KAM–renormalization-group for Hamiltonian systems with two degrees of freedom

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Abstract: We review a formulation of a renormalization-group scheme for Hamiltonian systems with two degrees of freedom. We discuss the renormalization flow on the basis of the continued fraction expansion of the frequency. The goal of this approach is to understand universal scaling behavior of critical invariant tori.

Keywords: renormalization group, breakup of invariant tori, KAM theory.

1 Introduction

Recently, renormalization-group ideas have been proposed to describe the breakup of invariant tori for Hamiltonian systems with two degrees of freedom [10, 8, 3, 4, 1]. Most of the numerical work has been done for the golden mean torus (mainly for practical convenience). In this article, we extend the renormalization-group transformation to more general frequencies.
We consider the following class of Hamiltonians with two degrees of freedom, quadratic in the actions $\underline{A} = (A_1, A_2)$:

$$H(\underline{A}, \underline{\varphi}) = \frac{1}{2} m(\underline{\varphi})(\underline{\Omega} \cdot \underline{A})^2 + [\underline{\omega}_0 + g(\underline{\varphi})\underline{\Omega}] \cdot \underline{A} + f(\underline{\varphi}),$$

where m, g, and f are even scalar functions of the angles $\underline{\varphi} = (\varphi_1, \varphi_2) \in T^2 = [0, 2\pi)^2$; the frequency vector of the considered torus $\underline{\omega}_0 = (\omega, -1)$ satisfies a Diophantine condition $|\underline{\omega}_0 \cdot \underline{\nu}| \geq \sigma |\underline{\nu}|^{-\tau}$ for all $\underline{\nu} \in Z^2$, and for some $\sigma > 0$ and $\tau > 1$. The rank of the matrix $\partial^2 H / \partial \underline{A}^2$ is equal to one (this matrix is proportional to the projection operator on the $\underline{\Omega}$-direction). Thus, det $[\partial^2 H / \partial \underline{A}^2] = 0$, i.e., the Hamiltonians we consider do not satisfy the twist condition. There is however a twist in a particular direction of the actions characterized by the vector $\underline{\Omega} = (1, \alpha)$, which is what is needed to obtain KAM stability of Diophantine tori for small perturbations [5].
If one takes g and f equal to zero, the equations of motion are

$$\frac{d\underline{A}}{dt} = -\frac{1}{2} \frac{\partial m}{\partial \underline{\varphi}} (\underline{\Omega} \cdot \underline{A})^2 , \qquad \frac{d\underline{\varphi}}{dt} = m(\underline{\varphi})(\underline{\Omega} \cdot \underline{A})\underline{\Omega} + \underline{\omega}_0 .$$

Then $\underline{A} = \underline{0}$ defines an invariant torus, and the motion on this torus is quasiperiodic with frequency vector $\underline{\omega}_0$. For g and f sufficiently small, a KAM theorem [5] proves the persistence of this torus for arbitrary m, provided that its mean value is nonzero. It is not necessary to have m small in order to prove the existence of the torus with frequency vector $\underline{\omega}_0$. This remark allows us to perform canonical transformations that stay within the class of Hamiltonians *quadratic* in the actions.
As one increases the perturbation (consisting of g and f), the torus gets deformed. This deformation is related to families of nearby periodic orbits that accumulate at the torus. This accumulation motivates the setup of a renormalization transformation combining a *rescaling* of phase space and an *elimination* of the irrelevant part at each scale.
An attractive (*trivial*) fixed point of the renormalization represents the phase where the torus exists. An hyperbolic (*nontrivial*) fixed point (or more generally an hyperbolic fixed set) corresponds to a

transition where the torus breaks up.
For the golden mean torus with frequency vector $\underline{\omega}_0 = (\gamma^{-1}, -1)$ where $\gamma = (1+\sqrt{5})/2$, numerical studies suggest that the critical surface is the stable manifold (of codimension 1) of a nontrivial fixed point. Also for quadratic irrationals (those with periodic continued fraction expansion), one expects a nontrivial fixed point for a certain renormalization operator. For nonperiodic continued fraction expansion, hyperbolic fixed sets instead of fixed points are conjectured [13, 11, 12, 14, 18, 15].

2 Renormalization transformation

We describe the renormalization scheme for a torus with arbitrary frequency vector $\underline{\omega}_0 = (\omega, -1)$ where $\omega \in]0,1[$. This renormalization relies upon the continued fraction expansion of ω:

$$\omega = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cdots}} \equiv [a_0, a_1, \ldots].$$

The best rational approximants are given by the truncations of this expansion $p_k/q_k = [a_0, a_1, \ldots, a_k = \infty]$. The corresponding periodic orbits with frequency vectors $\underline{\nu}_k = (q_k, p_k)$ (called "resonances" in what follows) accumulate at the invariant torus. This family of frequency vectors satisfies $|\underline{\omega}_0 \cdot \underline{\nu}_{k+1}| < |\underline{\omega}_0 \cdot \underline{\nu}_k|$ and $\lim_{k\infty} |\underline{\omega}_0 \cdot \underline{\nu}_k| = 0$, as it can be seen from the relation

$$\underline{\nu}_k = N_{a_0} \cdots N_{a_{k-1}} \underline{\nu}_0,$$

where $\underline{\nu}_0 = (1, 0)$ and N_{a_i} denotes the matrix

$$N_{a_i} = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, $\underline{\omega}_0 \cdot \underline{\nu}_k$ and $\underline{\omega}_0 \cdot \underline{\nu}_{k+1}$ are of opposite sign (as the stable eigenvalue of N_{a_i} is negative); thus the torus is approached from above and from below by the sequence of periodic orbits with frequency vectors $\{\underline{\nu}_k\}$.
We notice that if the continued fraction expansion is periodic with period s, then one can extract from the sequence $\{\underline{\nu}_k\}$, s families of periodic orbits which accumulate *geometrically*, with the same ratio, at the torus.
The word resonance refers to the fact that the small denominators $\underline{\omega}_0 \cdot \underline{\nu}_k$ that appear in the perturbation expansion are the smallest ones, i.e., $|\underline{\omega}_0 \cdot \underline{\nu}_k| < |\underline{\omega}_0 \cdot \underline{\nu}|$ for any $\underline{\nu} = (q, p)$ different from zero and $\underline{\nu}_k$, and such that $|q| < q_{k+1}$.
The main scale of the perturbation is defined by $\underline{\nu}_0$ for the torus with frequency ω. We will denote this scale by $[\underline{\nu}_0, \omega]$. The next smaller scale is $[\underline{\nu}_1, \omega]$. The renormalization transformation changes the coordinates such that the next smaller scale becomes the main one, i.e., the main scale is now $[\underline{\nu}_0, \omega']$ where $\omega' = [a_1, a_2, \ldots]$. As the frequency is changed (the continued fraction expansion is shifted to the left), the sequence of resonances is mapped into the sequence

$$\underline{\nu}'_k = N_{a_1} \cdots N_{a_{k-1}} \underline{\nu}_0.$$

The renormalization transformation is a map $(m, g, f, \omega, \alpha) \mapsto (m', g', f', \omega', \alpha')$, where α denotes the second component of $\underline{\Omega} = (1, \alpha)$.
The transformation $\mathcal{R}_{a_0}$ consists of four steps:
(1) A shift of the resonances constructed by the condition $\underline{\nu}_1 \mapsto \underline{\nu}_0$: we require that $\cos[(a_0, 1) \cdot \underline{\varphi}] = \cos[(1, 0) \cdot \underline{\varphi}']$. This change is done via a linear canonical transformation $(\underline{A}, \underline{\varphi}) \mapsto (N_{a_0}^{-1} \underline{A}, N_{a_0} \underline{\varphi})$. This step changes the frequency vector $\underline{\omega}_0$ into $\underline{\omega}'_0 = (\omega', -1)$, since $N_{a_0} \underline{\omega}_0 = -\omega \underline{\omega}'_0$.
(2) We rescale the energy (or time) by a factor ω^{-1}, and we change the sign of both phase space

coordinates $(\underline{A},\underline{\varphi}) \mapsto (-\underline{A},-\underline{\varphi})$, in order to have $\underline{\omega}_0'$ as the new frequency vector, i.e., the average of the term linear in the actions is of the form $\underline{\omega}_0' \cdot \underline{A}$. Furthermore, $\underline{\Omega} = (1,\alpha)$ is changed into $\underline{\Omega}' = (1,(a_0+\alpha)^{-1})$.
(3) Then we perform a rescaling of the actions: H is changed into $\hat{H}(\underline{A},\underline{\varphi}) = \lambda H(\underline{A}/\lambda,\underline{\varphi})$ with $\lambda = \lambda(H)$ such that the mean value of m is equal to 1, i.e., $\lambda = \omega^{-1}(a_0+\alpha)^2\langle m\rangle$.
(4) A canonical transformation that eliminates the nonresonant part of g and f.
The choice of which part of the perturbation is resonant or not is somewhat arbitrary. Recall that what is relevant is the accumulation of resonances; therefore a possible choice of irrelevant modes could include all modes except the resonances. However, it is desirable from a numerical point of view not to eliminate too many modes. A reasonable choice is the set $\mathcal{C}$ of integer vectors $\underline{\nu}$ such that $|\nu_2| > |\nu_1|$. We notice that the relation defining $\underline{\nu}_k = (q_k,p_k)$ shows that $q_k \geq p_k$ for $k \geq 0$, i.e., the resonances are not elements of $\mathcal{C}$.
From the form of the eigenvectors of N_{a_0}, one can see that every $\underline{\nu} \in Z^2 \setminus (0,0)$ goes into $\mathcal{C}$ after sufficiently many iterations of matrices N_{a_i} (as the unstable eigenvector of $N_{a_0}^{-1}$, which is $\underline{\omega}_0$, points into $\mathcal{C}$). In other terms, a resonant mode at some scale turns out to be a nonresonant one at a sufficiently smaller scale. We notice that $(0,0)$ is not an element of $\mathcal{C}$, i.e., it is resonant.
We eliminate completely all the nonresonant modes of g and f by a canonical transformation, connected to the identity, which is defined by iterating KAM-type transformations.
We denote I^- the projection operator on the nonresonant part:

$$I^- f(\underline{\varphi}) = \sum_{\underline{\nu}\in\mathcal{C}} f_\nu e^{i\nu\cdot\varphi},$$

and $\underline{\partial}$ the derivative with respect to the angles $\underline{\varphi}$: $\underline{\partial} = \partial/\partial\underline{\varphi}$.
The KAM iterations we perform are generated by functions linear in the actions, and it allows us, following Thirring [17], to remain quadratic in the actions at each step.
This can be seen by working with Lie transformations $\mathcal{U}_S : (\underline{A},\underline{\varphi}) \mapsto (\underline{A}',\underline{\varphi}')$ generated by

$$S(\underline{A},\underline{\varphi}) = Y(\underline{\varphi})\underline{\Omega}\cdot\underline{A} + Z(\underline{\varphi}) + a\underline{\Omega}\cdot\underline{\varphi},$$

characterized by two scalar functions Y and Z, and a constant a. The expression of the Hamiltonian in the new variables is obtained by

$$\begin{aligned} H' &= H \circ U_S = e^{\hat{S}} H \\ &= H + \{S,H\} + \{S,\{S,H\}\}/2! + \cdots, \end{aligned}$$

where $\{\,,\,\}$ is the Poisson bracket of two functions of the action and angle variables

$$\{S,H\} = \frac{\partial S}{\partial \underline{\varphi}} \cdot \frac{\partial H}{\partial \underline{A}} - \frac{\partial S}{\partial \underline{A}} \cdot \frac{\partial H}{\partial \underline{\varphi}}.$$

From this equation, one can see that H' is again quadratic in the actions: $\partial S/\partial\underline{\varphi}$ and $\partial H/\partial\underline{A}$ are linear in the actions, $\partial S/\partial\underline{A}$ is action-independent and $\partial H/\partial\underline{\varphi}$ is quadratic; therefore $\{S,H\}$ is quadratic in the actions, and in fact, quadratic in the $\underline{\Omega}\cdot\underline{A}$ variable.
The generating function is determined such that it eliminates the nonresonant modes of g and f. This cannot be defined directly, so one iterates an infinite number of steps of such transformations such that one iteration eliminates the order ε while producing an order ε^2. Following Ref. [4], we have the equations

$$\begin{aligned} &\underline{\omega}_0 \cdot \underline{\partial} Z + I^- f = const, \\ &\underline{\omega}_0 \cdot \underline{\partial} Y + I^- g + I^-(m\underline{\Omega}\cdot\underline{\partial} Z) + a\Omega^2 I^- m = 0, \\ &\langle m\rangle a\Omega^2 + \langle g\rangle + \langle m\underline{\Omega}\cdot\underline{\partial} Z\rangle = 0. \end{aligned}$$

The constant a corresponds to a translation in the action variables which has the purpose of eliminating the mean value of the linear term in the variable $\underline{\Omega} \cdot \underline{A}$. We notice that Y and Z only contain nonresonant modes, e.g., $Y = I^{-}Y$, and that Z (resp. Y) is chosen to reduce $I^{-}f$ (resp. $I^{-}g$) from ε to ε^2. These equations are solved by Fourier series:

$$Z(\underline{\varphi}) = \sum_{\underline{\nu} \in C} \frac{i}{\underline{\omega}_0 \cdot \underline{\nu}} f_{\underline{\nu}} e^{i\underline{\nu} \cdot \underline{\varphi}},$$

$$Y(\underline{\varphi}) = \sum_{\underline{\nu} \in C} \frac{i}{\underline{\omega}_0 \cdot \underline{\nu}} \left[g_{\underline{\nu}} + (m\underline{\Omega} \cdot \underline{\partial} Z)_{\underline{\nu}} + m_{\underline{\nu}} a \Omega^2 \right] e^{i\underline{\nu} \cdot \underline{\varphi}}.$$

For each iteration, we express the Hamiltonian in the new action and angle variables. This is done recursively, following Ref. [4]. This KAM-type iteration adds terms of order ε to m and to the resonant part of f and g.
We iterate this procedure in order to reduce completely the nonresonant part of f and g:

$$H' = H \circ U_H, \text{ where } U_H = U_{S_1} \circ U_{S_2} \circ \cdots U_{S_n} \circ \cdots,$$

where $I^{-}f = I^{-}g = 0$. We notice that step (4) does not change $\underline{\omega}_0$ and $\underline{\Omega}$ [as opposed to steps (1), (2), and (3)].
The transformation U_H is rigorously defined for a sufficiently small perturbation, consisting of f and g (see Ref. [10]), but the convergence in the whole domain of existence of the torus is a conjecture based on numerical observations.
In summary the renormalization transformation acts as follows: First, some of the resonant modes are turned into nonresonant ones (by a rescaling of phase space). Then a KAM-type iteration eliminates these nonresonant modes, while slightly changing the resonant ones.

3 Comments

The numerical implementation of the renormalization scheme for a given frequency ω shows that there are two main domains separated by a surface: one where the iteration converges to $f = g = 0$ and the other where it diverges to infinity.
The conjecture is that the boundary of the domain of convergence of the transformation, ∂D, coincides (at least locally, not too far away from the nontrivial fixed set) with the critical surface (where the torus is critical). This is by no means trivial since the transformation is based on properties that are valid for small f and g. However, for a one-parameter family, the numerical evidence of the coincidence between the critical coupling (where the torus breaks up, determined by Greene's criterion) and the value of the parameter where the iteration starts to diverge, indicates that we can expect ∂D to coincide with the critical surface, at least in a large region containing the nontrivial fixed set.
In this section, we analyze the renormalization flow on the basis of the continued fraction expansion of the frequency.

Quadratic irrational frequencies– We start by analyzing the effect on ω and α of s renormalization steps. Denote by $\{b_j\}$ the continued fraction expansion of α: $\alpha = [b_0, b_1, \ldots]$. The renormalization $\mathcal{R}_{a_{s-1}} \mathcal{R}_{a_{s-2}} \cdots \mathcal{R}_{a_0}$ changes $\omega = [a_0, a_1, \ldots]$ into $[a_s, a_{s+1}, \ldots]$, and α into $[a_{s-1}, a_{s-2}, \ldots, a_0, b_0, b_1, \ldots]$. If ω has a periodic continued fraction expansion of period s, i.e., $\omega = [(a_1, \ldots, a_s)_\infty]$, one expect to have a nontrivial fixed point on the critical surface of the renormalization transformation in which one step is defined by the composition $\mathcal{R}_{a_s} \mathcal{R}_{a_{s-1}} \cdots \mathcal{R}_{a_1}$. We notice that α converges to $[(a_s, \ldots, a_1)_\infty]$. Therefore $\underline{\Omega}$ converges to the unstable eigenvector of the matrix $N_{a_s} \cdots N_{a_1}$ (the stable eigenvector of this matrix is $\underline{\omega}_0$).
The nontrivial fixed point associated with ω defines a universality class that we characterize with

critical exponents such as the total rescaling of phase space (product of the s rescalings), and the unstable eigenvalue of the linearized map around the fixed point.
Two frequencies ω_1 and ω_2 having the same periodic tail (and different first entries) in their continued fraction expansions, belong to the same universality class. The initial integers in the continued fraction expansion are irrelevant.
Associated with this nontrivial fixed point, we also have nontrivial fixed sets related to the nontrivial fixed point by symmetries. Therefore these hyperbolic sets belong to the same universality class. According to Ref. [3], these sets are given by the intertwining relation

$$\mathcal{R}_{a_0} \circ \mathcal{T}_{\underline{\theta}} = \mathcal{T}_{N_{a_0}\underline{\theta}} \circ \mathcal{R}_{a_0},$$

where $\mathcal{T}_{\underline{\theta}}$ is defined as the translation $(\mathcal{T}_{\underline{\theta}} f)(\underline{\varphi}) = f(\underline{\varphi} + \underline{\theta})$. Applying the relation

$$\mathcal{R}_{a_s} \circ \cdots \circ \mathcal{R}_{a_1} \circ \mathcal{T}_{\underline{\theta}} = \mathcal{T}_{N_{a_s} \cdots N_{a_1}\underline{\theta}} \circ \mathcal{R}_{a_s} \circ \cdots \circ \mathcal{R}_{a_1}$$

to the fixed point $H_*(\underline{\varphi})$, we have

$$\mathcal{R}_{a_s} \circ \cdots \circ \mathcal{R}_{a_1} H_*(\underline{\varphi} + \underline{\theta}) = H_*(\underline{\varphi} + N_{a_s} \cdots N_{a_1}\underline{\theta}).$$

The map

$$\underline{\theta} \mapsto N_{a_s} \cdots N_{a_1}\underline{\theta} \bmod 2\pi,$$

gives the nature of the orbits to which the transformation converges.
For instance, for the golden mean case, one has a nontrivial fixed point and a nontrivial 3-cycle in the space of even Hamiltonians. For Hamiltonians without parity restriction, the nontrivial fixed sets can be labeled by the orbits of the Anosov map $N = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (N^2 is Arnold's cat map). For $\omega = [(1,2)_\infty]$, one has a nontrivial 2-cycle and two fixed points in the space of even Hamiltonians.
In the perturbative regime, there exists a geometrical accumulation of a certain sequence of periodic orbits with a "trivial" ratio (trivial in the sense that it is explicit, see comment in the previous section). The fact that it also happens in the critical case, but with a nontrivial ratio (which is the unstable eigenvalue of the linearized map around the nontrivial fixed point), implies universal self-similar properties of the critical torus [9, 16].
As already mentioned in Ref. [11], $\mathcal{R}_{a_i}$ becomes more and more singular as a_i increases. Therefore the picture we present is valid only for bounded a_i. For large a_i, there is, to our knowledge, no description of the renormalization flow and its properties, even from a numerical point of view.

Nonquadratic irrational frequencies– In that case, we cannot expect any geometrical accumulation of periodic orbits even if the Hamiltonian is close to integrable (perturbative regime). The renormalization has no fixed point, but instead its flow is related to a chaotic trajectory of the Gauss map

$$\omega \mapsto 1/\omega - [1/\omega],$$

where [] denotes the integer part. We notice that α follows the inverse of this trajectory.
For a typical $\omega \in]0,1[$, it is numerically impossible to figure out what could be the critical set, as one only works with finite precision (and thus a finite number of entries in the continued fraction, determined by iterating the Gauss map).
Instead, one should give a sufficient number of entries in the continued fraction expansion, and then iterate the renormalization transformation on the critical surface. These entries should be bounded in order to avoid singularities mentioned above.
On the critical surface, one can conjecture the existence of a critical strange attractor. But these hyperbolic sets should be conceptually different from the sets found in the quadratic irrational case (related to the nontrivial fixed point by symmetries): there is a continuous distribution of critical exponents (e.g., rescalings). The mean-rescaling (geometric mean value) and the largest Lyapunov

exponent characterize the universal class associated with ω.
In order to have information about a typical $\omega \in]0,1[$, an ergodic renormalization for random continued fractions has been proposed [13, 15]. It consists in determining universal parameters by averaging over a large number of random continued fractions, constructed following a given probability distribution for the coefficients.
The conjecture is that ergodic renormalization trajectories converge to a strange attractor (which contains all the nontrivial fixed sets obtained for quadratic irrationals). Lyapunov exponents and other quantities such as the mean-rescaling are universal. Related ideas have been proposed for circle maps in Refs. [11, 12].

Extension to higher dimensional systems– We have seen that the existence of a nontrivial fixed point was based on a *geometrical* accumulation of a certain sequence of periodic orbits. These were given by the truncations of the continued fraction expansion (for quadratic irrationals). For three degrees of freedom, we lack of a theory that generalizes the continued fractions. However, we can choose a torus such that it is a geometrical accumulation of periodics orbits, i.e., the sequence of resonances is generated by a *single* matrix (as it was the case for the golden mean). It allows us to define a renormalization-group transformation with a *fixed* frequency vector. Even in that case, preliminary studies [6] suggest that one can expect a strange attractor instead of a nontrivial fixed point. This feature depends strongly on the spectrum of the matrix which generates the resonances.

4 Analyticity properties of the nontrivial fixed point

As the renormalization transformation described in the previous sections does not reduce the nonresonant part of m (it only reduces g and f) in order to remain quadratic in the actions, the shift of the Fourier modes [step (1)] reduces the analyticity of m at each step. As a consequence, the nontrivial fixed point is expected to have a nonanalytic m. The numerical implementation of the transformation gives accurate computation of the critical exponents characterizing the universality class, despite the fact that m is nonanalytic at the nontrivial fixed point. At present, this fact is not well understood.
A method that can yield an analytic nontrivial fixed point is to eliminate the nonresonant part of m together with the one of g and f [1]. The main drawback of this method is that one cannot remain quadratic in the actions. This drastically complicates the numerical implementation of the transformation as one needs to work with N scalar functions instead of 3, N denoting the numerical truncation of the Taylor series in the actions.

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