The Global Minimum of the Effective Potential of the Many-Electron System with Delta-Interaction

by

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Abstract: We prove that the global minimum of the real part of the full effective potential of the many electron system with attractive delta interaction is in fact given by the BCS mean field configuration. This is a consequence of a simple bound which is obtained by applying Hadamard's inequality to the functional determinant. The second order Taylor expansion around the minimum is computed.

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I. Introduction and Results

In this article we consider the effective potential V of the nonrelativistic many electron system with attractive delta function interaction. In momentum space, it reads

$$V(\{\phi_q\}) = \sum_q |\phi_q|^2 - \log\det\begin{bmatrix} Id & \frac{ig}{\sqrt{\beta L^d}}\,C\phi^* \\ \frac{ig}{\sqrt{\beta L^d}}\,\bar{C}\phi & Id \end{bmatrix} \tag{1}$$
$$= \sum_q |\phi_q|^2 - \log\det\left[Id + \tfrac{g^2}{\beta L^d}\,\bar{C}\phi C\phi^*\right]$$

where $g^2 = \lambda > 0$ is an attractive coupling, $C = (\delta_{k,p}C_k)_{k,p}$ is a diagonal matrix with entries $C_k = \frac{1}{ik_0 - \varepsilon_{\mathbf{k}} + \mu}$, $\varepsilon_{\mathbf{k}} = \varepsilon_{-\mathbf{k}}$ being the single particle energy momentum relation and μ denoting the chemical potential. In the following we set $e_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ such that the Fermi surface is given by $e_{\mathbf{k}} = 0$. Furthermore ϕ is a short notation for the matrix $(\phi_{k-p})_{k,p}$. The momenta $k = (k_0, \mathbf{k}), p = (p_0, \mathbf{p})$ range over some finite subset of $\frac{\pi}{\beta}(2\mathbf{Z}+1) \times \left(\frac{2\pi}{L}\mathbf{Z}\right)^d$ if the system is kept in finite volume $[0, L]^d$ and at some small but positive temperature $T = \frac{1}{\beta} > 0$. To be specific, choose

$$k, p \in M_\nu := \left\{(k_0, \mathbf{k}) \in \tfrac{\pi}{\beta}(2\mathbf{Z}+1) \times \left(\tfrac{2\pi}{L}\mathbf{Z}\right)^d \,\middle|\, |e_{\mathbf{k}}| \le 1,\ |k_0| \le \nu\right\} \tag{2}$$

where $1 \ll \nu < \infty$ is some cuttoff. The momenta $q = (q_0, \mathbf{q})$ are given by $q \in \{k - p \,|\, k, p \in M_\nu\}$.

It is widely believed that the global minimum of the real part of V (in general, the determinant in (1) is complex) is given by the mean field configuration $\phi_x = const$ in coordinate space or $\phi_q \sim \delta_{q,0}$ in momentum space [1,2,3]. Although this is suggested by diagrammatic arguments [4], there was, to the authors knowledge, no rigorous proof of that. In Theorem 1 below we show that this result can be obtained by applying Hadamard's inequality (the absolute value of the determinant of n column vectors is less than the volume of the cube spanned by the n vectors) to the determinant in (1).

More precisely, all global minima of the real part

$$\mathrm{Re}V(\{\phi_q\}) = \sum_q |\phi_q|^2 - \log\left|\det\begin{bmatrix} Id & \frac{ig}{\sqrt{\beta L^d}}\,C\phi^* \\ \frac{ig}{\sqrt{\beta L^d}}\,\bar{C}\phi & Id \end{bmatrix}\right| \tag{3}$$

of V are given by

$$\phi_q = \delta_{q,0}\sqrt{\beta L^d}\,r_0\,e^{i\theta} \tag{4}$$

where $\theta \in [0, 2\pi]$ is an arbitrary phase and $\Delta^2 = \lambda r_0^2$ is a solution of the BCS equation

$$\tfrac{\lambda}{\beta L^d}\sum_{k\in M_\nu} \tfrac{1}{k_0^2 + e_{\mathbf{k}}^2 + \lambda r_0^2} = 1 \tag{5}$$

Equation (4) describes the BCS mean field configuration.

In Theorem 2 we expand V up to second order in $\xi = (\xi_{k-p})_{k,p}$ where

$$\xi_q = \phi_q - \delta_{q,0}\sqrt{\beta L^d}\, r_0\, e^{i\theta_0} = \begin{cases} (\rho_0 - \sqrt{\beta L^d}\, r_0) e^{i\theta_0} & \text{for } q = 0 \\ \rho_q e^{i\theta_q} & \text{for } q \neq 0. \end{cases} \tag{6}$$

We remark that a priori there is no need to introduce a small external field to fix the phase θ_0 and then to expand with respect to radial and tangential components as it is usually done like in [2] or [3]. The expansion around ξ gives

$$\begin{aligned} V(\{\phi_q\}) &= V_{\min} + 2\beta_0\,(\rho_0 - \sqrt{\beta L^d}\, r_0)^2 + \sum_{q\neq 0}(\alpha_q + i\gamma_q)\,\rho_q^2 \\ &\quad + \tfrac{1}{2}\sum_{q\neq 0}\beta_q\,|e^{-i\theta_0}\phi_q + e^{i\theta_0}\bar\phi_{-q}|^2 + O(\xi^3) \end{aligned} \tag{7}$$

where $V_{\min} \sim \beta L^d \Delta \log(1/\Delta)$ is identical the global minimum of the BCS effectiv potential

$$V_{\rm BCS}(\rho) = \beta L^d \left(\rho^2 - \tfrac{1}{\beta L^d}\textstyle\sum_k \log\left[1 + \frac{\lambda\rho^2}{k_0^2 + e_{\mathbf{k}}^2}\right]\right) \tag{8}$$

and the coefficients α_q, β_q are real and positive and γ_q is real. They are given in Theorem 2 below. In particular, $\alpha_q \sim q^2$ for small q.

The error term in (7) is not uniform in the coupling λ. Since $\beta_q \to 0$ and $\alpha_q + i\gamma_q \to 1$ for $\lambda \to 0$ (see (54)), equation (7) becomes in the limit $\lambda \to 0$

$$\sum_q |\phi_q|^2 = \sum_{q\neq 0} |\phi_q|^2 + O(\xi^3) \tag{9}$$

or $|\phi_0|^2 = O(\xi^3)$. The reason for this can already be seen if one expands $V_{\rm BCS}$ above around its minimum r_0. One obtains, abbreviating $E_k^2 := k_0^2 + e_{\mathbf{k}}^2 + \Delta^2$,

$$\begin{aligned} \tfrac{1}{\beta L^d}\big(V_{\rm BCS}(\rho) - V_{\min}\big) &= \rho^2 - r_0^2 - \tfrac{1}{\beta L^d}\textstyle\sum_k \log\left[1 + \frac{\lambda(\rho^2 - r_0^2)}{k_0^2 + e_{\mathbf{k}}^2 + \Delta^2}\right] \\ &= \left\{1 - \tfrac{\lambda}{\beta L^d}\textstyle\sum_k \frac{1}{E_k^2}\right\}(\rho^2 - r_0^2) + \tfrac{1}{2}\tfrac{1}{\beta L^d}\textstyle\sum_k \frac{\lambda^2}{E_k^4}(\rho^2 - r_0^2)^2 + O\big((\rho - r_0)^3\big) \\ &= \tfrac{1}{2}\tfrac{1}{\beta L^d}\textstyle\sum_k \frac{\lambda^2}{E_k^4}\left[4r_0^2(\rho - r_0)^2 + 4r_0(\rho - r_0)^3 + (\rho - r_0)^4\right] + O\big((\rho - r_0)^3\big) \end{aligned} \tag{10}$$

where in the last line the BCS equation (5) has been used. Since $\frac{1}{\beta L^d}\sum_k \frac{1}{E_k^4} \sim \frac{1}{\Delta^2} = \frac{1}{\lambda r_0^2}$, the quadratic term on the right hand side of (10) goes to 0, whereas the third and fourth order terms diverge.

Thus it is not clear to what extent one may draw conclusions from the quadratic approximation. Nevertheless, we write down the results for the partition function and the expectation value of the energy obtained by using (7). To this end we shortly recall the relation between the Hamiltonian and the functional integral representation of the model. One may look in [5] to see more explicit computations.

The Hamiltonian for the many-electron system with delta interaction in finite volume at some small but positive temperature $T = \frac{1}{\beta} > 0$ described in the grand canonical ensemble is given by

$$H = H_0 - \lambda H_{\rm int} = \tfrac{1}{L^d} \sum_{\mathbf{k}\sigma} (\varepsilon_{\mathbf{k}} - \mu) a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} - \tfrac{\lambda}{L^{3d}} \sum_{\mathbf{kpq}} a^+_{\mathbf{k}\uparrow} a^+_{\mathbf{q}-\mathbf{k}\downarrow} a_{\mathbf{q}-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow} \tag{11}$$

We are interested in the grand canonical partition function $Tr\, e^{-\beta H}$, in the correlation function

$$\tilde{\Lambda}(\mathbf{q}) = \tfrac{1}{L^{3d}} \sum_{\mathbf{k},\mathbf{p}} Tr[a^+_{\mathbf{k}\uparrow} a^+_{\mathbf{q}-\mathbf{k}\downarrow} a_{\mathbf{q}-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow} e^{-\beta H}] / Tr\, e^{-\beta H} \tag{12}$$

and in the expectation value $E_{\rm int} = \langle H_{\rm int} \rangle = \sum_{\mathbf{q}} \tilde{\Lambda}(\mathbf{q})$.

In terms of Grassmann integrals, the perturbation series for the normalized partition function $Z = Tre^{-\beta(H_0 - \lambda H_{\rm int})}/Tre^{-\beta H_0}$ is given by

$$Z = \int e^{\frac{\lambda}{(\beta L^d)^3} \sum_{k,p,q} \bar{\psi}_{k\uparrow} \bar{\psi}_{q-k\downarrow} \psi_{q-p\downarrow} \psi_{p\uparrow}} d\mu_C \tag{13}$$

and $\tilde{\Lambda}(\mathbf{q})$ is given by

$$\tilde{\Lambda}(\mathbf{q}) = \tfrac{1}{\beta} \sum_{q_0 \in \frac{2\pi}{\beta}\mathbf{Z}} \Lambda(q_0, \mathbf{q}) \tag{14}$$

where

$$\begin{aligned} \Lambda(q_0, \mathbf{q}) &= \tfrac{1}{(\beta L^d)^3} \sum_{k,p} \langle \bar{\psi}_{k\uparrow} \bar{\psi}_{q-k\downarrow} \psi_{q-p\downarrow} \psi_{p\uparrow} \rangle & (15) \\ &= \tfrac{1}{(\beta L^d)^3} \sum_{k,p} \tfrac{1}{Z} \int \bar{\psi}_{k\uparrow} \bar{\psi}_{q-k\downarrow} \psi_{q-p\downarrow} \psi_{p\uparrow}\, e^{\frac{\lambda}{(\beta L^d)^3} \sum_{k,p,q} \bar{\psi}_{k\uparrow} \bar{\psi}_{q-k\downarrow} \psi_{q-p\downarrow} \psi_{p\uparrow}} d\mu_C \\ &= \tfrac{d}{d\lambda_q}_{|\lambda_q = \lambda} \log \int e^{\frac{1}{(\beta L^d)^3} \sum_{k,p,q} \lambda_q \bar{\psi}_{k\uparrow} \bar{\psi}_{q-k\downarrow} \psi_{q-p\downarrow} \psi_{p\uparrow}} d\mu_C & (16) \end{aligned}$$

By making a Hubbard Stratonovich transformation, that is, by applying the formula ($\phi_q = u_q + iv_q$, $\bar{\phi}_q = u_q - iv_q$, $d\phi_q d\bar{\phi}_q := du_q dv_q$)

$$e^{\sum_q a_q b_q} = \int e^{\sum_q a_q \phi_q + b_q \bar{\phi}_q} e^{-\sum_q |\phi_q|^2} \prod_q \frac{d\phi_q d\bar{\phi}_q}{\pi}$$

with $a_q = ig_q(\beta L^d)^{-\frac{3}{2}} \sum_k \bar\psi_{k\uparrow}\bar\psi_{q-k\downarrow}$ and $b_q = ig_q(\beta L^d)^{-\frac{3}{2}} \sum_p \psi_{p\uparrow}\bar\psi_{q-p\downarrow}$, $g_q = \sqrt{\lambda_q}$, the exponent becomes quadratic in the Grassmann variables and the fermionic integral can be performed. The result is

$$Z(\{\lambda_q\}) = \int e^{-V(\{\phi_q\})} \prod_q \tfrac{d\phi_q d\bar\phi_q}{\pi} \tag{17}$$

where, if the λ_q's are not all equal, instead of (1) V is given by

$$V(\{\phi_q\}) = \sum_q |\phi_q|^2 - \log\det\left[\, Id + \tfrac{1}{\beta L^d} C(g\phi)^* \bar C(g\phi)\,\right] \tag{18}$$

where $(g\phi)$ stands for the matrix $(g_{k-p}\phi_{k-p})_{k,p}$. To perform the λ_q-derivative, one may change variables to obtain

$$\begin{aligned}
\Lambda(q) &= \tfrac{1}{Z}\tfrac{d}{d\lambda_q} \int e^{-\sum_q \frac{|\phi_q|^2}{\lambda_q}} \det\left[\, Id + \tfrac{1}{\beta L^d}\bar C\phi C\phi^*\,\right] \Pi_q \tfrac{d\phi_q d\bar\phi_q}{\lambda_q \pi} \\
&= \tfrac{1}{Z}\int \left(\tfrac{|\phi_q|^2}{\lambda_q^2} - \tfrac{1}{\lambda_q}\right) e^{-\sum_q \frac{|\phi_q|^2}{\lambda}} \det\left[\, Id + \tfrac{1}{\beta L^d}\bar C\phi C\phi^*\,\right] \prod_q \tfrac{d\phi_q d\bar\phi_q}{\lambda\pi} \\
&= \tfrac{1}{Z}\int \tfrac{|\phi_q|^2 - 1}{\lambda} e^{-\sum_q |\phi_q|^2} \det\left[\, Id + \tfrac{\lambda}{\beta L^d}\bar C\phi C\phi^*\,\right] \prod_q \tfrac{d\phi_q d\bar\phi_q}{\pi} \\
&= \tfrac{1}{Z}\int \tfrac{|\phi_q|^2 - 1}{\lambda} e^{-V(\{\phi_q\})} \prod_q \tfrac{d\phi_q d\bar\phi_q}{\pi}
\end{aligned} \tag{19}$$

For the ideal Fermi gas $\lambda = 0$ the above formula gives $\frac{0}{0}$. In that case one may use (15) to obtain

$$\begin{aligned}
\Lambda(q) &= \tfrac{1}{(\beta L^d)^3} \sum_{k,p} \langle \bar\psi_{k\uparrow}\psi_{p\uparrow}\rangle\langle\bar\psi_{q-k\downarrow}\psi_{q-p\downarrow}\rangle = \tfrac{1}{(\beta L^d)^3}\sum_{k,p} \beta L^d \delta_{k,p} C_k\, \beta L^d \delta_{q-k,q-p} C_{q-k} \\
&= \tfrac{1}{\beta L^d}\sum_k C_k C_{q-k}
\end{aligned} \tag{20}$$

which is the particle particle bubble.

If one substitutes V by the second order approximation

$$V_2(\{\phi_q\}) := V_{\min} + 2\beta_0\,(\rho_0 - \sqrt{\kappa}\, r_0)^2 + \sum_{q\neq 0}(\alpha_q + i\gamma_q)\rho_q^2 + \tfrac{1}{2}\sum_{q\neq 0}\beta_q\, |e^{-i\theta_0}\phi_q + e^{i\theta_0}\bar\phi_{-q}|^2 \tag{21}$$

the integrals in (17,19) become Gaussian and can be performed. The results are (the index "2" in the following means that V_2 instead of V has been used)

$$Z_2 = e^{-V_{\min}} \int_0^\infty e^{-2\beta_0^2(\rho_0 - \sqrt{\kappa} r_0)^2} 2\rho_0 d\rho_0 \prod_{\substack{q \\ q_0>0}} \tfrac{1}{\alpha_q^2 + \gamma_g^2 + 2\alpha_q\beta_q}\,, \tag{22}$$

$$\Lambda_2(q) = \tfrac{1}{\lambda}\left(\tfrac{\alpha_q + i\gamma_q + \beta_q}{\alpha_q^2 + \gamma_q^2 + 2\alpha_q\beta_q} - 1\right) \tag{23}$$

and, since $\gamma_{-q} = -\gamma_q$,

$$\begin{aligned} \varepsilon_{\mathrm{int},2} &:= \tfrac{1}{L^d}\langle H_{\mathrm{int}}\rangle_2 = \tfrac{1}{\beta L^d}\sum_q \Lambda_2(q) \\ &= \tfrac{1}{\lambda}\tfrac{1}{\beta L^d}\sum_q \left(\tfrac{\alpha_q+\beta_q}{\alpha_q^2+\gamma_q^2+2\alpha_q\beta_q} - 1\right) \geq \tfrac{1}{\lambda}\tfrac{1}{\beta L^d}\sum_q \left(\tfrac{1}{\alpha_q} - 1\right) \end{aligned} \tag{24}$$

In particular, since $\alpha_q \leq const\,\frac{q^2}{\Delta^2}$ for small q,

$$\varepsilon_{\mathrm{int},2} \geq \tfrac{1}{\lambda}\tfrac{1}{\beta L^d}\sum_q \left(const\,\tfrac{\Delta^2}{q^2} - 1\right) \tag{25}$$

which, for $L \to \infty$, is infrared singular for $d = 1$ and, since $q_0 = 0$ is an allowed value for positive temperature, also logarithmically divergent for small q in $d = 2$. A similar observation is made in [3].

II. Proof of Theorems

Theorem 1: *Let* ReV *be the real part of the effective potential for the many electron system with attractive delta interaction given by (3). Let* $e_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ *satisfy* $e_{\mathbf{k}} = e_{-\mathbf{k}}$ *and:* $\forall \mathbf{k}\ e_{\mathbf{k}} = e_{\mathbf{k}+\mathbf{q}} \Rightarrow \mathbf{q} = 0$*. Then all global minima of ReV are given by (4),*

$$\phi_q = \delta_{q,0}\sqrt{\beta L^d}\, r_0\, e^{i\theta}, \qquad \theta \in [0, 2\pi] \ \ \text{arbitrary} \tag{4}$$

where r_0 *is a solution of the BCS equation (5) or, equivalently, the global minimum of the function (8)*

$$\begin{aligned} V_{\mathrm{BCS}}(\rho) &= V(\{\delta_{q,0}\sqrt{\beta L^d}\rho\, e^{i\theta}\}) \\ &= \beta L^d\rho^2 - \sum_k \log\left[1 + \tfrac{\lambda\rho^2}{k_0^2 + e_{\mathbf{k}}^2}\right] = \beta L^d\rho^2 - \sum_{\mathbf{k}} \log\left[\tfrac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda\rho^2})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right] \end{aligned} \tag{8}$$

More specifically, there is the bound

$$\begin{aligned} \mathrm{Re}V(\{\phi_q\}) \geq V_{\mathrm{BCS}}(\|\phi\|) - \min_{k\in M_\nu}\Bigg\{ &\log\left[\prod_{q\neq 0}\left(1 - \tfrac{|\frac{\lambda}{\beta L^d}\sum_p \phi_p\bar\phi_{p+q}|^2}{(|a_k|^2+\lambda\|\phi\|^2)(|a_{k+q}|^2+\lambda\|\phi\|^2)}\right)^{\frac{1}{2}}\right] \\ &+ \log\left[\prod_{q\neq 0}\left(1 - \tfrac{\frac{\lambda}{\beta L^d}|\phi_q|^2|a_{k+q}-a_k|^2}{(|a_k|^2+\lambda\|\phi\|^2)(|a_{k+q}|^2+\lambda\|\phi\|^2)}\right)^{\frac{1}{2}}\right]\Bigg\} \end{aligned} \tag{26}$$

where $\|\phi\|^2 := \frac{1}{\beta L^d}\sum_q |\phi_q|^2$ *and* $|a_k|^2 := k_0^2 + e_{\mathbf{k}}^2$. *In particular,*

$$\mathrm{Re}V(\{\phi_q\}) \geq V_{\mathrm{BCS}}(\|\phi\|) \tag{27}$$

since the products in (26) are less or equal 1.

Proof: Suppose first that (26) holds. For each q, the round brackets in (26) are between 0 and 1 which means that $-\log(\Pi_q \cdots)$ is positive. Thus

$$\mathrm{Re}V(\{\phi_q\}) \geq V_{\mathrm{BCS}}(\|\phi\|) \geq V_{\mathrm{BCS}}(r_0) \tag{28}$$

which proves that $\phi_q = \delta_{q,0}\sqrt{\beta L^d} r_0\, e^{i\theta}$ are indeed global minima of ReV. On the other hand, if a configuration $\{\phi_q\}$ is a global minimum, then the logarithms in (26) must be zero for all $k \in M_\nu$ which in particular means that for all $q \neq 0$

$$\sum_p \phi_p \bar{\phi}_{p+q} = 0 \tag{29}$$

and for all $k \in M_\nu$ and $q \neq 0$

$$|\phi_q|^2 |a_{k+q} - a_k|^2 = |\phi_q|^2 [q_0^2 + (e_{\mathbf{k+q}} - e_{\mathbf{k}})^2] = 0 \tag{30}$$

which implies $\phi_q = 0$ for all $q \neq 0$. It remains to prove (26).

To this end, we write (recall that $C_k = \frac{1}{a_k} = \frac{1}{ik_0 - e_{\mathbf{k}}}$)

$$\det \begin{bmatrix} Id & \frac{ig}{\sqrt{\beta L^d}}\bar{C}\phi^* \\ \frac{ig}{\sqrt{\beta L^d}} C\phi & Id \end{bmatrix} = \det \begin{bmatrix} | & | \\ \vec{b}_k & \vec{b}'_{k'} \\ | & | \end{bmatrix} \tag{31}$$

where

$$\vec{b}_k = \begin{pmatrix} 1 \\ \\ \frac{ig}{\sqrt{\beta L^d}} \frac{\phi_{k-p}}{a_k} \end{pmatrix} \qquad \vec{b}'_{k'} = \begin{pmatrix} \frac{ig}{\sqrt{\beta L^d}} \frac{\bar{\phi}_{p-k'}}{\bar{a}_{k'}} \\ \\ 1 \end{pmatrix} \tag{32}$$

If $|\vec{b}_k|$ denotes the euclidean norm of $\vec{b}_k$, then we have

$$|\vec{b}_k|^2 = 1 + \tfrac{\lambda}{\beta L^d}\textstyle\sum_p \frac{|\phi_{k-p}|^2}{|a_k|^2} = 1 + \lambda \frac{\|\phi\|^2}{|a_k|^2} = |\vec{b}'_k|^2 \tag{33}$$

Therefore one obtains, if $\vec{e}_k = \frac{\vec{b}_k}{|\vec{b}_k|}$, $\vec{e}'_k = \frac{\vec{b}'_k}{|\vec{b}'_k|}$

$$\det \begin{bmatrix} | & | \\ \vec{b}_k & \vec{b}'_{k'} \\ | & | \end{bmatrix} = \prod_k \left\{ 1 + \lambda \frac{\|\phi\|^2}{|a_k|^2} \right\} \det \begin{bmatrix} | & | \\ \vec{e}_k & \vec{e}'_{k'} \\ | & | \end{bmatrix} \tag{34}$$

From this the inequality (27) already follows since the determinant on the right hand side of (34) is less or equal 1. To obtain (26), we choose a fixed but arbitrary momentum $t \in M_\nu$ and orthogonalize all vectors $\vec{e}_k$, $\vec{e}'_{k'}$ in the determinant with respect to e_t. That is, we write

$$\det \begin{bmatrix} | & | \\ \vec{e}_k & \vec{e}'_{k'} \\ | & | \end{bmatrix} = \det \begin{bmatrix} | & | \\ \vec{e}_k - (\vec{e}_k, \vec{e}_t)\vec{e}_t & \vec{e}'_{k'} - (\vec{e}'_{k'}, \vec{e}_t)\vec{e}_t \\ | & | \end{bmatrix} \tag{35}$$

Finally we apply Hadamard's inequality, $|\det F| \le \Pi_{j=1}^n |\vec{f}_j| = \left\{ \Pi_{j=1}^n \sum_{i=1}^n |f_{ij}|^2 \right\}^{\frac{1}{2}}$ if $F = (f_{ij})_{1 \le i,j \le n}$ is a complex matrix, to the determinant on the right hand side of (35). Since

$$|\vec{e}_k - (\vec{e}_k, \vec{e}_t)\vec{e}_t|^2 = 1 - |(\vec{e}_k, \vec{e}_t)|^2$$

one obtains

$$\begin{aligned} &\left| \det \begin{bmatrix} | & | \\ \vec{e}_k - (\vec{e}_k, \vec{e}_t)\vec{e}_t & \vec{e}'_{k'} - (\vec{e}'_{k'}, \vec{e}_t)\vec{e}_t \\ | & | \end{bmatrix} \right| \\ &\le |\vec{e}_t| \prod_{\substack{k \in M_\nu \\ k \ne t}} (1 - |(\vec{e}_k, \vec{e}_t)|^2)^{\frac{1}{2}} \prod_{k \in M_\nu} (1 - |(\vec{e}'_k, \vec{e}_t)|^2)^{\frac{1}{2}} \end{aligned} \tag{36}$$

or with (31) and (34)

$$\begin{aligned} \mathrm{Re} V(\{\phi_q\}) &= \sum_q |\phi_q|^2 - \log \left| \det \begin{bmatrix} Id & \frac{ig}{\sqrt{\beta L^d}} \phi^* \bar{C} \\ \frac{ig}{\sqrt{\beta L^d}} \phi C & Id \end{bmatrix} \right| \\ &= \beta L^d \|\phi\|^2 - \sum_k \log \left\{ 1 + \lambda \frac{\|\phi\|^2}{|a_k|^2} \right\} - \log \left| \det \begin{bmatrix} | & | \\ \vec{e}_k & \vec{e}'_{k'} \\ | & | \end{bmatrix} \right| \\ &= V_{\mathrm{BCS}}(\|\phi\|) - \log \left| \det \begin{bmatrix} | & | \\ \vec{e}_k & \vec{e}'_{k'} \\ | & | \end{bmatrix} \right| \\ &\ge V_{\mathrm{BCS}}(\|\phi\|) - \log \left\{ \prod_{\substack{k \in M_\nu \\ k \ne t}} (1 - |(\vec{e}_k, \vec{e}_t)|^2)^{\frac{1}{2}} \prod_{k \in M_\nu} (1 - |(\vec{e}'_k, \vec{e}_t)|^2)^{\frac{1}{2}} \right\} \end{aligned} \tag{37}$$

Finally one has

$$(\vec{b}_k, \vec{b}_t) = \sum_p \frac{ig}{\sqrt{\beta L^d}} \frac{\phi_{k-p}}{a_k} \overline{\frac{ig}{\sqrt{\beta L^d}} \frac{\phi_{t-p}}{a_t}}$$

which gives

$$|(\vec{e}_k, \vec{e}_t)|^2 = \frac{\left| \frac{\lambda}{\beta L^d} \sum_p \phi_{k-p} \bar{\phi}_{t-p} \right|^2}{(|a_k|^2 + \lambda \|\phi\|^2)\ (|a_t|^2 + \lambda \|\phi\|^2)} \tag{38}$$

and

$$(\vec{b}_k', \vec{b}_t) = \frac{ig}{\sqrt{\beta L^d}} \frac{\bar{\phi}_{t-k}}{\bar{a}_k} + \overline{\frac{ig}{\sqrt{\beta L^d}} \frac{\phi_{t-k}}{a_t}} = \frac{ig}{\sqrt{\beta L^d}} \bar{\phi}_{t-k} \left(\tfrac{1}{\bar{a}_k} - \tfrac{1}{\bar{a}_t} \right)$$

which gives

$$|(\vec{e}_k', \vec{e}_t)|^2 = \frac{\frac{\lambda}{\beta L^d} |\phi_{t-k}|^2 |a_t - a_k|^2}{(|a_k|^2 + \lambda\|\phi\|^2)\ (|a_t|^2 + \lambda\|\phi\|^2)} \tag{39}$$

Substituting (38,39) in (37) proves Theorem 1 □

Theorem 2: *Let V be the effective potential (1), let $\kappa = \beta L^d$ and let $\xi = (\xi_{k-p})_{k,p}$ be the matrix with entries*

$$\xi_q = \phi_q - \delta_{q,0} \sqrt{\kappa}\, r_0\, e^{i\theta_0} = \begin{cases} (\rho_0 - \sqrt{\kappa}\, r_0) e^{i\theta_0} & \text{for } q = 0 \\ \rho_q e^{i\theta_q} & \text{for } q \neq 0. \end{cases} \tag{40}$$

Then

$$V(\{\phi_q\}) = V_{\min} + 2\beta_0\, (\rho_0 - \sqrt{\kappa}\, r_0)^2 + \sum_{q\neq 0} (\alpha_q + i\gamma_q) \rho_q^2 + \tfrac{1}{2} \sum_{q\neq 0} \beta_q\, |e^{-i\theta_0} \phi_q + e^{i\theta_0} \bar{\phi}_{-q}|^2 + O(\xi^3) \tag{41}$$

where, if $E_k^2 = k_0^2 + e_{\mathbf{k}}^2 + \lambda r_0^2$,

$$\alpha_q = \tfrac{1}{2} \tfrac{\lambda}{\kappa} \sum_k \tfrac{q_0^2 + (e_{\mathbf{k}} - e_{\mathbf{k}-\mathbf{q}})^2}{E_k^2 E_{k-q}^2} > 0\,, \quad \beta_q = \tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^2 E_{k-q}^2} > 0\,,$$

$$\gamma_q = -\tfrac{\lambda}{\kappa} \sum_k \tfrac{k_0 e_{\mathbf{k}-\mathbf{q}} - (k_0 - q_0) e_{\mathbf{k}}}{E_k^2 E_{k-q}^2} \in \mathbf{R} \tag{42}$$

and

$$V_{\min} = \kappa \left(r_0^2 - \tfrac{1}{\kappa} \textstyle\sum_{\mathbf{k}} \log \left[\frac{\cosh(\frac{\beta}{2} \sqrt{e_{\mathbf{k}}^2 + \lambda r_0^2})}{\cosh \frac{\beta}{2} e_{\mathbf{k}}} \right] \right). \tag{43}$$

Proof: We abbreviate $\kappa = \beta L^d$ and write

$$V(\{\phi_q\}) = \sum_q |\phi_q|^2 - \log \frac{\det \begin{bmatrix} A & \frac{ig}{\sqrt{\kappa}} \phi^* \\ \frac{ig}{\sqrt{\kappa}} \phi & \bar{A} \end{bmatrix}}{\det \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}} \tag{44}$$

where $A = C^{-1} = (\delta_{k,p} a_k)_{k,p \in M_\kappa}$ and $a_k := 1/C_k = ik_0 - e_{\mathbf{k}}$. Then

$$V(\{\phi_q\}) - V(\{\sqrt{\kappa}\, \delta_{q,0}\, r_0 e^{i\theta_0}\}) = \sum_q \rho_q^2 - \kappa r_0^2 - \log \frac{\det \begin{bmatrix} A & \frac{ig}{\sqrt{\kappa}} \phi^* \\ \frac{ig}{\sqrt{\kappa}} \phi & \bar{A} \end{bmatrix}}{\det \begin{bmatrix} A & igr_0\, e^{-i\theta_0} \\ igr_0\, e^{i\theta_0} & \bar{A} \end{bmatrix}} \tag{45}$$

where $igr_0\,e^{i\theta_0} \equiv igr_0\,e^{i\theta_0}\,Id$ in the determinant above. Since

$$\begin{bmatrix} A & igr_0\,e^{-i\theta_0} \\ igr_0\,e^{i\theta_0} & \bar{A} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\bar{a}_k\,\delta_{k,p}}{|a_k|^2+\lambda r_0^2} & -\frac{igr_0\,e^{-i\theta_0}\,\delta_{k,p}}{|a_k|^2+\lambda r_0^2} \\ -\frac{igr_0\,e^{i\theta_0}\,\delta_{k,p}}{|a_k|^2+\lambda r_0^2} & \frac{a_k\,\delta_{k,p}}{|a_k|^2+\lambda r_0^2} \end{bmatrix}$$
$$\equiv \frac{1}{|a|^2+\lambda r_0^2}\begin{bmatrix} \bar{A} & -igr_0\,e^{-i\theta_0} \\ -igr_0\,e^{i\theta_0} & A \end{bmatrix} \tag{46}$$

and because of

$$\begin{bmatrix} A & \frac{ig}{\sqrt{\kappa}}\phi^* \\ \frac{ig}{\sqrt{\kappa}}\phi & \bar{A} \end{bmatrix} = \begin{bmatrix} A & igr_0\,e^{-i\theta_0} \\ igr_0\,e^{i\theta_0} & \bar{A} \end{bmatrix} + \begin{bmatrix} 0 & \frac{ig}{\sqrt{\kappa}}\phi^* - igr_0\,e^{-i\theta_0} \\ \frac{ig}{\sqrt{\kappa}}\phi - igr_0\,e^{i\theta_0} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A & ig\bar{\gamma} \\ ig\gamma & \bar{A} \end{bmatrix} + \begin{bmatrix} 0 & ig\,\xi^* \\ ig\,\xi & 0 \end{bmatrix} \tag{47}$$

where $\gamma = r_0\,e^{i\theta_0}$ and $\xi = (\xi_{k,p})$ is given by (40), the quotient of determinants in (45) is given by

$$\det\left[Id + \frac{1}{|a|^2+\lambda r_0^2}\begin{pmatrix} \bar{A} & -ig\bar{\gamma} \\ -ig\gamma & A \end{pmatrix}\begin{pmatrix} 0 & ig\,\xi^* \\ ig\,\xi & 0 \end{pmatrix}\right]$$
$$= \det\left[Id + \begin{pmatrix} \lambda\frac{\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi & ig\frac{\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^* \\ ig\frac{A}{|a|^2+\lambda r_0^2}\,\xi & \lambda\frac{\gamma}{|a|^2+\lambda r_0^2}\,\xi^* \end{pmatrix}\right] \tag{48}$$

Since

$$\log\det[Id+B] = Tr\,\log[Id+B] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}Tr\,B^n$$
$$= Tr\,B - \tfrac{1}{2}Tr\,B^2 + \tfrac{1}{3}Tr\,B^3 - + \cdots \tag{49}$$

one obtains to second order in ξ:

$$\log\det\left[Id + \begin{pmatrix} \lambda\frac{\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi & ig\frac{\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^* \\ ig\frac{A}{|a|^2+\lambda r_0^2}\,\xi & \lambda\frac{\gamma}{|a|^2+\lambda r_0^2}\,\xi^* \end{pmatrix}\right]$$
$$= Tr\begin{pmatrix} \lambda\frac{\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi & ig\frac{\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^* \\ ig\frac{A}{|a|^2+\lambda r_0^2}\,\xi & \lambda\frac{\gamma}{|a|^2+\lambda r_0^2}\,\xi^* \end{pmatrix} - \tfrac{1}{2}Tr\left\{\begin{pmatrix} \lambda\frac{\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi & ig\frac{\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^* \\ ig\frac{A}{|a|^2+\lambda r_0^2}\,\xi & \lambda\frac{\gamma}{|a|^2+\lambda r_0^2}\,\xi^* \end{pmatrix}^2\right\} + O(\xi^3)$$
$$= Tr\frac{\lambda\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi + Tr\frac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^* - \tfrac{1}{2}\left\{Tr\frac{\lambda\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi\frac{\lambda\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi + Tr\frac{ig\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^*\frac{igA}{|a|^2+\lambda r_0^2}\,\xi\right.$$
$$\left. + Tr\frac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^*\frac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^* + Tr\frac{igA}{|a|^2+\lambda r_0^2}\,\xi\frac{ig\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^*\right\} + O(\xi^3) \tag{50}$$

One has

$$\left(\frac{\lambda\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi\right)_{k,p} = \frac{\lambda\bar{\gamma}}{|a_k|^2+\lambda r_0^2}\,\xi_{k,p}\,, \quad \left(\frac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^*\right)_{k,p} = \frac{\lambda\gamma}{|a_k|^2+\lambda r_0^2}\,\bar{\xi}_{p,k}$$

$$\left(\tfrac{\lambda\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi\,\tfrac{\lambda\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi\right)_{k,k} = \sum_p \tfrac{\lambda\bar{\gamma}}{|a_k|^2+\lambda r_0^2}\,\xi_{k,p}\,\tfrac{\lambda\bar{\gamma}}{|a_p|^2+\lambda r_0^2}\,\xi_{p,k}$$

$$= \left(\tfrac{\lambda\bar{\gamma}\,\xi_{k,k}}{|a_k|^2+\lambda r_0^2}\right)^2 + \sum_{\substack{p\\ p\neq k}} \tfrac{\lambda\bar{\gamma}}{|a_k|^2+\lambda r_0^2}\,\tfrac{\lambda\bar{\gamma}}{|a_p|^2+\lambda r_0^2}\,\xi_{k,p}\,\xi_{p,k}$$

$$= \tfrac{1}{\kappa}\left(\tfrac{\lambda r_0(\rho_0-\sqrt{\kappa}r_0)}{|a_k|^2+\lambda r_0^2}\right)^2 + \tfrac{1}{\kappa}\sum_{q\neq 0} \tfrac{\lambda\bar{\gamma}}{|a_k|^2+\lambda r_0^2}\,\tfrac{\lambda\bar{\gamma}}{|a_{k-q}|^2+\lambda r_0^2}\,\phi_q\,\phi_{-q}$$

$$\left(\tfrac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^*\,\tfrac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^*\right)_{k,k} = \sum_p \tfrac{\lambda\gamma}{|a_k|^2+\lambda r_0^2}\,\bar{\xi}_{p,k}\,\tfrac{\lambda\gamma}{|a_p|^2+\lambda r_0^2}\,\bar{\xi}_{k,p}$$

$$= \left(\tfrac{\lambda\gamma\,\bar{\xi}_{k,k}}{|a_k|^2+\lambda r_0^2}\right)^2 + \sum_{\substack{p\\ p\neq k}} \tfrac{\lambda\gamma}{|a_k|^2+\lambda r_0^2}\,\tfrac{\lambda\gamma}{|a_p|^2+\lambda r_0^2}\,\bar{\xi}_{k,p}\,\bar{\xi}_{p,k}$$

$$= \tfrac{1}{\kappa}\left(\tfrac{\lambda r_0(\rho_0-\sqrt{\kappa}r_0)}{|a_k|^2+\lambda r_0^2}\right)^2 + \tfrac{1}{\kappa}\sum_{q\neq 0} \tfrac{\lambda\gamma}{|a_k|^2+\lambda r_0^2}\,\tfrac{\lambda\gamma}{|a_{k-q}|^2+\lambda r_0^2}\,\bar{\phi}_q\,\bar{\phi}_{-q}$$

and

$$\left(\tfrac{igA}{|a|^2+\lambda r_0^2}\,\xi\right)_{k,p} = \tfrac{iga_k}{|a_k|^2+\lambda r_0^2}\,\xi_{k,p}\,, \qquad \left(\tfrac{ig\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^*\right)_{k,p} = \tfrac{ig\bar{a}_k}{|a_k|^2+\lambda r_0^2}\,\bar{\xi}_{p,k}$$

$$\left(\tfrac{igA}{|a|^2+\lambda r_0^2}\,\xi\,\tfrac{ig\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^*\right)_{k,k} = \sum_p \tfrac{iga_k}{|a_k|^2+\lambda r_0^2}\,\xi_{k,p}\,\tfrac{ig\bar{a}_p}{|a_p|^2+\lambda r_0^2}\,\bar{\xi}_{k,p}$$

$$= -\lambda\tfrac{|a_k|^2}{(|a_k|^2+\lambda r_0^2)^2}|\xi_{k,k}|^2 - \lambda\sum_{\substack{p\\ p\neq k}} \tfrac{a_k}{|a_k|^2+\lambda r_0^2}\,\xi_{k,p}\,\tfrac{\bar{a}_p}{|a_p|^2+\lambda r_0^2}\,\bar{\xi}_{k,p}$$

$$= -\tfrac{\lambda}{\kappa}\tfrac{|a_k|^2}{(|a_k|^2+\lambda r_0^2)^2}(\rho_0-\sqrt{\kappa}r_0)^2 - \tfrac{\lambda}{\kappa}\sum_{q\neq 0} \tfrac{a_k}{|a_k|^2+\lambda r_0^2}\;\tfrac{\bar{a}_{k-q}}{|a_{k-q}|^2+\lambda r_0^2}\,\rho_q^2$$

$$\left(\tfrac{ig\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^*\,\tfrac{igA}{|a|^2+\lambda r_0^2}\,\xi\right)_{k,k} = \sum_p \tfrac{ig\bar{a}_k}{|a_k|^2+\lambda r_0^2}\,\bar{\xi}_{p,k}\,\tfrac{iga_p}{|a_p|^2+\lambda r_0^2}\,\xi_{p,k}$$

$$= -\lambda\tfrac{|a_k|^2}{(|a_k|^2+\lambda r_0^2)^2}|\xi_{k,k}|^2 - \lambda\sum_{\substack{p\\ p\neq k}} \tfrac{\bar{a}_k}{|a_k|^2+\lambda r_0^2}\,\bar{\xi}_{p,k}\,\tfrac{a_p}{|a_p|^2+\lambda r_0^2}\,\xi_{p,k}$$

$$= -\tfrac{\lambda}{\kappa}\tfrac{|a_k|^2}{(|a_k|^2+\lambda r_0^2)^2}(\rho_0-\sqrt{\kappa}r_0)^2 - \tfrac{\lambda}{\kappa}\sum_{q\neq 0} \tfrac{\bar{a}_k}{|a_k|^2+\lambda r_0^2}\;\tfrac{a_{k+q}}{|a_{k+q}|^2+\lambda r_0^2}\,\rho_q^2$$

Therefore (50) becomes

$$\log\det\left[Id + \begin{pmatrix} \lambda\frac{\bar{\gamma}}{|a|^2+\lambda r_0^2}\,\xi & ig\frac{\bar{A}}{|a|^2+\lambda r_0^2}\,\xi^* \\ ig\frac{A}{|a|^2+\lambda r_0^2}\,\xi & \lambda\frac{\gamma}{|a|^2+\lambda r_0^2}\,\xi^* \end{pmatrix}\right]$$

$$= Tr\,\tfrac{\lambda\bar\gamma}{|a|^2+\lambda r_0^2}\,\xi + Tr\,\tfrac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^* - \tfrac{1}{2}\Big\{ Tr\,\tfrac{\lambda\bar\gamma}{|a|^2+\lambda r_0^2}\,\xi\,\tfrac{\lambda\bar\gamma}{|a|^2+\lambda r_0^2}\,\xi + Tr\,\tfrac{ig\bar A}{|a|^2+\lambda r_0^2}\,\xi^*\,\tfrac{igA}{|a|^2+\lambda r_0^2}\,\xi$$

$$+ Tr\,\tfrac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^*\,\tfrac{\lambda\gamma}{|a|^2+\lambda r_0^2}\,\xi^* + Tr\,\tfrac{igA}{|a|^2+\lambda r_0^2}\,\xi\,\tfrac{ig\bar A}{|a|^2+\lambda r_0^2}\,\xi^*\Big\} + O(\xi^3)$$

$$= 2\tfrac{\lambda}{\kappa}\sum_k \tfrac{\sqrt{\kappa}\,r_0(\rho_0-\sqrt{\kappa}\,r_0)}{|a_k|^2+\lambda r_0^2}$$

$$- \tfrac{1}{2}\Big\{ \tfrac{1}{\kappa}\sum_k \Big(\tfrac{\lambda r_0(\rho_0-\sqrt{\kappa}r_0)}{|a_k|^2+\lambda r_0^2}\Big)^2 + \sum_{q\neq 0}\tfrac{1}{\kappa}\sum_k \tfrac{\lambda\bar\gamma}{|a_k|^2+\lambda r_0^2}\,\tfrac{\lambda\bar\gamma}{|a_{k-q}|^2+\lambda r_0^2}\,\phi_q\,\phi_{-q}$$

$$+ \tfrac{1}{\kappa}\sum_k \Big(\tfrac{\lambda r_0(\rho_0-\sqrt{\kappa}r_0)}{|a_k|^2+\lambda r_0^2}\Big)^2 + \sum_{q\neq 0}\tfrac{1}{\kappa}\sum_k \tfrac{\lambda\gamma}{|a_k|^2+\lambda r_0^2}\,\tfrac{\lambda\gamma}{|a_{k-q}|^2+\lambda r_0^2}\,\bar\phi_q\,\bar\phi_{-q} \tag{51}$$

$$-\tfrac{\lambda}{\kappa}\sum_k \tfrac{|a_k|^2}{(|a_k|^2+\lambda r_0^2)^2}(\rho_0-\sqrt{\kappa}r_0)^2 - \sum_{q\neq 0}\tfrac{\lambda}{\kappa}\sum_k \tfrac{\bar a_k}{|a_k|^2+\lambda r_0^2}\,\tfrac{a_{k+q}}{|a_{k+q}|^2+\lambda r_0^2}\,\rho_q^2$$

$$-\tfrac{\lambda}{\kappa}\sum_k \tfrac{|a_k|^2}{(|a_k|^2+\lambda r_0^2)^2}(\rho_0-\sqrt{\kappa}r_0)^2 - \sum_{q\neq 0}\tfrac{\lambda}{\kappa}\sum_k \tfrac{a_k}{|a_k|^2+\lambda r_0^2}\,\tfrac{\bar a_{k-q}}{|a_{k-q}|^2+\lambda r_0^2}\,\rho_q^2\Big\} + O(\xi^3)$$

Using the BCS equation (5), $\frac{\lambda}{\kappa}\sum_k \frac{1}{|a_k|^2+\lambda r_0^2} = 1$ and abbreviating

$$E_k^2 := |a_k|^2 + \lambda r_0^2 = k_0^2 + e_{\mathbf{k}}^2 + \lambda r_0^2$$

this becomes

$$2\sqrt{\kappa}r_0(\rho_0-\sqrt{\kappa}r_0)\cdot 1 - (\rho_0-\sqrt{\kappa}r_0)^2\tfrac{\lambda}{\kappa}\sum_k \tfrac{\lambda r_0^2-|a_k|^2}{E_k^4} + \sum_{q\neq 0}\rho_q^2\Big\{\tfrac{\lambda}{\kappa}\sum_k \tfrac{\bar a_k a_{k-q}}{E_k^2E_{k-q}^2}\Big\}$$

$$-\sum_{q\neq 0}\mathrm{Re}\left(e^{-2i\theta_0}\phi_q\phi_{-q}\right)\Big\{\tfrac{\lambda}{\kappa}\sum_k \tfrac{\lambda r_0^2}{E_k^2E_{k-q}^2}\Big\}$$

$$= 2\sqrt{\kappa}r_0(\rho_0-\sqrt{\kappa}r_0)\cdot 1 + (\rho_0-\sqrt{\kappa}r_0)^2 - (\rho_0-\sqrt{\kappa}r_0)^2 2\tfrac{\lambda}{\kappa}\sum_k \tfrac{\lambda r_0^2}{E_k^4}$$

$$+\sum_{q\neq 0}\rho_q^2\Big\{\tfrac{\lambda}{\kappa}\sum_k \tfrac{\bar a_k a_{k-q}}{E_k^2E_{k-q}^2}\Big\} - \sum_{q\neq 0}\mathrm{Re}\left(e^{-2i\theta_0}\phi_q\phi_{-q}\right)\Big\{\tfrac{\lambda}{\kappa}\sum_k \tfrac{\lambda r_0^2}{E_k^2E_{k-q}^2}\Big\}$$

$$= \rho_0^2 - \kappa r_0^2 - (\rho_0-\sqrt{\kappa}r_0)^2 2\tfrac{\lambda}{\kappa}\sum_k \tfrac{\lambda r_0^2}{E_k^4}$$

$$+\sum_{q\neq 0}\rho_q^2\Big\{\tfrac{\lambda}{\kappa}\sum_k \tfrac{\bar a_k a_{k-q}}{E_k^2E_{k-q}^2}\Big\} - \sum_{q\neq 0}\mathrm{Re}\left(e^{-2i\theta_0}\phi_q\phi_{-q}\right)\Big\{\tfrac{\lambda}{\kappa}\sum_k \tfrac{\lambda r_0^2}{E_k^2E_{k-q}^2}\Big\} \tag{52}$$

Therefore one obtains, recalling that $\xi_{k,p} = \frac{1}{\sqrt{\kappa}}\,\phi_{k-p} - \gamma\,\delta_{k,p}$,

$$V(\{\phi_q\}) - V(\{\sqrt{\kappa}\,\delta_{q,0}\,r_0 e^{i\theta_0}\}) = \sum_q \rho_q^2 - \kappa\,r_0^2 \;-\; \log\det\left[Id + \begin{pmatrix} \lambda\frac{\bar\gamma}{|a|^2+\lambda r_0^2}\,\xi & ig\frac{\bar A}{|a|^2+\lambda r_0^2}\,\xi^* \\ ig\frac{A}{|a|^2+\lambda r_0^2}\,\xi & \lambda\frac{\gamma}{|a|^2+\lambda r_0^2}\,\xi^* \end{pmatrix}\right]$$

$$
\begin{aligned}
&= \sum_{q\neq 0} \rho_q^2 + (\rho_0 - \sqrt{\kappa} r_0)^2 2\tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^4} - \sum_{q\neq 0} \rho_q^2 \left\{ \tfrac{\lambda}{\kappa} \sum_k \tfrac{\bar{a}_k a_{k-q}}{E_k^2 E_{k-q}^2} \right\} \\
&\quad + \sum_{q\neq 0} \mathrm{Re}\left(e^{-2i\theta_0} \phi_q \phi_{-q}\right) \left\{ \tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^2 E_{k-q}^2} \right\}
\end{aligned} \tag{53}
$$

Consider the coefficient of $\sum_{q\neq 0} \rho_q^2$. It is given by

$$
\begin{aligned}
1 - \tfrac{\lambda}{\kappa} \sum_k \tfrac{\bar{a}_k a_{k-q}}{E_k^2 E_{k-q}^2} &= \tfrac{1}{2}(1+1) - \tfrac{1}{2}\tfrac{\lambda}{\kappa} \sum_k \tfrac{2\bar{a}_k a_{k-q}}{E_k^2 E_{k-q}^2} \\
&= \tfrac{1}{2}\left(\tfrac{\lambda}{\kappa} \sum_k \tfrac{|a_k|^2 + \lambda r_0^2}{E_k^2 E_{k-q}^2} + \tfrac{\lambda}{\kappa} \sum_k \tfrac{|a_{k-q}|^2 + \lambda r_0^2}{E_k^2 E_{k-q}^2} \right) - \tfrac{1}{2}\tfrac{\lambda}{\kappa} \sum_k \tfrac{2\bar{a}_k a_{k-q}}{E_k^2 E_{k-q}^2} \\
&= \tfrac{1}{2}\tfrac{\lambda}{\kappa} \sum_k \tfrac{a_k \bar{a}_k - a_k \bar{a}_{k-q} - \bar{a}_k a_{k-q} + a_{k-q}\bar{a}_{k-q}}{E_k^2 E_{k-q}^2} + \tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^2 E_{k-q}^2} - \tfrac{1}{2}\tfrac{\lambda}{\kappa} \sum_k \tfrac{\bar{a}_k a_{k-q} - a_k \bar{a}_{k-q}}{E_k^2 E_{k-q}^2} \\
&= \tfrac{1}{2}\tfrac{\lambda}{\kappa} \sum_k \tfrac{(a_k - a_{k-q})(\bar{a}_k - \bar{a}_{k-q})}{E_k^2 E_{k-q}^2} + \tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^2 E_{k-q}^2} - i\tfrac{\lambda}{\kappa} \sum_k \tfrac{\mathrm{Im}(\bar{a}_k a_{k-q})}{E_k^2 E_{k-q}^2} \\
&= \tfrac{1}{2}\tfrac{\lambda}{\kappa} \sum_k \tfrac{q_0^2 + (e_{\mathbf{k}} - e_{\mathbf{k-q}})^2}{E_k^2 E_{k-q}^2} + \tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^2 E_{k-q}^2} - i\tfrac{\lambda}{\kappa} \sum_k \tfrac{k_0 e_{\mathbf{k-q}} - (k_0 - q_0) e_{\mathbf{k}}}{E_k^2 E_{k-q}^2} \\
&= \alpha_q + i\gamma_q + \beta_q
\end{aligned} \tag{54}
$$

Inserting (54) in (53), one gets

$$
\begin{aligned}
V(\{\phi_q\}) - V(\{\sqrt{\kappa}\,\delta_{q,0}\, r_0 e^{i\theta_0}\}) &= (\rho_0 - \sqrt{\kappa} r_0)^2 2\beta_0 + \sum_{q\neq 0} \rho_q^2 \left\{ 1 - \tfrac{\lambda}{\kappa} \sum_k \tfrac{2\bar{a}_k a_{k-q}}{E_k^2 E_{k-q}^2} \right\} \\
&\quad + \sum_{q\neq 0} \mathrm{Re}\left(e^{-2i\theta_0} \phi_q \phi_{-q}\right) \left\{ \tfrac{\lambda}{\kappa} \sum_k \tfrac{\lambda r_0^2}{E_k^2 E_{k-q}^2} \right\} \\
&= (\rho_0 - \sqrt{\kappa} r_0)^2 2\beta_0 + \sum_{q\neq 0} \rho_q^2 (\alpha_q + i\gamma_q) \\
&\quad + \sum_{q\neq 0} \rho_q^2 \beta_q + \sum_{q\neq 0} \tfrac{e^{-2i\theta_0} \phi_q \phi_{-q} + e^{2i\theta_0} \bar{\phi}_q \bar{\phi}_{-q}}{2} \beta_q
\end{aligned} \tag{55}
$$

Since $\beta_q = \beta_{-q}$, the last two q-sums in (55) may be combined to give

$$
\begin{aligned}
&\sum_{q\neq 0} \tfrac{\rho_q^2 + \rho_{-q}^2}{2} \beta_q + \sum_{q\neq 0} \tfrac{e^{-2i\theta_0} \phi_q \phi_{-q} + e^{2i\theta_0} \bar{\phi}_q \bar{\phi}_{-q}}{2} \beta_q \\
&\qquad = \tfrac{1}{2} \sum_{q\neq 0} (\phi_q \bar{\phi}_q + \phi_{-q} \bar{\phi}_{-q} + e^{-2i\theta_0} \phi_q \phi_{-q} + e^{2i\theta_0} \bar{\phi}_q \bar{\phi}_{-q}) \beta_q \\
&\qquad = \tfrac{1}{2} \sum_{q\neq 0} (e^{-i\theta_0} \phi_q + e^{i\theta_0} \bar{\phi}_{-q})(e^{i\theta_0} \bar{\phi}_q + e^{-i\theta_0} \phi_{-q}) \beta_q \\
&\qquad = \tfrac{1}{2} \sum_{q\neq 0} |e^{-i\theta_0} \phi_q + e^{i\theta_0} \bar{\phi}_{-q}|^2 \beta_q
\end{aligned} \tag{56}
$$

This proves Theorem 2 □

References

[1] J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz, *Fermionic Many Body Models*, in: CRM Proceedings and Lecture Notes Vol. 7, *Mathematical Quantum Theory I: Field Theory and Many Body Theory*, eds. J. Feldman, R. Froese, L. M. Rosen, 1994.

[2] J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz, *Ward Identities and a Perturbative Analysis of a U(1) Goldstone Boson in a Many Fermion System*, Helvetia Physica Acta 66, 1993, 498-550.

[3] T. Chen, J. Fröhlich, M. Seifert, *Renormalization Group Methods: Landau-Fermi Liquid and BCS Superconductor*, Proceedings of the Les Houches session *Fluctuating Geometries in Statistical Mechanics and Field Theory*, eds. F. David, P. Ginsparg, J. Zinn-Justin, 1994.

[4] J. Feldman, E. Trubowitz, *Perturbation Theory for Many Fermion Systems*, Helv. Phys. Acta 63, 1990, 156-260; *The Flow of an Electron Phonon System to the Superconducting State*, Helv. Phys. Acta 64, 1991, 214-357.

[5] D. Lehmann, *The Many-Electron System in the Forward, Exchange and BCS Approximation*, Communications in Mathematical Physics, 1998, to appear.