Theory of Circle Maps and the Problem of One-Dimensional Optical Resonator with a Periodically Moving Wall

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Abstract

We consider the electromagnetic field in a cavity with a periodically oscillating perfectly reflecting boundary and show that the mathematical theory of circle maps leads to several physical predictions. Notably, well-known results in the theory of circle maps (which we review briefly) imply that there are intervals of parameters where the waves in the cavity get concentrated in wave packets whose energy grows exponentially. Even if these intervals are dense for typical motions of the reflecting boundary, in the complement there is a positive measure set of parameters where the energy remains bounded.

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I. INTRODUCTION

In this paper, we consider the behavior of the electromagnetic field in a resonator one of whose walls is at rest and the other moving periodically. The main point we want to make is that several results in the mathematical literature of circle maps immediately yield physically important conclusions.

The problem at hand is mathematically equivalent to the study of the motion of vibrating strings with a periodically moving boundary [1,2], or the classical electromagnetic field in a periodically pulsating cavity [3,4]. It is connected with the vacuum quantum effects in such region [5,6]. The problem is also of practical importance, e.g., for the formation of short laser pulses [7].

The goal of this paper is to show that the problem of a classical wave with a periodically moving boundary can be easily reformulated in terms of the study of long term behavior of circle maps and, therefore, that many well known results in this theory lead to physical predictions. In particular, we give proofs of several results obtained numerically by Cole and Schieve [4] and others. Extensions of this approach, that will be discussed elsewhere, allow to reach conclusions for some quasi-periodic motions of small amplitude or possibly for non-homogeneous media.

In the case of more than one spatial dimension, the analogous problem [8] is much more complicated, so the predictions are not as clear as in the one-dimensional case and we will not discuss them further.

We emphasize that the mathematical theory presented is completely rigorous and, hence, the physical predictions made are general for the assumptions stated.

There are other intriguing relations for which we have no conceptual explanation. We observe that a calculation of Fulling and Davies [9] leads to the conclusion that the energy density radiated by a moving mirror is equal to the Schwarzian derivative of the motion of the mirror (for details see Sec. IV E). This Schwarzian derivative is a differential operator frequently used in the theory of one dimensional dynamical systems and particularly in the theory of circle maps.

The plan of the exposition is the following. In Sec. II we show how the physical problem can be formulated in terms of circle maps. Sec. III contains a brief exposition of the necessary facts from the theory of circle maps, in Sec. IV these facts are applied to the problem at hand and illustrated numerically, and in the conclusion we discuss the advantages of our approach.

II. PHYSICAL SETTING

A. Description of the system

We consider a one-dimensional optical resonator consisting of two parallel perfectly reflecting mirrors. For simplicity of notation, we will consider only the situation in which one of them is at rest at the origin of the x axis while the other one is moving periodically with period T. The case where the two mirrors are moving periodically with a common period can be treated in a similar manner. We assume that the resonator is empty, so that the speed of the electromagnetic waves in it is equal to the speed of light, c. The speed of the moving mirror cannot exceed c.

We note that the experimental situation does not necessarily require that there is a physically moving mirror. One experimental possibility – among others – would be to have a material that is a good conductor or not depending on whether a magnetic field of sufficient intensity is applied to it, and then have a magnetic field applied to it in a changing region. This induces reflecting boundaries that are moving with time. Note that the boundaries of this region could move even faster than c, hence the study of mirrors moving at a speed comparable to c is not unphysical (even if, presumably, one would also have to discuss corrections to the boundary conditions depending on the details of the experimental realizations).

We shall use dimensionless time t and length ℓ connected with the physical (i.e., dimensional) time t_{phys} and length ℓ_{phys} by $t := t_{\mathrm{phys}}/T$, $\ell := \ell_{\mathrm{phys}}/(cT)$.

Let the coordinate of the moving mirror be $x = a(t)$, where a is a C^k function ($k = 1, \ldots, \infty, \omega$) satisfying the conditions

$$a(t) > 0\,, \qquad |a'(t)| < 1\,, \qquad a(t+1) = a(t)\,. \tag{2.1}$$

The meaning of the first condition is that the cavity does not collapse, the second one means that the speed of the moving mirror cannot exceed the speed of light, and the third one is that the mirror's motion is periodic of period 1. An example which we will use for numerical illustrations is

$$a(t) = \frac{\alpha}{2} + \beta \sin 2\pi t \qquad \left(|\beta| < \frac{1}{2\pi}\,, \quad 0 < |\beta| < \frac{\alpha}{2}\right)\,. \tag{2.2}$$

Since there are no charges and no currents, we impose the gauge conditions $A_0 = 0$, $\nabla \cdot \mathbf{A} = 0$ on the 4-potential $A_\mu = (A_0, \mathbf{A})$ and obtain that $\mathbf{A}$ satisfies the homogeneous wave equation. We consider plane waves traveling in x-direction, so that without loss of generality, we assume that $\mathbf{A}(t,x) = A(t,x)\,\mathbf{e}_y$, and obtain that $A(t,x)$ must satisfy the homogeneous $(1+1)$-dimensional wave equation,

$$A_{tt}(t,x) - A_{xx}(t,x) = 0\,, \tag{2.3}$$

in the domain $\Sigma := \{(t,x) \in \mathbf{R}^2 \,|\, t_0 < t,\ 0 < x < a(t)\}$. It will also need to satisfy some boundary conditions that will be specified in Sec. II B, and appropriate initial conditions,

$$A(t_0,x) = \psi_1(x)\,, \qquad A_t(t_0,x) = \psi_2(x)\,. \tag{2.4}$$

Before discussing the boundary conditions and the method of solving the boundary-value problem in the domain Σ, let us discuss the way of solving (2.3) in the absence of spatial boundaries, i.e., in the domain $\{t_0 < t,\ x \in \mathbf{R}\}$. It is well-known that in this case, the solution of the problem (2.3), (2.4) at some particular space-time point (t, x) can be written as

$$A(t,x) = \Psi^-(x_0^-) + \Psi^+(x_0^+)\,, \tag{2.5}$$

where $x_0^\pm := x \pm (t - t_0)$, and Ψ^- and Ψ^+ are functions of one variable that are selected to match the initial conditions (2.4). The explicit expressions for $\Psi^\pm$ follow from the d'Alembert's formula (see, e.g., [10])

$$\Psi^\pm(s) = \frac{1}{2}\left[\psi_1(s) \pm \int_\zeta^s \psi_2(s')\,\mathrm{d}s'\right]\,, \tag{2.6}$$

where ζ is an arbitrary constant (the same for Ψ^+ and Ψ^-).

The representation (2.5) has a simple geometrical meaning: the value of $A(t,x)$ is a superposition of two functions, $\Psi^-(x_0^-)$ and $\Psi^+(x_0^+)$, the former being constant along the lines $\{x - t = \text{const}\}$, and the latter being constant along $\{x + t = \text{const}\}$. The disturbances at a space-time point (T, X) propagate in the space-time diagram along the lines $\{x - t = X - T\}$ and $\{x + t = X + T\}$ emanating from this point (in more physical terms, this corresponds to two rays moving to the right and to the left at unit speed); these lines are called *characteristics*, and the method of solving (2.3), (2.4) by using the representation (2.5) is called the *method of characteristics* (see, e.g., [10,11]).

B. Method of characteristics, boundary condition at the moving mirror, and Doppler shift at reflection

To obtain the boundary conditions at the stationary mirror, we note that the electric field, i.e., the temporal derivative of the vector potential, must vanish at this mirror, which yields the following "perfect reflection" boundary condition:

$$A_t(t,0) = 0\,. \tag{2.7}$$

The boundary condition at the moving mirror can be easily obtained by performing a Lorenz transformation from the laboratory frame to the inertial frame comoving with the moving mirror at some particular moment t. The temporal and spatial coordinates in the laboratory frame, t and x, are related to the ones in the instantaneously comoving inertial frame, t' and x', by

$$\begin{aligned} t - t_0 &= t'\cosh\zeta + x'\sinh\zeta \\ x - a(t_0) &= t'\sinh\zeta + t'\cosh\zeta\,, \end{aligned} \tag{2.8}$$

where $\tanh\zeta = a'(t)$. In the comoving frame, the boundary condition is $A_{t'}(0,0) = 0$, which, together with (2.8), yields

$$A_t(t, a(t)) + a'(t)\,A_x(t, a(t)) = 0\,. \tag{2.9}$$

The method of characteristics developed in (2.5) and (2.6) for situations with no boundaries can be adapted to provide rather explicit solutions for systems in spatially bounded space-time domains satisfying (2.9) at the boundaries (see, e.g., [12]).

The prescription is the following. The solution of the boundary value problem (2.3), (2.4), (2.7), (2.9) in the domain Σ is a superposition of two functions that are constant on the straight pieces of the characteristics and change their sign at each reflection. To find $A(t,x)$, one has to consider the two characteristics, γ^- and γ^+, passing through (t,x), and propagate them backwards in time (according to the rule that, upon reaching a mirror, they change direction of propagation) until they reach the line $\{\text{time} = t_0\}$ at the points (t_0, x_0^-) and (t_0, x_0^+), resp. – see Fig. 1. Then $A(t,x)$ is given by

$$A(t,x) = (-1)^{N_-}\Psi^-(x_0^-) + (-1)^{N_+}\Psi^+(x_0^+)\,, \tag{2.10}$$

where $N_\mp$ are the number of reflections of $\gamma^\mp$ on the way back from (t,x) to $(t_0, x_0^\mp)$. In Sec. II C we will give explicit formulae for $x_0^\mp$ and $A(t,x)$ in terms in circle maps.

Indeed, because the solution (2.10) is the sum of two functions constant along the straight pieces of the characteristics, the wave equation is satisfied in the interior. Also, the initial conditions are easily verified because for $t - t_0$ small, x_0^- and x_0^+ are close to x [see (2.17)].

To check that this prescription also satisfies the boundary conditions, we need another argument. Consider the space-time diagram of the reflection of the field between two infinitesimally close characteristics reflected by the moving mirror at time θ, shown in Fig. 2. The world line of the mirror is denoted by m, the angle δ between it and the time direction is connected with the mirror's velocity at reflection by $\tan\delta = a'(\theta)$. The Doppler factor at reflection, $D(\theta)$, is defined as the ratio of the spatial distances Δ and Δ' between the characteristics before and after reflection:

$$D(\theta) := \frac{\Delta}{\Delta'} = \tan\left(\frac{\pi}{4} - \delta\right) = \frac{1-\tan\delta}{1+\tan\delta} = \frac{1-a'(\theta)}{1+a'(\theta)}\,. \tag{2.11}$$

Thus, the absolute values of the temporal and spatial derivatives of the field increase by a factor of $D(\theta)$ after reflection. This implies that if in the space-time domain between the two characteristics, the values of the corresponding derivatives of the field before reflection are denoted by A_t and A_x, then after reflection they will become $-D(\theta)\mathsf{A}_t$ and $D(\theta)\mathsf{A}_x$, resp. Hence, in the space-time domain of the overlap the derivatives of the field will be

$$\begin{aligned} A_t(\theta, a(\theta)) &= \mathsf{A}_t - D(\theta)\mathsf{A}_t \\ A_x(\theta, a(\theta)) &= \mathsf{A}_x + D(\theta)\mathsf{A}_x\,. \end{aligned} \tag{2.12}$$

Now, we will show that the modified method of characteristics is consistent with the boundary condition (2.9). We note that $\mathsf{A}_t = -\mathsf{A}_x$, which simply means that before reflection the rays are moving to the right at unit speed. If we multiply the second equation of (2.12) by $a'(\theta) = \frac{1-D(\theta)}{1+D(\theta)}$ [which follows from (2.11)] and add it to the first, we obtain exactly the boundary condition (2.9).

The same prescription gives a solution of the Dirichlet problem $A(t,0) = A(t,a(t)) = 0$. Similar methods can be developed for other boundary conditions.

We note that the method of characteristics also yields information in the important case when the medium is inhomogeneous and perhaps time dependent. This is a physically natural problem since in many applications we have cavities filled with optically active media whose characteristics are changed by external perturbations. In this case, the method of characteristics does not yield an exact solution as above but rather, it is the main ingredient of an iterative procedure [11]. Physically, what happens is that in inhomogeneous media, the waves change shape while propagating in contrast with the propagation without change in shape in homogeneous media (2.5). We plan to come to this problem in a near future.

C. Using circle maps to solve the boundary-value problem (2.3), (2.4), (2.7), (2.9) in the domain Σ

We now reformulate the method of characteristics into a problem of circle maps.

We consider a particular characteristic and denote by $\{\tau_n\}$ the times at which it reaches the stationary mirror and $\{\theta_n\}$ the times at which it reaches the oscillating one; let $\ldots < \tau_n < \theta_n < \tau_{n+1} < \theta_{n+1} < \ldots$. Note that, with this notation,

$$\begin{aligned} \tau_n &= \theta_n - a(\theta_n) = (\mathrm{Id} - a)(\theta_n) \\ \tau_{n+1} &= \theta_n + a(\theta_n) = (\mathrm{Id} + a)(\theta_n)\,. \end{aligned} \tag{2.13}$$

Therefore

$$\begin{aligned} \tau_{n+1} &= (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1}(\tau_n) =: F(\tau_n) \\ \theta_{n+1} &= (\mathrm{Id} - a)^{-1} \circ (\mathrm{Id} + a)(\theta_n) =: G(\theta_n)\,. \end{aligned} \tag{2.14}$$

We refer to F and G as the time advance maps. They allow to compute the time of reflection on one side in terms of the time of the previous reflection on the same side. The conditions (2.1) on the range of a and a' guarantee that $(\mathrm{Id} - a)$ is invertible and that F and G are C^k (by the implicit function theorem).

When the function a is 1-periodic, F and G satisfy

$$F(t+1) = F(t) + 1\,, \qquad G(t+1) = G(t) + 1\,. \tag{2.15}$$

These relations mean that $F(t)$ and $G(t)$ depend only on the fractional part of t. In physical terms, we characterize a reflection of a ray by the *phase* of the oscillating mirror when the impact takes place, i.e., by the time of reflection modulo 1; if we know the phase at one reflection, we can compute the phase at the next impact. Mathematically, this means that F and G can be regarded as lifts of maps from $S^1 \equiv \mathbf{R}/\mathbf{Z}$ to S^1 (see Sec. III).

We want to argue that the study of the dynamics of the circle maps (2.14) leads to important conclusions for the physical problem, which we will take up after we collect some information about the mathematical theory of circle maps. In particular, many results in the mathematical literature are directly relevant for physical applications. This is natural because the long term behavior of the solution can be obtained by repeated application of the time advance maps [see (2.18)].

We call attention to the fact that

$$G = (\mathrm{Id} + a)^{-1} \circ F \circ (\mathrm{Id} + a) = (\mathrm{Id} - a)^{-1} \circ F \circ (\mathrm{Id} - a)\,, \tag{2.16}$$

so that

$$G^n = (\mathrm{Id} + a)^{-1} \circ F^n \circ (\mathrm{Id} + a) = (\mathrm{Id} - a)^{-1} \circ F^n \circ (\mathrm{Id} - a)\,.$$

In dynamical systems theory this is usually described as saying that the maps F and G are "conjugate" (see Sec. III C). In our situation, this comes from the fact that F and G are physically equivalent descriptions of the relative phase of different successive reflections: F advances the τ variables while G advances the θ's, and the θ's are related to the τ's by (2.13).

Now, we use circle maps to derive an explicit formula for the solution of the boundary-value problem (2.3), (2.4), (2.7), (2.9) in the domain Σ. Let us trace back in time the characteristics γ^- and γ^+ coming "from the past" to the space-time point (t, x) – see Fig. 1. Let $\theta_0^\pm := (a \mp \mathrm{Id})^{-1}(t - x)$ be the last moments the characteristics $\gamma^\pm$ are reflected by the moving mirror, and let $\theta_{-k}^\pm := G^{-k}(\theta_0^\pm)$. After N_+, resp. N_-, reflections on the way backwards in time (out of which n_+, resp. n_-, are from the moving mirror), the characteristic γ^+, resp. γ^-, crosses the line $\{\text{time} = t_0\}$. The spatial coordinate of the intersection of $\gamma^\pm$ and $\{\text{time} = t_0\}$ can be easily seen to be

$$x_0^\pm = h(\theta_{-n_\pm}^\pm, t_0) := \left|(\mathrm{Id} - a)(\theta_{-n_\pm}^\pm) - t_0\right|. \tag{2.17}$$

Thus, the formula for the vector potential is

$$\begin{aligned} A(t,x) = (-1)^{N_-}\Psi^- \circ h\left(G^{-n_-} \circ (a + \mathrm{Id})^{-1}(t - x),\, t_0\right) \\ +(-1)^{N_+}\Psi^+ \circ h\left(G^{-n_+} \circ (a - \mathrm{Id})^{-1}(t - x),\, t_0\right)\,. \end{aligned} \tag{2.18}$$

D. Energy of the field

The method of characteristics gives a very illuminating picture of the mechanism of the change of the field energy,

$$E(t) = \int_0^{a(t)} \mathcal{T}^{00}(t,x) = \frac{1}{8\pi}\int_0^{a(t)} \left[A_t(t,x)^2 + A_x(t,x)^2\right]\,\mathrm{d}x\,, \tag{2.19}$$

due to the distortion of the wave at reflection from the moving mirror. Indeed, consider the change of the energy of a very narrow wave packet at reflection from the moving mirror at time θ. Since at reflection the temporal and the spatial distances decrease by a factor of

$D(\theta)$, $|A_t|$ and $|A_x|$ will increase by a factor of $D(\theta)$. Therefore, the integrand of the energy integral will increase $D(\theta)^2$ times, while the support of the integrand (i.e., the spatial width of the wave packet at time t) will shrink by a factor of $D(\theta)$. Hence, the energy of the wave packet after reflection will be $D(\theta)$ times greater than its energy before reflection.

In the general case, one can use (2.18) and obtain the energy of the system at time t. For the sake of simplicity, we will give the formula only under the assumption that at time t all the rays are going to the right, i.e., assuming that the vector potential is of the form $A(t,x) = (-1)^{N_-}\Psi^-(x_0^-)$. Let us introduce the "local Doppler factor"

$$D(t_0,\, x_0^-;\, t) := \left|\frac{\partial}{\partial t} h(\theta_{n_-}^-,\, t_0)\right| = \frac{1 - a'(\theta_{-n_-}^-)}{1 + a'(\theta_0^-)}\, (G^{-n_-})'(\theta_0^-)\,. \tag{2.20}$$

It has the physical meaning of the ratio of the frequencies of the incident wave and the wave at time t [cf. (2.11)]. Note that $D(t_0,\, x_0^-;\, t)$ is equal to the derivative of G^{-n_-} multiplied by a factor which is bounded and bounded away from 0 independently of n_- [due to the fact that $|a'(t)| < 1$]. From (2.18) and (2.17) we obtain that the square of $D(t_0,\, x_0^-;\, t)$ is the ratio of the energy density $\mathcal{T}^{00}(t,x)$ and the initial energy density, $\mathcal{T}^{00}(t_0, x_0^-)$:

$$\mathcal{T}^{00}(t,x) = 2\, \left|(\Psi^-)'(x_0^-)\right|^2\, D(t_0,\, x_0^-;\, t)^2 = \mathcal{T}^{00}(t_0, x_0^-)\, D(t_0,\, x_0^-;\, t)^2\,.$$

On the other hand, $D(t_0,\, x_0^-;\, t)$ is connected with the Jacobian of the change of coordinates $x_0^- \mapsto x$ by

$$\left|\frac{\partial x}{\partial x_0^-}\right| = \left|\frac{\partial x_0^-}{\partial x}\right|^{-1} = D(t_0,\, x_0^-;\, t)^{-1}\,.$$

Hence, the energy of the system at time t is

$$E(t) = \int_0^{a(t)} \mathcal{T}^{00}(t_0, x_0^-)\, D(t_0,\, x_0^-;\, t)\, \mathrm{d}x_0^-\,. \tag{2.21}$$

Note that since the local Doppler factor squared is the ratio of the energy densities at two consecutive reflection points, then it satisfies the following multiplicative property. Let (t_1, x_1^-), (t_2, x_2^-), $\ldots$, (t_k, x_k^-) be space-time points on the characteristic connecting (t_0, x_0^-) and (t,x), such that at all of them the rays are going to the right, and let $t_0 < t_1 < \ldots < t_k < t$. Then

$$D(t_0,\, x_0^-;\, t) = D(t_0,\, x_0^-;\, t_1)\, D(t_1,\, x_1^-;\, t_2)\, \cdots\, D(t_{k-1},\, x_{k-1}^-;\, t_k)\, D(t_k,\, x_k^-;\, t)\,.$$

As can be seen from (2.20), these multiplicative properties are closely related to the chain rule for diffeomorphisms,

$$(G^n)'(\theta) = G'(G^{n-1}(\theta))\, G'(G^{n-2}(\theta))\, \cdots\, G'(\theta)\,. \tag{2.22}$$

The mathematical theory of dynamical systems contains many results about derivatives of highly iterated maps as above (2.22). In Sec. IV C we will be able to translate some of them into asymptotic properties of the field energy.

A simple and intuitively clear formula for the rate of change of the field energy can be obtained by using (2.19), (2.3), (2.7), (2.9), and integrating by parts:

$$\begin{aligned} E'(t) &= \frac{1}{4\pi}\int_0^{a(t)} (A_t A_{xx} + A_x A_{tx})\,\mathrm{d}x + \frac{1}{8\pi}a'(t)A_x(t,a(t))^2 \\ &= \frac{1}{4\pi}A_t(t,a(t))A_x(t,a(t)) + \frac{1}{8\pi}a'(t)A_x(t,a(t))^2 \\ &= -\frac{1}{8\pi}a'(t)A_x(t,a(t))^2\,. \end{aligned}$$

From this, we realize that the force experienced by the wall is $\frac{1}{8\pi}A_x(t,a(t))^2$.

E. The inverse problem: determining the mirror's motion given the circle map

It is important to know whether the notion of a "typical" G is the same as the notion of a "typical" a or a "typical" F (in the mathematical literature people speak about "generic" maps, and in physical literature about "universal" maps). We do not know the answer to this question, and here we will give some arguments showing that the answer is not obvious. In this paper we will not use "generic" or "universal". Rather we will make explicit the non-degeneracy assumptions so that they can be checked in the concrete examples. In Sec. IV D we will show that some universal properties for families of circle maps do not apply to G constructed according to (2.14) with $a(t) = \bar{a} + \varepsilon b(t)$.

While the function a can be expressed in terms of F as $a = \frac{F-\mathrm{Id}}{2}\circ\left(\frac{F+\mathrm{Id}}{2}\right)^{-1}$, the relation between G and a is much harder to invert. We should have

$$a(\theta) + a(G(\theta)) = \widetilde{G}(\theta)\,, \tag{2.23}$$

where $\widetilde{G}(\theta) := G(\theta) - \theta$, so for any n,

$$a(\theta) = \widetilde{G}(\theta) - \widetilde{G}(G(\theta)) + \cdots + (-1)^n\widetilde{G}(G^n(\theta)) + (-1)^{n+1}a(G^{n+1}(\theta))\,.$$

Hence, if $G^{2k}(\theta_0) = \theta_0 \pmod 1$, a necessary condition for the existence of a is that

$$\sum_{i=0}^{2k-1}(-1)^i\widetilde{G}(G^i(\theta_0)) = 0\,. \tag{2.24}$$

An example of a G where the above condition is not satisfied can be readily constructed. We note that if a map fails to satisfy (2.24) and if $(G^{2k})'(\theta_0) \neq 1$, then all small perturbations will also fail to satisfy (2.24). Thus, there are whole neighborhoods of maps that cannot be realized as G for a moving mirror.

On the other hand, given very simple G's, it is easy to construct infinitely many a's that satisfy (2.23) and that therefore lead to the same G. For example, for $G(\theta) = \theta + \frac{1}{2}$, (2.23) amounts to $a(\theta + \frac{1}{2}) + a(\theta) = \frac{1}{2}$. If we prescribe a for θ in $[0, \frac{1}{2}]$, then this equation determines a on $[\frac{1}{2}, 1]$ (the only care needs to be exercised so that the two determinations of

a match at $\theta = \frac{1}{2}$). A similar construction works when G permutes several intervals – if we prescribe $a(\theta)$ in an interval I, (2.23) determines $a(\theta)$ in $G(I)$.

In the case when G is conjugate to an irrational rotation, $G = h^{-1} \circ R_\alpha \circ h$, then (2.23) is equivalent to

$$a \circ h^{-1} \circ R_\alpha + a \circ h^{-1} = h^{-1} \circ R_\alpha - h^{-1} .$$

Then $a \circ h^{-1}$ can be determined using Fourier analysis, setting $h^{-1}(\theta) = \theta + \sum_{k=-\infty}^{\infty} \widehat{\tau}_k e^{2\pi i k\theta}$, $a \circ h^{-1}(\theta) = \theta + \sum_{k=-\infty}^{\infty} \widehat{\psi}_k e^{2\pi i k\theta}$, which leads to

$$\left(e^{2\pi i k\alpha} + 1\right) \widehat{\psi}_k = \left(e^{2\pi i k\alpha} - 1\right) \widehat{\tau}_k . \tag{2.25}$$

Let us assume that $|k\alpha - n - \frac{1}{2}| \geq \mathrm{const}\, |k|^{-\nu}$ for some $\nu \geq 1$ (a condition of this type is called a Diophantine condition – see definition III.4), and that h^{-1} has r derivatives (which implies that its Fourier coefficients $\widehat{\tau}_k$ satisfy $|\widehat{\tau}^k| \leq \mathrm{const}\, |k|^{-r}$). Then if $r > \nu + 2$, then the coefficients $\widehat{\psi}_k$ define a smooth function (for more details see, e.g., [13, Sec. XIII.4]). Of course, once we know $a \circ h^{-1}$, then, since h^{-1} depends only on G and is therefore determined, we can obtain a.

In summary, there are maps G that do not come from any a at all, come from infinitely many a's, or come from one and only one a. The maps F can always be obtained from one and only one a.

III. MAPS OF THE CIRCLE

In this section we recall briefly some facts from the theory of the dynamics of the orientation preserving homeomorphisms (OPHs) and diffeomorphisms (OPDs) of the circle S^1, following closely the book of Katok and Hasselblatt [14, Ch. 11, 12]; see also [15] and [13]. This is a very rich theory and we will only recall the facts that we will need in the physical application.

We shall identify S^1 with the quotient $\mathbf{R}/\mathbf{Z}$ and use the universal covering projection

$$\pi : \mathbf{R} \to S^1 \equiv \mathbf{R}/\mathbf{Z} : x \mapsto \pi(x) := x \,(\mathrm{mod}\, 1) .$$

Another way of thinking about S^1 is identifying it with the unit circle in $\mathbf{C}$, and using the universal covering projection $x \mapsto e^{2\pi i x}$.

Let $f : S^1 \to S^1$ be an OPH and $F : \mathbf{R} \to \mathbf{R}$ be its *lift* to $\mathbf{R}$, i.e., a map satisfying $f \circ \pi = \pi \circ F$. The fact that f is an OPH implies that $F(x+1) = F(x) + 1$ for each $x \in \mathbf{R}$, which is equivalent to saying that $F - \mathrm{Id}$ is 1-periodic. The lift F of f is unique up to an additive integer constant. If a point $x \in S^1$ is q-periodic, i.e., $f^q(x) = x$, then $F^q(x) = x + p$ for some $p \in \mathbf{N}$.

A. Rotation number

A very important number to associate to a map of the circle is its rotation number, introduced by Poincaré. It is a measure of the average amount of rotation of a point along an orbit.

Definition III.1 *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism and $F : \mathbf{R} \to \mathbf{R}$ a lift of f. Define*

$$\tau_0(F) := \lim_{n\to\infty} \frac{F^n(x) - x}{n}, \qquad \tau(f) := \tau_0(F) \pmod 1 \tag{3.1}$$

and call $\tau(f)$ a rotation number *of f.*

It was proven by Poincaré that the limit in (3.1) exists and is independent of x. Hence, $\tau(f)$ is well defined.

The rotation number is a very important tool in classifying the possible types of behavior of the iterates of the OPHs of S^1. The simplest example of an OPH of S^1 is the *rotation* by α on $S^1 \equiv \mathbf{R}/\mathbf{Z}$, $r_\alpha : x \mapsto x + \alpha \pmod 1$ (corresponding to a rotation by $2\pi\alpha$ radians on S^1 thought of as the unit circle in $\mathbf{C}$). The map $R_\alpha : x \mapsto x + \alpha$ is a lift of r_α, and $\tau(r_\alpha) = \alpha \pmod 1$. In the case of r_α there are two possibilities:

(a) If $\tau(r_\alpha) = p/q \in \mathbf{Q}$, then $R^q_{p/q}(x) = x + p$ for each $x \in \mathbf{R}$, so every point in S^1 is q-periodic for $r_{p/q}$. If p and q are relatively prime, q is the minimal period.

(b) If $\tau(r_\alpha) \notin \mathbf{Q}$, then r_α has no periodic points; every point in S^1 has a dense orbit. Thus, the α- and ω-limit sets of any point $x \in S^1$ are the whole S^1, which is usually described as saying that S^1 is a *minimal set* for r_α. [Recall that $\alpha(x)$ is the set of the points at which the orbit of x accumulates in the past, and $\omega(x)$ those points where it accumulates in the future.]

B. Types of orbits of OPHs of the circle

To classify the possible orbits of OPHs of the circle, we need the following definition (for the particular case $f : S^1 \to S^1$).

Definition III.2 *(a) On orbit $\mathcal{O}$ of f is called* homoclinic *to an invariant set $T \in S^1 \setminus \mathcal{O}$ if $\alpha(x) = \omega(x) = T$ for any $x \in \mathcal{O}$.*

(b) An orbit $\mathcal{O}$ of f is said to be heteroclinic *to two disjoint invariant sets T_1 and T_2 if $\mathcal{O}$ is disjoint from each of them and $\alpha(x) = T_1$, $\omega(x) = T_2$ for any $x \in \mathcal{O}$.*

With this definition, the possible types of orbits of circle OPHs were classified by Poincaré [16] as follows (for a modern pedagogical treatment see, e.g., [14, Sec. 11.2]):

(1) For $\tau(f) = p/q \in \mathbf{Q}$, all orbits of f are of the following types:

  (a) a periodic orbit with the same period as the rotation $r_{p/q}$ and ordered in the same way as an orbit of $r_{p/q}$;

(b) an orbit homoclinic to the periodic orbit if there is only one periodic orbit;

(c) an orbit heteroclinic to two different periodic orbits if there are two or more periodic orbits.

(2) When $\tau(f) \notin \mathbf{Q}$, the possible types of orbits are:

(a) an orbit dense in S^1 that is ordered in the same way as an orbit of $r_{\tau(f)}$ (as are the two following cases);

(b) an orbit dense in a Cantor set;

(c) an orbit homoclinic to a Cantor set.

We also note that in cases 2(b) and 2(c), the Cantor set that has a dense orbit is unique and can be obtained as the set of accumulation points of any orbit.

C. Poincaré and Denjoy theorems

Because of the simplicity of the rotations it is natural to ask whether a particular OPH of S^1 is equivalent in some sense to a rotation. To state the results, we give a precise definition of "equivalence" and the important concept of topological transitivity.

Definition III.3 *(a) Let $f : M \to M$ and $g : N \to N$ be C^m maps, $m \geq 0$. The maps f and g are* topologically conjugate *if there exists a homeomorphism $h : M \to N$ such that $f = h^{-1} \circ g \circ h$.*

(b) The map g is a topological factor *of f (or f is* semiconjugate *to g) if there exists a surjective continuous map $h : M \to N$ such that $h \circ f = g \circ h$; the map h is called a* semiconjugacy.

(c) A map $f : M \to M$ is topologically transitive *provided the orbit, $\{f^k(x)\}_{k\in\mathbf{Z}}$, of some point x is dense in M.*

The meaning of the conjugacy is that g becomes f under a change of variables, so that from the point of coordinate independent physical quantities, f and g are equivalent. The meaning of the semiconjugacy is that, embedded in the dynamics of f, we can find the dynamics of g.

The following theorem of Poincaré [16] was chronologically the first theorem classifying circle maps.

Theorem III.1 (Poincaré Classification Theorem) *Let $f : S^1 \to S^1$ be an OPH with irrational rotation number. Then:*

(a) if f is topologically transitive, then f is topologically conjugate to the rotation $r_{\tau(f)}$;

(b) *if f is not topologically transitive, then there exists a non-invertible continuous monotone map $h : S^1 \to S^1$ such that $h \circ f = r_{\tau(f)} \circ h$; in other words, f is semiconjugate to the rotation $r_{\tau(f)}$.*

If we restrict ourselves to considering not OPHs, but OPDs of the circle, we can say more about the conjugacy problem. An important result in this direction was the theorem of Denjoy [17].

Theorem III.2 (Denjoy Theorem) *A C^1 OPD of S^1 with irrational rotation number and derivative of bounded variation is topologically transitive and hence (according to Poincaré theorem) topologically conjugate to a rotation. In particular, every C^2 OPD $f : S^1 \to S^1$ is topologically conjugate to $r_{\tau(f)}$.*

We note that this condition is extremely sharp. For every $\varepsilon > 0$ there are $C^{2-\varepsilon}$ maps with irrational rotation number and semiconjugate but not conjugate to a rotation (see [13, Sec. X.3.19]).

D. Smoothness of the conjugacy

So far we have discussed only the conditions for existence of a conjugacy h to a rotation, requiring h to be only a homeomorphism. Can anything more be said about the differentiability properties of h in the case of smooth or analytic maps of the circle? As we will see later, this is a physically important question since physical quantities such as energy density depend on the smoothness of the conjugacy. To answer this question precisely, we need two definitions.

Definition III.4 *A number ρ is called* Diophantine *of type (K, ν) (or simply of type ν) for $K > 0$ and $\nu \geq 1$, if $\left|\rho - \frac{p}{q}\right| > K\,|q|^{-1-\nu}$ for all $\frac{p}{q} \in \mathbf{Q}$. The number ρ is called* Diophantine *if it is Diophantine for some $K > 0$ and $\nu \geq 1$. A number which is not Diophantine is called a* Liouville number.

It can be proved that for $K \to 0$, the set of all Diophantine numbers of type (K, ν) has Lebesgue measure as close to full as desired.

Definition III.5 *A function f is said to be $C^{m-\delta}$ where $m \geq 1$ is an integer and $\delta \in (0, 1)$, if it is C^{m-1} and its $(m-1)$st derivative is $(1-\delta)$-Hölder continuous, i.e.,*

$$\left|D^{m-1}f(x) - D^{m-1}f(y)\right| < \text{const}\ |x - y|^{1-\delta}\ .$$

The first theorem answering the question about the smoothness of the conjugacy was the theorem of Arnold [18]. He proved that if the analytic map $f : S^1 \to S^1$ is sufficiently close (in the sup-norm) to a rotation and $\tau(f)$ is Diophantine of type $\nu \geq 1$, then f is analytically conjugate to the rotation $r_{\tau(f)}$, i.e., there exists an analytic function $h : S^1 \to S^1$ such that

$h \circ f = r_{\tau(f)} \circ h$. The iterative technique applied by Arnold was fruitfully used later in the proof of the celebrated Kolmogorov-Arnold-Moser (KAM) theorem – see, e.g., [19]. Arnold's result was extended to the case of finite differentiability by Moser [20]. In such a case, the Diophantine exponent ν has to be related to the number of derivatives one assumes for the map.

Arnold's theorem is local, i.e., it is important that f is close to a rotation. Arnold conjectured that any analytic map with a rotation number in a set of full measure is analytically conjugate to a rotation. Herman [13] proved that there exists a set $\mathcal{A} \subset [0,1]$ of full Lebesgue measure such that if $f \in C^k$ for $3 \leq k \leq \omega$ and $\tau(f) \in \mathcal{A}$, then the conjugacy is $C^{k-2-\varepsilon}$ for any $\varepsilon > 0$. The set $\mathcal{A}$ is characterized in terms of the growth of the partial quotients of the continued fraction expansions of its members; all numbers in $\mathcal{A}$ are Diophantine of order ν for any $\nu \geq 1$. His result was improved by Yoccoz [21] who showed that if $f \in C^k$, $3 \leq k \leq \omega$, $\tau(f)$ is a Diophantine number of type $\nu \geq 1$, and $k > 2\nu - 1$, then there exists a $C^{k-\nu-\varepsilon}$ conjugacy h between f and $r_{\tau(f)}$ for any $\varepsilon > 0$, and by several others. The best result on smooth conjugacy we know of, is the following version of Herman's theorem as extended by Katznelson and Ornstein [22].

Theorem III.3 (Herman, Katznelson and Ornstein) *Assume that f is a C^k circle OPD whose rotation number is Diophantine of order ν, and $k > \nu + 1$. Then the homeomorphism h which conjugates f with the rotation $r_{\tau(f)}$ is of class $C^{k-\nu-\varepsilon}$ for any $\varepsilon > 0$.*

There are examples of $C^{2-\varepsilon}$ maps with a Diophantine rotation number arbitrarily close to a rotation and not conjugated by an absolutely continuous function to a rotation – see, e.g., [23].

E. Devil's staircase, frequency locking, Arnold's tongues

Let $\{f_\alpha\}_{\alpha \in A}$ be a one-parameter family of circle OPHs such that $f_\alpha(x)$ is increasing in α for every x. Then the function $\alpha \mapsto \tau(f_\alpha)$ is non-decreasing. (Since the maps are only defined modulo an integer and so is the rotation number, some care needs to be taken to define increasing and non-decreasing when some of the objects we are considering change integer parts. What is meant precisely is that if one takes the numbers with their integer parts, they can be made increasing or non-decreasing. This is done in detail in [14, Sec. 11.1], and we will dispense with making it explicit since it does not lead to confusion.)

For such a family the following fact holds: if $\tau(f_\alpha) \notin \mathbf{Q}$, then $\alpha \mapsto \tau(f_\alpha)$ is strictly increasing locally at α; on the other hand, if f_α has rational rotation number and the periodic point is attracting or repelling (i.e., there is a neighborhood of the point that gets mapped into itself by forwards or backwards iteration), then $\alpha \mapsto \tau(f_\alpha)$ is locally constant at this particular value of α, i.e., for all α' sufficiently close to α, $\tau(f_{\alpha'}) = \tau(f_\alpha)$. The local constancy of the function $\alpha \mapsto \tau(f_\alpha)$ is known as *frequency (phase, mode) locking*. Note that, since the rotation number is continuous, when it indeed changes, it has to go through rational numbers. The described phenomenon suggests the following definition.

Definition III.6 *A monotone continuous function* $\psi : [0,1] \to \mathbf{R}$ *is called a* devil's staircase *if there exists a family* $\{I_\xi\}_{\xi\in\Xi}$ *of disjoint open subintervals of* $[0,1]$ *with dense union such that* ψ *takes constant values on these subintervals. (We call attention to the fact that the complement of the intervals in which the function is constant can be of positive measure.)*

The devil's staircase is said to be complete *if the union of all intervals* I_ξ *has a full Lebesgue measure.*

A very common way of phase locking for differentiable mappings arises when the map we consider has a periodic point and that the derivative of the return map at the periodic point is not equal to 1. By the implicit function theorem, such a periodic orbit persists, and the existence of a periodic orbit implies that the rotation number is locally constant. At the end of the phase locking interval the map has derivative one and experiences a saddle-node (tangent) bifurcation.

We note that, unless certain combinations of derivatives vanish (see, e.g., [24]), the saddle-node bifurcation happens in a universal way. That is, there are analytic changes of variables sending one into another. This leads to quantitative predictions. For example, the Lyapunov exponents of a periodic orbit should behave as a square root of the distance of the parameter to the edge of the phase locking interval.

Of course, other things can happen in special cases: the fixed point may be attractive but only neutrally so, there may be an interval of fixed points, the family may be such that there are no frequency locking intervals (e.g., the rotation). Nevertheless, all these conditions are exceptional and can be excluded in concrete examples by explicit calculations. (For example, if the family of maps is analytic but not a root of the identity, it is impossible to have an interval of fixed points.)

In the example we will consider, we will not perform a complete proof that a devil's staircase occurs, but rather we will present numerical evidence. In particular, the square root behavior of the Lyapunov exponent with the distance to the edge of the phase locking interval seems to be verified.

Let us now consider two-parameter families of OPDs of the circle, $\{\phi_{\alpha,\beta}\}$, depending smoothly on α and β. Assume that when $\beta = 0$, the maps of the family are rotations by α, i.e., $\phi_{\alpha,0} = r_\alpha$. We will call β the *nonlinearity parameter*. Assume also that $\partial\phi_{\alpha,\beta}/\partial\alpha > 0$. An example of this type is the family studied by Arnold [18],

$$\eta_{\alpha,\beta} : S^1 \to S^1 : x \mapsto \eta_{\alpha,\beta}(x) := x + \alpha + \beta \sin 2\pi x \,(\mathrm{mod}\, 1)\,, \tag{3.2}$$

where $\alpha \in [0,1)$, $\beta \in (0, 1/2\pi)$.

The rotation number τ is a continuous map in the uniform topology, and $\phi_{\alpha,\beta}$ is a continuous function of α and β, so the function $(\alpha,\beta) \mapsto \tau(\phi_{\alpha,\beta}) =: \tau_\beta(\alpha)$ depends continuously on α and β. The map τ_β is non-decreasing; for $\beta > 0$, τ_β is locally constant at each α for which $\tau_\beta(\alpha)$ is rational and strictly increasing if $\tau_\beta(\alpha)$ is irrational. Thus, τ_β is a devil's staircase.

Since τ_β is strictly increasing for irrational values of $\tau_\beta(\alpha)$, the set $I_\nu := \{(\alpha,\beta) \,|\, \tau_\beta(\alpha) = \nu\}$ for an irrational $\nu \in [0,1]$ is a graph of a continuous function. For a rational ν, I_ν has a non-empty interior and is bounded by two continuous curves. The wedges between these two curves are often referred to as *Arnold's tongues.*

The fact that $\tau(\phi_{\alpha,0}) = \tau(r_\alpha) = \alpha$ implies that for $\beta = 0$, the set of α's for which there is frequency locking coincides with the rational numbers between 0 and 1, so its Lebesgue measure is zero. When $\beta > 0$, its Lebesgue measure is positive. The width of the Arnold's tongues for small β for the Arnold's map (3.2) is investigated, e.g., in [25]. Much of this analysis carries out for more general functions such as the ones we encounter in the problem of the periodically pulsating resonator.

The total Lebesgue measure of the frequency locking intervals, $m(\{\tau_\beta^{-1}(\nu) \,|\, \nu \in \mathbf{Q} \cap [0,1]\})$, becomes equal to 1 when the family of circle maps consists of maps with a horizontal point (so that the map, even if having a continuous inverse, fails to have a differentiable one) – see [26,27] for numerical results and [28] for analytical proof. With the Arnold's map $\eta_{\alpha,\beta}$ this happens when $\beta = 1/2\pi$. In our case this happens when the mirror goes at one instant at the speed of light.

We note also that the numerical papers [29,26,27,30] contain not only conjectures about the measure of the phase locking intervals but, perhaps more importantly, conjectures about scaling relations that hold "universally". In particular, the dimension of the set of parameters not covered by the phase locking intervals should be the same for all non-degenerate families. These universality conjectures are supported not only by numerical evidence but also by a renormalization group picture – see, e.g., [31] and the references therein. These universality predictions have been verified in several physical contexts. Notably in turbulence by Glazier and Libchaber [32].

As we will see in Sec. IV D, we do not expect that the families obtained in (2.14) for mirrors oscillating with different amplitudes belong to the same universality class as typical mappings, but they should have universality properties that are easy to figure out from those of the above references.

F. Distribution of orbits

For the physical problem at hand it is also important to know how the iterates of the circle map $x \mapsto g(x) := G(x)(\bmod 1)$ are distributed. As we shall see in lemma IV.1, if the iterates of g are well distributed (in an appropriate sense), the energy of the field in the resonator does not build up. The distribution of an orbit is conveniently formalized by using the concept of invariant measures. We recall that a measure μ on X is *invariant* under the measurable map $f : X \to X$ if $\mu(f^{-1}(A)) = \mu(A)$ for each measurable set A.

Given a point $x \in S^1$, the frequency of visit of the orbit of x to $I \subset S^1$ can be defined by

$$\mu_x(I) := \lim_{n\to\infty} \frac{\#\{i \,|\, 0 \le i \le n \text{ and } f^i(x) \in I\}}{n} . \tag{3.3}$$

It is easy to check that if for every interval I, the limit (3.3) exists, it defines an invariant measure describing the frequency of visit of the orbit of x. Therefore, if there are orbits which have asymptotic frequencies of visit, we can find invariant measures.

A trivial example of the existence of such measures is when x is periodic. In such a case, the measure μ_x is a sum of Dirac delta functions concentrated on the periodic orbit. The measure of an interval is proportional to the number of points in the orbit it contains. We also note that it is easy to construct two-dimensional systems [e.g., $(r,\theta) \mapsto (1+0.1(r-1),\, \theta+(r-1)^2\sin(\theta)^2)$] for which the limits like the one in (3.3) do not exist except for measures concentrated on the fixed points, so that even the existence of such equidistributed orbits is not obvious.

There are also relations going in the opposite direction – if invariant measures exist, they imply the existence of well distributed orbits. We recall that the Krylov-Bogolyubov theorem [14, Thm. 4.1.1] asserts that any continuous map on a compact metrizable space has an invariant probability measure. Moreover, the Birkhoff ergodic theorem [14, Thm. 4.1.2] implies that given any invariant measure μ, the set of points for which μ_x as in (3.3) does not exist has measure zero.

Certain measures have the property that $\mu_x = \mu$ for μ-almost all points. These measures are called *ergodic*. (There are several equivalent definitions of ergodicity and this is one of them.) From the physical point of view, we note that a measure is ergodic if all the points in the measure are distributed according to it. For maps of the circle, there are several criteria that allow to conclude that a map is ergodic.

For rotations of the circle with an irrational rotation number we recall the classical Kronecker-Weyl equidistribution theorem [14, Thm. 4.2.1] which shows that any irrational rotation is uniquely ergodic, i.e., has only one invariant measure – the Lebesgue measure m. (Such uniquely ergodic maps are, obviously, ergodic because, by Birkhoff ergodic theorem, the limiting distribution has to exist almost everywhere, but, since there is only one invariant measure, all these invariant distributions have to agree with the original measure.) Thus, the iterates of any $x \in S^1$ under an irrational rotation are uniformly distributed on the circle.

For general non-linear circle OPDs the situation may be quite different. As an example, consider Arnold's map $\eta_{\alpha,\beta}$ (3.2). If it is conjugate to an irrational rotation by h, i.e., $\eta_{\alpha,\beta} = h^{-1} \circ r_{\tau(\eta_{\alpha,\beta})} \circ h$, then there is a unique invariant probability measure μ defined for each measurable set A by $\mu(A) := m(h(A))$. This implies that if I is an interval in S^1, then the frequency with which a point x visits I is equal to $\mu(I)$.

On the other hand, if $\tau(\eta_{\alpha,\beta}) = p/q \in \mathbf{Q}$, then all orbits are periodic or asymptotic to periodic. Thus, the only possible invariant measure is concentrated at the periodic points and therefore singular, if the periodic points are isolated. Let us now assume that α is very close to $\tau_\beta^{-1}(p/q)$, but does not belong to it. Then $\eta_{\alpha,\beta}$ has no periodic orbits, but still there exists a point x which is "almost periodic", i.e., the orbits linger for an extremely long time near the points x, $\eta_{\alpha,\beta}(x)$, $\cdots$, $\eta_{\alpha,\beta}^{q-1}(x)$. So that, even if the invariant measure is absolutely continuous, one expects that it is nevertheless quite peaked around the periodic orbit – see Fig. 5. The behavior of such maps is described quantitatively by the "intermittency theory" [33].

The continuity properties of the measures of the circle are not so easy to ascertain.

Nevertheless, there are certain results that are easy to establish.

Of course, in the case that we have a rational rotation number and isolated periodic orbits, some of them attracting and some of them repelling, the only possible invariant measures are measures concentrated in the periodic orbits.

For the irrational rotation number case, the Kronecker-Weyl theorem implies that all the maps with an irrational rotation number – since they are semi-conjugate to a rotation by Poincaré theorem – are uniquely ergodic. In the situations where Herman's theorem applies, this measure will have a smooth density since it is the push-forward of Lebesgue measure by a smooth diffeomorphism.

We also recall that by Banach-Alaoglu theorem and the Riesz representation theorem, the set of Borel probability measures is compact when we give it the topology of $\mu_n \to \mu \iff \mu_n(A) \to \mu(A)$ for all Borel measurable sets A. (This convergence is called weak-* convergence by functional analysts and convergence in probability by probabilists.)

Lemma III.1 *If λ^* is a parameter value for which f_{λ^*} admits only one invariant measure μ_{λ^*}, given μ_{λ_i} invariant measures for f_{λ_i}, with $\lambda_i \to \lambda^*$, then μ_i converges in the weak-* sense to μ_{λ^*}.*

Note that we are not assuming that f_{λ_i} are uniquely ergodic. In particular, the lemma says that in the set of uniquely ergodic maps, the map that a parameter associates the invariant measure is continuous if we give the measures the topology of weak-* convergence.

Proof. Let $\mu_{\lambda_{i_k}}$ be a convergent subsequence. The limit should be an invariant measure for f_{λ^*}. Hence, it should be μ_{λ^*}. It is an easy point set topology lemma that for functions taking values in a compact metrizable space, if all subsequences converge to the same point, then this point is a limit. The space of measures with weak-* topology is metrizable because by Riesz representation theorem is the dual of the space of continuous functions with sup-norm, which is metrizable.

We also point out that as a corollary of KAM theory [18] we can obtain that for non-degenerate families, if we consider the parameter values for which the rotation number is Diophantine with uniform constants, the measures are differentiable jointly on x and in the parameter. (For the differentiability in the parameter, we need to use Whitney differentiability or, equivalently, declare that there is a family of densities differentiable both in x and in λ that agrees with the densities for these values of λ.)

On the other hand, we point out that there are situations where the invariant measure is not unique (e.g., a rational rotation or a map with more than one periodic orbit). In such cases, it is not difficult to approximate them by maps in such a way that the invariant measure is discontinuous in the weak-* topology as a function of the parameter. The discontinuity of the measures with respect to parameters, as we shall see, has the physical interpretation that, by changing the oscillation parameters by arbitrarily small amounts, we can go from unbounded growth in the energy to the energy remaining bounded.

IV. APPLICATION TO THE RESONATOR PROBLEM

Now we return to the problem of a one-dimensional optical resonator with a periodically moving wall to discuss the physical implications of circle maps theory, and illustrate with numerical results in an example.

A. Circle maps in the resonator problem

If we take $a(t)$ to depend on two parameters, α and β, as in (2.2), then, as we saw in Sec. II C, the time between the consecutive reflections at the mirrors can be described in terms of the functions $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$ defined by (2.14). These maps are lifts of circle maps that we will denote by $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$. The restriction on the range of β in (2.2) implies that $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$ are analytic circle OPDs. Therefore, we can apply the results about the types of orbits of OPHs of S^1, Poincaré and Denjoy theorems, as well as the smooth conjugacy results and the facts about the distribution of orbits.

In an application where the motion of the mirror [i.e., $a(t)$] is given, one needs to compute $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$ (2.14), which cannot be expressed explicitly from $a(t)$ but they require only to solve one variable implicit equation. In the numerical computations we used the subroutine ZEROIN [34] to solve implicit equations. If $y = F_{\alpha,\beta}(t)$ and $z = G_{\alpha,\beta}(t)$, then for $a(t)$ given by (2.2), y and z are given implicitly by

$$\begin{aligned} -y + t + \alpha + 2\beta \sin[\pi(y+t)] &= 0 \\ -z + t + \alpha + \beta\,[\sin(2\pi t) + \sin(2\pi z)] &= 0\,. \end{aligned}$$

Given t, we can find y, z applying ZEROIN.

B. Rotation number, phase locking

In this section, our goal is to translate the mathematical predictions from the theory of circle maps into physical predictions for the resonator problem.

The theory of circle maps guarantees that the measure of the frequency locking intervals for $g_{\alpha,\beta}$ is small when β is small and becomes 1 when $\beta = 1/2\pi$. The theory also guarantees for analytic maps that, unless a power of the map is the identity, the frequency locking intervals are non-trivial. For the example that we have at hand, it is very easy to verify that this does not happen and, therefore, we can predict that there will be frequency locking intervals and that as the amplitude of the oscillations of the moving mirror increases so that the maximum speed of the moving mirror reaches the speed of light, the devil's staircase becomes complete. Fig. 3 shows a part of the complete devil's staircase – the situation which happens when the maps $g_{\alpha,\beta}$ and $f_{\alpha,\beta}$ lose their invertibility, i.e., for $\beta = 1/2\pi$.

We also recall that the theory of circle maps makes predictions about what happens for non-degenerate phase locking intervals. Namely, for parameters inside the phase locking interval the map has a periodic fixed point and the Lyapunov exponent is smaller than 0, while at the edges of the phase locking interval the map experiences a non-degenerate saddle-node bifurcation – provided that certain combinations of the derivatives do not vanish [24].

We note that for parameters for which the map is in non-degenerate frequency locking, i.e., $\tau(g_{\alpha,\beta}) = p/q$ and the attractive periodic point of period q has a negative Lyapunov exponent, $\{G^{nq}_{\alpha,\beta}(x)\}_{n=0}^{\infty}$ will converge exponentially to the fixed point for all x in a certain interval, according to the results about the types of orbits of circle maps (Sec. III B). The whole circle can be divided into such intervals and a finite number of periodic points. Therefore, the graph of $G^{nq}_{\alpha,\beta}$, and hence of $g^{nq}_{\alpha,\beta}$, will look – up to errors exponentially small in n – like a piecewise-constant function with values (up to integers) in the fixed points of $g^{q}_{\alpha,\beta}$ – see Fig. 4. The fact that certain functions tend to piecewise-constant functions for large values of the argument (which follows from what we found about $G^{nq}_{\alpha,\beta}$ for large n) was observed numerically for particular motions of the mirror in [6,4]. In physical terms, this means that the rays will be getting closer and closer together, so with the time the wave packets will become narrower and narrower and more and more sharply peaked. The number of wave packets is equal to q. The number of reflections from the moving mirror per unit time will tend to the inverse of the rotation number. In the next section we discuss how this yields an increase of the field energy which happens exponentially fast on time.

The fact that for $\tau(g_{\alpha,\beta}) \in \mathbf{Q}$ the rays approach periodic orbits, is also interesting from a quantum mechanical point of view due to the relation between the periodic orbits in a classical system and the energy levels of the corresponding quantum system, given by the Gutzwiller's trace formula (see, e.g., [35]).

We also note that we expect that slightly away from the edges of a phase locking interval, the invariant density will be sharply peaked around the points in which it was concentrated in the phase locking intervals. This is described by the "intermittency theory" [33].

To observe numerically in our example what happens when α enters or leaves a frequency locking interval, we set $N_{\beta}(\nu) := \{\alpha \in [0,1) \,|\, \tau(g_{\alpha,\beta}) = \nu\}$. Fig. 5 represents the Radon-Nikodym derivative $\mathrm{d}\mu/\mathrm{d}m$ of the invariant probability measure μ with respect to the Lebesgue measure m [i.e., of the density of the invariant measures, which, as we saw in (3.3), is the frequency of visit of the iterates]. The figure shows $\mathrm{d}\mu/\mathrm{d}m$ for α close to the left end of $N_{0.1}(1/6)$. When α approaches (from the left) the left end of $N_{0.1}(1/6)$, $\mathrm{d}\mu/\mathrm{d}m$ becomes sharply peaked at some points, and when α enters the frequency locking interval, the invariant measure becomes singular ($g_{\alpha,0.1}$ undergoes tangent bifurcation at $\alpha = 0.253977\ldots$). All seems to be consistent with the conjecture that all the frequency-locking intervals in the family (away of $\beta = 0$) are non-degenerate, i.e., that at the boundaries of the phase locking intervals the map satisfies the hypothesis of the saddle-node bifurcation theorem.

C. Doppler shift

One of the most interesting parts of the applications of circle map theory is the ease with which we can describe the effect on the energy after repeated reflections.

Recall that in Sec. II D, we found the time dependence of the field energy under the assumption that at time t all rays are going to the right. This assumption is not very restrictive in the case of a rational rotation number since, as we found in Sec. IV B, the field develops wave packets that become narrower with time, so (2.20) and (2.21) hold for the asymptotic behavior of the energy. Note that (2.20) expresses the Doppler shift factor in terms of the derivatives of the map G. This gives a very close relation between the dynamics and the behavior of the wave packets.

Proposition IV.1 *Let α and β be such that $\tau(g_{\alpha,\beta}) = p/q$, and that the map $G := G_{\alpha,\beta}$ has a stable periodic orbit $\Theta_q = \{\theta_1, \ldots, \theta_q\}$ such that $(G^q)'(\theta_1) < 1$. Assume that the initial electromagnetic field in the cavity is not zero at some space-time point for which the phase of the first reflection from the moving mirror is in the basin of attraction of Θ_q.*

Then the energy of the field in the resonator will be asymptotically increasing at an exponential rate:

$$E(t) \sim \exp\left\{\frac{\ln D(\Theta_q)}{p}\, t\right\} . \tag{4.1}$$

Proof. First notice that the number of reflections from the moving mirror per unit time reaches a well defined limit (one and the same for all rays) – the inverse of the rotation number. Secondly, as was discussed in Sec. II, at reflection from the moving mirror at phase θ, a wave packet becomes narrower by a factor of $D(\theta)$ (2.11), which leads to a $D(\theta)$ times increase in its energy. Asymptotically, the phases at reflection will approach the stable periodic orbit $\Theta_q = \{\theta_1, \ldots, \theta_q\}$ of $g_{\alpha,\beta}$. The Doppler factors at reflection will tend correspondingly to $\{D(\theta_1), \ldots, D(\theta_q)\}$ (2.11). Hence, in time p each ray will undergo q reflections from the moving mirror, the total Doppler shift factor along the periodic orbit Θ_q being

$$D(\Theta_q) := \prod_{i=1}^{q} D(\theta_i) = \prod_{i=1}^{q} \frac{1 - a'(\theta_i)}{1 + a'(\theta_i)} .$$

On the other hand, the definition of the map G as the advance in the time between successive reflections from the moving mirror yields $\theta_i = G^{i-1}(\theta_1)$. The chain rule applied to the explicit expression (2.14) for G yields

$$(G^{q-1})'(\theta_1) = \prod_{j=1}^{q-1} G'(\theta_j) = \prod_{j=1}^{q-1} \frac{1 + a'(\theta_j)}{1 - a'(\theta_{j+1})} ,$$

which gives the following expression for $D(\Theta_q)$ [cf. (2.20)]:

$$D(\Theta_q) := \frac{1 - a'(\theta_1)}{1 + a'(\theta_q)} \left[(G^{q-1})'(\theta_1)\right]^{-1} = \frac{1 - a'(\theta_1)}{1 + a'(\theta_q)} (G^{1-q})'(\theta_q) . \tag{4.2}$$

Hence, the energy density grows by a factor of $D(\Theta_q)^2$. Since after q reflections the wave packet is concentrated in a length $D(\Theta_q)$ times smaller, the total energy grows by a factor of $D(\Theta_q)$ in p units of time, which implies (4.1).

The quantities $(G^n)'(\theta)$ that appear in (4.2) have been studied intensively in dynamical systems theory since they control the growth of infinitesimal perturbations of trajectories. Similarly, they are factors that multiply the invariant densities when they get transported, as we will see in (4.3).

We found numerically the total Doppler factors $D(\Theta_q)$ for some particular choices of the parameters. In Fig. 6, $\log_{10} D(\Theta_6)$ is shown for different values of β and for $\alpha \in N_\beta(1/6)$. Obviously, the maximum value of $D(\Theta_6)$ depends strongly on β, becoming infinite for $\beta = 1/2\pi$ and some $\alpha \in N_{1/2\pi}(1/6)$. For smaller values of β, the Doppler factor is much smaller. Moreover, the width of the frequency locking intervals for small β is small, so the probability of hitting a frequency locking interval with arbitrarily chosen α and β is small. [The likelihood of frequency locking for the Arnold's map (3.2) is studied numerically in [27].]

In the case when Herman's theorem apply, the derivatives of G^n are bounded independently of n, which causes the energy of the system to be bounded for all times, which is proved in the following proposition.

Proposition IV.2 *If $G_{\alpha,\beta}$ is such that it satisfies the hypothesis of Herman's theorem, then the energy density remains bounded for all times.*

Proof. In such a case $G_{\alpha,\beta} = h^{-1} \circ R \circ h$ with h differentiable and R a rotation by $\tau(g_{\alpha,\beta})$. Therefore $G^n_{\alpha,\beta} = h^{-1} \circ R^n \circ h$ and

$$(G^n_{\alpha,\beta})'(\theta) = (h^{-1})'(R^n \circ h(\theta))\, (R^n)'(h(\theta))\, h'(\theta) = (h^{-1})'(R^n \circ h(\theta))\, h'(\theta)$$

because $(R^n)' = 1$. The two factors in the right-hand side of the above equation are bounded uniformly in θ and n. Thus, the "local Doppler factors" (2.20) will be bounded, which implies the boundedness of the energy (2.21).

There is an interesting connection between the invariant densities of the system and the growth of the electromagnetic energy density.

Recall that if a density μ is invariant, $\mu(G(\theta)) = \mu(\theta)/G'(\theta)$. Hence, if the density μ never vanishes, $G'(\theta) = \mu(\theta)/\mu(G(\theta))$ and, therefore, $(G^i)'(\theta) = \mu(\theta)/\mu(G^i(\theta))$. Let us assume that there is only one characteristic passing through the space-time point (t, x), and this characteristic is going to the right. Then, using the notations of Sec. II C, we can write the energy density at (t, x) as [cf. (2.20)]

$$\mathcal{T}^{00}(t,x) = \left[\frac{1 - a'(\theta^-_{-n_-})}{1 + a'(\theta^-_0)} \, \frac{\mu(G^{n_-}(\theta^-_{-n_-}))}{\mu(\theta^-_{-n_-})} \right]^2 \mathcal{T}^{00}(t_0, x^-_0)\,. \tag{4.3}$$

In the general case [with two characteristics through (x, t)], one can use (2.18) and (2.19) to prove the following result:

Lemma IV.1 *If a system has an invariant density μ which is bounded away from zero, then the electromagnetic energy density of a C^1 initial data is smaller than $C\mu^2$ for all times.*

In the cases that Herman's theorem applies, there is an invariant density bounded away from zero (and also bounded). Hence, we conclude that there are values of the amplitude of mirror's oscillations for which the energy density of the field remains bounded. This set is typically a Cantor set interspersed with values for which the energy increases exponentially.

Some other results about the behavior of the energy with respect to time and parameters are obtained in [2].

We call attention to the fact that [18] contains examples of analytic maps whose rotation numbers are very closely approximated by rationals and that are arbitrarily close to a rotation such that they preserve no invariant density and, therefore, are not smoothly conjugate to a rotation.

It is also known that for all rotation numbers one can construct $C^{2-\varepsilon}$ maps arbitrarily close to rotations with this rotation number and such that they do not preserve any invariant measures [23]. It is a testament to the ubiquity of these maps that these questions were motivated and found applications in the theory of classification of C^* algebras.

D. The behavior for small amplitude and universality

We note that, even if all the motions of the mirror lead to a circle map as in (2.15), it does not seem clear to us that all the maps of the circle can appear as F, G for a certain a. This makes it impossible to conclude that the theory of generic circle maps applies directly to obtain conclusions for a generic motion of the mirror. Of course, all the conclusions of the general theory that apply to all maps of the circle apply to our case. Those conclusions that require non-degeneracy assumptions will need that we verify the assumptions. Nevertheless, the very developed mathematical theory of generic or universal circle maps cannot be applied without caution to maps that appear as the result of generic or universal oscillations of the mirror.

One aspect that we have found makes a big difference with the generic theory is the situation where the mirror oscillates with small amplitude, i.e., $a_\varepsilon(t) = \bar{a} + \varepsilon b(t)$ with b a periodic function of zero average and period 1, and $\varepsilon \ll 1$. The first parameter, $\bar{a}$, is the average length of the resonator, while $\varepsilon = 0$ is called the "nonlinearity parameter" for obvious reasons. If we denote by $F_{\bar{a},\varepsilon}$ and $G_{\bar{a},\varepsilon}$ the corresponding 2-parametric families of maps of the circle constructed according to (2.14), then we have, for three times differentiable families,

$$\begin{aligned} F_{\bar{a},\varepsilon}(t) &= t + 2\bar{a} + 2\varepsilon b(t+\bar{a}) + 2\varepsilon^2 b'(t+\bar{a}) b(t+\bar{a}) + O(\varepsilon^3)\,, \\ G_{\bar{a},\varepsilon}(t) &= t + 2\bar{a} + \varepsilon[b(t) + b(t+2\bar{a})] + \varepsilon^2 b'(t+2\bar{a})[b(t) + b(t+2\bar{a})] + O(\varepsilon^3)\,. \end{aligned} \tag{4.4}$$

Note that the term of order ε has vanishing average. As we will immediately show, this property causes that some well known generic properties of families of circle mappings do not hold for families of maps constructed as in (2.14).

Indeed, if we consider the expressions for small amplitude developed in (4.4), we can write the maps as

$$H_\varepsilon(t) = t + 2\bar{a} + \varepsilon H_1(t) + \varepsilon^2 H_2(t) + O(\varepsilon^3)\,.$$

Since the conclusions of the theory of circle maps are independent of the coordinate system chosen, it is natural to try to choose a coordinate system where these expressions are as simple as possible. Hence, we choose $h_\varepsilon(t) := t + \varepsilon\eta(t)$, a perturbation of the identity, and consider $h_\varepsilon^{-1} \circ H_\varepsilon \circ h_\varepsilon$, which is just H_ε in another system of coordinates, related to the original one by h_ε. Then, up to terms of order ε^3, we have

$$\begin{aligned} h_\varepsilon^{-1} \circ H_\varepsilon \circ h_\varepsilon(t) &= t + 2\bar{a} + \varepsilon\left[\eta(t) - \eta(t+2\bar{a}) + H_1(t)\right] \\ &\quad + \varepsilon^2\left\{\eta'(t+2\bar{a})\eta(t+2\bar{a}) - \eta'(t+2\bar{a})\left[\eta(t) + H_1(t)\right] + H_1'(t)\eta(t) + H_2(t)\right\}\,. \end{aligned} \tag{4.5}$$

We would like to choose η in such a way that the ε term is not present. Note that since $\int \eta(t+2\bar{a})\,\mathrm{d}t = \int \eta(t)\,\mathrm{d}t$, this is impossible unless $\int H_1(t)\,\mathrm{d}t = 0$. When $\int H_1(t)\,\mathrm{d}t = 0$, H_1 is smooth and $2\bar{a}$ is Diophantine, a well-known result (see, e.g., [13, Sec. XIII.4]) shows that in such a case we can obtain one η satisfying

$$\eta(t) - \eta(t+2\bar{a}) + H_1(t) = 0 \tag{4.6}$$

and $\overline{\eta} = 0$. [Such η is conventionally obtained by using Fourier coefficients. Note that in Fourier coefficients, (4.6) amounts to $\hat{\eta}_k(e^{2\pi i k 2\bar{a}} - 1) = \widehat{(H_1)}_k$. If H_1 is smooth, the Fourier coefficients decrease fast and if $2\bar{a}$ is Diophantine, then $(e^{2\pi i k 2\bar{a}} - 1)^{-1}$ does not grow too fast. For more details we refer to the reference above.]

Since for the functions $F_{\bar{a},\varepsilon}$ and $G_{\bar{a},\varepsilon}$ the term of order ε has a zero average, we can transform these functions into lifts of rotations plus $O(\varepsilon^2)$. This implies, in particular, that their rotation number is $\tau(F_{\bar{a},\varepsilon}) = \tau(G_{\bar{a},\varepsilon}) = 2\bar{a} + O(\varepsilon^2)$. One could wonder if it would be possible to continue the process and eliminate also to order ε^2.

If we look at the ε^2 terms in (4.5), we see that $\overline{\eta'(t)\eta(t)} = 0$, and, when η is chosen as in (4.6),

$$\eta'(t+2\bar{a})[\eta(t) + H_1(t)] = \eta'(t+2\bar{a})\eta(t+2\bar{a})\,,$$

which also has average zero. Therefore, a necessary condition for the ε^2 term in $h_\varepsilon^{-1} \circ H_\varepsilon \circ h_\varepsilon(t)$ to be zero is $\overline{H_1'(t)\eta(t)} + \overline{H_2(t)} = 0$.

For the $F_{\bar{a},\varepsilon}$ in (4.4) we see that F_2 has zero average. Nevertheless, the term $F_1'(t)\eta(t)$ does not in general have average zero as can be seen in examples. Hence, we see that the rotation number indeed changes by an order which is $O(\varepsilon^2)$ and not higher in general. This property is not generic for families of circle maps starting with a rotation $2\bar{a}$ and it puts them outside of the universality classes considered in [29,31], etc., since the correspondence between rotation numbers and parameters is not the same.

According to the geometric picture of renormalization developed in [31], the space of circle maps is divided into slices of rational rotation numbers, which are – in appropriate sense – parallel. In that language – in which we think of families of circle maps as curves in the space of mappings – the families of advance maps $F_{\bar{a},\varepsilon}$ and $G_{\bar{a},\varepsilon}$ (for fixed $\bar{a}$) have second order tangency to the foliation of rational rotation numbers rather than being transversal. Hence, the scaling predicted by universality theory should be true for ε^2 in place of ε. We have not verified this prediction, but we expect to come back to it soon.

E. Schwarzian derivative in the problem of moving mirrors

Fulling and Davies [9] calculated the energy-momentum tensor in the two-dimensional quantum field theory of a massless scalar field influenced by the motion of a perfectly reflecting mirror (see also [36]). They obtained that the "renormalized" vacuum expectation value of the energy density radiated by a moving mirror into initially empty space is

$$\mathcal{T}^{00}(u) = -\frac{1}{24\pi}\left[\frac{F'''(u)}{F'(u)} - \frac{3}{2}\left(\frac{F''(u)}{F'(u)}\right)^2\right] ,$$

where $u = t - x$, and F is related to the law of the motion of the mirror, $x = a(t)$, by (2.14). The right-hand side of this equation is nothing but (up to a constant factor) the Schwarzian derivative of F – a differential operator which naturally appears in complex analysis, e.g., it is invariant under a fractional linear transformation; vanishing Schwarzian derivative of a function is the necessary and sufficient condition that the function is fractional linear transformation, etc. More interestingly, the Schwarzian derivative has been used as an important tool in the proof of several important theorems in the theory of circle maps – see, e.g., [21,37]. In the light of the connection between the solutions of the wave equation in a periodically pulsating domain and the theory of circle maps it is not impossible that this is not just a coincidence.

V. CONCLUSION

Using the method of characteristics for solving the wave equation, we reformulated the problem of studying the electromagnetic field in a resonator with a periodically oscillating wall into the language of circle maps. Then we used some results of the theory of circle maps in order to make predictions about the long time behavior of the field. We found that many results in the theory of circle maps have a directly observable physical meaning. Notably, for a typical family of mirror motions we expect that the electromagnetic energy grows exponentially fast in a dense set of intervals in the parameters. Nevertheless, it remains bounded for all times for a Cantor set of parameters that has positive measure.

There are several advantages of the approach presented here. First, it allows us to understand better the time evolution of the electromagnetic field in the resonator and the mechanism of the change in the field energy. Second, the predictions are based on the general theory of circle maps so they are valid for any periodic motion of the mirror; let us also emphasize that our method is non-perturbative. Last, but not least, for a given motion of the mirror, one can easily make certain predictions about the behavior of the field by simply calculating the rotation number of the corresponding circle map, and without solving any partial differential equations.

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FIGURES

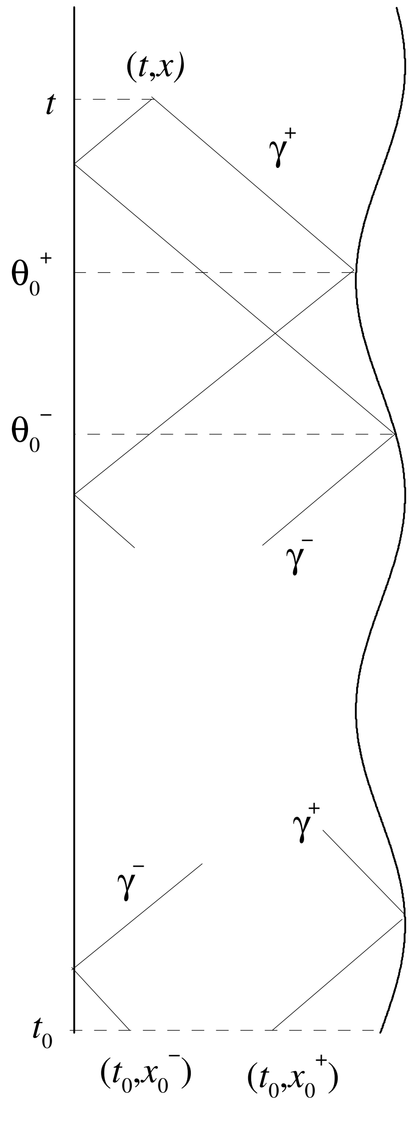

FIG. 1. Finding $A(t,x)$ by the method of characteristics.

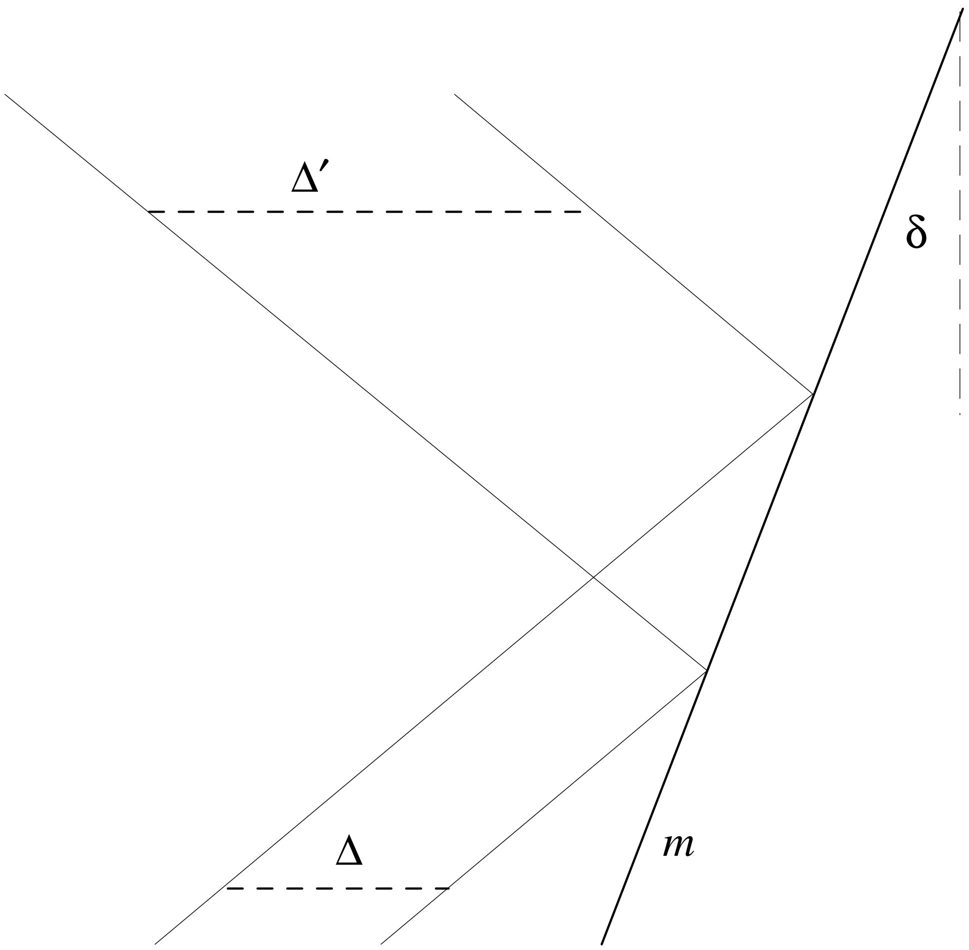

FIG. 2. Reflection by the moving mirror.

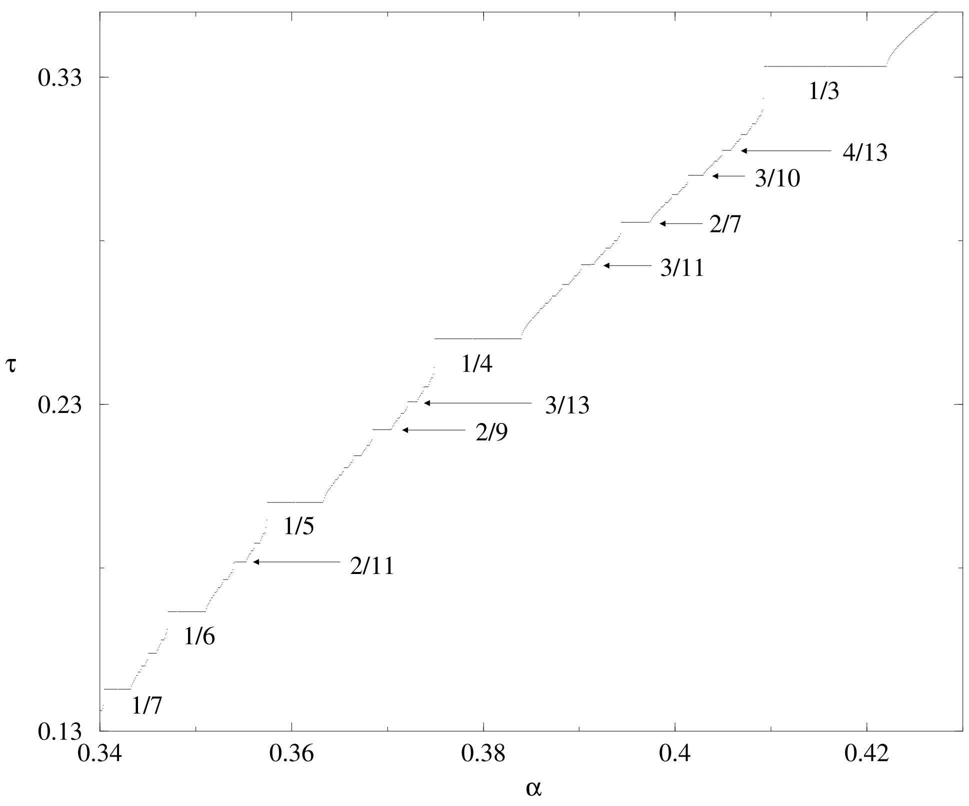

FIG. 3. A part of the graph of $\tau(g_{\alpha,1/2\pi})$ vs. α.

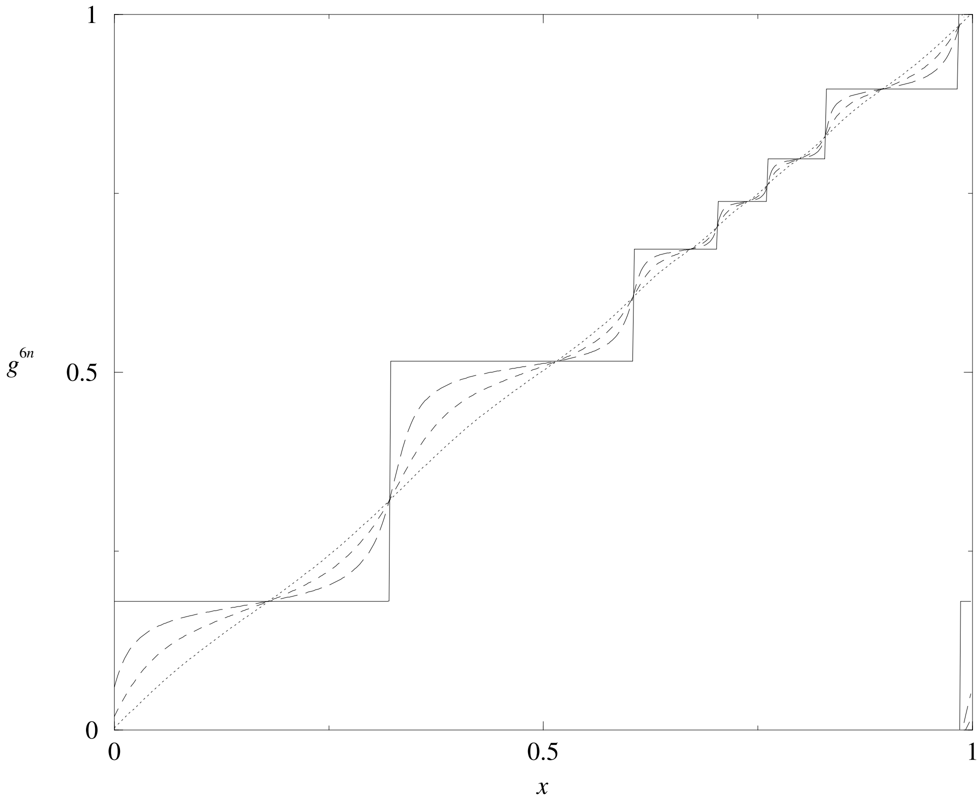

FIG. 4. Development of the piecewise-constant structure of $g^{6n}_{0.2545,\,0.1}$ (the rotation number of $g_{0.2545,\,0.1}$ is 1/6). Graphs of $g^{6n}_{0.2545,\,0.1}$ are plotted for $n = 1$ (dotted line), $n = 5$ (dashed line), $n = 10$ (long dashed line), $n = 100$ (solid line).

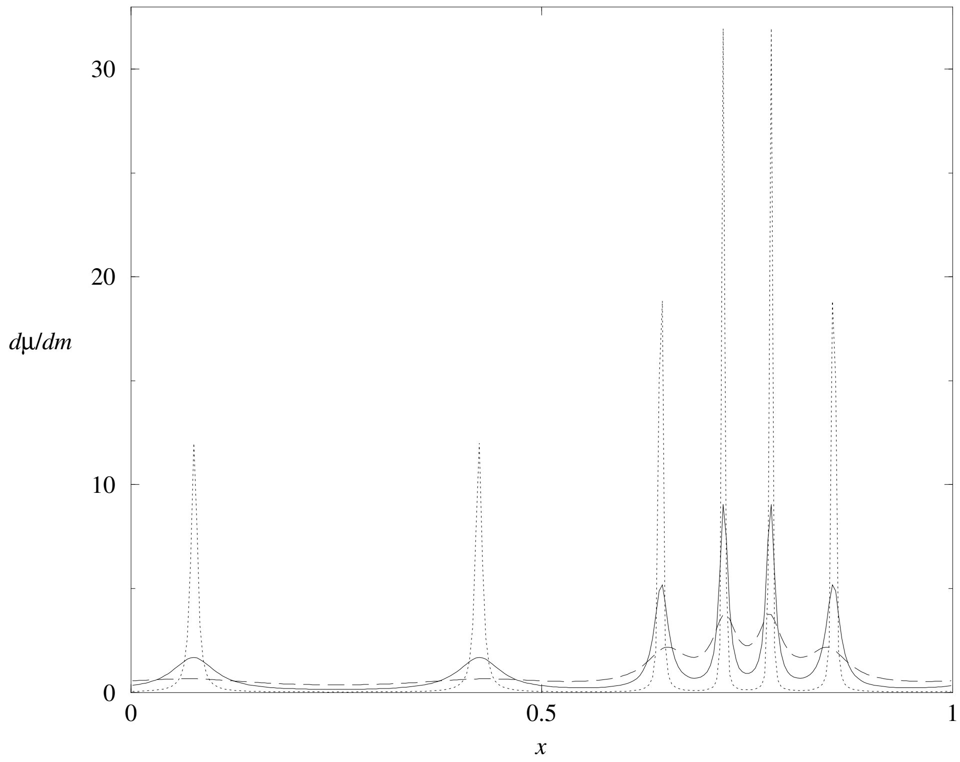

FIG. 5. Density of the invariant measures for $\beta = 0.1$ and $\alpha = 0.253$ (dashed line), $\alpha = 0.2539$ (solid line), and $\alpha = 0.253975$ (dotted line).

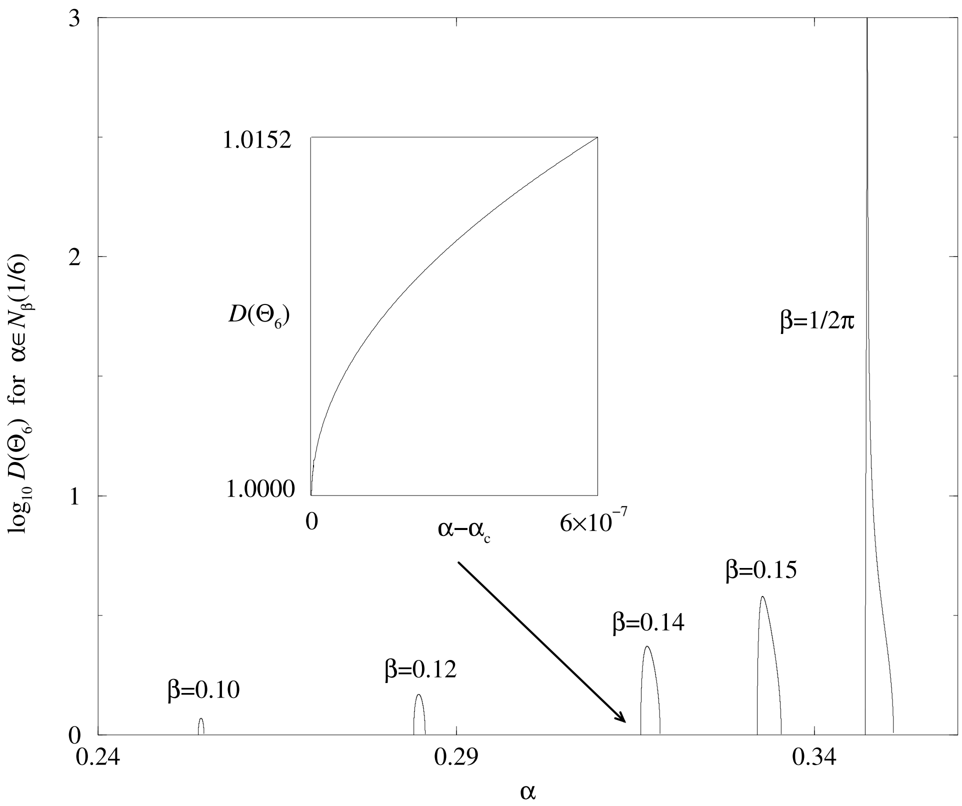

FIG. 6. A log-linear graph of the total Doppler factor for $g_{\alpha,\beta}$ in the phase locking interval of rotation number $1/6$ for different β. The insert [linear-linear graph of $D(\Theta_6)$ vs. $\alpha - \alpha_c$] calls attention to the square-root behavior at edges; α_c is the value of α at the left end of $N_{0.14}(1/6)$.