AN EXAMPLE OF PURE STABILITY FOR THE WAVE EQUATION WITH MOVING BOUNDARY *

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Abstract

This paper studies the behavior of the energy of the wave equation solution in a finite moving domain $0 < x < a(t)$ with $a(t)$ assumed to move slower than the light and periodically. Moreover a is continuous, piecewise linear with two independent parameters. The boundary conditions are of Dirichlet type. We show that the energy is always bounded and give results which involve only the independent parameters. The proof is based on the determination of the invariant measure of a homeomorphism of the circle related to a.

*To appear in Journal of Mathematical Analysis and Applications, in press.
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1 Introduction

Since the pioneering work of Balazs [1] on the study of the wave equation in a time-dependent domain, there have been several improvements (e.g. Cooper [2]). The mechanism of this problem was discovered by Cooper and Koch [3] (through the spectrum of the evolution operator on one period) and Dittrich, Duclos, and Gonzalez [4] (with direct estimates on the energy). A detailed survey with extensions is given in Gonzalez [5]. The common difficulty to these recent results is the lack of transparency: they rely on the arithmetic properties of a rotation number $\rho(F)$ of a homeomorphism of the circle F depending only on a. And, in general, one is not able to express analytically $\rho(F)$ w.r.t. the parameters of a (especially if $\rho(F)$ is irrational).
In this article we give results, for a particular boundary a, which depend only on the parameters. To do so, we compute the invariant measure of F for irrational $\rho(F)$. The main result is that the energy is always bounded; i.e., the model is stable in the sense introduced by [4] (see Definition 3.1). This may be considered as a surprise since instability, i.e., the energy is not bounded, is generic as is shown in Dittrich, Duclos, and Gonzalez [4, Prop. 3.29] for at least C^1-boundaries.
Let a be a strictly positive real function which is continuous, periodic, piecewise linear with no more than two slopes (α, $\beta \in (-1,1)$). Let $a_0 := a(0)$. It is assumed that there exists a relation between α, β and a_0, so that only two parameters are independent. In the proof, without loss of generality, a_0 will be considered as fixed. Moreover, we can assume that α and β are different. Otherwise a would be constant, which is a trivial case. The problem we consider is the *Dirichlet problem*:

$$\begin{aligned}
\varphi_{tt} - \varphi_{xx} &= 0\ , \quad t \in \mathbb{R}, \quad 0 < x < a(t)\ , && (1)\\
\varphi(x,0) &= \varphi_0(x)\ , \quad 0 < x < a(0)\ , && (2)\\
\varphi_t(x,0) &= \varphi_1(x)\ , \quad 0 < x < a(0)\ , && (3)\\
\varphi(0,t) &= 0\ , \quad t \in \mathbb{R}\ , && (4)\\
\varphi(a(t),t) &= 0\ , \quad t \in \mathbb{R}\ . && (5)
\end{aligned}$$

Notice that the *Neumann problem* for which

$$\begin{aligned}
\varphi_x(0,t) &= 0\ , \quad t \in \mathbb{R}\ ,\\
\varphi_x(a(t),t) &= 0\ , \quad t \in \mathbb{R}
\end{aligned}$$

are required instead of (4) and (5) would lead to similar results with minor changes for the constants (see [4] for the analysis of these boundary conditions). Since it will not play a role in the mathematical analysis, the wave velocity of φ is normalized to 1. In addition, by a rescaling in the parameters, we can also take the period of a equal to 1; this will simplify our notations. The energy of the field φ is given by the standard expression

$$E(t) := \frac{1}{2}\int_0^{a(t)} \left(|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2\right)\, dx\ . \qquad (6)$$

Let $\Omega := \{(x,t) \in \mathbb{R}^2; 0 < x < a(t)\}$. By [4, Theorem 2.12], there exists a unique weak solution $\varphi \in H^1_{0,loc}(\Omega)$ of the Dirichlet problem (1)-(5). It is given on Ω by $\varphi(x,t) = f(t+x) - f(t-x)$ with $f \in H^1_{loc}(\mathbb{R})$. The initial conditions (2)-(3) yield f on $[-a(0), a(0)]$ and the boundary condition (5) allows to write the following functional equation: $f \circ F = f$ on $\mathbb{R}$, where $F(x) := (x - a(x))^{-1} + a((x-a(x))^{-1})$ (the functions $x \mapsto x + a(x)$ and $x \mapsto x - a(x)$ are invertible by assumption on a). This equation is a way to construct the solution φ of the problem. Details on the function F are given in Section 3.1. For a more detailed approach, the reader is referred to Gonzalez [5]. The energy becomes

$$\forall t \in \mathbb{R}, \quad E(t) = \int_{t-a(t)}^{t+a(t)} |f'(x)|^2 \, dx.$$

2 Main results

Let a and E be defined as in Introduction.

Theorem 2.1 *The energy E is always bounded. More precisely, let p, $q \in \mathbb{N}^*$, let $l_1 := \frac{1+\alpha}{1-\alpha}$, $l_2 := \frac{1+\beta}{1-\beta}$, $l := \min\{l_1, l_2\}$, $L := \max\{l_1, l_2\}$. Then the two statements hold:*

(1)

$$\frac{\ln l_1}{\ln \frac{l_1}{l_2}} = \frac{p}{q} \Longleftrightarrow \forall (\varphi_0, \varphi_1) \in H^1_0((0,a(0))) \times L^2((0,a(0))), \quad E(t+p) = E(t). \tag{7}$$

(2) *If* $\frac{\ln l_1}{\ln \frac{l_1}{l_2}} \in \mathbb{R} \setminus \mathbb{Q}$, *then:*

$$\forall t \in \mathbb{R}, \quad \frac{l}{L^2} E(0) \leq E(t) \leq \frac{L}{l^2} E(0). \tag{8}$$

Remark 2.2 Note that the results of this Theorem depend only on the parameters α, β of a. As already noticed in the Introduction, the boundedness of the energy is a suprising fact and is specific to this boundary. It is easy to find piecewise linear boundaries such that the energy is unbounded (for instance, a non constant with $\frac{p}{2} \in \operatorname{Ran} a$: see [4, Remark 3.26]). The bounds of (8), as far as we know, are the first ones to be obtained for irrational rotation numbers. A lower bound was already known (see [4, Theorem 3.1]) for any well-defined boundary a, but here the constant is improved.

3 Proof

3.1 Notations and known results

The notations and definitions used in this paper are those from [4]. For convenience, we recall here what is necessary.

Let X be either the set $\mathbb{Z}$ (the integers) or $\mathbb{N}$ (the nonnegative integers) or $\mathbb{Q}$ (the rational numbers) or $\mathbb{R}$ (the real numbers). Then $X^* := X \setminus \{0\}$, $X_+ := \{x \in X;\ x \geq 0\}$, and $X_+^* := X^* \cap X_+$. Denote by $\mathbb{T}$ the 1-dimensional torus (the circle of unit length) and by X either $\mathbb{T}$ or $\mathbb{R}$. Let $C^0(\mathbb{T})$ be the space of the continuous periodic functions on $\mathbb{R}$. For a measurable function $F : X \to \mathbb{R}$, we shall denote by $F_{\min}$ and $F_{\max}$ its essential infimum and its essential supremum respectively. Let $\mathrm{Lip}(X)$ be the space of Lipschitz continuous functions. We shall denote the Lipschitz constant of a function F by

$$L(F) := \sup_{x,y \in X, x \neq y} \left| \frac{F(x) - F(y)}{x - y} \right| .$$

Let $\pi : \mathbb{R} \to \mathbb{T}$, $x \mapsto x + \mathbb{Z}$, be the canonical projection. For any continuous map $\overline{F} : \mathbb{T} \to \mathbb{T}$, the function F satisfying $\overline{F} \circ \pi = \pi \circ F$ is called a lift of $\overline{F}$ to $\mathbb{R}$. Denote by $\mathrm{Diff}^0(\mathbb{R})$ the homeomorphisms on $\mathbb{R}$. One calls $\mathcal{D}^0(\mathbb{T})$ the set of lifts of the orientation-preserving homeomorphisms of $\mathbb{T}$, i.e., $\mathcal{D}^0(\mathbb{T}) := \left\{ F \in \mathrm{Diff}^0(\mathbb{R});\ F - \mathrm{Id} \in C^0(\mathbb{T}) \right\}$, and $F \in \mathcal{D}^0(\mathbb{T})$ is a Lipschitz homeomorphism if F and F^{-1} are Lipschitz continuous.

For any $F \in \mathcal{D}^0(\mathbb{T})$, the rotation number $\rho(F)$ is defined by:

$$\rho(F) := \lim_{n \to +\infty} \frac{F^n(x) - x}{n} , \quad x \in \mathbb{R} , \tag{9}$$

where $F^n := F \circ F \circ \ldots \circ F$ is the n-th iterate of F. In Herman [6, Proposition II.2.3, p. 20], the limit (9) is proven to exist (it is a real number independent of x) and to be uniform w.r.t. x. If in the sequel $\rho(F) = \frac{p}{q}$ for $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$, it is always assumed that p and q are relatively primes. A point $x_0 \in \mathbb{R}$ is said to be a periodic point of period $q \in \mathbb{N}^*$ of $F \in \mathcal{D}^0(\mathbb{T})$ if there exists $p \in \mathbb{N}$ such that $F^q(x_0) = x_0 + p$. If $q = 1$, x_0 is said to be a fixed point. One can show (e.g. Herman [6, Prop. II.5.3, p. 24]) that the existence of a periodic point x_0 for $F \in \mathcal{D}^0(\mathbb{T})$, $F^q(x_0) = x_0 + p$, is equivalent to $\rho(F) = \frac{p}{q} \in \mathbb{Q}$. It means that if the rotation number is irrational then there are no periodic points.

Let Id be the identity on $\mathbb{R}$ and $a \in \mathrm{Lip}(\mathbb{T})$, $L(a) \in [0, 1)$. Define $h := \mathrm{Id} - a$, $k := \mathrm{Id} + a$ on $\mathbb{R}$. It is easy to see that h, k, h^{-1}, k^{-1} are Lipschitz homeomorphisms of the circle and $L(k), L(h) \leq 1 + L(a)$, $L(h^{-1}), L(k^{-1}) \leq \frac{1}{1-L(a)}$. Hence $k \circ h^{-1}, h \circ k^{-1}$ are also Lipschitz homeomorphisms of the circle and $L(k \circ h^{-1}), L(h \circ k^{-1}) \leq \frac{1+L(a)}{1-L(a)}$.
In the sequel, it is useful to keep in mind that:

$$F := (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1} \tag{10}$$

$$\text{iff} \quad a = \left(\frac{F - \mathrm{Id}}{2} \right) \circ \left(\frac{F + \mathrm{Id}}{2} \right)^{-1} . \tag{11}$$

In this kind of problem, [4] proved that there is a deep interplay between the invariant measures of homeomorphisms of the circle and the behavior of the energy. Let us recall some definitions. Let $F \in C^0(\mathbb{T})$. The measure μ is said to be an invariant measure of F iff μ belongs to the set of probability measures on $\mathbb{T}$ (i.e. $\mu \in (C^0(\mathbb{T}))'$ the dual space of $C^0(\mathbb{T})$, $\mu \geq 0$, and $\mu(\mathbb{T}) = 1$) and for every μ-measurable set A, $\mu(F^{-1}(A)) = \mu(A)$. According to Krylov and Bogolyubov's theorem, for any continuous map of $\mathbb{T}$, there exists at least one invariant measure. For the particular case of $F \in \mathcal{D}^0(\mathbb{T})$: if $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$, the invariant measure μ is unique. To our knowledge very few is known on analytic computations of the rotation number and *a fortiori* the invariant measure. If $\rho(F) \in \mathbb{Q}$, the invariant measure is, in general, not unique and it may be atomic. For a detailed study of homeomorphisms of the circle, the reader is referred to Herman [6].

We recall the following definition.

Definition 3.1 *We say that the model is stable if E is bounded from below and from above by strictly positive constants. Equivalently the model is unstable if the limit superior (resp. limit inferior) of E is infinite (resp. 0).*

For instance, Theorem 2.1 shows that the model of this paper is always stable. ¿From the paper of Dittrich, Duclos and Gonzalez [4], we will use the two following results.

Theorem [4, Th. 3.3]
(1) Let $a \in \mathrm{Lip}(\mathbb{T})$, $L(a) \in [0, 1)$; i.e., the boundary a is a Lipschitz continuous periodic and strictly positive function on $\mathbb{R}$ with a Lipschitz constant strictly less than 1. Assume that $F^q = \mathrm{Id} + p$ for some $p, q \in \mathbb{N}^*$. Then the energy E is p-periodic on $\mathbb{R}$: $E(t + p) = E(t)$ for any $t \in \mathbb{R}$.
(2) Let $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0, 1)$. Assume that for any initial conditions $(\varphi_0, \varphi_1) \in H_0^1((0, a(0))) \times L^2((0, a(0)))$, the energy E is p-periodic for some $p \in \mathbb{R}_+^*$. Then the boundary a is p-periodic and there exists an integer q, $q \geq 2$, such that $F^q = \mathrm{Id} + p$.

Theorem [4, Th. 3.7]
Let $a \in \mathrm{Lip}(\mathbb{T})$, $L(a) \in [0, 1)$. Assume that the rotation number of F is irrational and denote by μ the unique invariant measure of the projection $\overline{F}$ of F onto the circle. If there exist strictly positive and finite constants $\lambda_1, \lambda_2 \in \mathbb{R}$ such that the Radon-Nikodym derivative of μ w.r.t. the Lebesgue measure satisfies: $\lambda_1 \leq \frac{d\mu}{dm} \leq \lambda_2$, then

$$\forall t \in \mathbb{R}, \quad \frac{\lambda_1}{\lambda_2}\frac{E(0)}{F'_{\max}} \leq E(t) \leq \frac{\lambda_2}{\lambda_1}\frac{E(0)}{F'_{\min}}.$$

Note that criteria of instability (for rational rotation numbers as well as for irrational) are also given in [4].

3.2 Proof

We give first the expression of a, derive F, and prove four important lemmas. Let $a \in C^0(\mathbb{T})$, $a > 0$, be such that

$$a(t) := \begin{cases} \alpha t + \frac{\alpha(1-\alpha)(1+\beta)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)} \\ \beta t - \beta + \frac{\alpha(1-\beta^2)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(3+\beta)-2\beta}{2(\alpha-\beta)}, \end{cases}$$

with α, $\beta \in (-1, 1)$. Since $a > 0$, then $\alpha \neq \beta$. Notice that only two paramaters, namely α and β, are independent. The definition of a (which at first sight is not the simplest one) is chosen such that F is directly given on $[0, 1)$: since the rest of the proof relies on F, this choice is a simplication for the formalism, but there is no loss of generality.
To compute F we use the relation (10). Let $l_1 := \frac{1+\alpha}{1-\alpha}$, $l_2 := \frac{1+\beta}{1-\beta}$. Obviously, $l_1, l_2 > 0$ and $l_1 \neq l_2$. Then:

$$F(x) := k \circ h^{-1}(x) = \begin{cases} l_1 x + F_0 & \text{if } 0 \leq x \leq x_0 \\ l_2 x + F_0 + 1 - l_2 & \text{if } x_0 < x < 1, \end{cases}$$

with $F_0 := \frac{l_2(l_1-1)}{l_1-l_2}$, $x_0 := \frac{1-l_2}{l_1-l_2}$. It can be seen that $F_0, x_0 \in [0, 1)$.
Extended through the formula: $F(x+1) = F(x)+1$ for any $x \in \mathbb{R}$, F becomes the lift of a circle homeomorphism. Note that under the assumptions on α and β, F cannot have fixed points. But if $l_1 = l_2$ (i.e., $\alpha = \beta$), then all points are fixed. In the next lemmas we give precise details on the dynamics of F.

Lemma 3.2

$$\rho(F) = \frac{\ln l_1}{\ln \frac{l_1}{l_2}}. \tag{12}$$

Proof: Assume that $\rho(F)$ is irrational. Then there exists a continuous non decreasing function g such that $g \circ F = g + \rho(F)$ and $g(x) = \int_0^x d\mu$, where μ is the unique invariant measure of $\overline{F}$.
In particular, $\rho(F) = g \circ F(0) - g(0) = \mu((0, F_0))$. Since $\overline{F}$ is μ-invariant, $\mu((0, F_0)) = \mu(\overline{F}^{-1}((0, F_0))) = \mu((x_0, 1))$. By Denjoy's proposition (e.g., Herman [6, Prop. VI.1.1]),

$$\begin{aligned} \int_{\mathbb{T}} \ln F' \, d\mu &= 0 \\ &= \int_0^{x_0} \ln l_1 \, d\mu + \int_{x_0}^1 \ln l_2 \, d\mu \\ &= (1 - \rho(F)) \ln l_1 + \rho(F) \ln l_2. \end{aligned}$$

The Lemma is proved for irrational rotation numbers. Since $\rho(F)$ is a continuous function as F_0 varies, it follows that (12) holds for rational rotation numbers, too. ∎

Lemma 3.3 *If the rotation number of F, $\rho(F)$, is irrational, there exists a lift of a Lipschitz circle homeomorphism g such that g is a conjugacy of F, i.e., $g \circ F = g + \rho(F)$.*

Proof: Since F is a special case of Herman [6, Sec. VI.7.3], the existence of such a conjugacy g can be proved by reproducing the proof of Herman [6, Prop. 7.7.1]. Herman's proposition asserts that the conjugacy is a Lipschitz circle homeomorphism iff $\frac{\beta}{\beta+1} \in \mathbb{Z}\rho(F) \pmod 1$ with $\beta := -\frac{\ln l_2}{\ln l_1}$. Clearly $\beta = \frac{1}{\rho(F)} - 1$. Thus $\frac{\beta}{\beta+1} = 1 - \rho(F)$, which concludes the lemma. ∎

Lemma 3.4 *Let $p, q \in \mathbb{N}^*$. Then:*

$$\rho(F) = \frac{p}{q} \Longleftrightarrow F^q = \mathrm{Id} + p.$$

Proof: Obviously if $F^q = \mathrm{Id} + p$ for some $p, q \in \mathbb{N}^*$, then $\rho(F) = \frac{p}{q}$.
Let $x_{-1} := \overline{F}^{-1}(x_0)$.
If $F_0 > x_0$,

$$\overline{F}^2(x) = \begin{cases} l_1 l_2 x + \overline{F}(F_0) & \text{if } x \in [0, x_{-1}] \\ l_2^2 x + \overline{F}(F_0) - l_2^2 & \text{if } x \in (x_{-1}, 1). \end{cases}$$

If $F_0 = x_0$, $\overline{F}^2(x) = l_1 l_2 x + \overline{F}(F_0)$ if $x \in [0, 1)$.
If $F_0 < x_0$,

$$\overline{F}^2(x) = \begin{cases} l_1^2 x + \overline{F}(F_0) & \text{if } x \in [0, x_{-1}] \\ l_1 l_2 x + \overline{F}(F_0) - l_1 l_2 & \text{if } x \in (x_{-1}, 1). \end{cases}$$

Since $\overline{F}^2(x_{-1}) \in \{0, 1\}$, it follows that $\overline{F}^2$ is of the same kind of circle homeomorphism as $\overline{F}$ (i.e., piecewise linear with no more than two different slopes, and two independent parameters). By an obvious induction, the same conclusion holds for $\overline{F}^n$. Hence, if the rotation number $\rho(F)$ is rational and equal to $\frac{p}{q}$ (p and q are supposed to be positive integers and relatively primes), there exists at least one periodic point. But it is possible only if all points of $\mathbb{T}$ are periodic (see the remark before Lemma 3.2). Then there exists positive integers p' and q' such that $F^{q'} = \mathrm{Id} + p'$. Thus, $\rho(F) = \frac{p'}{q'}$. We deduce that $p' = p$ and $q' = q$ and finally $F^q = \mathrm{Id} + p$. ∎

Lemma 3.5 *If the rotation number of F is irrational, then the Radon-Nikodym derivative of the invariant measure μ of F w.r.t. the Lebesgue measure is given by:*

$$\frac{d\mu}{dx} = \frac{1}{\ln \frac{l_1}{l_2}} \frac{1}{x + \frac{l_2}{l_1 - l_2}} \quad a.e. \tag{13}$$

Proof: By Lemma 3.3, we are allowed to differentiate the conjugacy relation: $F' \cdot g' \circ F = g'$ a.e. Since $g' = \frac{d\mu}{dx}$ a.e., solving the system:

$$\begin{cases} l_1 g'(l_1 x + F_0) = g'(x) & \text{a.e. in } (0, x_0) \\ l_2 g'(l_2 x + F_0 - l_2) = g'(x) & \text{a.e. in } (x_0, 1) \end{cases}$$

gives the density of the invariant measure of F.
The value of the rotation number obtained in Lemma 3.2 suggests us to look for solutions of the form: $g(x) = g_0 \ln|x+g_1| + g_2$ on $[0,x_0]$ and $g(x) = g_3 \ln|x+g_4| + g_5$ on $(x_0,1)$ (with $g_0,g_1,g_2,g_3,g_4,g_5 \in \mathbb{R}$ and $g_1,g_4 \notin [-1,0]$). The preceding system yields by a straightforward computation $g_1 = g_4 = \frac{l_2}{l_1-l_2}$. If $l_1 > l_2$, then $g_1,g_4 > 0$, and if $l_1 < l_2$, $\frac{l_2}{l_1-l_2} \in [-1,0] \Leftrightarrow l_1 \leq 0, l_2 \geq 0$, which is impossible by assumption. Thus, $g_1,g_4 \notin [-1,0]$. Then the conditions $g(0)=0$, $g(1)=1$ and the continuity of g at x_0 imply that $g_0 = g_3 = \frac{1}{\ln\frac{l_1}{l_2}}$ and $g_2 = g_5 = -\ln|g_1|$. It is easy to check that $\rho(F) = 1-g(x_0) = \frac{\ln l_1}{\ln\frac{l_1}{l_2}}$.
The formula (13) is now obvious. ∎

By Lemma 3.2, Lemma 3.4, and Dittrich, Duclos, and Gonzalez [4, Th. 3.3] (see in Subsection 3.1), the relation (7) follows.
By Lemma 3.5, if $l_1 > l_2$:

$$\frac{1}{\ln\frac{l_1}{l_2}}\frac{l_1-l_2}{l_1} \leq \frac{d\mu}{dx} \leq \frac{1}{\ln\frac{l_1}{l_2}}\frac{l_1-l_2}{l_2},$$

$F'_{\min} = l_2$, $F'_{\max} = l_1$, and if $l_1 < l_2$:

$$\frac{1}{\ln\frac{l_1}{l_2}}\frac{l_1-l_2}{l_2} \leq \frac{d\mu}{dx} \leq \frac{1}{\ln\frac{l_1}{l_2}}\frac{l_1-l_2}{l_1},$$

$F'_{\min} = l_1$, $F'_{\max} = l_2$. Then Lemma 3.2 and Dittrich, Duclos, and Gonzalez [4, Th. 3.7] (see in Subsection 3.1) give (8).

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