ac}) = 0, \quad \rho_{sc}(\mathbb{R} \setminus M'_{sc}) = 0, \tag{2.1}$$

where

$$\begin{aligned} M_{ac} &= \{E \in \mathbb{R} : \ Lu = Eu \text{ has no subordinate solution at } +\infty, \text{ or} \\ &\qquad\qquad\quad\ Lu = Eu \text{ has no subordinate solution at } -\infty\}, && (2.2) \\ M'_{sc} &= \{E \in \mathbb{R} : \ \text{There exists a solution of } Lu = Eu \text{ which is} \\ &\qquad\qquad\quad\ \text{subordinate at } +\infty \text{ and at } -\infty, \text{ but not in } L^2(\mathbb{R})\}. && (2.3) \end{aligned}$$

Defining also

$$\begin{aligned} M_{sc} = \{E \in \mathbb{R} : \ & \text{There exists a solution } u_+ \text{ of } Lu = Eu \text{ which is} \\ & \text{subordinate at } +\infty \text{ and } u_+ \notin L^2(0,\infty), \text{ or} \\ & \text{there exists a solution } u_- \text{ of } Lu = Eu \text{ which is} \\ & \text{subordinate at } -\infty \text{ and } u_- \notin L^2(-\infty,0)\}, \end{aligned} \tag{2.4}$$

one has $M'_{sc} \subset M_{sc}$, and thus

$$\rho_{sc}(\mathbb{R} \setminus M_{sc}) = 0. \tag{2.5}$$

Theorem 2.1 *Let V_ω be an ergodic potential such that the Lyapunov exponent $\gamma(E)$ is strictly positive for almost all $E \in \mathbb{R}$. Let M^ω_{ac} and M^ω_{sc} be the sets defined by (2.2) and (2.4) for $H_\omega = -d^2/dx^2 + V_\omega$, then*
(a) $|M^\omega_{ac}| = 0$ almost surely,
(b) $|M^\omega_{sc}| = 0$ almost surely.

Remarks: (i) The idea of using subordinacy theory in proofs of localization is due to Minami [25].

(ii) Theorem 2.1(a) and (2.1) imply that $\sigma_{ac}(H_\omega) = \emptyset$ almost surely, i.e. $\Sigma_{ac} = \emptyset$. The proof of Theorem 2.1(a) below actually shows the Pastur-Ishii theorem $\Sigma_{ac} \subset \overline{\{E : \gamma(E) = 0\}}^{ess}$. That this follows easily from subordinacy theory was noted in [2].

Proof of Theorem 2.1: Positivity of the Lyapunov exponent implies by the Osceledec-Ruelle theorem (e.g. [5]) that for almost every E there exist exponentially decaying solutions u_+ and u_- of $H_\omega u = Eu$ at $+\infty$ respectively $-\infty$ for a.e. ω. By Fubini 'a.e. E' and 'a.e. ω' can be exchanged. For these ω one has $|M^\omega_{ac}| = |M^\omega_{sc}| = 0$ since an E for which u_+ and u_- as above exist is neither in M^ω_{ac} nor in M^ω_{sc} (this uses that L^2-solutions are subordinate and that, up to a constant, at most one solution can be subordinate). $\square$

For the rest of this section we look at the cases where H_ω is either H^D_ω or H^P_ω. To this end, define the random variables ξ_n, $n \in \mathbb{Z}$, by

$$\xi_n = \begin{cases} d_n & \text{in case of the displacement model,} \\ p_n & \text{in case of the Poisson model,} \end{cases}$$

with distribution as given in Section 1.

For a given sequence $\xi = (\xi_n)$ let $H_\xi^D := -d^2/dx^2 + \sum_n f(x-n-\xi_n)$ and $H_\xi^P := -d^2/dx^2 + \sum_i f(x-x_i)$, where x_i are the unique numbers determined by (1.4) with $\xi_n = p_n$ (and $x_0 = 0$). Let either $X = D$ or $X = P$ and M_{sc}^ξ the set defined by (2.4) with $L = H_\xi^X$.

The existence of solutions as in (2.4) only depends on asymptotic properties of the potential. In particular, the set M_{sc}^ξ does not depend on (ξ_0, ξ_1), i.e. there exists a set $M_{sc} = M_{sc}((\xi_n)_{n\in\mathbb{Z}\setminus\{0,1\}})$ such that $M_{sc}^\xi = M_{sc}$ for all (ξ_0, ξ_1).

The results of the remaining Sections 3 to 6 of this paper are summarized in

Theorem 2.2 *Let $X = D$ or $X = P$. Then there exists a countable closed set $M \subset \mathbb{R}$ (only depending on the single site potential f) with the following property:*

If $(\xi_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ is given such that $|M_{sc}| = 0$ and $\mathbb{P}^{0,1}$ denotes the joint distribution of (ξ_0, ξ_1), then for every $E_0 \in \mathbb{R}\setminus M$ there exists $\varepsilon > 0$ such that

$$\mathbb{P}^{0,1}\left\{\sigma_{sc}(H_\xi^X) \cap (E_0-\varepsilon, E_0+\varepsilon) = \emptyset\right\} > 0. \tag{2.6}$$

From here the proofs of our main results Theorem 1.1 and Theorem 1.2 are completed as follows: By Theorem 2.1 one has $|M_{sc}(\omega)| = 0$ for a.e. ω, i.e. $|M_{sc}| = 0$ for a.e. $(\xi_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$. Theorem 2.2, the independence of (ξ_0, ξ_1) from $(\xi_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$, and Fubini imply

$$\mathbb{P}\{\sigma_{sc}(H_\omega^X) \cap (E_0-\varepsilon, E_0+\varepsilon) = \emptyset\} > 0.$$

Since $\sigma_{sc}(H_\omega^X) = \Sigma_{sc}^X$ for a.e. ω this implies $\Sigma_{sc}^X \cap (E_0-\varepsilon, E_0+\varepsilon) = \emptyset$. This holds for arbitrary E_0 in the open set $\mathbb{R}\setminus M$, thus $\Sigma_{sc}^X \subset M$. The countable set M can not support any continuous spectrum, thus $\Sigma_{sc}^X = \emptyset$. Exponential decay of eigenfunctions at the Lyapunov rate holds almost surely by standard arguments, cf. [21, Theorem A1].

This leaves us with having to prove Theorem 2.2, which will be done separately for $X = P$ and $X = D$. In fact, we will find in both cases that Theorem 2.2 holds for a discrete set M.

3 Some Inverse Theory

The following Theorem 3.1 contains the key observation which will lead to the proof of Theorem 2.2 and thus of localization for $X = P$, i.e. the Poisson model (see Section 4). Theorem 3.1 combined with Corollary 3.2 will play an equally important role in the treatment of the displacement model in Section 5.

To prove Theorem 2.2 for $X = P$ we need to study the operators H_ξ^P with all but the parameters $\xi_0 = p_0$ and $\xi_1 = p_1$ fixed. For this matter we introduce the two-parameter family $H_{a,b} = -d^2/dx^2 + V_{a,b}$ of self-adjoint operators in $L^2(\mathbb{R})$, where for $a \geq 0$ and $\infty \geq b \geq 0$

$$V_{a,b}(x) = \begin{cases} V(x), & x < 0 \\ 0, & 0 \leq x < a \\ q(x-a), & a \leq x < a+D \\ 0, & a+D \leq x < a+D+b \\ V(x-(a+D+b)), & a+D+b \leq x. \end{cases} \tag{3.1}$$

Self-adjointness (in the sense of SL-theory) is guaranteed by requiring $q \in L^1(0, D)$ and $V \in L^1_{loc}(\mathbb{R})$ real-valued and such that the differential expression $-d^2/dx^2 + V$ is limit point at $+\infty$ and $-\infty$. As in the counterexample given in Section 1 it is still possible to construct specific V and q such that for some $E < 0$ there is a solution of $-u'' + V_{a,b}u = Eu$ which satisfies

(i) $u(x) = Ce^{\sqrt{|E|}x}$ or $u(x) = Ce^{-\sqrt{|E|}x}$ in $[0, a]$, and

(ii) $u(x) = C'e^{\sqrt{|E|}x}$ or $u(x) = C'e^{-\sqrt{|E|}x}$ in $[a + D, a + D + b]$, and

(iii) $u \in L^2(\mathbb{R})$.

Thus E will be an eigenvalue of $H_{a,b}$ for *all* a and b, excluding the possibility of proving spectral averaging. However, it turns out that for fixed V and $q \neq 0$ this can happen for not more than a discrete set of negative E's. This is the content of Theorem 3.1 (note that (i) and (ii) above are equivalent to $u'(a)/u(a) \in \{-\sqrt{|E|}, \sqrt{|E|}\}$ and $u'(a+D)/u(a+D) \in \{-\sqrt{|E|}, \sqrt{|E|}\}$).

Theorem 3.1 *Let $q \in L^1(0, D)$ be real-valued with $q \neq 0$. Then the set $M_0 \subset (-\infty, 0]$ of all those $E \leq 0$ such that there exists a non-trivial solution u of $-u'' + qu = Eu$ on $[0, D]$ which satisfies*

$$\frac{u'(0)}{u(0)} \in \left\{-\sqrt{|E|}, \sqrt{|E|}\right\} \quad \text{and} \quad \frac{u'(D)}{u(D)} \in \left\{-\sqrt{|E|}, \sqrt{|E|}\right\} \tag{3.2}$$

is discrete.

(3.2) represents four different "spectral parameter dependent boundary value problems" for $-u'' + qu = Eu$, according to the different sign combinations of $\pm\sqrt{|E|}$. The Theorem says that if at least one of these boundary value problems has non-discrete negative "spectrum", then $q = 0$. (Of course, for the sign combinations $+-$ and $-+$ not even $q = 0$ gives negative spectrum.) The case $+-$, and only this one, is obvious since in this case M_0 coincides with the negative eigenvalues of $-d^2/dx^2 + q$ on $L^2(\mathbb{R})$.

Proof: We get a periodic potential Q on $\mathbb{R}$ by extending q periodically. Let T be the corresponding self-adjoint operator in $L^2(\mathbb{R})$. For $\alpha \in \mathbb{C}$ let f_α be the solution of $-u'' + Qu = -\alpha^2 u$ with initial conditions $f_\alpha(0) = 1$ and $f_\alpha'(0) = \alpha$. This gives entire functions $F^-(\alpha) := f_\alpha'(D) - \alpha f_\alpha(D)$ and $F^+(\alpha) := f_\alpha'(D) + \alpha f_\alpha(D)$. Suppose that M is not discrete. It follows $F^- \equiv 0$ or $F^+ \equiv 0$. We are going to show $Q = 0$, i.e. $q = 0$, which is a contradiction.

Case $F^- = 0$: We have $f_\alpha'(D) = \alpha f_\alpha(D)$ for all $\alpha \in \mathbb{C}$. Thus all functions f_α are eigenfunctions of the corresponding transfer-matrix problem for T. For $\alpha \in \mathbb{R} \setminus \{0\}$, corresponding to negative energies, f_α is real-valued. Therefore the corresponding eigenvalue $f_\alpha(D)$ is real. This gives $\sigma(T) \subset [0, \infty)$ by Floquet theory.

For $\alpha \in i\mathbb{R} \setminus \{0\}$, corresponding to positive energies, f_α is not a multiple of a real-valued function. The corresponding eigenvalue (except it is -1 or 1) therefore can not be real. Again by Floquet theory, this yields $(0, \infty) \subset \sigma(T)$, i.e. $\sigma(T) = [0, \infty)$. By a classical result of Borg [1], this characterizes $Q = 0$.

Case $F^+ \equiv 0$: This means that $f_\alpha'(D) = -\alpha f_\alpha(D)$ for all $\alpha \in \mathbb{C}$, giving $f_\alpha'(2D) = \alpha f_\alpha(2D)$. By the arguments in the first case we again get $Q = 0$. □

Corollary 3.2 *Let $q \in L^1(0, D)$ be real-valued, $q \neq 0$, $u_0(\cdot, \theta, E, q)$ the solution of the initial value problem*

$$-u'' + qu = Eu, \quad u(0) = \sin\theta, \quad u'(0) = \cos\theta \tag{3.3}$$

and M_1 the set of all those $E \in \mathbb{R}$ such that

$$u_0'(D, \theta, E, q)^2 + Eu_0(D, \theta, E, q)^2 = \cos^2\theta + E\sin^2\theta \tag{3.4}$$

for all $\theta \in [0, \pi)$. Then M_1 is a discrete set.

Note that (3.4) is equivalent to

$$\int_0^D q(x)u_0(x,\theta,E,q)u_0'(x,\theta,E,q)\,dx = 0. \tag{3.5}$$

The Corollary may be reformulated as follows: If the set of E's such that (3.4) respectively (3.5) holds for all θ has an accumulation point, then $q = 0$.

Proof: Suppose that M_1 is not discrete. Then it follows from analyticity of (3.4) in E (for fixed θ) that $M_1 = \mathbb{R}$. In particular, for arbitrary $E < 0$ and $\theta = \operatorname{arccot}(\pm\sqrt{|E|})$, i.e. $u'(0)/u(0) \in \{\pm\sqrt{|E|}\}$, it follows from (3.4) that $Eu(D)^2 + u'(D)^2 = 0$, i.e. $u'(D)/u(D) \in \{\pm\sqrt{|E|}\}$. By Theorem 3.1 this is only possible for $q = 0$. Contradiction. □

4 Spectral Averaging for the Poisson Model

Let $H_{a,b} = -d^2/dx^2 + V_{a,b}$ be the two-parameter family of self-adjoint operators defined by (3.1). We assume that $q \neq 0$. By $\rho_{a,b}$ we denote the Weyl-Titchmarsh spectral measure for $H_{a,b}$. Theorem 2.2 in the case $X = P$ will follow from Theorem 4.1 below, which establishes spectral averaging for the family $H_{a,b}$ away from $M_0 \cup \{0\}$, M_0 the set whose discreteness was proven in Theorem 3.1.

Theorem 4.1 *For fixed numbers $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$ and arbitrary Borel sets $B \subset \mathbb{R}$ define*

$$\rho(B) = \int_{b_1}^{b_2}\int_{a_1}^{a_2} \rho_{a,b}(B)\,da\,db. \tag{4.1}$$

Then the Borel measure ρ is absolutely continuous on $\mathbb{R} \setminus (M_0 \cup \{0\})$.

At positive energies it will in fact not be necessary to average over both parameters simultaneously, i.e. it will turn out that $\int_{a_1}^{a_2} \rho_{a,b}(B)\,da$ defines an absolutely continuous measure on $(0,\infty)$ for every b, which immediately implies that ρ is absolutely continuous on $(0,\infty)$. This part of our result is basically identical to Proposition 3 of [32], where a slightly different one-parameter model was used. We will comment on this briefly at the end of the section and for now concentrate on the more involved proof of the fact that ρ is absolutely continuous on $(-\infty, 0) \setminus M_0$.

For this it is sufficient to show the following: For every $E_0 \in (-\infty,0) \setminus M_0$ there exist $\varepsilon > 0$ and $C > 0$ such that

$$\int f(E)\,d\rho(E) \leq C \int f(E)dE \tag{4.2}$$

for every continuous non-negative f with support in $(E_0 - \varepsilon, E_0 + \varepsilon)$. Thus we in fact establish local Lipshitz continuity of ρ in $(-\infty,0) \setminus M_0$.

The proof of (4.2) will use Prüfer variables. For the solution u of $-u'' + Qu = Eu$ with $u(c) = \sin\theta$ and $u'(c) = \cos\theta$ let

$$r_c(x,\theta,E,Q) = (u(x)^2 + u'(x)^2)^{1/2}$$

be the Prüfer amplitude and

$$\phi_c(x,\theta,E,Q) = \arctan\frac{u(x)}{u'(x)} \quad \left(\text{resp. } \operatorname{arccot}\frac{u'(x)}{u(x)}\right)$$

be the Prüfer angle. Uniqueness of ϕ_c is achieved by requiring that $\phi_c(c,\theta,E,Q)=\theta$ and that $\phi_c(\cdot,\theta,E,Q)$ is continuous.

The estimate (4.2) will be a consequence of

Lemma 4.2 *For every $E_0\in(-\infty,0)\setminus M_0$ there exist $\varepsilon>0$ and $C=C(a_1,a_2,b_1,b_2)>0$ such that*

$$\int_{b_1}^{b_2}\int_{a_1}^{a_2} r_0(a+D+b+N,\theta,E,V_{a,b})^{-2}\,da\,db \le C \tag{4.3}$$

uniformly in $N>0$, $\theta\in[0,\pi)$ and $E\in[E_0-\varepsilon,E_0+\varepsilon]$.

Proof: The definition of the Prüfer variables yields

$$\begin{aligned}
& r_0(a+D+b+N,\theta,E,V_{a,b})^{-2}\\
&= r_0(a,\theta,E,0)^{-2} r_a(a+D+b+N,\phi_0(a,\theta,E,0),E,V_{a,b})^{-2}\\
&\le C_1 r_0(D+b+N,\phi_0(a,\theta,E,0),E,V_{0,b})^{-2}
\end{aligned} \tag{4.4}$$

locally uniformly in (a,θ,E). Here it was used that r_0 does not vanish and is continuous in (a,θ,E).

In order to get an estimate for the a-integral in (4.3) we would like to use (4.4) and substitute $\alpha(a):=\phi_0(a,\theta,E,0)$ for a. The problem arises that $\alpha(a)$ is stationary in a for $\theta=\pm\operatorname{arccot}\sqrt{|E|}$, causing a singularity of the Jacobian. This leads to the necessity of a different estimate for $r_0(a+D+b+N,\theta,E,V_{a,b})^{-2}$ if θ is close to $\pm\operatorname{arccot}\sqrt{|E|}$ (see (4.11) below). For these θ it holds by Theorem 3.1 (and a continuity argument) that $\phi_0(a+D,\theta,E,V_{a,b})\neq\pm\operatorname{arccot}\sqrt{|E|}$, since we only study energies $E\in(-\infty,0)\setminus M_0$. Thus, in this case, $\phi_0(a+D+b,\theta,E,V_{a,b})$ is non-stationary in b, which will allow us to estimate (4.3) by exchanging the order of integration, i.e. integrating over b first. Here are the details:

From $E_0\in(-\infty,0)\setminus M_0$ it follows that

$$\phi_0(D,\pm\operatorname{arccot}\sqrt{|E_0|},E_0,q)\notin\left\{\pm\operatorname{arccot}\sqrt{|E_0|}\right\}+\pi\mathbb{Z}.$$

This and the continuity of $\phi_0(D,\theta,E,q)$ in θ and E imply the existence of $\varepsilon>0$ such that

$$\phi_0(D,\theta,E,q)\notin\left\{\pm\operatorname{arccot}\sqrt{|E|}\right\} \tag{4.5}$$

for $E\in[E_0-\varepsilon,E_0+\varepsilon]$ and $\operatorname{dist}\Big(\theta,\{\pm\operatorname{arccot}\sqrt{|E|}\}\Big)\le\varepsilon$. From the "phase space" structure of the solutions of $-u''=Eu$ for $E<0$ (in particular the asymptotic character of the lines $u'=\sqrt{|E|}u$ and $u'=-\sqrt{|E|u}$) it follows that there exists an $\varepsilon'=\varepsilon'(a_2)>0$ such that

$$\left|\phi_0(a,\theta,E,0)+\operatorname{arccot}\sqrt{|E|}\right|\le\varepsilon \tag{4.6}$$

if $E\in[E_0-\varepsilon,E_0+\varepsilon]$, $a\in[a_1,a_2]$ and $\left|\theta+\operatorname{arccot}\sqrt{|E|}\right|\le\varepsilon'$ and such that

$$\left|\phi_0(a,\theta,E,0)-\operatorname{arccot}\sqrt{|E|}\right|\le\varepsilon \tag{4.7}$$

if $E\in[E_0-\varepsilon,E_0+\varepsilon]$, $a\in[a_1,a_2]$ and $\left|\theta-\operatorname{arccot}\sqrt{|E|}\right|\le\varepsilon'$.

For fixed but arbitrary $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$ the estimate (4.3) will now be shown separately for $\theta \in T_1 = T_1(E)$ and $\theta \in T_2 = T_2(E)$, where

$$T_1 := \left\{ \theta : \left| \theta - \operatorname{arccot} \sqrt{|E|} \right| \geq \varepsilon' \text{ and } \left| \theta + \operatorname{arccot} \sqrt{|E|} \right| \geq \varepsilon' \right\},$$

$$T_2 := \left\{ \theta : \left| \theta - \operatorname{arccot} \sqrt{|E|} \right| \leq \varepsilon' \text{ or } \left| \theta + \operatorname{arccot} \sqrt{|E|} \right| \leq \varepsilon' \right\}.$$

First assume that $\theta \in T_1$.

Then $\frac{d}{da}\phi_0(a,\theta,E,0) = \cos^2\phi_0 + E\sin^2\phi_0 \neq 0$ for all a. Thus by continuity

$$\left| \frac{d}{da}\phi_0(a,\theta,E,0) \right| \geq C_2 > 0$$

uniformly in $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$, $\theta \in T_1$ and $a \in [a_1, a_2]$.

This can be used to substitute $\alpha(a) := \phi_0(a,\theta,E,0)$ for a after integrating (4.4). Since $\alpha([a_1,a_2]) \subset [\alpha_1, \alpha_2]$ locally uniformly in (θ, E) it follows that

$$\begin{aligned} \int_{a_1}^{a_2} r_0(a+D+b+N,\theta,E,V_{a,b})^{-2}\,da &\leq \frac{C_1}{C_2}\int_{\alpha_1}^{\alpha_2} r_0(D+b+N,\alpha,E,V_{0,b})^{-2}\,d\alpha \\ &\leq \frac{C_1}{C_2}(|\alpha_2 - \alpha_1| + \pi). \end{aligned} \tag{4.8}$$

Here the last step follows by subdividing $[\alpha_1, \alpha_2]$ into intervals of length $\leq \pi$ and using the well known formula

$$\int_\theta^{\theta+\pi} r_0(x,\alpha,E,Q)^{-2}\,d\alpha = \pi \quad \text{for all } \theta, x, E, Q, \tag{4.9}$$

see e.g. Corollary 12 of [32]. Thus

$$\int_{b_1}^{b_2}\int_{a_1}^{a_2} r_0(\ldots)^{-2}\,da\,db \leq \frac{C_1}{C_2}(b_2 - b_1)(|\alpha_2 - \alpha_1| + \pi), \tag{4.10}$$

which completes the proof of Lemma 4.2 for the case $\theta \in T_1$.

In the remaining case $\theta \in T_2$ we will do the b-integration in (4.3) first. We start by estimating

$$\begin{aligned} & r_0(a+D+b+N,\theta,E,V_{a,b})^{-2} \\ &= r_0(a+D+b,\theta,E,V_{a,b})^{-2} \\ &\quad \times r_{a+D+b}(a+D+b+N,\phi_0(a+D+b,\theta,E,V_{a,b}),E,V_{a,b})^{-2} \\ &\leq C_3 r_0(N,\phi_0(a+D+b,\theta,E,V_{a,\infty}),E,V)^{-2} \end{aligned} \tag{4.11}$$

locally uniformly in (a,b,θ,E). In order to substitute $\beta = \phi_0(a+D+b,\theta,E,V_{a,\infty})$ for b it will be used that

$$\frac{d\beta}{db}(b) = \cos^2\phi_0(a+D+b,\theta,E,V_{a,\infty}) + E\sin^2\phi_0(a+D+b,\theta,E,V_{a,\infty}) \tag{4.12}$$

and that by "additivity" of the Prüfer angle

$$\phi_0(a+D+b,\theta,E,V_{a,\infty}) = \phi_0(b,\phi_0(D,\phi_0(a,\theta,E,0),E,q),E,0). \tag{4.13}$$

It holds by (4.5) and (4.6) resp. (4.7) that $\phi_0(D, \phi_0(a, \theta, E, 0), E, q) \notin \{\pm\text{arccot}\sqrt{|E|}\}$ for $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$, $\theta \in T_2$ and $a \in [a_1, a_2]$. Thus by (4.13) we get $\phi_0(a + D + b, \theta, E, V_{a,\infty}) \notin \{\pm\text{arccot}\sqrt{|E|}\}$ for $b \in [b_1, b_2]$. (4.12) yields $\frac{d\beta}{db} \neq 0$ and thus by continuity

$$\left|\frac{d\beta}{db}\right| \geq C_4 > 0$$

uniformly in the parameter range considered for (E, θ, a, b). From here the proof of Lemma 4.2 for $\theta \in T_2$ is completed as in the case $\theta \in T_1$, integrating over b first, using (4.9) (on β), and finally integrating over a. □

Proof of Theorem 4.1. Let $\rho_{-\infty}^{0,\theta}$ be the Weyl-Titchmarsh spectral measure for the selfadjoint realization of $-d^2/dx^2 + V$ on $(-\infty, 0)$ with boundary condition $u(0)\cos\theta - u'(0)\sin\theta = 0$. By a formula going back to work of Carmona (compare [5, Theorem III.3.6], for a proof under the assumptions used here see [31, Appendix B]) we have for every a and b that

$$d\rho_{a,b}(E) = \mathrm{w} - \lim_{N\to\infty} \frac{1}{\pi} \int_0^\pi r_0(a + D + b + N, \theta, E, V_{a,b})^{-2}\, d\rho_{-\infty}^{0,\theta}(E)\, d\theta.$$

Thus for continuous, non-negative f with support in $(E_0 - \varepsilon, E_0 + \varepsilon)$ we have

$$\begin{aligned}
&\int f(E)\, d\rho(E) \\
&= \int_{b_1}^{b_2} \int_{a_1}^{a_2} \int f(E)\, d\rho_{a,b}(E)\, da\, db \\
&= \int_{b_1}^{b_2} \int_{a_1}^{a_2} \lim_{N\to\infty} \frac{1}{\pi} \int_0^\pi \int f(E) r_0(a + D + b + N, \theta, E, V_{a,b})^{-2}\, d\rho_{-\infty}^{0,\theta}(E)\, d\theta\, da\, db.
\end{aligned} \tag{4.14}$$

The estimate (4.3) in Lemma 4.2, in particular its uniformity in θ, N and E, allow to exchange the (a, b)-integration and the N-limit in (4.14) by Fatou's lemma and to conclude

$$\begin{aligned}
\int f(E)\, d\rho(E) &\leq \frac{C}{\pi} \int_0^\pi \int f(E)\, d\rho_{-\infty}^{0,\theta}(E)\, d\theta \\
&= C \int f(E)\, dE,
\end{aligned}$$

where finally spectral averaging over θ was used, see [19].

It remains to comment on the changes, in fact simplifications, in the above proof which show that ρ is absolutely continuous on $(0, \infty)$. One has $\frac{d}{da}\phi_0(a, \theta, E, 0) = \cos^2\phi_0 + E\sin^2\phi_0 > 0$ for all $E > 0$, θ and a (No singular θ-values!). Thus, if $0 < E_1 < E_2 < \infty$, then by continuity $\frac{d}{da}\phi_0(a, \theta, E, 0) \geq C > 0$ uniformly in $E \in [E_1, E_2]$, $\theta \in [0, \pi)$, and $a \in [a_1, a_2]$. Proceeding as in the proof of Lemma 4.2, first integrating a and then b, yields (4.3) uniformly in $N > 0$, $\theta \in [0, \pi)$, and $E \in [E_1, E_2]$. This implies local Lipshitz continuity of ρ in $(0, \infty)$ as before. □

Proof of Theorem 2.2 in the case $X = P$:

Let $(\xi_n)_{n\in\mathbb{Z}\setminus\{0,1\}} = (p_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ be fixed and such that $|M_{sc}| = 0$, and let $p_0 \geq s$ and $p_1 \geq 2s$. Then $H_\xi^P = H_{a,b}$, where the potential V in (3.1) is determined by $(p_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ and $a = p_0 - s$, $b = p_1 - 2s$, $D = 2s$, $q(x) = f(x - s)$.

If M_0 is the set from Theorem 4.1, then $M := M_0 \cup \{0\}$ is discrete. Pick arbitrary $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$. Then by Theorem 4.1

$$\int_{b_1}^{b_2} \int_{a_1}^{a_2} \rho_{a,b}(M_{sc} \cap (\mathbb{R} \setminus M))\, da\, db = \rho(M_{sc} \cap (\mathbb{R} \setminus M)) = 0.$$

Thus $\rho_{a,b}(M_{sc} \cap (\mathbb{R} \setminus M)) = 0$ for (Lebesgue) almost every $(a,b) \in [a_1, a_2] \times [b_1, b_2]$. We infer from (2.5) that $\rho^{sc}_{a,b}(\mathbb{R} \setminus M) = 0$ and thus $\sigma_{sc}(H_{a,b}) \cap (\mathbb{R} \setminus M) = \emptyset$ for almost every $(a,b) \in [a_1, a_2] \times [b_1, b_2]$. Since p_0 and p_1 have absolutely continuous distribution with density $\alpha e^{-\alpha t}$ it follows that

$$\mathbb{P}^{0,1}\left\{\sigma_{sc}(H^P_\xi) \cap (\mathbb{R} \setminus M) = \emptyset\right\} \geq \int_{a_1+s}^{a_2+s} \alpha e^{-\alpha t} dt \cdot \int_{b_1+2s}^{b_2+2s} \alpha e^{-\alpha t} dt > 0.$$

This implies (2.6) since $\mathbb{R} \setminus M$ is open. □

5 Spectral Averaging for the Displacement Model

To prove Theorem 2.2 in the case $X = D$ we aim at Theorem 5.4 below, which is an analog to Theorem 4.1. However, the statements and proofs are somewhat more involved. Again, we exclude a set M of "singular" energies, see (5.1) below. For $E_0 \in \mathbb{R} \setminus M$ we will establish spectral averaging in a neighborhood of E_0 for the family $-d^2/dx^2 + \tilde{V}_{a,b}$ defined by (5.7) below. The averaging will be done over $(a,b) \in [a_0 - \delta, a_0 + \delta] \times [b_0 - \delta, b_0 + \delta]$ with $a_0, b_0 \in (0, 1-D)$. By picking $\delta > 0$ sufficiently small we can guarantee that for every initial angle θ either the Prüfer angle $\phi_0(1, \theta, \tilde{V}_{a,b})$ is non-stationary in a or $\phi_0(2, \theta, \tilde{V}_{a,b})$ is non-stationary in b (for fixed a). In fact, this will require to exclude a finite set (depending on E_0 and a_0) of possible choices for b_0, see Lemma 5.2 below. After these preparations, Lemma 5.3 below will be proven in the same way as the corresponding Lemma 4.2 in the Poisson case.

Let $q \in L^1(0, D)$ be real valued and non-zero, and $u_0(\cdot, \theta, E, q)$ the solution defined in Corollary 3.2. The set

$$\begin{aligned} M \;:=\; & \Big\{E > 0 :\, u_0'(D, \theta, E, q)^2 + E u_0(D, \theta, E, q)^2 \\ & \quad = \cos^2\theta + E \sin^2\theta \text{ for all } \theta \in \mathbb{R}\Big\} \\ & \cup \quad \{0\} \quad \cup \\ & \Big\{E < 0 :\, \text{There is a solution } u \text{ of } -u'' + qu = Eu \text{ such that} \\ & \quad \frac{u'(0)}{u(0)}, \frac{u'(D)}{u(D)} \in \{-\sqrt{|E|}, \sqrt{|E|}\}\Big\} \end{aligned} \tag{5.1}$$

is discrete in $\mathbb{R}$ because $M \cap (-\infty, 0]$ is discrete in $(-\infty, 0]$ by Theorem 3.1 and $M \cap [0, \infty)$ is discrete by Corollary 3.2. Note that $M_1 \subset M$, where M_1 is the set from Corollary 3.2.

Let the Prüfer amplitude $r_c(x, \theta, E, Q)$ and the Prüfer angle $\phi_c(x, \theta, E, Q)$ be defined as in Section 4. We will often drop the parameter E. For $0 < D < 1$, $q \in L^1(0, D)$ real-valued and non-zero, and $a \in (0, 1-D)$ let

$$\tilde{V}_a(x) := \begin{cases} q(x-a), & a \leq x \leq a + D, \\ 0, & \text{elsewhere in } [0,1]. \end{cases} \tag{5.2}$$

Lemma 5.1

$$\begin{aligned}\frac{d}{da}\phi_0(1,\theta,\tilde{V}_a) &= -\cos^2\phi_0(1,\theta,\tilde{V}_a) - E\sin^2\phi_0(1,\theta,\tilde{V}_a)\\ &\quad + \frac{\cos^2\phi_0(a,\theta,0)+E\sin^2\phi_0(a,\theta,0)}{r_a^2(1,\phi_0(a,\theta,0),\tilde{V}_a)}\end{aligned}$$

Proof: We have

$$\begin{aligned}\phi_0(1,\theta,\tilde{V}_a) &= \phi_{a+D}(1,\phi_0(D,\phi_0(a,\theta,0),q),0)\\ &= \phi_{\alpha(a)}(1,\beta(a),0),\end{aligned}$$

where $\alpha(a)=a+D$ and $\beta(a)=\phi_0(D,\phi_0(a,\theta,0),q)$. Setting also $f(\alpha,\beta)=\phi_\alpha(1,\beta,0)$ we get

$$\frac{d}{da}\phi_0(1,\theta,\tilde{V}_a) = \frac{d}{da}f(\alpha(a),\beta(a)) = \frac{\partial f}{\partial\alpha} + \beta'\frac{\partial f}{\partial\beta}. \tag{5.3}$$

By a well known formula (e.g. [32, Proposition 11]) it holds that

$$\frac{\partial f}{\partial\beta} = \frac{1}{r_\alpha^2(1,\beta,0)}. \tag{5.4}$$

We have $\frac{d}{da}\phi_0(a,\theta,0) = \cos^2\phi_0(a,\theta,0)+E\sin^2\phi_0(a,\theta,0)$. This and the argument used in (5.4) yields

$$\beta'(a) = \frac{\cos^2\phi_0(a,\theta,0)+E\sin^2\phi_0(a,\theta,0)}{r_0^2(D,\phi_0(a,\theta,0),q)}. \tag{5.5}$$

Finally, to calculate $\partial f/\partial\alpha$, we use that

$$\phi_\alpha(1,\beta,0) = -\phi_1(\alpha,\pi-\beta,0) \pmod{\pi}.$$

Thus,

$$\begin{aligned}\frac{\partial f}{\partial\alpha} &= -\cos^2\phi_1(\alpha,\pi-\beta,0) - E\sin^2\phi_1(\alpha,\pi-\beta,0)\\ &= -\cos^2\phi_\alpha(1,\beta,0) - E\sin^2\phi_\alpha(1,\beta,0).\end{aligned} \tag{5.6}$$

Inserting (5.4), (5.5) and (5.6) in (5.3) concludes the proof. □

Let $0<D<1$, $q\in L^1(0,D)$ real valued and non-zero, and $a,b\in(0,1-D)$. Let

$$\tilde{V}_{a,b}(x) = \begin{cases} q(x-a) & \text{for } a\le x\le a+D,\\ q(x-1-b) & \text{for } 1+b\le x\le 1+b+D,\\ 0 & \text{elsewhere in } [0,2].\end{cases} \tag{5.7}$$

Fix an $E_0\in\mathbb{R}\setminus M$. Also fix $a\in(0,1-D)$.

Lemma 5.2 *There exists a finite subset $M(a,E_0)\subset(0,1-D)$ with the following property: If $b\in(0,1-D)\setminus M(a,E_0)$, then there exist closed sets $T_1\subset\mathbb{R}$ and $T_2\subset\mathbb{R}$ such that*
(i) $T_1\cup T_2=\mathbb{R}$,
(ii) $\frac{d}{da}\phi_0(1,\theta,\tilde{V}_a,E_0)\neq 0$ for all $\theta\in T_1$,
(iii) $\frac{d}{db}\phi_0(2,\theta,\tilde{V}_{a,b},E_0)\neq 0$ for all $\theta\in T_2$.

Proof: Lemma 5.1 and $r_0(1,\theta,\tilde{V}_a) = r_0(a,\theta,0) r_a(1,\phi_0(a,\theta,0),\tilde{V}_a)$ imply that

$$
\begin{aligned}
& \frac{d}{da}\phi_0(1,\theta,\tilde{V}_a) = 0 \\
\iff \quad & u_0'(D,\phi_0(a,\theta,0),q)^2 + E_0 u_0(D,\phi_0(a,\theta,0),q)^2 \\
& = \cos^2\phi_0(a,\theta,0) + E_0 \sin^2\phi_0(a,\theta,0),
\end{aligned} \tag{5.8}
$$

where it was also used that $u_0'(\cdot,\theta,\tilde{V}_a)^2 + E_0 u_0(\cdot,\theta,\tilde{V}_a)^2$ is constant in $[a+D,1]$.

Since $E_0 \in \mathbb{R}\setminus M$ and therefore $E_0 \in \mathbb{R}\setminus M_1$ it follows from analyticity in α that

$$N(E_0) := \{\alpha \in \mathbb{R} : \ u_0'(D,\alpha,q)^2 + E_0 u_0(D,\alpha,q)^2 = \cos^2\alpha + E_0\sin^2\alpha\}$$

is a discrete and, obviously, π-periodic subset of $\mathbb{R}$. The function $\alpha(\theta) = \phi_0(a,\theta,0)$ is strictly increasing in θ and such that $\alpha(\theta+\pi) = \alpha(\theta)+\pi$ for all θ. This implies that

$$N(a,E_0) := \{\theta \in \mathbb{R} : \ \phi_0(a,\theta,0) \in N(E_0)\}$$

is discrete and π-periodic. By (5.8) we have

$$\frac{d}{da}\phi_0(1,\theta,\tilde{V}_a) \neq 0 \quad \text{for } \theta \in \mathbb{R}\setminus N(a,E_0). \tag{5.9}$$

Let

$$M(a,E_0) := \left\{ b \in (0,1-D) : \ \frac{d}{db}\phi_0(2,\theta,\tilde{V}_{a,b}) = 0 \text{ for some } \theta \in N(a,E_0) \right\}.$$

As in Lemma 5.1 one gets

$$
\begin{aligned}
\frac{d}{db}\phi_0(2,\theta,\tilde{V}_{a,b}) &= -\cos^2\phi_0(2,\theta,\tilde{V}_{a,b}) - E_0\sin^2\phi_0(2,\theta,\tilde{V}_{a,b}) \\
&\quad + \frac{u_0'(1+b,\theta,\tilde{V}_a)^2 + E_0 u_0(1+b,\theta,\tilde{V}_a)^2}{r_0^2(2,\theta,\tilde{V}_{a,b})}
\end{aligned} \tag{5.10}
$$

and as in (5.8) this implies

$$
\begin{aligned}
& \frac{d}{db}\phi_0(2,\theta,\tilde{V}_{a,b}) = 0 \\
\iff \quad & u_0'(D,\phi_0(b,\eta,0),q)^2 + E_0 u_0(D,\phi_0(b,\eta,0),q)^2 \\
& = \cos^2\phi_0(b,\eta,0) + E_0\sin^2\phi_0(b,\eta,0),
\end{aligned} \tag{5.11}
$$

where $\eta = \phi_0(1,\theta,\tilde{V}_a)$. This holds if and only if $\phi_0(b,\eta,0) \in N(E_0)$. For fixed θ and thus fixed η the set

$$\{b \in \mathbb{R} : \ \phi_0(b,\eta,0) \in N(E_0)\}$$

is discrete. (This is clear for $E_0 > 0$. If $E_0 < 0$ then $\phi_0(b,\eta,0)$ is either strictly monotoneous in b or $\phi_0(b,\eta,0) = \eta$ constantly in b. The latter would require $\cot\eta \in \pm\sqrt{|E_0|}$ and can not happen in combination with $\phi_0(b,\eta,0) \in N(E_0)$, since (5.11) would yield a contradiction to $E_0 \notin M$.) From this and π-periodicity one gets that

$$\tilde{M}(a,E_0) := \{b \in \mathbb{R} : \ \phi_0(b,\phi_0(1,\theta,V_a),0) \in N(E_0) \quad \text{for some } \theta \in N(a,E_0)\}$$

is discrete. Thus $M(a,E_0) = \tilde{M}(a,E_0) \cap (0,1-D)$ is finite.

For $b \in (0, 1-D) \setminus M(a, E_0)$ one has $\frac{d}{db}\phi_0(2, \theta, \tilde{V}_{a,b}) \neq 0$ for all $\theta \in N(a, E_0)$. Continuity and periodicity of $\frac{d}{db}\phi_0(2, \theta, \tilde{V}_{a,b})$ in θ (compare (5.10)) implies the existence of $\varepsilon > 0$ such that $\frac{d}{db}\phi_0(2, \theta, \tilde{V}_{a,b}) \neq 0$ for all $\theta \in T_2 := \{\theta : \operatorname{dist}(\theta, N(a, E_0)) \leq \varepsilon\}$. By (5.9) one has $\frac{d}{da}\phi_0(1, \theta, \tilde{V}_a) \neq 0$ for $\theta \in T_1 := \{\theta : \operatorname{dist}(\theta, N(a, E_0)) \geq \varepsilon\}$. This completes the proof since T_1 and T_2 are closed with $T_1 \cup T_2 = \mathbb{R}$. $\square$

Let $V \in L^1_{loc}(\mathbb{R})$ be real-valued and such that $-\frac{d^2}{dx^2} + V$ is limit point at $+\infty$ and $-\infty$. Extend the definition of $\tilde{V}_{a,b}$ in (5.7) to the whole line by setting $\tilde{V}_{a,b}(x) = V(x)$ for $x < 0$ and for $x > 2$.

Lemma 5.3 *Fix $E_0 \in \mathbb{R} \setminus M$. Let $a_0 \in (0, 1-D)$ and $b_0 \in (0, 1-D)$ be given such that there exist closed sets $T_1 \subset \mathbb{R}$ and $T_2 \subset \mathbb{R}$ which satisfy (i), (ii) and (iii) of Lemma 5.2. Then there exist $\delta > 0$, $\varepsilon > 0$ and $C > 0$ such that*

$$\int_{a_0-\delta}^{a_0+\delta} \int_{b_0-\delta}^{b_0+\delta} r_0(N, \theta, E, \tilde{V}_{a,b})^{-2}\, db\, da \leq C \tag{5.12}$$

uniformly in $N > 0$, $\theta \in [0, \pi)$ and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$.

Proof: It is sufficient to assume $N > 2$. We will treat the two cases $\theta \in T_1$ and $\theta \in T_2$ separately. First, assume $\theta \in T_1$.

Similar to (4.4) we have

$$\begin{aligned} r_0(N, \theta, \tilde{V}_{a,b})^{-2} &= r_0(1, \theta, \tilde{V}_a)^{-2} r_1(N, \phi_0(1, \theta, \tilde{V}_a), \tilde{V}_{a,b})^{-2} \\ &\leq C_1 r_0(N-1, \phi_0(1, \theta, \tilde{V}_a), \tilde{V}_b)^{-2} \end{aligned} \tag{5.13}$$

locally uniformly in (a, θ, E). Let $\alpha := \phi_0(1, \theta, \tilde{V}_a)$. Then $\frac{d\alpha}{da} = \frac{d}{da}\phi_0(1, \theta, \tilde{V}_a) \neq 0$ at (a_0, E_0). Continuity in (a, θ, E) implies that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\left|\frac{d\alpha}{da}\right| \geq C_2 > 0$$

uniformly in $\theta \in T_1$, $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$ and $a \in [a_0 - \delta, a_0 + \delta]$. Substituting α for a and using the argument from the proof of (4.8) and (4.10) we conclude that

$$\int_{b_0-\delta}^{b_0+\delta} \int_{a_0-\delta}^{a_0+\delta} r_0(N, \theta, \tilde{V}_{a,b})^{-2}\, da db \leq C$$

uniformly in $N > 2$, $\theta \in T_1$ and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$.

Next, consider the case $\theta \in T_2$, in which we will do the b-integration in (5.12) first. We have

$$\begin{aligned} r_0(N, \theta, \tilde{V}_{a,b})^{-2} &= r_0(2, \theta, \tilde{V}_{a,b})^{-2} r_2(N, \phi_0(2, \theta, \tilde{V}_{a,b}), \tilde{V}_{a,b})^{-2} \\ &\leq C_3 r_0(N-2, \phi_0(2, \theta, \tilde{V}_{a,b}), V(\cdot + 2))^{-2} \end{aligned}$$

locally uniformly in (a, b, θ, E). Setting $\beta(b) := \phi_0(2, \theta, \tilde{V}_{a,b})$ it holds that $\frac{d\beta}{db} \neq 0$ at (a_0, b_0, E_0). We now use continuity of $\frac{d}{db}\phi_0(2, \theta, \tilde{V}_{a,b})$ in (a, b, θ, E) and continue to argue very similar to the case $\theta \in T_1$, doing the b-integration first. We conclude the existence of $\varepsilon > 0$ and $\delta > 0$ such that

$$\int_{a_0-\delta}^{a_0+\delta} \int_{b_0-\delta}^{b_0+\delta} r_0(N, \theta, \tilde{V}_{a,b})^{-2}\, db\, da \leq C' < \infty$$

uniformly in $N > 2$, $\theta \in T_2$ and $E \in [E_0 - \varepsilon, E_0 + \varepsilon]$.

This implies (5.12) with the smaller of the two values found for ε respectively δ and the larger of the constants C and C'. $\square$

Finally, denoting by $\tilde{\rho}_{a,b}$ the spectral measure of $-d^2/dx^2 + \tilde{V}_{a,b}$, the next result follows from Lemma 5.3 in the same way as Theorem 4.1 followed from Lemma 5.2:

Theorem 5.4 *For $E_0 \in \mathbb{R} \setminus M$, a_0, b_0, δ and ε as in Lemma 5.3 and Borel sets B define*

$$\tilde{\rho}(B) = \int_{a_0-\delta}^{a_0+\delta} \int_{b_0-\delta}^{b_0+\delta} \tilde{\rho}_{a,b}(B)\, db\, da. \tag{5.14}$$

Then the Borel measure $\tilde{\rho}$ is absolutely continuous in $(E_0 - \varepsilon, E_0 + \varepsilon)$.

Proof of Theorem 2.2 in the case $X = D$: Let $(\xi_n)_{n\in\mathbb{Z}\setminus\{0,1\}} = (d_n)_{n\in\mathbb{Z}\setminus\{0,1\}}$ be fixed such that $|M_{sc}| = 0$. Then for arbitrary $d_0, d_1 \in [-d_{max}, d_{max}]$ the operator H_ξ^D is, up to translation by 1/2, given by $-d^2/dx^2 + \tilde{V}_{a,b}$, where $\tilde{V}_{a,b}(x) = \sum_n f(x - \frac{1}{2} - n - d_n)$ for $x \in \mathbb{R} \setminus [0, 2]$ and $\tilde{V}_{a,b}(x)$ is defined by (5.7) with $q(x) := f(x - (\frac{1}{2} - d_{max}))$, $a := d_{max} + d_0$, $b := d_{max} + d_1$ if $x \in [0, 2]$.

Let M be the discrete set defined in (5.1) and $E_0 \in \mathbb{R} \setminus M$. Let μ be the distribution of the random variables d_0 respectively d_1 and μ_{ac} its absolutely continuous part. Pick a_0 such that $a_0 - d_{max} \in \operatorname{supp}\mu_{ac}$ and b_0 such that $b_0 \notin M(a_0, E_0)$, the finite set from Lemma 5.2, and $b_0 - d_{max} \in \operatorname{supp}\mu_{ac}$. This is possible since $\operatorname{supp}\mu_{ac}$ is infinite.

Lemma 5.2 shows that Lemma 5.3 and thus Theorem 5.4 apply with this choice of E_0, a_0 and b_0. Thus the measure $\tilde{\rho}$ defined by (5.14) is absolutely continuous in $(E_0 - \varepsilon, E_0 + \varepsilon)$ for suitable $\varepsilon > 0$ and $\delta > 0$. Thus $\tilde{\rho}_{a,b}(M_{sc} \cap (E_0 - \varepsilon, E_0 + \varepsilon)) = 0$ and therefore $\sigma_{sc}(-d^2/dx^2 + \tilde{V}_{a,b}) \cap (E_0 - \varepsilon, E_0 + \varepsilon) = \emptyset$ for almost every $(a, b) \in [a_0 - \delta, a_0 + \delta] \times [b_0 - \delta, b_0 + \delta]$. Since $a_0 - d_{max}, b_0 - d_{max} \in \operatorname{supp}\mu_{ac}$ this implies that $\mathbb{P}^{0,1}\{\sigma_{sc}(H_\xi^D) \cap (E_0 - \varepsilon, E_0 + \varepsilon)\} > 0$. $\square$

6 A second proof of localization for the Poisson model

Consider a Poisson process on $\mathbb{R}$ with density $\alpha > 0$ and let $I \subset \mathbb{R}$ be a non-trivial finite interval. The event that I contains exactly one Poisson point X has positive probability $\alpha|I|e^{-\alpha|I|}$. Conditioned on this event, the point X has uniform distribution on I, see e.g. [12, p.352f]. Thus we have some 'displacement-type randomness' in the Poisson potential V_ω^P. Using also the independence properties of the Poisson process, this allows us to apply the methods developed in the context of the displacement model and thereby to provide an alternate proof of localization for the Poisson model. We briefly outline this proof below, mainly for the reason that uniform conditional distribution of Poisson points generalizes to dimension $d > 1$, while the approach of using the distances p_n from (1.4) does not. This may provide a strategy for proving localization in the multidimensional Poisson model.

For the Poisson potential $V_\omega^P(x) = \sum_i f(x - x_i(\omega))$ with $\operatorname{supp} f \subset [-s, s]$ as defined in Section 1, choose $L > 2s$ and $d \in (0, \frac{L}{2} - s)$. Let $\Omega_0 \subset \Omega$ be the event that none of the intervals $[-s, \frac{L}{2} - d]$, $[\frac{L}{2} + d, \frac{3L}{2} - d]$ and $[\frac{3L}{2} + d, 2L + s]$ contain any Poisson points, while each of the intervals $[\frac{L}{2} - d, \frac{L}{2} + d]$ and $[\frac{3L}{2} - d, \frac{3L}{2} + d]$ contains exactly one Poisson point. By (1.3) and independence of the number of points in disjoint intervals we have $\mathbb{P}(\Omega_0) > 0$.

For $\omega \in \Omega_0$, let $X_1(\omega)$ and $X_2(\omega)$ be the unique Poisson points in $[\frac{L}{2} - d, \frac{L}{2} + d]$ and $[\frac{3L}{2} - d, \frac{3L}{2} + d]$, respectively.

On Ω_0, the potentials $V^P|_{\mathbb{R}\setminus[0,2L]}$ and $V^P|_{[0,2L]} = f(\cdot - X_1) + f(\cdot - X_2)$ are independent with respect to the conditional probability $\mathbb{P}_{\Omega_0}$, since $f(\cdot - X_1) + f(\cdot - X_2)$ does not overlap with $f(\cdot - x_i)$ for any of the remaining points $x_i \in \mathbb{R} \setminus [-s, 2L+s]$.

Also, X_1 and X_2 are independent and uniformly distributed with respect to $\mathbb{P}_{\Omega_0}$, i.e. for $I_1 \subset [\frac{L}{2} - d, \frac{L}{2} + d]$ and $I_2 \subset [\frac{3L}{2} - d, \frac{3L}{2} + d]$ it holds that

$$\mathbb{P}_{\Omega_0}(X_1 \in I_1, X_2 \in I_2) = \frac{|I_1|}{2d} \cdot \frac{|I_2|}{2d}.$$

This allows to apply the results of Section 5 (the unit intervals $[0,1]$ and $[1,2]$ replaced by $[0,L]$ and $[L,2L]$) and prove an analog of Theorem 2.2: For fixed $V^P|_{\mathbb{R}\setminus[0,2L]}$ with $|M_{sc}| = 0$ (M_{sc} the support of $\sigma_{sc}(H_\omega^P)$ for arbitrary X_1, X_2), it holds that

$$\mathbb{P}^{X_1,X_2}\left\{\sigma_{sc}(H_\omega^P) \cap (E_0 - \varepsilon, E_0 + \varepsilon) = \emptyset\right\} > 0 \tag{6.1}$$

for $E_0 \in \mathbb{R} \setminus M$ and suitable $\varepsilon > 0$. Here $\mathbb{P}^{X_1,X_2}$ is the distribution of (X_1, X_2) on Ω_0, M the same discrete set as used in the proof of Theorem 2.2 for $X = D$ in Section 5.

As in Section 2 it follows from positivity of the Lyapunov exponent that $|M_{sc}| = 0$ for $\mathbb{P}_{\Omega_0}$-a.e. $V^P|_{\mathbb{R}\setminus[0,2L]}$. Thus, (6.1) and independence of $V^P|_{\mathbb{R}\setminus[0,2L]}$ and $V^P|_{[0,2L]}$ yield

$$\mathbb{P}_{\Omega_0}\left\{\sigma_{sc}(H_\omega^P) \cap (E_0 - \varepsilon, E_0 + \varepsilon) = \emptyset\right\} > 0.$$

We conclude $\mathbb{P}\{\sigma_{sc}(H_\omega^P) \cap (E_0 - \varepsilon, E_0 + \varepsilon) = \emptyset\} > 0$ since $\mathbb{P}(\Omega_0) > 0$, implying $\Sigma_{sc}^P = \emptyset$ as at the end of Section 2. This completes the proof. $\square$

References

[1] G. Borg: Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. *Acta Math.* **78**, 1 – 96 (1946)

[2] D. Buschmann: A Proof of the Ishii-Pastur Theorem by the Method of Subordinacy. *Univ. Iagel. Acta Math.* **34**, 29 – 34 (1997)

[3] R. Carmona: Exponential localization in one-dimensional disordered systems. *Duke Math. J.* **49**, 191 – 213 (1982)

[4] R. Carmona, A. Klein and F. Martinelli: Anderson Localization for Bernoulli and Other Singular Potentials. *Commun. Math. Phys.*, **108**, 41 – 66 (1987)

[5] R. Carmona and J. Lacroix: Spectral theory of random Schrödinger operators. Birkhäuser, Basel–Berlin 1990

[6] E.A. Coddington and N. Levinson: Theory of ordinary differential equations. McGraw-Hill, New York 1955

[7] J.-M. Combes and P.D. Hislop: Localization for continuous random Hamiltonians in d-dimensions. *J. Funct. Anal.* **124**, 149 – 180 (1994)

[8] J.M. Combes, P.D. Hislop and E. Mourre: Spectral averaging, perturbation of singular spectrum, and localization. *Trans. Amer. Math. Soc.* **348**, 4883 – 4894 (1996)

[9] H. von Dreifus and A. Klein: A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **124**, 285 – 299 (1989)

[10] D.J. Gilbert: On Subordinacy and Analysis of Schrödinger Operators with two Singular Endpoints. *Proc. Royal Soc. Edinburgh* **112A**, 213 – 229 (1989)

[11] I.Ya. Goldsheidt, S. Molchanov and L. Pastur: A pure-point spectrum of the stochastic one-dimensional Schrödinger equation. *Funct. Anal. Appl.* **11**, 1 – 10 (1977)

[12] B.V. Gnedenko: The Theory of Probability, 4th Edition, Chelsea Publishing Company, New York, 1967

[13] A. Figotin and A. Klein: Localization of classical waves I: Acoustic waves. *Commun. Math. Phys.* **180**, 439 – 482 (1996)

[14] W. Kirsch, S. Kotani and B. Simon: Absence of absolutely continuous spectrum for some one dimensional random but deterministic potentials. *Ann. Inst. Henri Poincaré* **42**, 383 – 406 (1985)

[15] W. Kirsch, P. Stollmann and G. Stolz: Anderson Localization for Random Schrödinger Operators with Long Range Interactions. To appear in *Commun. Math. Phys.*

[16] F. Klopp: Localization for Semiclassical Continuous Random Schrödinger Operators II: the Random Displacement Model. *Helv. Phys. Acta* **66**, 810 – 841 (1993)

[17] F. Klopp: Localization for some continuous random Schrödinger operators. *Commun. Math. Phys.* **167**, 553 – 569 (1995)

[18] S. Kotani: Lyapunov indices determine absolutely continuous spectra of stationary one dimensional Schrödinger operators. In: Stochastic Analysis, ed. K. Ito, 225 – 247, North Holland, Amsterdam 1984

[19] S. Kotani: Lyapunov exponents and spectra for one-dimensional random Schrödinger operators. In: Random Matrices and their Applications, eds. J.E. Cohen, H. Kesten and C.M. Newman, American Mathematical Society, *Contemporary Math.* **50**, 277 – 286, Providence 1986

[20] S. Kotani: One-Dimensional Random Schrödinger Operators and Herglotz Functions. In: Probabilistic Methods in Mathematical Physics, eds. K. Ito and N. Ikeda, 219 – 250, Academic Press, Boston 1987

[21] S. Kotani and B. Simon: Localization in General One-Dimensional Random Systems. *Commun. Math. Phys.* **112**, 103 – 119 (1987)

[22] H. Kunz and B. Souillard: Sur le spectre des opérateurs aux différences finies aléatoires. *Commun. Math. Phys.* **79**, 201 – 249 (1980)

[23] N. Minami: An Extension of Kotani's Theorem to Random Generalized Sturm-Liouville Operators. *Commun. Math. Phys.* **103**, 387 – 402 (1986)

[24] N. Minami: An extension of Kotani's theorem to random generalized Sturm-Liouville operators II. In: Stochastic processes in classical and quantum systems, 411 – 419, *Lecture Notes in Physics* **262**, Springer 1986

[25] N. Minami: Random Schrödinger operators with a constant electric field. *Ann. Inst. Henri Poincaré* **56**, 307 – 344 (1992)

[26] S. Molchanov: Lectures on random media. In: Lectures on probability theory (Saint-Flour, 1992), 242–411, *Lecture Notes in Math.* **1581**, Springer 1994.

[27] L. Pastur and A. Figotin: Spectra of Random and Almost-Periodic Operators. Springer-Verlag, 1991

[28] M. Reed and B. Simon: Methods of Modern Mathematical Physics, IV. Analysis of Operators, Academic Press, New York 1978

[29] B. Simon: Localization in General One Dimensional Random Systems, I. Jacobi Matrices. *Commun. Math. Phys.* **102**, 327 – 336 (1985)

[30] B. Simon and T. Wolff: Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Commun. Pure Appl. Math.* **39**, 75 – 90 (1986)

[31] G. Stolz: Spectral theory of Schrödinger operators with potentials of infinite barriers type. Habilitationsschrift, Frankfurt 1994

[32] G. Stolz: Localization for random Schrödinger operators with Poisson potential. *Ann. Inst. Henri Poincaré* **63**, 297 – 314 (1995)