Hardy inequalities for magnetic Dirichlet forms

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Abstract

It is known that the classical Hardy inequality fails in $\mathbb{R}^2$. We show that under certain non-degeneracy conditions on vector potentials, the Hardy inequality becomes possible for the corresponding magnetic Dirichlet form.
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0. Introduction. Let $\mathbf{a}$ be a magnetic vector potential and let ρ be a non-negative function on $\mathbb{R}^d$, $d \geq 2$. In this short note we consider the Hardy type estimates

$$\int_{\mathbb{R}^d} \rho |u|^2 dx \leq C h(\mathbf{a})[u] := C \int_{\mathbb{R}^d} |(i\nabla + \mathbf{a})u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^d), \tag{1}$$

where the constant C might depend on $\mathbf{a}$, d and ρ but not on u. If $\mathbf{a} = 0$, $\rho = |x|^{-2}$ and $d \geq 3$, then (1) coincides with the classical Hardy inequality where $C = C(d) = 4(d-2)^{-2}$. It is also known from Kato's inequality [K] (see also [AHS]), that (1) for $\mathbf{a} = 0$ implies the same inequality (with the same constant C) for $\mathbf{a} \neq 0$.

If $d = 2$ then the classical Hardy inequality is no longer true. The standard form of this inequality can be given by (1) with $\rho(x) = |x|^{-2}(1 + \log^2 |x|)^{-1}$, $\mathbf{a} = 0$ and under some additional assumptions on u. For example we have

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2(1 + \log^2 |x|)} dx \leq C \int_{\mathbb{R}^2} |\nabla u|^2 dx, \qquad \int_{\{|x|=1\}} u(x)\, dx = 0. \tag{2}$$

It was observed in [S1] that the logarithmic factor in (2) is needed only for functions u depending on $|x|$ and can be removed for functions u satisfying $\int_{\{|x|=r\}} u(x)\, dx = 0$ for any $r > 0$.

The main result of this paper shows that by introducing a non-trivial magnetic field $\mathbf{a}$, we sometimes are able to remove the unpleasant logarithmic factor in (2) and prove the inequality

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} dx \leq C \int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \tag{3}$$

without any other additional assumptions on u. An important example where (3) can be used is the study of the negative spectrum of two-dimensional Schrödinger operators (see [BL], [S2] and [LN]). In [LN] the authors were forced to implant the Hardy term $|x|^{-2}$ into their class of Schrödinger operators in order to obtain a CLR-type inequality. The inequality (3) automatically gives this term and therefore leads to natural applications to the corresponding magnetic Schrödinger operator.

Another application of our results concerns the problem of the existence of resonance states. In particular, (3) implies that the class of vector potentials $\mathbf{a}$ considered in this paper, "takes off" the resonance state at 0 for the corresponding two-dimensional magnetic Schödinger operator. This is, however, a partial case of a more general result obtained in [W], where it was proved that any non-trivial vector potential $\mathbf{a}$ removes the resonance state at 0. This fact follows from the following Hardy-type inequality

$$\int_{|x|<1} |u|^2\, dx \leq C \int_{\mathbb{R}^2} \omega |(i\nabla + \mathbf{a})u|^2\, dx\,, \qquad C = C(\omega, \mathbf{a})\,,$$

holding for any $\mathbf{a}$ which cannot be gauged away (see [W]) and any positive weight ω for which $\omega + \omega^{-1}$ is locally bounded.

Notice that the spectral properties of the operator K_r defined in (6) and the corresponding decompositions (9), (10) were used in [LS] when studying the spectral asymptotics of magnetic two-dimensional Schrödinger operators with respect to a small coupling constant at the corresponding vector potential.

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1. The main result. Most of our discussions will be described in polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{S}^1$. We denote by $e_r = x/r$ the radial unit vector and by e_θ the unit vector which completes e_r up to the oriented orthonormal basis at $x \neq 0$. We work in the transversal (or Poincaré) gauge (see [T, Section 8.4.2]). In this gauge the radial component of the vector potential $\mathbf{a}$ is equal to zero and therefore

$$(\mathbf{a}, e_r) = 0 \qquad \text{and} \qquad (\mathbf{a}, e_\theta) =: a.$$

Applying Stokes' formula we find

$$\Phi(r) := (2\pi)^{-1} \int_{\mathbb{S}} a(r, \theta)\, r\, d\theta = (2\pi)^{-1} \int_{|x|<r} b(x)\, dx, \qquad b = \operatorname{curl} \mathbf{a}, \tag{4}$$

where by $\Phi = \Phi(r)$ we denote the normalized magnetic flux through the disk $B(0,r) = \{x \in \mathbb{R}^2 : |x| < 1\}$. The quadratic form $h(\mathbf{a})$ introduced in (1) can be written in a more convenient form by using the polar coordinate

$$h(\mathbf{a})[u] = \int_0^\infty \int_0^{2\pi} \left(|u'_r|^2 + r^{-2} |iu'_\theta + r\, a\, u|^2 \right) r\, d\theta\, dr. \tag{5}$$

When studying the form (5) we use spectral properties of the selfadjoint differential operator K_r defined on $H^1(\mathbb{S})$ by

$$K_r \varphi = i \frac{\partial}{\partial \theta} \varphi + r a \varphi, \qquad 0 \le \theta < 2\pi. \tag{6}$$

The spectrum of this operator is discrete and its eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$ and the complete orthonormal system of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{Z}}$ are given by

$$\lambda_k = \lambda_k(r) = k + (2\pi)^{-1}\, r \int_0^{2\pi} a(r,\theta)\, d\theta = k + \Phi(r) \tag{7}$$

and

$$\varphi_k = \varphi_k(r,\theta) = (2\pi)^{-1/2} e^{-i(\theta\, \lambda_k(r) - r \int_0^\theta a(r,\eta)\, d\eta)}. \tag{8}$$

For any function $u \in L^2(\mathbb{R}^2)$ we introduce the following decomposition

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} u_k(r) \varphi_k(r,\theta). \tag{9}$$

Obviously if $u \in H^1(\mathbb{R}^2)$, then by Parseval's identity

$$h(\mathbf{a})[u] = \int_{\mathbb{R}^2} |u'_r|^2\, dx + \sum_{k\in\mathbb{Z}} \int_0^\infty |\lambda_k(r)|^2 |u_k(r)|^2\, r^{-1} dr. \tag{10}$$

Let $\varepsilon \in (0, 1/2)$. Denote

$$M(\varepsilon) = \{r > 0 : \min_{k\in\mathbb{Z}} |k - \Phi(r)| < \varepsilon\}. \tag{11}$$

If Φ is a continuous function (which normally follows from our assumptions), then the set $M(\varepsilon)$ consists of not more than a countable number of open intervals.

Theorem 1 *Let* $\mathbf{a}$ *be continuous on* $\mathbb{R}^2$ *and* curl $\mathbf{a} \in L^1_{\rm loc}(\mathbb{R}^2)$. *Assume that there exist* $A = A(\varepsilon) > 0$, $\varepsilon \in (0,1/2)$ *and a finite or infinite number of open intervals* $I_j = (\alpha_j, \beta_j)$, $j = 1, \ldots, N$, $N \le \infty$, *possibly accumulating at infinity, such that*

1) $M(\varepsilon) \subset \cup_{j=1}^N I_j$,

2) $\beta_{j-1} < \alpha_j < \beta_j < \infty, \quad j = 1, \ldots, N$,

3) $|I_j| = \beta_j - \alpha_j \le A \min\{1 + \alpha_j, \alpha_j - \beta_{j-1}, \alpha_{j+1} - \beta_j\}, \quad j = 1, \ldots, N$,

where formally $\beta_0 := \beta_1 - \alpha_2$. *Then the following magnetic Hardy-type inequality holds*

$$\int_{\mathbb{R}^2} \frac{|u|^2}{1+|x|^2}\, dx \le C \int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})\, u|^2\, dx, \qquad u \in C_0^\infty(\mathbb{R}^2), \tag{12}$$

where $C = C(\varepsilon, A)$.

Remark. The condition 3) forbids the flux Φ (see (4)) to stabilize at integers. For example, if the magnetic field b is compactly supported and its total flux is not an integer, then the number of intervals I_j is finite, all the conditions 1)-3) of Theorem 1 are satisfied and the inequality (12) holds true.

Before proving Theorem 1 we would like to illustrate the conditions 1)-3) by giving the following example: Assume that for each $j \in \mathbb{N}$ there exists a gauge such that $\mathbf{a}(x) = 0$ for $x \in \{\alpha_j < |x| < \beta_j\}$, where $\beta_j < \alpha_{j+1}$ and $\beta_j/\alpha_j \to \infty$ as $j \to \infty$. Introduce

$$\psi_j(x) = \min\{\ln_+(|x|\alpha_j^{-1}),\, 1,\, \ln_+(\beta_j |x|^{-1})\}, \quad j \in \mathbb{N}.$$

Then $h(\mathbf{a})[\psi_j] = h(0)[\psi_j] \le 4\pi$, while $\int |\psi_j|^2 (1+|x|^2)^{-1}\, dx \to \infty$ as $j \to \infty$ and thus the inequality (12) fails. In this case the magnetic flux Φ satisfies $\Phi(r) \in \mathbb{Z}$ if $\alpha_j < r < \beta_j$, which is the course of the magnetic field being trivial on long intervals.

2. Proof of Theorem 1.

Fix now a smooth function χ such that $0 \le \chi(t) \le 1$, $\chi(t) = 1$ for $0 \le t \le 1$ and $\chi(t) = 0$ for $t < -\frac{1}{2A}$ or $t > 1 + \frac{1}{2A}$. Denote

$$\chi_j(r) := \chi(|I_j|^{-1}(r - \alpha_j)) \quad \text{and} \quad \psi(r) := \sum_{j=1}^N \chi_j(r).$$

¿From the condition 3) it follows that the supports of the functions χ_j are disjoint. The function $1-\psi$ "cuts off" the "bad" set where the eigenvalues λ_k introduced in (7) are less than ε. Thus $\psi(r) \ne 1$ implies $|\lambda_k(r)| > \varepsilon$ and we conclude

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^2} \frac{|u|^2}{1+|x|^2}\, dx &\le \int_{\mathbb{R}^2} \frac{|\psi u|^2}{1+|x|^2} dx + \int_0^\infty \frac{(1-\psi(r))^2}{1+r^2} \sum_{k\in\mathbb{Z}} |u_k(r)|^2\, r\, dr \\ &\le \int_{\mathbb{R}^2} \frac{|\psi u|^2}{1+|x|^2}\, dx + \varepsilon^{-2} \int_0^\infty \sum_{k\in\mathbb{Z}} \lambda_k^2(r) |u_k(r)|^2 \frac{dr}{r} \\ &\le \int_{\mathbb{R}^2} \frac{|\psi u|^2}{1+|x|^2}\, dx + \epsilon^{-2} h(\mathbf{a})[u]. \end{aligned} \tag{13}$$

It remains to estimate the term $\int (1+|x|^2)^{-2}|\psi u|^2 dx$. Since the supports of the functions χ_j are disjoint, it is sufficient to consider their contributions separately. We now use that if $u \in H^1(\alpha,\beta)$, $u(\beta)=0$, $\beta > \alpha \geq 0$, then

$$\int_\alpha^\beta |u(r)|^2\, r\, dr \leq 2^{-1}(\beta-\alpha)^2 \int_\alpha^\beta |u'(r)|^2\, r\, dr$$

For any fixed $\theta \in \mathbb{S}$ we have $\operatorname{supp} \chi_j u(\cdot,\theta) \subset (\alpha_j - |I_j|/2A, \beta_j + |I_j|/2A)$ and thus the latter inequality implies

$$\begin{aligned} &\int |\chi_j(r) u(r,\theta)|^2\, r\, dr \leq 2^{-1}|I_j|^2(1+A^{-1})^2 \int \left|\frac{\partial}{\partial r}(\chi_j(r)u(r,\theta))\right|^2 r\, dr \\ &\leq 2^{-1}(1+A^{-1})^2 \left(|I_j|^2 \int_{\chi_j \neq 0} \left|\frac{\partial u}{\partial r}\right|^2 r\, dr + (\max|\chi'|^2) \int_{\chi_j' \neq 0} |u|^2\, r\, dr \right). \end{aligned} \tag{14}$$

The condition 3) gives us, in particular, $|I_j| < 1+\alpha_j$. If we integrate the inequality (14) over $\mathbb{S}$ and estimate the values of $|x|$ according to the two side inequalities $\alpha_j - (2A)^{-1} < |x| < \beta_j + (2A)^{-1} = \alpha_j + |I_j| + (2A)^{-1}$ as $x \in \operatorname{supp}\chi_j$, then we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|\chi_j u|^2}{1+|x|^2} dx &= \int_0^{2\pi}\int_0^\infty \frac{|\chi_j u|^2}{1+|r|^2}\, r\, dr\, d\theta \\ &\leq C_1 \int_0^{2\pi}\int_{\chi_j\neq 0} \left|\frac{\partial u}{\partial r}\right|^2 r\, dr d\theta + C_2 \int_{\chi_j' \neq 0} \sum_{k\in\mathbb{Z}} |u_k(r)|^2 \frac{dr}{r}, \end{aligned}$$

where the constants $C_l = C_l(A)$, $l=1,2$. By using again $\chi_{j_1}\chi_{j_2} = 0$ for $j_1 \neq j_2$ and the inequality $|\lambda_k(r)| > \varepsilon$ as $\chi_k'(r) \neq 0$, we conclude

$$\begin{aligned} \sum_{j=1}^N \int_{\mathbb{R}^2} \frac{|\chi_j u|^2}{1+|x|^2}\, dx &\leq C_1 \int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial r}\right| dx + C_2 \varepsilon^{-2} \int_0^\infty \sum_{j=1}^N \lambda_j^2(r)|u_j(r)|^2 \frac{dr}{r} \\ &\leq C_3 h(\mathbf{a})[u], \end{aligned}$$

where $C_3 = \max(C_1, C_2\varepsilon^{-2})$. This together with (13) completes the proof. □

3. A local Hardy inequality We can now easily obtain a version of Hardy's inequality for a set of functions with supports in a bounded set. By analogy with (11) we introduce

$$L(\varepsilon) = \{r > 0 : \min_{k\in\mathbb{Z}} |k - \Phi(r)| < \varepsilon\}, \quad \varepsilon \in (0, 1/2).$$

Theorem 2 *Let* $\mathbf{a}$ *be continuous on* $\mathbb{R}^2 \setminus \{0\}$ *and* $\operatorname{curl} \mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\{0\})$. *Assume that there exist* $A = A(\varepsilon) > 0$, $\varepsilon \in (0,1/2)$ *and a finite or infinite number of open intervals* $I_j = (\alpha_j, \beta_j)$, $j = 1,\dots,N$, $N \leq \infty$, *possibly accumulating at zero, such that*

1) $L(\varepsilon) \subset \cup_{j=1}^N I_j$,

2) $0 < \beta_{j+1} < \alpha_j < \beta_j < \infty, \quad j = 1,\dots,N$,

3) $|I_j| = \beta_j - \alpha_j \leq A \min\{\alpha_j, \alpha_j - \beta_{j+1}, \alpha_{j-1} - \beta_j\}, \quad j=1,\dots,N$,

where $\alpha_0 := \alpha_1 + \beta_1 - \beta_2$. *Then the following magnetic Hardy-type inequality holds*

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\, dx \leq C \int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})\, u|^2\, dx, \qquad u \in C_0^\infty(B(0,R)\setminus\{0\}), \tag{15}$$

where $C = C(\varepsilon, A, R)$.

Proof. Let us consider the change of variables $x = 1/y$, $v(y) = u(1/y)$. Then the class of function $C_0^\infty(B(0,r_0)\setminus\{0\})$ maps onto $C_0^\infty(\mathbb{R}^2 \setminus \bar{B}(0,1/r_0))$, all the conditions of Theorem 2 become equivalent to the corresponding conditions of Theorem 1 and the inequality (15) turns into (12). The theorem is proved. □

Remark. The conditions of Theorem 2 are satisfied only if the magnetic field $b = \operatorname{curl}\mathbf{a}$ has a singularity at $x=0$. Otherwise for any $\varepsilon > 0$ and $r_0 > 0$ there exists $\delta > 0$ such that $[0,\delta) \subset L(\varepsilon, r_0)$. This contradicts the condition 2) which states, in particular, that all the intervals I_j are separated from 0.

Combining Theorems 1 and 2 we obtain a result concerning the inequality (3).

Corollary 1 *Under the conditions of Theorems 1 and 2 the two-dimensional magnetic Hardy inequality* (3) *holds.*

4. Aharonov-Bohm-type magnetic fields. Finally we would like to give here a simple example. Let $\mathbf{a}$ be an Aharonov-Bohm-type magnetic field, namely,

$$(\mathbf{a}, e_\theta) = a(r,\theta) = \frac{\Psi(\theta)}{r}, \qquad \Psi \in L^\infty(\mathbb{S}). \tag{16}$$

In this case the corresponding magnetic field b is equal to zero everywhere except $x=0$. Denote by $\bar\Psi$ the mean value of the function Ψ over $\mathbb{S}$

$$\bar\Psi = (2\pi)^{-1} \int_0^{2\pi} \Psi(\theta)\, d\theta.$$

Theorem 3 *Let us assume that the vector potential* $\mathbf{a}$ *is given by* (16) *and* $\bar\Psi \neq k$, *for any* $k \in \mathbb{Z}$. *Then*

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\, dx \leq A \int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})u|^2\, dx, \qquad u \in C_0^\infty(\mathbb{R}^2\setminus\{0\}),$$

where $A = (\min_{k\in\mathbb{Z}} |k - \bar\Psi|)^{-2}$. *The constant* A *is sharp.*

Proof. In this case the eigenvalues defined in (7) are independent of r and equal $\lambda_k = k + \bar{\Psi}$. Then using (9) and (10) we obtain

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\, dx \le A \int_0^\infty \sum_{k\in\mathbb{Z}} \lambda_k^2 |u_k(r)|^2 \frac{dr}{r} \le A\, h(\mathbf{a})[u].$$

Let us assume that the constant A is achieved at k_0, $A = |k_0 + \bar{\Psi}|^{-2}$. Then it easy to see that the constant A is sharp on the class of functions $\phi(r)\exp(i\theta k_0)$, where $\phi \in C_0^\infty(0,\infty)$. The proof is complete. □

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