Change of stability for symmetric bifurcating solutions in the Ginzburg-Landau equations

Catherine Bolley
Ecole Centrale de Nantes
BP 92101
44321 Nantes Cedex 03 - France
and
Bernard Helffer
Département de Mathématiques
Université Paris-Sud
F-91405 0rsay - France

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Abstract

We consider the bifurcating solutions for the Ginzburg-Landau equations when the superconductor is a film of thickness d submitted to an external magnetic field. We refine some results obtained in our article [BoHe5] on the stability of bifurcating solutions starting from normal solutions.
We prove, in particular, the existence of curves $d \to \kappa_0(d)$, defined for large d and tending to $2^{-1/2}$ when $d \to +\infty$ and $\kappa \to d_1(\kappa)$, defined for small κ and tending to $\sqrt{5}$ when $\kappa \to 0$, which separate the sets of pairs (κ, d) corresponding to different behaviors of the symmetric bifurcating solutions. By this way, we give in particular a complete answer to the question of stability of symmetric bifurcating solutions in the asymptotics κ fixed-d large or d fixed-κ small.

1 Introduction

In continuation with our previous paper [BoHe5], we consider the Ginzburg-Landau equations for a film submitted to an exterior magnetic field.
Let d be the thickness of the slide, κ the Ginzburg-Landau parameter and $h \in]0, +\infty[$ a constant proportional to the exterior magnetic field; these equations can then be written,

$$
(GL)_d \quad \begin{cases} (a) \ - \kappa^{-2} f'' - f + f^3 + A^2 f = 0 \quad \text{in }]-d/2, d/2[\ , \\ (b) \ f'(\pm d/2) = 0\ , \\ (c) \ - A'' + f^2 A = 0 \quad \text{in }]-d/2, d/2[\ , \\ (d) \ A'(\pm d/2) = h\ , \end{cases} \tag{1.1}
$$

where f and A in $H^2(]-d/2, d/2[)$. The function f is a real wave function and A represent, in this model, the only non zero component of the interior magnetic potential. They characterize the state of the material when it is a film (see [GiLa], [Gi1]).

A solution of $(GL)_d$ is a triple $(f, A;\ h)$ in $H^2(]-d/2, d/2[) \times H^2(]-d/2, d/2[)$ $\times]0, +\infty[$ satisfying $(GL)_d$.
For given $\kappa > 0$, $h \geq 0$ and $e \in I\!R$, the triple $(f, A;\ h) = (0, h(x+e);\ h)$ is clearly a solution of $(GL)_d$. These solutions are called *normal solutions*. A *superconducting solution* is a solution $(f, A;\ h)$ of $(GL)_d$, such that f is not identically 0.

We are interested in this paper by bifurcating superconducting solutions starting from normal solutions.
A continuous curve $\varepsilon \rightarrow (f(\ \cdot\ ,\varepsilon),\ A(\ \cdot\ ,\ \varepsilon);\ h(\varepsilon))$ will be called a *bifurcating curve of solutions starting from a normal solution* $(0, h_0(x+e);\ h_0)$, with $h_0 > 0$ and $e \in I\!R$, if there exists $\varepsilon_0 > 0$ such that :

i) $(f(\ \cdot\ ,\ \varepsilon),\ A(\ \cdot\ ,\ \varepsilon);\ h(\varepsilon))$ is a solution for $|\varepsilon| \leq \varepsilon_0$,

ii) this solution is superconducting for $\varepsilon \neq 0$,

iii) the map $\varepsilon \rightarrow (f(\ \cdot\ ,\ \varepsilon),\ A(\ \cdot\ ,\ \varepsilon);\ h(\varepsilon))$ is continuous

from $[-\varepsilon_0, \varepsilon_0]$ to $H^2(]-d/2, d/2[) \times H^2(]-d/2, d/2[) \times I\!R$,
and satisfies

$$
(f(\ \cdot\ ,0),\ A(\ \cdot\ ,\ 0);\ h(0)) = (0,\ h_0(x+e);\ h_0)\ .
$$

In that case, the normal solution $(0, h_0(x+e); h_0)$ is called a bifurcating point for the problem $(GL)_d$.

We say that a bifurcating solution $(f(\,\cdot\,,\varepsilon), A(\,\cdot\,,\varepsilon); h(\varepsilon))$, with $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, is *symmetric* if $f(\,\cdot\,,\varepsilon)$ is even and $A(\,\cdot\,,\varepsilon)$ is odd on $[-d/2, d/2]$, and that it is *asymmetric* otherwise.

The existence of symmetric or asymmetric bifurcating solutions starting from normal solutions has been studied in our previous studies in [Bo1], [Bo2], [BoHe3]. Necessary conditions for the existence can also be found in the articles [Od1] and [Od2].

We have studied in [BoHe5] the structure and the stability of the bifurcating solutions starting from normal solutions. Our study gave precise results in several asymptotic regimes like κd small or κd large for the symmetric solutions, or in the regime κd large for the asymmetric solutions, excluding in each regime a small set where the study was not completed. Our purpose, in this paper, is to complete our preceding work in the case of the symmetric bifurcating solutions by describing the transition between the various regimes. An analogous study in the case of the asymmetric solutions is probably possible.

The plan of this study is the following.
In Section 2, after recalling some of the results on bifurcating solutions obtained mainly in [BoHe5], we present the main results of this paper.
Section 3 describes the scalings used in the next sections and gives some useful formulas.
Section 4 is devoted to the study of the symmetric bifurcating solutions when κd is large.
Section 5 gives an analogous study when κd is small.
In Section 6, we add some remarks on the stability of the symmetric bifurcating solutions when κd is large or small.

2 Main results on bifurcating solutions

Before giving the principal results of this paper, we recall some properties of the symmetric bifurcating solutions.

2.1 Previous results on bifurcating solutions

The occurence of bifurcating points for $(GL)_d$ is analyzed by considering the spectral properties of a system obtained by linearization of the Ginzburg-Landau equations at a normal solution $(0,\, h_0(x+e);\, h_0)$, that is the problem

$$\left\{ \begin{array}{l} -\,\kappa^{-2}\phi'' + h_0^2\,(x+e)^2\,\phi = \tau\,\phi \quad \text{in }]-d/2, d/2[\\ \phi'(\pm d/2) = 0\ , \\ \phi = \phi(\ \cdot\ ; \kappa, d, h_0) \in H^2(]-d/2, d/2[)\ , \end{array} \right. \tag{2.1}$$

(see [BoHe5] for details).
We consider, as in our previous paper, pairs (ϕ, τ) solutions of (2.1) depending on κ, d, h_0 and e, such that ϕ is strictly positive and satisfies the normalization condition

$$||\phi||_{L^2(]-d/2,d/2[)} = 1\ . \tag{2.2}$$

$\tau = \tau(\kappa, d, h_0, e)$ is consequently the lowest eigenvalue of the operator.
We have proved (see [BoHe1] (Proposition 0.1)) that, for all $\kappa > 0$, $d > 0$ and $e \in I\!R$, there exists a unique $h_0 = \bar{h}(\kappa, d, e)$ such that

$$\tau(\kappa, d, e, \bar{h}(\kappa, d, e)) = 1\ . \tag{2.3}$$

Theorem 2.1 in [BoHe3] gives sufficient conditions on h_0 and e for the existence of solutions bifurcating from $(0, h_0(x+e);\, h_0)$. But, because in this paper, we only improve preceding results in the case of symmetric bifurcating solutions, we restrict, in the following, our reminders to these particular solutions.
We first remark that the corresponding bifurcating points are also symmetric. This is equivalent to the condition $e = 0$, so that a bifurcating point will be in the form $(0, h_0\,x;\, h_0)$ and we shall, in general, skip the e-dependence of h_0, τ, ϕ and write $\bar{h}(\kappa, d)$, $\tau(\kappa, d, \bar{h}(\kappa, d))$, $\phi(\ \cdot\ ; \kappa, d)\cdots$ etc.

We have then proved :

Theorem 2.1 *(Theorem 2.1 in [BoHe3] with e = 0) :*
Let $(\kappa, d) \in]0,+\infty[^2$ *and let* $h_0 \in I\!R \times]0,+\infty[$ *satisfying*

$$\begin{array}{ll} (a) & h_0 = \bar{h}(\kappa, d)\ , \\ (b) & \dfrac{\partial^2 \tau}{\partial e^2}(\kappa, d, h_0) \neq 0\ . \end{array} \tag{2.4}$$

Then, there exists a constant $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\kappa, d) > 0$ *and a* C^∞ *curve,* $]-\tilde{\varepsilon}_0, \tilde{\varepsilon}_0[\ni \varepsilon \to (f(\ \cdot\ , \varepsilon), A(\ \cdot\ , \varepsilon); h(\varepsilon))$ *of superconducting bifurcating solutions such that,*

$$\begin{array}{rll} f(x,\varepsilon) & = \varepsilon\, f_0(x) + \varepsilon^3\, f_1(x) + o(\varepsilon^3) & in\ H^2(]-d/2, d/2[)\ , \\ A(x,\varepsilon) & = A_0(x) + \varepsilon^2\, A_1(x) + o(\varepsilon^2) & in\ H^2(]-d/2, d/2[)\ , \\ h(\varepsilon) & = h_0 + \varepsilon^2\, h_1 + o(\varepsilon^2)\ , & \end{array} \tag{2.5}$$

where f_0 *is the principal normalized positive eigenfunction* ϕ *defined by (2.1), with* $h_0 = \bar{h}(\kappa, d)$*, so that* $\tau = 1$*, and* $A_0 = h_0\, x$*.*
Moreover, there exist constants $\varepsilon_0 = \varepsilon_0(\kappa, d) > 0$ *with* $\varepsilon_0 \leq \tilde{\varepsilon}_0$*, and* $\gamma_0 = \gamma_0(\kappa, d) > 0$ *such that, for* $0 < |\varepsilon| \leq \varepsilon_0$*, the solution* $(f(\cdot, \varepsilon), A(\cdot, \varepsilon); h(\varepsilon))$ *is the unique solution of* $(GL)_d$ *such that*

$$\begin{array}{llll} i) & ||f(\ \cdot\ , \varepsilon)||_{H^2(]-d/2,d/2[)} \leq \gamma_0 & , \quad ii) & ||A(\ \cdot\ , \varepsilon) - A_0||_{H^2(]-d/2,d/2[)} \leq \gamma_0\ , \\ iii) & |h(\varepsilon) - h_0| \leq \gamma_0 & , \quad iv) & (f(\ \cdot\ , \varepsilon),\ f_0)_{L^2(]-d/2,d/2[)} = \varepsilon\ . \end{array}$$

We notice that the functions f_i, A_i and the coefficients h_i (for $i = 0, 1$) depend, as the function ϕ, on κ and d. We have in particular :

$$f_0(\ \cdot\ ) = f_0(\ \cdot\ ; \kappa, d) \quad , \quad A_0(\ \cdot\ ) = A_0(\ \cdot\ ; \kappa, d) \quad , \quad h_1 = h_1(\kappa, d)\ . \tag{2.6}$$

2.2 Main results on the structure of the bifurcating solutions

The first purpose of this paper is to study the sign of h_1 which is the second term in the expansion of $h(\varepsilon)$ defined in (2.5), and to prove the existence, for d large enough, of a curve separating the set of the pairs (κ, d) such that $h_1 > 0$ and the set of pairs such that $h_1 < 0$.

We prove, in particular, in this paper the following theorem :

Theorem 2.2 [d **large,** κd **large**] *:*
There exists a constant $d_0 > 0$ *and a* C^1 *function* $\kappa_0 : d \to \kappa_0(d)$ *defined for* $d \geq d_0$*, satisfying*

$$\lim_{d \to +\infty} \kappa_0(d) = 2^{-1/2} ,$$

and such that, for $\kappa = \kappa_0(d)$ *and* $d \geq d_0$*, we have :*

$$h_1(\kappa_0(d), d) = 0 ,$$

where $h_1(\kappa, d)$ *is defined by (2.5).*
Moreover, there exists a constant $a_1 > 0$ *such that for all pairs* (κ, d) *with* $\kappa d \geq a_1$*,* $d \geq d_0$ *and* $\kappa \neq \kappa_0(d)$*, we have*

$$(\kappa_0(d) - \kappa) \cdot h_1(\kappa, d) > 0 .$$

As a corollary, we get for the bifurcating solutions,

Corollary 2.3 *: There exists* $a_1 > 0$ *such that, for all pairs* (κ, d) *with* $\kappa d \geq a_1$*,* $d \geq d_0$ *and* $\kappa \neq \kappa_0(d)$*, and for* $h_0 = \bar{h}(\kappa, d)$*, there exists* $\varepsilon_1 = \varepsilon_1(\kappa, d) > 0$ *such that the curve of bifurcating solutions starting from the normal solution* $(0, h_0 x; h_0)$ *verifies, for* $0 < |\varepsilon| \leq \varepsilon_1$*,*

$$(\kappa_0(d) - \kappa) \cdot (h(\varepsilon) - h_0) > 0 .$$

We studied the sign of h_1 in [BoHe5], but our study lefts the sign of h_1 undetermined in a domain defined, for any $\eta > 0$, by $\Omega^1_\eta \equiv \{(\kappa, d) \; ; \; |\kappa - 2^{-1/2}| \leq \eta \; , \; \kappa d \geq a_1\}$ where a_1 is some large constant depending on $\eta > 0$. Theorem 2.2 completes then this preceding study in Ω^1_η.

We get also (see Section 4) an analogous result when κd is small, which gives the existence, for κ small, of a curve $\kappa \to d_1(\kappa)$ separating the values of the parameters κ and d where $h_1(\kappa, d) > 0$ of those where $h_1(\kappa, d) < 0$.
We prove the following theorem which improves Theorem 5.8 in [BoHe5],

Theorem 2.4 [κ **small,** κd **small**] *:*
There exists a constant $\kappa_1 > 0$ *and a* C^1 *function* $\kappa \to d_1(\kappa)$ *defined for* $0 < \kappa \leq \kappa_1$*, satisfying*

$$\lim_{\kappa \to 0} d_1(\kappa) = \sqrt{5} ,$$

and such that, for the pairs (κ, d) *with* $0 < \kappa \leq \kappa_1$ *and* $d = d_1(\kappa)$*, we have :*

$$h_1(\kappa, d_1(\kappa)) = 0 \ ,$$

where $h_1(\kappa, d)$ *is defined in (2.5).*
Moreover, there exists a constant $a_2 > 0$ *such that for* (κ, d) *verifying* $\kappa d \leq a_2$*,* $0 < \kappa \leq \kappa_1$ *and* $d \neq d_1(\kappa)$*,*

$$(d_1(\kappa) - d) \cdot h_1 > 0 \ .$$

As in the case when κd is large, we deduce a corresponding result for the bifurcating solutions :

Corollary 2.5 *: There exists* $a_2 > 0$ *such that for* (κ, d) *verifying* $\kappa d \leq a_2$*,* $0 < \kappa \leq \kappa_1$*,* $d \neq d_1(\kappa)$ *and for* $h_0 = \bar{h}(\kappa, d)$*, there exists* $\varepsilon_1 = \varepsilon_1(\kappa, d)$ *such that the unique curve of bifurcating solutions starting from the normal solution* $(0, h_0\, x;\, h_0)$ *verifies, for* $0 < |\varepsilon| \leq \varepsilon_1$ *,*

$$(d_1(\kappa) - d) \cdot (h(\varepsilon) - h_0) > 0 \ . \tag{2.7}$$

The sign of h_1 was also left undetermined in [BoHe5] Theorem 5.8, in a domain
$\Omega^2_\eta \equiv \{(\kappa, d) \ ; \ |d - \sqrt{5}| \leq \eta \ , \ \kappa d \leq a_2\}$, for some $a_2 > 0$ depending on η.
Theorem 2.4 allows us to complete this result in Ω^2_η.

3 Study of $h_1(\kappa, d)$ in other scalings

In order to prove Theorem 2.2, we rewrite the problem in other units as in our preceding paper [BoHe5].

3.1 Scalings

We have used, in preceding papers and in particular in [BoHe5], two different scalings for studying (2.1). We shall get, by using once again these units, the main results of this article. Let us recall these scalings and explicit, what was only implicit in [BoHe5], the inverse transformation .

i) The first scaling, is given by :

$$y = \sqrt{\kappa\, h_0}\; x\ , \quad a = \frac{d}{2}\, \sqrt{\kappa\, h_0}\ , \quad \mu = \frac{\kappa\, \tau}{h_0}\ . \tag{3.1}$$

It has the advantage to give an operator which is independent of the parameters.
If P is the harmonic oscillator

$$P \equiv -\frac{d^2}{dy^2} + y^2\ , \tag{3.2}$$

the spectral problem ((2.1)-(2.2)) becomes, indeed, for $a > 0$,

$$\left\{ \begin{array}{l} P\, \bar{f} = \mu\, \bar{f} \quad \text{ in }]-a, a[\ , \\ (\partial_y \bar{f})(\pm a; a) = 0\ , \\ \bar{f}(\ \cdot\ ; a) \in H^2(]-a, a[)\ , \end{array} \right. \tag{3.3}$$

where $\bar{f}$ is considered as a function of y and a, and μ as a function of a and where we denote by $\partial_y \bar{f}$ the partial derivative of $\bar{f}$ with respect to y.
The eigenfunction $\bar{f}$ is assumed to be normalized so that

$$||\bar{f}(\ \cdot\ ; a)||_{L^2(]-a,a[)} = 1\ . \tag{3.4}$$

We have then $\mu = \mu(a)$ and $\bar{f} = \bar{f}(y; a)$ and get the relation :

$$\bar{f}(y; a)\ =\ (\kappa\, h_0)^{-1/4}\ \phi(x; \kappa, d, h_0)\ , \tag{3.5}$$

with the parameters related by (3.1).

ii) The second scaling gives an interval independent of the parameters. It is defined by

$$u = \frac{x}{d}\ , \quad a = \frac{d}{2}\sqrt{\kappa h}\ , \quad \bar{\lambda} = \kappa^2\, d^2\, \tau\ . \tag{3.6}$$

It will be used when κd is small or for differentiating $\bar{f}$ with respect to a.
The spectral problem becomes,

$$\left\{ \begin{array}{l} -\dfrac{\partial^2 \bar{\phi}}{\partial u^2} + 16\, a^4\, u^2\, \bar{\phi} = \bar{\lambda}\, \bar{\phi} \quad \text{ in }]-1/2, 1/2[\ , \\ (\partial_u \bar{\phi})(\pm 1/2; a) = 0\ , \\ \bar{\phi}(\ \cdot\ ; a) \in H^2(]-1/2, 1/2])\ , \end{array} \right. \tag{3.7}$$

where $\bar{\phi} = \bar{\phi}(u;a)$ and $\bar{\lambda} = \bar{\lambda}(a)$. The eigenfunction $\bar{\phi}$ is normalized so that

$$|| \bar{\phi}(\ \cdot \ ; a) ||_{L^2(]-1/2,1/2[)} = 1 \ . \tag{3.8}$$

and we then get the relation:

$$\bar{\phi}(u;a) = d^{1/2} \ \phi(x; \kappa, d, h) \ .$$

iii) The scalings (3.1) and (3.6) are connected by the change of variables and parameters

$$y = 2\,a\,u \ , \quad \mu(a) = \frac{\bar{\lambda}(a)}{4a^2} = \frac{\kappa^2\, d^2\, \tau}{4a^2} \ , \tag{3.9}$$

and we have ,

$$\bar{\phi}(u;a) \ = \ (2a)^{1/2} \ \bar{f}(y;a) \ . \tag{3.10}$$

Remark 3.1 *: These two scalings will be used when* $h_0 = \bar{h}(\kappa, d)$*.*
In that case, the normalized eigenfunction ϕ *of ((2.1)-(2.2)) depends only on* x*,* κ *and* d*, and we have :*

$$\mu(a) = \kappa \left(\bar{h}(\kappa, d)\right)^{-1} \quad \text{and} \quad \bar{\lambda}(a) = \kappa^2\, d^2 \ . \tag{3.11}$$

We shall give the inverse transformation in the following subsection.

3.2 Variations of the parameters - an inverse scaling

The results stated in Subsection 2.2 concern asymptotics when the product κd is large or small. The following lemma gives an equivalent by using the parameter a.

Lemma 3.2 *: (cf. [BoHe5], Subsection 2.2 b))*
Let $\bar{\lambda}(a)$ *be the principal eigenvalue of (3.7), then* $a \to \bar{\lambda}(a)$ *is a strictly increasing function of* a *on* $]0, +\infty[$ *which verifies :*

$$\bar{\lambda}(0) = 0 \quad \text{and} \quad \bar{\lambda}(+\infty) = +\infty \ .$$

More precisely, when $a \to 0$*,*

$$\bar{\lambda}(a) = \frac{4}{3}\, a^4 + \mathcal{O}(a^8) \ , \tag{3.12}$$

and when $a \to +\infty$,

$$\bar{\lambda}(a) = 4\,a^2 + \mathcal{O}\left(\exp(-\frac{5}{8}\,a^2)\right) . \tag{3.13}$$

It results from Lemma 3.2 that the function $a \to \bar{\lambda}(a)$ is one to one from $[0,+\infty[$ onto $[0,+\infty[$ and that we can define a function α on $[0,+\infty[$ by :

$$\alpha \ : \ s \in]0,+\infty[\to \ \alpha(s) = \bar{\lambda}^{-1}(s^2) . \tag{3.14}$$

The function α is also one to one from $[0,+\infty[$ onto $[0,+\infty[$ and is strictly increasing from 0 towards $+\infty$.

Now, when $h_0 = \bar{h}(\kappa, d)$, we have, from (3.11), $\bar{\lambda}(a) = \kappa^2\,d^2$. We then get :

Lemma 3.3 *: Let α be the function defined on $[0,+\infty[$ by (3.14).*
When $h_0 = \bar{h}(\kappa,d)$ for $(\kappa,d) \in]0,+\infty[^2$, then, a depends only on the product κd with :

$$a(\kappa,d) = \alpha(\kappa\,d) . \tag{3.15}$$

Moreover, the function $s \to \alpha(s)$ is a C^∞ function on $]0,+\infty[$ verifying :
i)

$$\alpha(s) = \left(\frac{3}{4}\right)^{\frac{1}{4}}\ (s)^{\frac{1}{2}}\ (1 + \mathcal{O}(s^2)) \quad \textit{when } s \to 0 .$$

and
ii)

$$\alpha(s) = \frac{1}{2}\,s\ \left(1 + \mathcal{O}\left(\frac{1}{s^2}\ \exp(-\ \frac{5}{32}s^2)\right)\right) \quad \textit{when } s \to +\infty . \tag{3.16}$$

The lemma results from Lemma 3.2 and from the relation $\kappa\,d = (\bar{\lambda}(a))^{1/2}$ recalled in (3.11).
The regularity of α results from the regularity of the principal eigenvalue with respect to the coefficients the equation (3.7).

This inverse transformation was not explicited in [BoHe5], so this result completes Section 3 in [BoHe5].

3.3 Formulas for h_1

We recall that h_1 is defined in (2.6) and that it is a function of κ and d. It is given by the following formula (see [BoHe3] Section 2 or [MiKe]):

$$\begin{aligned} & 2\,\frac{h_1(\kappa,d)}{\bar{h}(\kappa,d)} \int_{-\frac{d}{2}}^{\frac{d}{2}} (A_0(x;\kappa,d))^2\ (f_0(x;\kappa,d))^2\,dx \\ & \qquad = -\int_{-\frac{d}{2}}^{\frac{d}{2}} (f_0(x;\kappa,d))^4 dx + 2\int_{-\frac{d}{2}}^{\frac{d}{2}} (\frac{\partial A_{1,0}}{\partial x}(x;\kappa,d))^2\,dx\ , \end{aligned} \tag{3.17}$$

where, with the notations of [BoHe3], $A_1(\ \cdot\ ;\kappa,d)$ defined in (2.5) is splitted as

$$A_1(x;\kappa,d) = a_1(\kappa,d) + A_{1,0}(x;\kappa,d) + h_1(\kappa,d)\,x, \quad \text{with } a_1(\kappa,d)\in I\!R\ ,$$

and where $A_{1,0}(\ \cdot\ ;\kappa,d)$ is the unique solution in $H^2(]-d/2,d/2[)$ of the following problem

$$\left\{ \begin{array}{l} -\dfrac{\partial^2 A_{1,0}}{\partial x^2} + f_0^2\ A_0 = 0 \quad \text{in }]-d/2,d/2[\ , \\ \dfrac{\partial A_{1,0}}{\partial x}(\pm d/2;\kappa,d) = 0\ , \\ A_{1,0}(0;\kappa,d) = 0\ . \end{array} \right. \tag{3.18}$$

As in [BoHe5], we shall study h_1 as function of the parameters κ and a by using the scaling (3.1) : we shall write $\widetilde{h_1}(\kappa,a)$ with :

$$\widetilde{h_1}(\kappa,a) = h_1(\kappa,\kappa^{-1}\bar{\lambda}^{1/2}(a)) \quad \text{for } (\kappa,a)\in(I\!R^{+*})^2\ ,$$

by using (3.14) and (3.15).
In this scaling, $\widetilde{h_1}$ is given by :

$$2\,\frac{\widetilde{h_1}(\kappa,a)}{\kappa} \int_{-a}^{a} \bar{B}_0(y;a)^2\ \bar{f}(y;a)^2\ dy = \frac{\kappa}{\sqrt{\mu(a)}} \cdot \widetilde{\Lambda}(\kappa,a)\ , \tag{3.19}$$

where

$$\widetilde{\Lambda}(\kappa,a) \equiv -\int_{-a}^{a} \bar{f}(y;a)^4\ dy\ + 2\,\kappa^{-2}\int_{-a}^{a}\left(\frac{\partial \bar{B}_{1,0}}{\partial y}(y;a)\right)^2\ dy\ , \tag{3.20}$$

and where[1] $\bar{B}_0$ and $\bar{B}_{1,0} \in H^2(]-a,a[)$ are defined by

$$\left\{\begin{array}{l} \bar{B}_0(y) = \sqrt{\kappa\ (\bar{h}(\kappa,d))^{-1}}\ A_0(x;\kappa,d)\ =\ y\ , \\ \text{and} \\ \bar{B}_{1,0}(y;a) = \kappa\ A_{1,0}(x;\kappa,d)\ . \end{array}\right. \qquad \text{for } y \in]-a,a[\ .$$

We verify that the function $\bar{B}_{1,0}$ depends only on y and a. It is indeed the solution of the following equation which results from (3.18) :

$$\left\{\begin{array}{l} \dfrac{\partial^2 \bar{B}_{1,0}}{\partial y^2}(y;a) = y\ \bar{f}(y;a)^2 \quad \text{for } y \in]-a,a[\ , \\ \dfrac{\partial \bar{B}_{1,0}}{\partial y}(\pm a;a) = 0\ , \\ \bar{B}_{1,0}(0;a) = 0\ , \\ \bar{B}_{1,0}(\ \cdot\ ;a) \in H^2(]-a,a[)\ . \end{array}\right. \tag{3.21}$$

3.4 Approximations of eigenfunctions for κd large

Accurate approximations for the eigenelements of the spectral problem (3.3) will also be needed. They have been calculated in [Bo2], when a tends to $+\infty$, by using a semi-classical analysis. Let us now describe these results.

In the spirit of B. Helffer-J. Sjöstrand [HeSj], we have constructed an approximation of $\bar{f}$ and of μ valid when a tends to $+\infty$. For this purpose, we have considered, instead of (3.3), the problem on $I\!R$ with $\mu = 1$, that is

$$Pf = f \quad \text{in } I\!R\ ,$$

and a basis $\{\phi_1, \phi_2\}$, of solutions of this equation chosen such that (see Y. Sibuya [Si])

$$\phi_1(y) = \exp(-\frac{y^2}{2}) \quad \text{for } y \in I\!R\ , \tag{3.22}$$

$$\phi_2(y) = \exp(\frac{y^2}{2}) \cdot \frac{1}{y} \cdot \left(1 + \mathcal{O}\left(\frac{1}{y^2}\right)\right) \qquad \text{as } y \to +\infty\ . \tag{3.23}$$

[1] See also Formula (6.14) in [BoHe5], where the function $\bar{B}_0$ is defined by $\bar{B}_0(y;a,\kappa) = A_0(x;\kappa,d)$, with $y = \sqrt{\kappa\, h_0}$ and $a = \alpha(\kappa d)$, giving a function depending *a priori* on y, κ and d, then on y, κ and a.

and, in this case,

$$\phi_2'(y) = \exp(\frac{y^2}{2}) \cdot \left(1 + \mathcal{O}\left(\frac{1}{y}\right)\right) \quad \text{as } \; y \to +\infty \; . \tag{3.24}$$

We have proved, using technics from [HeSj], the following result :

Proposition 3.4 *(see Proposition 2.9 in [Bo2]) :*
There exist positive constants C_1, C_2 and a_0 such that for $a \geq a_0$,
i)

$$|\mu(a) - 1| \leq C_1 \; \exp(-\frac{11}{16} \; a^2) \; . \tag{3.25}$$

ii)

$$\sup_{y \in [-a,a]} |\bar{f}(y) - \bar{f}^{app}(y)| \leq C_2 \; \exp(-\frac{5}{8} \; a^2) \; , \tag{3.26}$$

where $\bar{f}^{app}$ is an even approximation of $\bar{f}$ in the form

$$\bar{f}^{app}(y) = \beta(a) \exp(-\frac{y^2}{2}) + \rho(a) \cdot \phi_2(|y|) \cdot \Xi(\frac{y}{a}) \; , \tag{3.27}$$

with

$$\beta(a) = \pi^{-1/4} + \mathcal{O}(a \; \exp(-a^2)) \quad \text{as } \; a \to +\infty \; , \tag{3.28}$$

$$\rho(a) = \pi^{-1/4} \; a \; \exp(-a^2) \cdot \left(1 + \mathcal{O}\left(\frac{1}{a^2}\right)\right) \; , \tag{3.29}$$

and where Ξ is an even $C^\infty(I\!R)$ cutoff function s.t.

$$\Xi(x) = 0 \text{ if } \; |x| \leq 1/2 \; ; \quad \Xi(x) = 1 \;\; \text{if } \; |x| \geq 3/4 \; .$$

Remark 3.5 *: The coefficients $\beta(a)$ and $\rho(a)$ are defined in such a way that :*

$$(\partial_y \bar{f}^{app})(\pm a; a) = 0 \; , \tag{3.30}$$

and

$$||\, \bar{f}^{app}(\;\cdot\; ; a) \,||_{L^2(]-a,a[)} = 1 \; . \tag{3.31}$$

4 Cancellation of $\widetilde{h_1}(\kappa, a)$ when a is large

4.1 Reduction to an implicit Theorem

Let us consider the bifurcating solutions starting from symmetric normal solutions like $(0,\ h_0\, x;\ h_0)$ when $h_0 = \bar{h}(\kappa, d)$ for some pair (κ, d) of parameters. As proved in [BoHe5], when $\dfrac{\partial^2 \tau}{\partial e^2}(\kappa, d, h_0) \neq 0$, there exists a unique curve of bifurcating solutions starting from $(0,\ h_0\, x;\ h_0)$, and this is in particular the case when κd is large enough (or small enough) (see Theorem 3.4 in [BoHe5]). Therefore, using Lemma 3.3, this is the case when a is large (or small) enough. Let us consider, in this section, the case when a is large.

For proving Theorem 2.2, we shall show, in the next section, the existence of a curve $d \to \kappa_0(d)$, defined for d large enough, and such that

$$\left\{ \begin{array}{l} h_1(\kappa_0(d), d) = 0\ , \\ \lim\limits_{d \to +\infty}\ \kappa_0(d) =\ 2^{-1/2}\ , \end{array} \right. \tag{4.1}$$

which is the boundary line between $h_1^{-1}(]0, +\infty[)$ and $h_1^{-1}(]-\infty, 0[)$. Following the ideas of our paper [BoHe5], we first study h_1 in the scaling (3.1) by using (3.19); this will be a first step of the proof of Theorem 2.2.

According to (3.19) and the study of Subsection 3.3, we remark that

$$\widetilde{h_1}(\kappa, a)\ =\ 0 \iff \widetilde{\Lambda}(\kappa, a)\ =\ 0\ .$$

So, our purpose is to study the zero set of the map $(\kappa, a) \to \widetilde{\Lambda}(\kappa, a)$.

We prove the following proposition,

Proposition 4.1 *: There exists a constant $a_0 > 0$ and a C^1 function $r_0\ :\ a \to r_0(a)$ defined for $a \geq a_0$, verifying*

$$\lim_{a \to +\infty} r_0(a) = 2^{-1/2}\ , \tag{4.2}$$

and such that, for $a \geq a_0$,

i) $\widetilde{\Lambda}(r_0(a), a) = 0\ ,$
ii) for $\kappa \neq r_0(a)\ :\ (r_0(a) - \kappa) \cdot \widetilde{\Lambda}(\kappa, a) > 0\ .$

This result improves Lemma 5.3 in [BoHe5] and then using (3.19) improves Proposition 5.4 of the same paper.

4.2 Approximations of some partial derivatives with respect to a or y

The proof of Proposition 4.1 will use an Implicit Function Theorem and then needs estimates for the first partial derivative of $\bar{f}(\ \cdot\ ; a)$ with respect to a.

Let us prove,

Proposition 4.2 : *There exist constants $C > 0$ and $a_0 > 0$ such that for $a \geq a_0$,*

$$|| \frac{\partial \bar{f}}{\partial a}(\ \cdot\ ; a) ||_{C^1([-a,a])} \leq C\, a^5 \exp(-\frac{1}{2}a^2)\ . \tag{4.3}$$

Proof :

Let us write an equation satisfied by $y \to \dfrac{\partial \bar{f}}{\partial a}(y; a)$.

By differentiating the equation (3.3) with respect to a in any fixed interval of $]-a, a[$, we get :

$$\left\{ \begin{array}{l} -\dfrac{\partial^2}{\partial y^2} \left(\dfrac{\partial \bar{f}}{\partial a}\right)(y; a) + y^2\, \dfrac{\partial \bar{f}}{\partial a}(y; a) - \mu(a)\, \dfrac{\partial \bar{f}}{\partial a}(y; a) \\ \qquad\qquad\qquad = \mu'(a)\ \bar{f}(y; a) \quad \text{in }]-a, a[\ , \\ \dfrac{\partial \bar{f}}{\partial a}(\ \cdot\ ; a) \in H^2(]-a, a[)\ . \end{array} \right. \tag{4.4}$$

Now, using the normalization condition $||\, \bar{f}(\ \cdot\ ; a)\, ||_{L^2(]-a,a[)} = 1$, we get :

$$2 \int_a^a \bar{f}(y; a)\, \frac{\partial \bar{f}}{\partial a}(y; a)\, dy\ +\ [\bar{f}(a; a)^2 - \bar{f}(-a; a)^2] = 0\ ,$$

and, because $\bar{f}(\ \cdot\ ; a)$ is even :

$$\left(\frac{\partial \bar{f}}{\partial a}(\ \cdot\ ; a),\ \bar{f}(\ \cdot\ ; a)\right)_{L^2(]-a,a[)} = 0\ . \tag{4.5}$$

It remains to compute at $y=\pm a$, the first partial derivative of $\bar{f}(\ \cdot\ ;a)$ with respect to a.
By using that, for any $a>0$ $\dfrac{\partial \bar{f}}{\partial y}(\pm a;a)=0$, we get, for example at $y=a$ (and an analogous result at $y=-a$) ;

$$\frac{\partial^2 \bar{f}}{\partial y\,\partial a}(a;a)=-\frac{\partial^2 \bar{f}}{\partial y^2}(a;a)=(\mu(a)-a^2)\bar{f}(a;a)\ . \tag{4.6}$$

This computation can be justified by using an extension of $\bar{f}$ into a function $(y,a)\to\bar{g}(y;a)$, which is a C^2 function on $I\!R\times]0,+\infty[$, and then by differentiating $\bar{g}$ with respect to a.
Now, from Proposition 3.4, we get an estimate of $\bar{f}(a;a)$ when a tends to $+\infty$, and then from (4.6) :

$$\frac{\partial^2 \bar{f}}{\partial y\,\partial a}(\pm a;a)=2\,\pi^{-1/4}\ \left(1-a^2+\mathcal{O}\left(a^{-2}\right)\right)\ \exp\left(-\frac{1}{2}a^2\right)\ . \tag{4.7}$$

This means in particular that the function $y\to\dfrac{\partial \bar{f}}{\partial a}(y;a)$ does not satisfy the Neumann conditions at $\pm a$. For getting estimates on this function, we add, for large y, some even function in such a way that the sum will satisfy these Neumann conditions.
Let us consider the function $\bar{w}(\ \cdot\ ;a)$ defined by :

$$\bar{w}(y;a)=\frac{\partial \bar{f}}{\partial a}(y;a)+\gamma(a)\ \phi_2(|y|)\,\Xi(\frac{y}{a})\quad \text{for } y\in[-a,a]\ ,$$

where ϕ_2 and Ξ are defined in Proposition 3.4 and where

$$\gamma(a)=-\ \left(\frac{\partial^2 \bar{f}}{\partial y\,\partial a}(a;a)\right)\cdot\left(\phi_2'(a)\right)^{-1}=\frac{(a^2-\mu(a))\ \bar{f}(a;a)}{\phi_2'(a)}\ .$$

This choice of $\gamma(a)$ gives :

$$\frac{\partial \bar{w}}{\partial y}(\pm a;a)=0\ .$$

Moreover, by using once again the approximation $\bar{f}^{app}(\ \cdot\ ;a)$ of $\bar{f}(\ \cdot\ ;a)$ given in Proposition 3.4, we get :

$$\gamma(a)=2\pi^{-1/4}\ a^2\ \exp(-a^2)\ \left(1+\mathcal{O}(a^{-1})\right)\quad \text{when } a\to+\infty\ . \tag{4.8}$$

The new function $\bar{w}(\ \cdot\ ;a)$ satisfies the Neumann conditions at $\pm a$, but not the orthogonality condition necessary for our estimates, so we add to $\bar{w}(\ \cdot\ ;a)$ another term belonging to the kernel of $P-I$. Let us define :

$$\bar{Z}(y;a) = \bar{w}(y;a) + \nu(a)\ \bar{f}(y;a)\ ,$$

with

$$\nu(a) = -\,\gamma(a) \int_{-a}^{a} \phi_2(|y|)\ \Xi(\frac{y}{a})\ \bar{f}(y;a)\ dy\ ,$$

then :

$$\Big(\bar{Z}(\ \cdot\ ;a), \bar{f}(\ \cdot\ ;a)\Big)_{L^2(]-a,a[)} = 0\ .$$

We get, using (3.23) and Proposition 3.4 :

$$\begin{aligned} \nu(a) = &-2\,\gamma(a) \int_{a/2}^{a} \Xi(\frac{y}{a})\ \Big[\beta(a)\ \phi_1(y)\,\phi_2(y) + \rho(a)\ \phi_2(y)\ \Xi(\frac{y}{a})\Big]\ dy \\ &+\mathcal{O}\left(a\,\gamma(a)\exp(-\frac{5}{8}a^2)\,\phi_2(a)\right) \quad \text{when } a\to+\infty \end{aligned} \tag{4.9}$$

and then

$$\nu(a) = \mathcal{O}(\gamma(a)) \quad \text{when } a \to +\infty\ . \tag{4.10}$$

We have proved that the function $\bar{Z}(\ \cdot\ ;a)$ satisfies the following equation :

$$\left\{ \begin{array}{l} -\dfrac{\partial^2 \bar{Z}}{\partial y^2} + y^2\ \bar{Z}\ - \mu(a)\ \bar{Z} = \mu'(a)\ \bar{f}\ - \gamma(a)\ (y^2-1)\ \phi_2(|y|)\ \Xi(\dfrac{y}{a}) \\ \qquad\qquad - \dfrac{\gamma(a)}{a}\ \phi_2'(|y|)\ \Xi'(\dfrac{y}{a}) -\ 2\,sign(a)\dfrac{\gamma(a)}{a^2}\ \phi_2(|y|)\ \Xi''(\dfrac{y}{a}) \quad \text{in }]-a,a[\ , \\ \dfrac{\partial \bar{Z}}{\partial y}(\pm a;a) = 0\ , \\ \Big(\bar{Z}(\ \cdot\ ;a), \bar{f}(\ \cdot\ ;a)\Big)_{L^2(]-a,a[)} = 0\ , \\ \bar{Z}(\ \cdot\ ;a) \in H^2(]-a,a[)\ . \end{array} \right. \tag{4.11}$$

For getting an estimate of $\bar{Z}(\ \cdot\ ;a)$, we need an estimate on $\mu'(a)$.

Lemma 4.3 *:* **Approximation of $\mu'(a)$ when $a \to +\infty$**
When a tends to $+\infty$, the principal eigenvalue $\mu(a)$ of (3.7) verifies :

$$\mu'(a) = \ a\pi^{-1/2}\ \exp(-a^2)\cdot\Big(1+\mathcal{O}(a^{-2})\Big)\ . \tag{4.12}$$

Proof : Let us consider the equation (4.11).
A compatibility relation gives, by the Heilman-Feynman formula :

$$\mu'(a) + 2 = 4\,a \int_{-a}^{a} y\,\bar{f}(y;a)\,\frac{\partial \bar{f}}{\partial y}(y;a)\,dy\;, \tag{4.13}$$

because $\bar{f}(\;\cdot\;;a)$ is normalized in $L^2(]-a,\,a[)$.
Using Proposition 3.4, we then get, when a tends to $+\infty$,

$$\mu'(a)) = 4\,a\;\bar{f}^2(a;a)\;, \tag{4.14}$$

and then the lemma.

Let us now define the spaces,

$$G^a \equiv \{\phi \in L^2(]-a,a[)\;;\;(\phi,\bar{f}(.;a))_{L^2(]-a,a[)}\,\}$$

and

$$F^a \equiv \{\phi \in H^2(]-a,a[) \bigcap G^a\;;\;\phi'(\pm a) = 0\,\}\;.$$

It results from (4.4)-(4.5)- (4.6) that :

$$\begin{aligned} &||\,\bar{Z}(\;\cdot\;;a)\,||_{H^2(]-a,a[)} \leq ||\,(P-\mu(a))^{-1}\,||_{\mathcal{L}(G^a,F^a)} \\ &\quad\cdot\Big[|\mu'(a)|\cdot||\,\bar{f}\,||_{L^2(]-a,a[)}\;+2\,a^3\,\gamma(a)\phi_2(a) + 2\,\gamma(a)\,(\phi_2'(a)+\phi_2(a))\Big]\;, \end{aligned}$$

using that, for some positive constant C and for $a > 0$,

$$||\,(P-\mu(a)\,I)^{-1}\,||_{\mathcal{L}(G^a,F^a)} \leq C\,a^2\;,$$

Using also the estimates of Proposition 3.4 and Lemma 4.3 on $\mu'(a)$, we deduce that, when $a \to +\infty$:

$$||\,\bar{Z}(\;\cdot\;;a)\,||_{H^2(]-a,a[)} \leq C\,a^5\exp(-\frac{1}{2}a^2)\;. \tag{4.15}$$

Using now that $\bar{Z}(\;\cdot\;;a)$ and $\dfrac{\partial \bar{f}}{\partial a}(\;\cdot\;;a)$ are linked by the following relation :

$$\bar{Z}(y;a) = \frac{\partial \bar{f}}{\partial a}(y;a) + \gamma(a)\;\phi_2(|y|)\,\Xi(\frac{y}{a}) + \nu(a)\;\bar{f}(y;a) \quad \text{for } y \in [-a,a]\;,$$

we get the proposition.

We shall also need later $\frac{\partial a}{\partial \kappa}(\kappa, d)$ and $\frac{\partial a}{\partial d}(\kappa, d)$ for κ and $d > 0$.

Lemma 4.4 *:* **Approximation of $\frac{\partial a}{\partial \kappa}(\kappa, d)$ and of $\frac{\partial a}{\partial d}(\kappa, d)$**
When a (or $\kappa\, d$) tends to $+\infty$:

$$\begin{aligned} \frac{\partial a}{\partial \kappa}(\kappa, d) &= \frac{a(\kappa, d)}{\kappa} \cdot \left(1 + \mathcal{O}\left(a^2 \exp(-\frac{5}{8}a^2)\right)\right) \ , \\ \frac{\partial a}{\partial d}(\kappa, d) &= \frac{\kappa}{2} \cdot \left(1 + \mathcal{O}\left(a^2 \exp(-\frac{5}{8}a^2)\right)\right) \ . \end{aligned}$$

Proof :
The two partial derivatives of a are given by differentiating the relation $\bar{\lambda}(a) = \kappa^2 d^2$.
We get, by derivation with respect to κ :

$$\frac{\partial a}{\partial \kappa}(\kappa, d) = 2\,\kappa d^2\ (\bar{\lambda}'(a))^{-1} = \frac{2\,\bar{\lambda}(a)}{\kappa}\ \left(\bar{\lambda}'(a)\right)^{-1} .$$

Now, using (3.13) and Lemma 3.3, we get the first part of the lemma.

Let us now differentiate $\bar{\lambda}(a(\kappa, d))$ with respect to d ; we get :

$$\frac{\partial a}{\partial d}(\kappa, d) = 2\,\kappa^2 d\ \left(\bar{\lambda}'(a)\right)^{-1} = 2\,\kappa\,(\bar{\lambda}(a))^{1/2}\ \left(\bar{\lambda}'(a)\right)^{-1} ,$$

and then, using still (3.13) and Lemma 3.3, we get the second part of the lemma.

4.3 Proof of Proposition 4.1

Let us now prove Proposition 4.1. It will be done in several steps.

Step 1 : Extension of the function $\widetilde{\Lambda}$.
By studying the equation (3.7) and (3.5), then (3.21) and (3.20), we remark

that we can extend $\bar{f}(\ \cdot\ ; a)$ as an even function of a to the negative a, and $\bar{B}_{1,0}(\ \cdot\ ; a)$ and $\widetilde{\Lambda}(\kappa, a)$ as odd functions of a.
Moreover, from computations in [BoHe5] (Formula (5.13)), $\widetilde{\Lambda}(\kappa, a)$ satisfies, for a large enough,

$$\widetilde{\Lambda}(\kappa, a) = 2^{-3/2}\,\pi^{-1/2}\,\kappa^{-2}\,\left[1 - 2\,\kappa^2 + (1+\kappa^2)\,\mathcal{O}\left(a^2\exp(-\frac{5\,a^2}{8})\right)\right]\,. \tag{4.16}$$

We then define a parameter $\bar{a} = a^{-1}$ and a function $\widetilde{\Lambda}_2$ of κ and $\bar{a}$, for $(\kappa, \bar{a}) \in]0, +\infty[\times I\!R$ by

$$\left\{\begin{array}{ll} (a) & \widetilde{\Lambda}_2(\kappa, \bar{a}) \equiv \widetilde{\Lambda}(\kappa, a) \quad \text{when } (\kappa, \bar{a}) \in]0, +\infty[\times I\!R \setminus \{0\}\,, \\ (b) & \widetilde{\Lambda}_2(\kappa, 0) \equiv 2^{-3/2}\,\pi^{-1/2}\,\kappa^{-2}\,(1 - 2\,\kappa^2) \quad \text{when } \kappa > 0\,. \end{array}\right. \tag{4.17}$$

It results from $(4.17)_{(b)}$ that

$$\widetilde{\Lambda}_2(2^{-1/2}, 0) = 0\,, \tag{4.18}$$

and we search to apply the Implicit Function Theorem to the function $\widetilde{\Lambda}_2(\kappa, \bar{a})$ at the point $(\kappa, \bar{a}) = (2^{-1/2}, 0)$ in order to get the existence, for $\bar{a}$ small, of a unique C^1 function $\bar{a} \to \bar{r}_0(\bar{a})$ such that

$$\left\{\begin{array}{l} \widetilde{\Lambda}_2(\bar{r}_0(\bar{a}), \bar{a}) = 0\,, \\ \bar{r}_0(0) = 2^{-1/2}\,. \end{array}\right. \tag{4.19}$$

The function r_0, defined for a positive large enough, by

$$r_0(a) = \bar{r}_0(\bar{a})\,,$$

will be proved to satisfy Proposition 4.1.

Step 2 : The partial derivatives of $\widetilde{\Lambda}_2$.

For applying the Implicit Function Theorem, we need to prove that $\widetilde{\Lambda}_2$ is continuously differentiable in a neighborhood of the point $(\kappa, \bar{d}) = (2^{-1/2}, 0)$, and that $\dfrac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(2^{-1/2}, 0)$ is different from 0.

The last point is immediate from $(4.17)_{(b)}$. We have, indeed :

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(2^{-1/2}, 0) = -\,\pi^{-1/2}\,. \tag{4.20}$$

a) Regularity of $\widetilde{\Lambda}_2(\kappa, \bar{a})$ on $]0, +\infty[\times I\!R^*$:
The functions $\bar{f}(\;\cdot\;; a)$ and $\partial_y \bar{B}_{1,0}(\;\cdot\;; a)$ being independent of κ and admitting continuous partial derivatives with respect to the parameter a, we deduce that $\widetilde{\Lambda}_2(\kappa, \bar{a})$ is a C^1 function with respect to $(\kappa, \bar{a})$ in $]0, +\infty[\times]0, +\infty[$ and then, by parity, in $]0, +\infty[\times I\!R^*$.
It remains to study the regularity of $\widetilde{\Lambda}_2$ on the half straight-line $D_0 \equiv \{(\kappa, 0)\;;\; \kappa > 0\}$. This is studied now.

b) i) Continuity of $\widetilde{\Lambda}_2(\kappa, \bar{a})$ on D_0.
The continuity of $\widetilde{\Lambda}_2$ on $\{(\kappa, 0)\;;\; \kappa > 0\}$ results from (4.16) and $(4.17)_{(b)}$.

ii) Continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, 0)$.

Let us prove the continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}$ at $(\kappa, 0)$ for $\kappa > 0$.

We have, from $(4.17)_{(a)}$ and (3.20), and with $a = \bar{a}^{-1} \neq 0$,

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a}) = -4\,\kappa^{-3} \int_{-a}^{a} (\frac{\partial \bar{B}_{1,0}}{\partial y}(y; a))^2\, dy \tag{4.21}$$

Now, using the computations of [BoHe5] (proof of Lemma 5.3) which give the last integral, we get for positive $\bar{a}$ small enough and $\kappa > 0$,

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a}) = -\,\kappa^{-3}\, 2^{-1/2}\, \pi^{-1/2} \cdot \left[1 + \mathcal{O}\left(\bar{a}^{-2} \exp(-\frac{5}{8}\bar{a}^{-2})\right)\right]\;, \tag{4.22}$$

with a $\mathcal{O}$ uniform in κ. The result is extended to the negative $\bar{a}$ with $|\bar{a}|$ small enough, by parity. Consequently, for $\bar{a} \neq 0$:

$$\lim_{(\tilde{\kappa}, \bar{a}) \to (\kappa, 0)} \frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\tilde{\kappa}, \bar{a}) = -\,\kappa^{-3}\, 2^{-1/2}\, \pi^{-1/2}\,.$$

Thanks to $(4.17)_{(b)}$, the continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}$ at $(\kappa, 0)$ follows.

iii) The partial derivative $\dfrac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, 0)$.

¿From the relations (4.16)-(4.17), the partial derivative of $\widetilde{\Lambda}_2$ with respect to $\bar{a}$ calculated at $(\kappa, \bar{a}) = (\kappa, 0)$, is given by the limit when η tends to 0 of

$$\frac{\widetilde{\Lambda}_2(\kappa,\eta) - \widetilde{\Lambda}_2(\kappa,0)}{\eta} = 2^{-3/2}\,\pi^{-1/2}\,(\kappa^{-2}+1)\,\mathcal{O}\left(\bar{\eta}^{-3}\ \exp(-\frac{5}{8}\,\bar{\eta}^{-2})\right) .$$

Therefore, $\dfrac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, \bar{a})$ exists and

$$\forall \kappa > 0\ , \quad \frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, 0) = 0\ . \tag{4.23}$$

Expression of $\dfrac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a})$ for $\tilde{\kappa} > 0$ and $\bar{a} \in I\!R^*$.

For the continuity of the partial derivative $\dfrac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}$ at $(\kappa, 0)$ for $\kappa > 0$, we need more computations.
Using (4.17), (3.20) and the parity of the different functions, we have,

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a}) \;=\; -\,a^2\,\frac{\partial \widetilde{\Lambda}}{\partial a}(\tilde{\kappa}, a)\ , \tag{4.24}$$

with

$$\begin{aligned}\frac{\partial \widetilde{\Lambda}}{\partial a}(\tilde{\kappa}, a) = &-2\bar{f}(a;a)^4 + 2\,\tilde{\kappa}^{-2}\,(\frac{\partial \bar{B}_{1,0}}{\partial y}(a;a))^2\\ &-4\int_{-a}^{a}\bar{f}(y;a)^3\,\frac{\partial \bar{f}}{\partial a}(y;a)\,dy + 4\,\tilde{\kappa}^{-2}\int_{-a}^{a}\frac{\partial \bar{B}_{1,0}}{\partial y}(y;a)\,\frac{\partial^2 \bar{B}_{1,0}}{\partial a\,\partial y}(y;a)\,dy\ .\end{aligned} \tag{4.25}$$

Equations (4.24) and (4.25) allow us to use preceding computations performed when $a \to +\infty$.

Estimates for the first terms in (4.25).

In (4.25), the parameter $\tilde{\kappa}$ appears only in factor before the second and the third terms. So, we study only terms depending on a.

The semi-classical analysis performed in [Bo2] and partially recalled in Proposition 3.4, gave estimates for the three first terms. These terms satisfy, when a tends to $+\infty$,

$$\bar{f}(a;a)^4 = \mathcal{O}(\exp(-2a^2)) \,, \tag{4.26}$$

$$\frac{\partial \bar{B}_{1,0}}{\partial y}(a;a)^2 = \mathcal{O}\left(a^2 \exp(-\frac{5}{4}a^2)\right) \,. \tag{4.27}$$

and, as consequence of Lemma 4.2, using also that $\bar{f}$ is bounded ,

$$\int_{-a}^{a} \bar{f}^3(y;a)\, \frac{\partial \bar{f}}{\partial a}(y;a)\, dy = \mathcal{O}\left(a^2\, \exp(-\frac{1}{2}a^2)\right) \quad \text{when } a \to +\infty \,. \tag{4.28}$$

The partial derivative $\dfrac{\partial^2 \bar{B}_{1,0}}{\partial a\, \partial y}$.

Let us now consider the partial derivative $\dfrac{\partial \bar{B}_{1,0}}{\partial a}(\ \cdot\ ; a)$.
The equation (3.21) giving $\bar{B}'_{1,0}$ by integration implies :

$$\frac{\partial^2 \bar{B}_{1,0}}{\partial a\, \partial y}(y;a) = a\, \bar{f}(a;a)^2 + \int_{-a}^{y} 2t\, \bar{f}(t,a)\, \frac{\partial \bar{f}}{\partial a}(t;a)\, dt \,.$$

Therefore, using this relation and the usual approximation of $\bar{f}$ (Proposition 3.4), we get, as a tends to $+\infty$, uniformly on $[-a,a]$,

$$\frac{\partial^2 \bar{B}_{1,0}}{\partial a\, \partial y}(y;a) = \mathcal{O}\left(a^2\, \exp(-\frac{1}{2}a^2)\right) \,. \tag{4.29}$$

Moreover, computations in [BoHe5] (proof of Lemma 5.3) give,

$$\frac{\partial \bar{B}_{1,0}}{\partial y}(y;a) = -2^{-1}\pi^{-1/2}\, \exp(-y^2) + \mathcal{O}\left(a\, \exp(-\frac{5}{8}a^2)\right) \,. \tag{4.30}$$

Consequently, when $a \to +\infty$,

$$\int_{-a}^{a} \frac{\partial \bar{B}_{1,0}}{\partial y}\, \frac{\partial^2 B_{1,0}}{\partial a\, \partial y}\, dy = +\mathcal{O}\left(a^3\, \exp(-\frac{1}{2}a^2)\right) \,. \tag{4.31}$$

In this step, the terms, and then the $\mathcal{O}$, do not depend on κ.

Coming back to (4.24) and (4.25), we get, from the preceding steps (4.26), (4.27), (4.28) and (4.31), an estimate of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a})$.
We have proved :

Lemma 4.5 *: For $(\tilde{\kappa}, \bar{a}) \in]0, +\infty[\times I\!R^*$ and $\bar{a}$ small enough :*

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a}) \;=\; \mathcal{O}\left(\bar{a}^{-5} \exp(-\frac{1}{2\bar{a}^2})\right) \;,$$

where the $\mathcal{O}$ is uniform for $\tilde{\kappa}$ in a compact set of $]0, +\infty[$.

Continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}$ at $(\kappa, 0)$.

It results from Lemma 4.5 that, for all $\kappa > 0$, $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a})$ tends to 0 as $(\tilde{\kappa}, \bar{a})$ tends to $(\kappa, 0)$. Using (4.23), we get the continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}$ at any point $(\kappa, 0)$.

d) Conclusion.

The preceding steps imply that the Implicit Function Theorem can be applied to $\widetilde{\Lambda}_2$ at $(2^{-1/2}, 0)$. It results the following lemma :

Lemma 4.6 *There exists a constant $\bar{a}_0 > 0$ and a C^1 function $\bar{a} \to \bar{r}_0(\bar{a})$ defined, for $|\bar{a}| \leq \bar{a}_0$ such that (4.19) is satisfied, that is such that :*

$$\left\{ \begin{array}{l} \widetilde{\Lambda}_2(\bar{r}_0(\bar{a}), \bar{a}) = 0 \;, \\ \bar{r}_0(0) = 2^{-1/2} \;. \end{array} \right. \tag{4.32}$$

As a consequence, we get the existence of a C^1 function $a \to r_0(a)$ defined for $a \geq \bar{a}_0^{-1}$, by

$$r_0(a) = \bar{r}_0(\frac{1}{a}) \;,$$

such that (4.2) and Part i) of Proposition 4.1 are satisfied.

For proving ii), we remark, using $(4.17)_{(b)}$ with the parameter $\bar{a} = a^{-1}$, that for $\kappa > 0$,

$$\widetilde{\Lambda}_2(\kappa, 0) \cdot (\bar{r}_0(0) - \kappa) > 0 \;,$$

with $\bar{r}_0(0) = 2^{-1/2}$.
By continuity of $\widetilde{\Lambda}_2$, it results that, for $\kappa > 0$ and $\bar{a}$ small enough,

$$\widetilde{\Lambda}_2(\kappa, \bar{a}) \cdot (\bar{r}_0(\bar{a}) - \kappa) > 0 \ .$$

Part ii) in Proposition 4.1 follows.

4.4 Proof of Theorem 2.2

Let us now prove Theorem 2.2 which is written in the physical parameters κ and d. The proof is a consequence of Proposition 4.1 where the parameters κ and a were used.

The first problem is to prove the existence of a curve $d \to \kappa_0(d)$ defined for large d and $\kappa_0(d)$ near $2^{-1/2}$, such that, for $(\kappa, d) = (\kappa_0(d), d)$, the parameter $h_1(\kappa, d)$ cancels. We have studied this problem in the previous section in the parameters κ and a for large a where we have used that $h_1 = \widetilde{h}_1(\kappa, a)$ is equal to 0 if and only if $\widetilde{\Lambda}(\kappa, a) = 0$.
But, from Lemma 3.3, we have

$$a(\kappa, d) = \alpha(\kappa d) \ ,$$

so that $\widetilde{\Lambda}$ can be written as a function of κ and d : we then define

$$\Lambda(\kappa, d) = \widetilde{\Lambda}(\kappa, \alpha(\kappa d)) \quad \text{for } (\kappa, d) \in (I\!R^{+*})^2 \ .$$

The existence of a curve $r_0 \, : \, a \to r_0(a)$ (see Lemma 4.6) defined for $a \geq a_0$ such that $\widetilde{\Lambda}(r_0(a), a) = 0$ for $a \geq a_0$ will give us the existence of κ_0 for large d such that (4.1) is satisfied.

The curve $(r_0(a), a)$ which cancels $\widetilde{\Lambda}$ is given by the relation :

$$\kappa \ = \ r_0(a) \quad \text{with } \ a = \alpha(\kappa d) \ ,$$

so that we get a relation between κ and d which does also Λ cancel.

We then let :

$$F(\kappa, d) \equiv \kappa \ - \ (r_0 \circ \alpha)(\kappa d) \ , \tag{4.33}$$

and we remark that :

$$\lim_{d\to+\infty} F(2^{-1/2}, d) = 0 \ .$$

It is then easy to apply an Implicit Function Theorem to the function F at the point $(2^{-1/2}, +\infty)$ (or, if we prefer, at the point $(2^{-1/2}, 0)$ to a function $(\kappa, \bar{d}) \to \bar{F}(\kappa, \bar{d})$ defined by $\bar{F}(\kappa, \bar{d}) = F(\kappa, d^{-1})$ with $\bar{d} = d^{-1}$ and extended to null and negative $\bar{d}$), for getting the existence of a C^1-curve $\kappa_0 \ : \ d \to \kappa_0(d)$ defined for large d and such that $h_1(\kappa_0(d), d) = 0$ for large d.
The continuity and differentiability of F in a neighborhood of $(2^{-1/2}, \infty)$ comes, indeed, from the composition of the two functions $a \to r_0(a)$ and $s \to \alpha(s)$ by using that, from Lemma 4.6, r_0 is continuously differentiable in a neighborhood of ∞. The continuity and the differentiability of α for large s ($\kappa > 0$ is close to $2^{-1/2}$ and d is large) comes from Lemma 3.3.
We have also to verify that $\frac{\partial F}{\partial \kappa}(2^{-1/2}, d)$ is different from zero when d tends to 0.

Let us show that $\frac{\partial F}{\partial \kappa}(2^{-1/2}, d) = 1 + \mathcal{O}\left(d^3 \exp(-\frac{1}{4}d^2)\right)$ **as** $d \to +\infty$.
We have, by differentiating (4.33) and with $\sigma = 2^{-1/2} d$:

$$\frac{\partial F}{\partial \kappa}(2^{-1/2}, d) \ = \ 1 - \ d\, r_0'(\alpha(\sigma))\ \alpha'(\sigma) \ ,$$

with, from Lemma 3.3, $\alpha(\sigma) \ = \ 2^{-3/2} d (1 + \mathcal{O}\left(d^{-2} \exp(-\frac{5}{64} d^2)\right)$ when $d \to \infty$.

But,

$$r_0'(a) = - \ \frac{1}{a^2}\, \bar{r}_0'(a^{-1}) \ ,$$

and

$$\bar{r}_0'(\bar{a}) = - \ \left(\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, \bar{a})\right) \cdot \left(\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a})\right)^{-1} \ ,$$

so that, using (4.22) and Lemma 4.5, we get :

$$r_0'(\alpha(\sigma)) \ = \mathcal{O}\left(\alpha(\sigma)^3 \exp(-\frac{1}{2}\alpha(\sigma)^2)\right) \ .$$

Now, from Lemma 3.3, we have, when $d \to +\infty$:

$$r_0'(\alpha(\sigma)) = \mathcal{O}\left(d^3 \exp(-\frac{1}{4}\, d^2)\right) ,$$

and

$$\alpha'(s) = 2s \left(\bar{\lambda}'\right)^{-1} = \frac{1}{2} \left(1 + \mathcal{O}\left(d^{-2} \exp(-\frac{5}{64} d^2)\right)\right) .$$

Therefore,

$$\frac{\partial \bar{F}}{\partial \kappa}(2^{-1/2}, d) \;=\; 1 + \mathcal{O}\left(d^3 \exp(-\frac{1}{4} d^2)\right) \quad \text{as } d \to +\infty . \tag{4.34}$$

Conclusion :
The Implicit Function Theorem applied to F at the point $(2^{-1/2}, 0)$, gives the existence of a constant $d_0 > 0$ and of a C^1 curve $\kappa_0 \;:\; d \to \kappa_0(d)$ defined for $d \geq d_0$, such that :

$$F(\kappa_0(d), d) = 0 \quad \text{with} \lim_{d \to +\infty} \kappa_0(d) \;=\; 2^{-1/2} .$$

We then get Theorem 2.2.

5 The sign of h_1 when κd is small

Let us now consider the case when κd is small.
Theorem 5.9 in [BoHe5] establishes, in that case, the existence of a family of bifurcating solutions starting from the normal solution $(0, h_0 x; h_0)$ where $h_0 = \bar{h}(\kappa, d)$, and parametrized, in a neighborhood of the normal solution, by[2] :

$$\sigma = \kappa d \;,\; d \;,\; \bar{\varepsilon} = d^{-1/2}\, \varepsilon \;\text{ and } u = \frac{x}{d} \in [-\frac{1}{2}, \frac{1}{2}] ,$$

for $\bar{\varepsilon}$ small enough.
These solutions are described by $(g(\,\cdot\,; \sigma, d, \bar{\varepsilon}),\, V(\,\cdot\,; \sigma, d, \bar{\varepsilon});\, \eta(\sigma, d, \bar{\varepsilon}))$ where

$$\begin{array}{ll} g(u;\, \sigma, d, \bar{\varepsilon}) = f(x; \kappa, d, \varepsilon) \;; & \\ V(u;\, \sigma, d, \bar{\varepsilon}) = A(x;\, \kappa, d, \varepsilon) & \quad \text{with } u \in [-\frac{1}{2}, \frac{1}{2}] \\ \eta(\sigma, d, \bar{\varepsilon}) = d\, h(\kappa, d, \varepsilon) \,. & \end{array} \tag{5.1}$$

[2]In the proof of Theorem 5.9 in [BoHe5], we have sometimes to read σ^2 instead of σ

¿From [BoHe5], this family of bifurcating solutions admits an expansion in power of $\bar{\varepsilon}$ such that for $\bar{\varepsilon}$ small :

$$
\begin{aligned}
&g(\ \cdot\ ;\sigma,d,\bar{\varepsilon}) = \bar{\varepsilon}\, g_0(\ \cdot\ ;\sigma,d) + \bar{\varepsilon}\,\tilde{g}(\ \cdot\ ;\sigma,d,\bar{\varepsilon})\ ,\\
&V(\ \cdot\ ;\sigma,d,\bar{\varepsilon}) = V_0(\ \cdot\ ;\sigma,d) + \bar{\varepsilon}^2\, V_1(\ \cdot\ ;\sigma,d) + o(\bar{\varepsilon}^2) \quad \text{in } H^2(]-1/2,1/2[)\ ,\\
&\partial_u V(\pm 1/2;\ \sigma,d,\bar{\varepsilon}) = \eta(\sigma,d,\bar{\varepsilon}) = \eta_0(\sigma,d) + \bar{\varepsilon}^2\ d\,\bar{\bar{h}}_1(\sigma,d) + o(\bar{\varepsilon}^2)\ ,
\end{aligned}
$$

with

$$
(g_0(\ \cdot\ ;\sigma,d), \tilde{g}(\ \cdot\ ;\sigma,d))_{L^2(]-1/2,1/2[)} = 0
$$

and

$$
\tilde{g}(\ \cdot\ ;\sigma,d) = \sum_{j=1}^{p} \bar{\varepsilon}^{2p}\, g_j(\ \cdot\ ;\sigma,d) + o(\bar{\varepsilon}^{2p}) \quad \text{in } H^2(]-1/2,1/2[)\ ,
$$

and where $g_0(\ \cdot\ ;\sigma,d)$ is the normalized principal eigenfunction of (3.7) in the new parameters.

Let us define

$$
\bar{\bar{h}}_0(\sigma,d) = \bar{h}(d^{-1}\sigma,d) \quad \text{and } \bar{\bar{h}}_1(\sigma,d) = h_1(d^{-1}\sigma,d)\ ,
$$

we then get :

$$
\eta_0(\sigma,d) = d\,\bar{\bar{h}}_0(\sigma,d)\ .
$$

Moreover, in [BoHe5] (Theorem 5.8), we have got results on the sign of $\bar{\bar{h}}_1(\sigma,d)$, outside the domain Ω^2_η defined in Section 2. Our purpose, in this subsection, is first to study the sign of $\bar{\bar{h}}_1(\sigma,d)$ in this domain, and then to come back to the initial units (κ and d) for getting Theorem 2.4.

We get from the expansion of the bifurcating solutions a formula analogous to (3.19) in our new scaling:

$$
\begin{aligned}
&2\,\frac{\bar{\bar{h}}_1(\sigma,d)}{\bar{\bar{h}}_0(\sigma,d)} \int_{-\frac{1}{2}}^{\frac{1}{2}} V_0(u;\sigma,d)^2\ g_0(u;\sigma,d)^2\ du\\
&\qquad = -\int_{-\frac{1}{2}}^{\frac{1}{2}} g_0(u;\sigma,d)^4\ du + 2\,d^{-2}\int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{\partial V_{1,0}}{\partial u}(u;\sigma,d))^2\ du\ ,
\end{aligned}
\tag{5.2}
$$

where $V_{1,0}(u;\sigma,d) = A_{1,0}(x;\,\kappa,d)$, so that $V_{1,0}(\;\cdot\;;\sigma,d)$ satisfies

$$\left\{ \begin{array}{l} -\dfrac{\partial^2 V_{1,0}}{\partial u^2} + d^2\, g_0(\;\cdot\;;\sigma,d)^2\, V_0(\;\cdot\;;\sigma,d) = 0 \quad \text{in }]-1/2,1/2[\;, \\ \partial_u V_{1,0}(\pm 1/2;\sigma,d) = 0\;, \\ V_{1,0}(\pm 1/2;\sigma,d) = 0\;. \end{array} \right. \tag{5.3}$$

We proceed now as in the preceding section. We denote by $\bar{\bar{\Lambda}}_2$ the right-hand side of (5.2). It is a function of $\sigma \geq 0$ and $d > 0$ given by

$$\bar{\bar{\Lambda}}_2(\sigma,d) \equiv -\int_{-\frac{1}{2}}^{\frac{1}{2}} (g_0(u;\sigma,d))^4\; du + 2\, d^{-2} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{\partial V_{1,0}}{\partial u}(u;\sigma,d))^2\; du\;, \tag{5.4}$$

but we remark that the Ginzburg-landau equations parametrized by σ and d can be extended to negative values of σ in such a way that $g_0(\;\cdot\;;\sigma,d)$ is an even function of σ. It results from (5.4), using also (5.3) that $\bar{\bar{\Lambda}}_2(\sigma,d)$ appears then also as an even function of σ.

We have $\bar{\bar{h}}_1(\sigma,d) = 0$ if and only if $\bar{\bar{\Lambda}}_2(\sigma,d) = 0$, and we are interested by solving the problem near $d = \sqrt{5}$.
When $\sigma = 0$, we remark that

$$V_{1,0}(u;0,d) = \frac{\sqrt{3}}{3}\, d^2\, u^3 - \frac{\sqrt{3}}{4}\, d^2\, u\;,$$

so that

$$\bar{\bar{\Lambda}}_2(0,d) = -1 + \frac{d^2}{5} \quad \text{for } \; d > 0\;, \tag{5.5}$$

We now apply the Implicit Function Theorem to $\bar{\bar{\Lambda}}_2$ at $(\sigma,d) = (0,\sqrt{5})$ for getting the existence of a curve $\sigma \rightarrow \bar{d}_1(\sigma)$ defined for σ in a neighborhood of 0, and such that

$$\left\{ \begin{array}{l} \bar{\bar{\Lambda}}_2(\sigma,\bar{d}_1(\sigma)) = 0\;, \\ \bar{d}_1(0) = \sqrt{5}\;. \end{array} \right. \tag{5.6}$$

The regularity of the function $\bar{\bar{\Lambda}}_2$ with respect to (σ,d) in $[0,+\infty[\times]0,+\infty[$ will result from the regularity of the functions g_0 and $\partial_u V_{1,0}$. But $g_0(\;\cdot\;;\sigma,d)$

verify :

$$\begin{cases} -\dfrac{\partial^2 g_0}{\partial u^2} + \sigma^2\, \eta_0(\sigma,d)^2\, u^2\, g_0 = \sigma^2\, g_0 \quad \text{in } \left]-1/2,1/2\right[\,, \\ (\partial_u g_0)(\pm 1/2;\sigma,d) = 0\,, \\ g_0(\ \cdot\ ;\sigma,d) > 0\,, \\ g_0(\ \cdot\ ;\sigma,d) \in H^2(]-1/2,1/2[)\,, \end{cases} \tag{5.7}$$

and $\partial_u V_{1,0}$ verifies (5.3). These are Sturm-Liouville type equations, so, we get by standard theorems that g_0 and $\partial_u V_{1,0}$ are C^∞ with respect to the parameters σ and d in a neighborhood of $(0,\sqrt{5})$.
Consequently, $\bar{\bar{\Lambda}}_2$ is a C^∞ function in (σ,d) for (σ,d) in a neighborhood of $(0,\sqrt{5})$.

We now just observe that, according to (5.5) :

$$\frac{\partial \bar{\bar{\Lambda}}_2}{\partial d}(0,\sqrt{5}) = \frac{2\sqrt{5}}{5} \neq 0\,. \tag{5.8}$$

So that the Implicit Function Theorem implies the existence of a C^1 curve $\sigma \to \bar{d}_1(\sigma)$ defined for σ small and satisfying (5.6).

Using now the relation $\sigma = \kappa d$, we can proceed as in the proof of Theorem 2.2 for getting the existence of a C^1 curve $\kappa \to d_1(\kappa)$ defined for κ small and satisfying Theorem 2.4.
The last part of the theorem and Corollary 2.5 follow from the estimate, proved for κd small, in [BoHe5] Formula (5.24) :

$$h(\kappa,d,\epsilon) = \bar{h}(\kappa,d) - \frac{\sqrt{3}}{d^2}\left(1 - \frac{d^2}{5} + (1+d^2)\ \mathcal{O}(\kappa^2 d^2)\right)\ \epsilon^2 + \mathcal{O}_{\kappa,d}(\epsilon^3)\,. \tag{5.9}$$

6 Remarks on the stability of the symmetric bifurcating solutions

6.1 Local stability for the problem $(GL)_d$

Let (κ,d) be in $]0,+\infty[)^2$. A solution $(\hat{f},\hat{A};\hat{h})$ of the Ginzburg-Landau equations $(GL)_d$ is said *locally stable* if at fixed $\hat{h}$ it gives a local minimum of

the GL functional $(\Delta G)_h(f,A)$ attached to $(GL)_d$, with respect to the pairs (f,A). Else, it is said *unstable*.
We recall that the GL functional $(\Delta G)_h(f,A)$ is defined on $H^1(]-d/2,d/2[)^2$ by (see for example [BoHe5])

$$(f,A) \to (\Delta G)_h(f,A) = \int_{-\frac{d}{2}}^{\frac{d}{2}} [\, \kappa^{-2} f'^2 - f^2 + \frac{1}{2} f^4 + A^2 f^2 + (A'-h)^2 \,]\, dx \quad (6.1)$$

It results from [BoHe5] (Theorem 4.5) that, for κd large,the symmetric bifurcating solutions are unstable in the sense given before. This instability is completely determined for κd large enough, whatever the sign of $h_1(\kappa,d)$. So, the preceding sections are not useful for this study.

But, when κd is small, Theorem 2.4 allows us to improve Theorem 4.3 of [BoHe5] by the following theorem,

Theorem 6.1 *:*
There exist constants a_0 and a_1 such that, for (κ,d) in $]0,+\infty[)^2$ verifying $\kappa d \leq a_0$ and $0<\kappa \leq a_1$, and for h_0 satisfying (2.4), the following properties are satisfied,
i) When $d<d_1(\kappa)$, the bifurcating solutions starting from the normal solution $(0, h_0\, x; h_0)$ are locally stable.
i) When $d>d_1(\kappa)$, the bifurcating solutions starting from the normal solution $(0, h_0\, x; h_0)$ are unstable,
where $d_1(\kappa)$ is the function defined in Theorem 2.4.

Proof:
We come back to the proof of Theorem 4.3 in [BoHe5].
The study of the stability of the bifurcating solutions is reduced to the study of the sign of the two lowest eigenvalues, denoted $\lambda^{(1)}(\varepsilon,\kappa,d)$ and $\lambda^{(2)}(\varepsilon,\kappa,d)$ when ε is small, for the Neumann realization of the operator :

$$H^2(]-d/2,d/2[)^2 \ni \begin{pmatrix} g \\ b \end{pmatrix} \longmapsto \begin{pmatrix} -\kappa^{-2} g'' + (A^2+3f^2-1)\, g + 2\, A\, f\, b \\ -b'' + f^2\, b + 2A\, f\, g \end{pmatrix}, \quad (6.2)$$

with $A = A(\ \cdot\ ,\kappa,d,\varepsilon)$, $f = f(\ \cdot\ ,\kappa,d,\varepsilon)$.

When $\varepsilon = 0$, 0 is the lowest eigenvalue; it has multiplicity *two*.
When $|\varepsilon|>0$, using a perturbation analysis, we have seen that the sign of

the eigenvalues $\lambda^{(1)}(\varepsilon,\kappa,d)$ and $\lambda^{(2)}(\varepsilon,\kappa,d)$, is given, for ε small, by the sign of two reals denoted $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ which depend on κ and d.
We have proved that $\lambda_2^{(2)}(\kappa,d) > 0$ when κd is small (see [BoHe5], Lemma 7.3 i)), and that the sign of $\lambda_2^{(1)}(\kappa,d)$ is given by the sign of $h_1(\kappa,d)$ (see (7.6) in [BoHe5]).
Therefore, Theorem 6.1 results from Theorem 2.4.

6.2 Local stability for a reduced symmetric problem

In some other studies [BoHe2], [BoHe4], as was done in previous articles from [Gi2] or [Ch], we have restricted the research of the solutions of the Ginzburg-Landau equations to the set of the symmetric solutions. This assumption leads us to restrict the domain of the GL functional to a subset of $(H^1(]-d/2,d/2[))^2$ corresponding to these solutions.
This subset is defined, for fixed κ and d, by

$$\begin{aligned}\mathcal{H}_{sym} = \{(f,A) \in (H^1(]-d/2,d/2[))^2\ ;\\ \forall x \in]-d/2,d/2[\,,\ (f(-x),A(-x)) = (f(x),-A(x))\ \}\ .\end{aligned}$$

By considering this problem, the stability of a GL solution is sometimes different from the preceding case. This is in particular the case for the symmetric bifurcating solutions when κd is large. We have proved in [BoHe5] (Theorem 4.5) that the symmetric bifurcating solutions are unstable for κd large enough (with respect to the GL functional $(\Delta G)_h$). We now prove the following theorem.

Theorem 6.2 *:*
There exist constants $d_0 > 0$ and $a_0 > 0$ such that, for (κ,d) in $(]0,+\infty[)^2$ verifying $\kappa d \geq a_0$ and $d \geq d_0$ and for $h_0(\kappa,d)$ satisfying (2.4), the following properties are satisfied .
i) When $\kappa > \kappa_0(d)$, the bifurcating solutions starting from $(0,h_0\,x;h_0)$ are locally stable by respect to the functional $(\Delta G)_h$, in restriction to the set $\mathcal{H}_{sym}$.
ii) When $\kappa < \kappa_0(d)$, they are unstable,
where $\kappa_0\ :\ d \to \kappa_0(d)$ is the map defined in Theorem 2.2 .

Proof:
It is sufficient to prove that the bifurcating solutions give local minima for

$(\Delta G)_h$, when h is fixed. So, as in the proof of [BoHe5] (Theorem 4.5) we study the Hessian of the functional in restriction to $\mathcal{H}_{sym}$ corresponding to the symmetric solutions .
The study of the local stability is then reduced to the analysis of the spectrum of the self adjoint operator attached to this problem . This leads us to study the corresponding linearized GL equations (6.2) at the triple $(f(.,\varepsilon,\kappa,d), A(\cdot,\varepsilon,\kappa,d); h(\varepsilon,\kappa,d))$, but when the solutions (g,b) are searched in $\mathcal{H}_{sym}\bigcap(H^2(]-d/2,d/2[))^2$.

By the choice $h_0(\kappa,d)=\bar{h}(\kappa,d)$, the lowest eigenvalue for this problem, when $\varepsilon=0$, is equal to *zero* with multiplicity *one* (but *two* for the full problem). The corresponding eigenspace is generated by

$$\bar{u}_1(\;\cdot\;;\kappa,d)=\begin{pmatrix} f_0(\;\cdot\;;\kappa,d)\\ 0\end{pmatrix}\;,$$

where $f_0(\;\cdot\;;\kappa,d)$ is the normalized eigenfunction of (2.1) where $\tau=1$.
By regular perturbation analysis, we get,

Lemma 6.3 *:*
Let $d>0$, $\kappa>0$ and h_0 satisfying (2.4). Then, there exists ε_0 s.t. for $0<|\varepsilon|\leq\varepsilon_0$, the eigen-elements corresponding to the first lowest eigenvalue of (6.2) with (g,b) in $\mathcal{H}_{sym}\bigcap(H^2(]-d/2,d/2[))^2$, are described by

$$\left\{\begin{array}{l} \mu^{(1)}(\varepsilon,\kappa,d)=\varepsilon^2\mu_2^{(1)}(\kappa,d)+o(\varepsilon^2)\;,\\ g^{(1)}(x,\varepsilon,\kappa,d)=f_0(x;\kappa,d)+o(\varepsilon)\quad \textit{in } H^2(]-d/2,d/2[)\;,\\ b^{(1)}(x,\varepsilon,\kappa,d)=2\,\varepsilon\,A_{1,0}(x;\kappa,d)+o(\varepsilon^2)\;,\end{array}\right. \tag{6.3}$$

with

$$\mu_2^{(1)}(\kappa,d)=-\,4\,\frac{h_1(\kappa,d)}{h_0(\kappa,d)}\int_{-\frac{d}{2}}^{\frac{d}{2}}(A_0(x;\kappa,d))^2\,(f_0(x;\kappa,d))^2\,dx\;. \tag{6.4}$$

For proving Theorem 6.2, it is then sufficient to remark that, using the symmetry of $f_0(\;\cdot\;;\kappa,d)$ and $A_0(\;\cdot\;;\kappa,d)$, we have :

$$\lambda_2^{(1)}(\kappa,d)=\mu_2^{(1)}(\kappa,d)\;,$$

and then to apply Theorem 2.2. We have, indeed, $h_1(\kappa,d)<0$ when $\kappa>\kappa_0(d)$ and $h_1(\kappa,d)>0$ when $\kappa<\kappa_0(d)$.

Conclusion : This study gives a complete answer to the stability of bifurcating symmetric solutions when κd is large or small. We still have no information on the sign of h_1 in domains of the type $\{(\kappa, d)\ ;\ a_1 < \kappa d < a_2\}$ for $(a_1, a_2) \in (I\!R^{+*})^2$.

References

[Bo1] C. Bolley : Familles de branches de bifurcations dans les équations de Ginzburg-Landau, M^{2}AN, Vol. 25, n$^{\mathrm{o}}$ 3. p. 307-335, (1991).

[Bo2] C. Bolley : Modélisation du champ de retard à la condensation d'un supraconducteur par un problème de bifurcation. M^{2}AN, Vol. 26, n$^{\mathrm{o}}$ 2, p. 235-287 (1992).

[BoHe1] C. Bolley, B. Helffer : An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material. Annales de l'Institut Henri Poincaré (Section Physique Théorique), Vol. 58, n$^{\mathrm{o}}$ 2, p. 189-233 (1993).

[BoHe2] C. Bolley, B. Helffer : Rigorous results for the Ginzburg-Landau equations associated to a superconducting film in the weak κ-limit. Reviews in Math. Physics, Vol. 8, n$^{\mathrm{o}}$ 1, p. 43-83 (1996).

[BoHe3] C. Bolley, B. Helffer : Rigorous results on the Ginzburg-Landau models in a film submitted to an exterior parallel magnetic field. Part I. Nonlinear Studies, Vol. 3, n$^{\mathrm{o}}$ 1, p. 1-29 (1996).

[BoHe4] C. Bolley, B. Helffer : The Ginzburg-Landau equations in a semi-infinite superconducting film in the large κ limit. Eur. J. Appl. Math., Vol. 8, p. 347-367 (1997).

[BoHe5] C. Bolley, B. Helffer : Stability of bifurcating solutions for the Ginzburg-Landau equations. To appear in Reviews in Math. Physics (1998).

[Ch] S. J. Chapman : Nucleation of superconductivity in decreasing fields. I and II. Eur. J. Appl. Math., Vol. 5, n$^{\mathrm{o}}$ 4, p. 449-468, p. 469-494 (1994).

[CHLO] S. J. Chapman, S. D. Howison, J. B. McLeod and J. R. Ockendon : Normal-superconducting transitions in Ginzburg-Landau theory. Proc. Roy. Soc. Edin. 119A, p. 117-124 (1991).

[Gi1] V. L. Ginzburg : On the theory of superconductivity. Nuovo Cimento 2, p. 1234 (1955).

[Gi2] V. L. Ginzburg : On the destruction and the onset of superconductivity in a magnetic field. Soviet Physics JETP 7, 78 (1958).

[GiLa] V. L. Ginzburg, L. D. Landau : On the theory of superconductivity. Zh. Eksperim. i teor. Fiz. 20, p. 1064-1082 (1950).
English translation Men of Physics: L. D. Landau, I, Ed. by D. Ter Haar, Pergamon Oxford, p. 138-167 (1965).

[HeSj] B. Helffer, J. Sjöstrand : Multiple wells in the semiclassical limit I, Comm. in P.D.E., Vol. 9(4), p. 337-408 (1984).

[MiKe] M. H. Millman, J. B. Keller : Perturbation theory of nonlinear boundary-value problems. Journal of Mathematical Physics, Vol. 10, n^{o} 2, (February 1969).

[Od1] F. Odeh : Existence and bifurcation theorems for the Ginzburg-Landau equations. J. of mathematical physics, Vol. 8, n^{o} 12, (December 1967).

[Od2] F. Odeh : A bifurcation problem in superconductivity. In bifurcation theory and nonlinear eigenvalue problems, Edited by J. B. Keller, S. Antman, W. A. Benjamin, Inc. p. 99-112 (1969).

[Si] Y. Sibuya : Global theory of a second linear differential equation with a polynomial coefficient, North-Holland (1975).