Splitting of a small separatrix loop near the saddle-center bifurcation in area-preserving maps

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Abstract

When the saddle-center bifurcation occurs in an analytic family of area-preserving maps, first a parabolic fixed point appears at the origin and then this point bifurcates, creating an elliptic and hyperbolic fixed point. Separatrices of the hyperbolic fixed point form a small loop around the elliptic point. In general the separatrices intersect transversaly and the splitting is exponentially small with respect to the perturbation parameter. We derive an asymptotic formula, which describes the splitting, and study the properties of the preexponential factor.

1 Introduction

It is well known, that the splitting of separatrices plays an important role in creation of chaotic motions in hamiltonian dynamics. For many interesting analytic systems this phenomenon is exponentially small with respect to a perturbation parameter. Under quite general conditions this is true for the following problems:

a) high frequency perturbations of autonomous systems;

b) a (simple) resonance in near integrable hamiltonian systems with 2 degrees of freedom;

c) close to identity maps;

d) bifurcations of periodic points in families of area-preserving maps.

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In some aspects these problems are equivalent. For example, for all of them an exponentially small upper bound for the splitting may be obtained from Neishtadt averaging theorem [14]. On the other hand lower bounds are quite sensitive to the form of the equations and in general a direct reduction is not possible. But methods, developed for one type of the problems, may be usually adapted to study the others.

In the case of the exponentially small splitting the standard tool—Poincaré-Arnold-Melnikov method [13, 9]—requires a special justification, and in some cases its naive application gives no correct estimates for the splitting. In particular, this is true in the case of the Chirikov standard map

$$SM : (x, y) \mapsto (x + y + \varepsilon \sin x,\ y + \varepsilon \sin x)\,,$$

where ε is a small positive parameter. This map was widely used as a model for the motion near a resonance for hamiltonian systems with two degrees of freedom [3]. After a proper scaling in the vertical direction separatrices of the standard map look like the pendulum separatrix. Unlike the pendulum the separatrices of the standard map split [3, 10], i.e., they intersect each other transversaly. The splitting angle, α, at a symmetrical primary homoclinic point is exponentially small: $\alpha \approx \omega/4$,

$$\omega = \frac{4\pi|\Theta_1|}{h^2}\,\mathrm{e}^{-\pi^2/\mathrm{h}} + \mathrm{O}\left(\mathrm{e}^{-\pi^2/\mathrm{h}}\right)\,,$$

where $h \approx \sqrt{\varepsilon}$ is the logarithm of the multiplicator of the hyperbolic fixed point [12, 7, 6]. The factor Θ_1 is a remarkable part of this formula. Its value $|\Theta_1| = 1\,118.827\,765\,95...$ was found from a study of the semistandard map [12]. There are many independent methods for its evaluation (see e.g. [15, 16]), nevertheless its nature is not completely clear, i.e., it is not known if there is a finite expression for this constant in terms of classical constants like π, e, etc. A set of similar constants was found in the study of some other area-preserving maps [5, 8].

In the present paper we study the splitting of a small separatrix loop, which appears after the Bogdanov-Tackens (saddle-center) bifurcation in an analytic family of area-preserving maps. We discovered that in general the splitting is described by the same formula as in the case of the standard map, but with different values of the factor Θ_1. This constant describes the splitting of complex invariant manifolds associated to a parabolic fixed point. The constant may be interpreted as an analytic invariant (with respect to canonical coordinate changes) of the area-preserving map near the parabolic fixed point. It is very similar to classification constants of Ecalle's theory of resurgent functions [4]. Ecalle proved that for any value of the classification constant there is a corresponding analytic (tangent to identity) map. Moreover, one may arbitrarily prescribe for this map any finite number of Tailor coefficients. In particular, this implies that the classification is not defined by a finite number of Taylor coefficient, but instead depends on the whole map. Although the present case is not covered by original

Ecalle's work, since in our case the linear part of the map at the parabolic fixed point is different (non diagonalizable).

First, we describe the bifurcation and give the asymptotic formula for the splitting. Then we

2 Asymptotical formula for the splitting

We consider the dynamical system generated by iterations of an analytic area-preserving map of the form

$$F(\vec{x},\varepsilon) = F_0(\vec{x}) + \varepsilon F_1(\vec{x}) + \dots \qquad \vec{x} = (x,y)\,.$$

Let us assume that for $\varepsilon = 0$ a parabolic fixed point appears at the origin, and for small positive ε the point bifurcates giving rise to a pair of a hyperbolic and elliptic fixed point. Separatrices associated with the hyperbolic fixed point form a small loop around the elliptic fixed point (Fig. 1 represents a typical picture of separatrices). Although it is not visible on the picture, in general the stable and unstable separatrices do not coincide, but intersect transversaly. The splitting is exponentially small with respect to ε [1].

Let us formulate the assumptions on the function $F(\vec{x},\varepsilon)$. Let $U \subset \mathbf{C}^2$ be an open neighborhood of the origin and let $F(\vec{x},\varepsilon)$ be real-analytic in a complex neighborhood of $U \times \{|\varepsilon| < \varepsilon_0\}$. Moreover,

$$\begin{aligned} F_0(x,y) &= (x + y + \dots,\ y - b_2x^2 + \dots)\,, && (1)\\ F_1(0,0) &= (a_0, b_0)\,, && (2) \end{aligned}$$

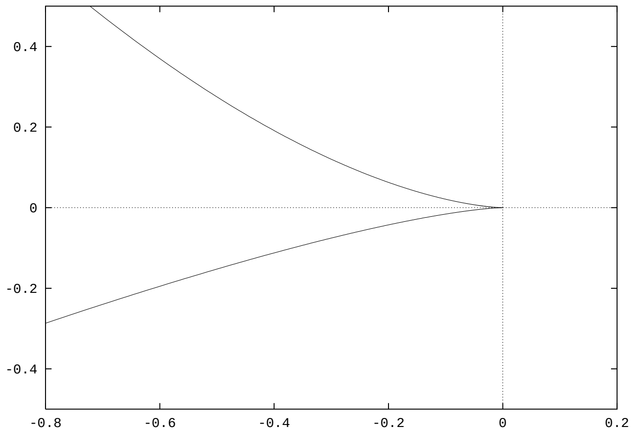
where the constants $b_0, b_2 > 0$. The dots stand for the sum of all quadratic and higher order terms of the Tailor expansion of F_0 except for the term proportional to x^2 in the second component, which was written explicitly.

Under these assumptions for any small positive ε the map $F(\vec{x},\varepsilon)$ has a hyperbolic fixed point $\vec{p}_\varepsilon = \left(-\sqrt{\varepsilon b_0/b_2},\ 0\right) + O(\varepsilon)$ with the multiplicators λ and λ^{-1}, $\lambda = 1 + \sqrt[4]{4\varepsilon b_0 b_2} + O(\sqrt{\varepsilon})$. In this problem

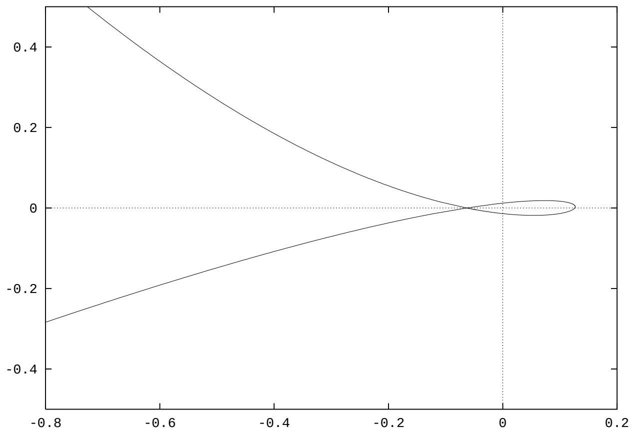
$$h = \log\lambda = \sqrt[4]{4b_0b_2\varepsilon} + O(\sqrt{\varepsilon}) \qquad (3)$$

is a natural small parameter. Using it instead of ε we simplify considerably the form of perturbation series.

We will study the splitting at a primary homoclinic point (a first intersection of the separatrices). To measure the splitting we use Lazutkin's homoclinic invariant, equal to the area of a parallelogram defined by two properly chosen vectors tangent to the stable and unstable separatrices at the homoclinic point [8, 7]. Its formal definition can be found in the next section. The value of the homoclinic invariant is proportional to the sinus of the splitting angle. The latter



a)



b)

Figure 1: Creation of a separatrix loop in the saddle-center bifurcation

is not a natural quantity for the hamiltonian dynamics. It is clear, that by the same method it is possible to get asymptotic formulas for other values, i.e. for splitting angles and areas of lobes bounded by separatrices segments between two homoclinic points (see e.g [6]).

Theorem 1 *For all sufficiently small positive ε the map $F(\vec{x},\varepsilon)$ has a primary homoclinic point with the homoclinic invariant*

$$\omega = \frac{4\pi|\Theta_1(F_0)|}{h^2} \mathrm{e}^{-2\pi^2/\mathrm{h}} + \mathrm{O}\left(\mathrm{e}^{-2\pi^2/\mathrm{h}}\right).$$

The proof of the theorem is quite similar to the case of the standard map described in details in [6]. So we only outline the main steps, and concentrate the attention on the factor $\Theta_1(F_0)$. Note, that the method also provides a complete asymptotic expansion for the homoclinic invariant, which has the form of the exponent times a series in powers of h^2. The formula above represents the main term of this expansion.

The area-preserving Hénon map

$$(x, y) \mapsto (x + y - x^2 + \varepsilon,\ y - x^2 + \varepsilon)$$

satisfies the conditions of the theorem. The splitting of the separatrices for the Hénon map was studied in [5, 8]. A high-precision value $|\Theta_1| = 2.474\,425\,593\,55 \cdot 10^6$ was obtained in [2].

The parallels with Ecalle's resurgent function theory [4] proposes the following conjecture.

Conjecture. *For any integer $n \geq 2$ and any complex constant C there is an area-preserving map $\tilde{F}_0$ with the same n-jet as F_0 and $\Theta_1(\tilde{F}_0) = C$.*

This conjecture is strongly supported by the numerical experiments described in the last section of the present paper.

3 Expansions for the separatrices

It is comparatively easy to check that the hyperbolic fixed point p_ε and the small parameter ε may be represented as convergent series in powers of h^2. The first terms of these expansions are provided in the previous section. According to Adomar-Perron theorem the hyperbolic fixed point has one dimensional stable and unstable manifolds, which are usually called separatrices. It is convenient to represent the separatrices in a parametric form using the solutions $\vec{x}(t) = \vec{x}^{\pm}(t)$ of the equation

$$\vec{x}(t + h) = F(\vec{x}(t), \varepsilon) \tag{4}$$

supplied with the boundary condition

$$\lim_{t\to\sigma\infty} \vec{x}^{\sigma}(t) = \vec{p}_\varepsilon\,, \qquad \sigma \in \{-,+\}\,.$$

The sign + corresponds to the stable separatrix and − corresponds to the unstable one. The parametrization is defined up to a substitution $t \mapsto t + t_0$, where t_0 is an arbitrary constant, provided one requires $2\pi \mathrm{i}$ periodicity of the parametrization in t or, equivalently, that $\vec{x}^{\sigma}(t)$ may be expanded in convergent series in powers of $\mathrm{e}^{-\sigma t}$. Standard arguments, based on the preservation of the area and on the first order approximation of the separatrices, show that separatrices do intersect. The freedom in the definition of the parametrization may be used to ensure that $t = 0$ corresponds to a primary homoclinic point, in particular, $\vec{x}^-(0) = \vec{x}^+(0)$.

The homoclinic invariant, ω, is defined by the equality $\omega = \Omega(\dot{\vec{x}}^-(0), \dot{\vec{x}}^+(0))$, where $\Omega = \mathrm{d}x \wedge \mathrm{d}y$ is the canonical area form on the plane and the dot denotes the derivative with respect to t. Since F is area-preserving, ω takes the same value at all points of one homoclinic trajectory, and its value does not change after canonical coordinate changes. Moreover, its definition is independent from the normalization of the separatrices parametrizations.

The homoclinic invariant is closely related with the splitting angle α:

$$\sin\alpha = \frac{\omega}{\|\dot{\vec{x}}^-(0)\| \, \|\dot{\vec{x}}^+(0)\|} \, .$$

An approximate expression for the denominator may be easily found after finding the first terms of the series (5).

It is not difficult to see by the standard scaling, that the height of the separatrix loop on Fig. 1 b) is of the order of h^2, while its width is of the order of h^3. This proposes the following form for the expansion for $\vec{x}(t) = (x, y)(t, h)$

$$x(t,h) = h^2 \sum_{k=0}^{\infty} h^k X_k(t) \, , \qquad y(t,h) = h^3 \sum_{k=0}^{\infty} h^k Y_k(t) \, . \tag{5}$$

Substituting these series into the equation (4), taking into account (3) and collecting the terms of the main order in h we obtain

$$\dot{X}_0 = Y_0 \, , \qquad \dot{Y}_0 = \frac{1}{4b_2} - b_2 X_0^2 \, .$$

This equation is nonlinear and has a homoclinic solution

$$X_0(t) = \frac{1}{2b_2}\left(\frac{3}{\cosh^2(t/2)} - 1\right) , \qquad Y_0(t) = -\frac{3}{2b_2}\frac{\sinh(t/2)}{\cosh^3(t/2)} \, . \tag{6}$$

Collecting the terms, which have equal order in h, it is not difficult to check that all other equations may be written in the form

$$\dot{X}_k = Y_k + g_1 \, , \qquad \dot{Y}_k = \left(\frac{3}{\cosh^2(t/2)} - 1\right) X_k + g_2 \, , \qquad k \geq 1 \, ,$$

where the functions g_1 and g_2 are polynomial in X_l, Y_l, $1 \leq l \leq k-1$, and their derivatives. These equations are linear with respect to (X_k, Y_k) and may be solved explicitly. The system has a solution, such that $X_k(t)$ converge to constants as $t \to -\infty$. It may be shown that the solution converge to the same constant as $t \to +\infty$. Consequently, any partial sum of the series (5) converges to a constant as $t \to -\infty$ and $t \to +\infty$. If we retain a finite number of terms, we get a closed loop. This loop approximates the actual separatrix with the error of the order of the first skipped term. We say that the series (5) represent a "formal" separatrix. In particular, this shows that the splitting is less than any power of the small parameter.

In order to detect the splitting we study analytical continuation of the separatrices. In the case of the standard map this procedure was described with more details in [10, 6]. The series (5) also approximate the analytical continuation of the parametrizations on the complex values of t. When t approaches the singularities of the coefficients the error increases. The coefficients are 2πi-periodic and real-analytic. Because of these symmetries it is sufficient to study the separatrices in a neighborhood of one singularity only, namely $t_* = \mathrm{i}\pi$. It is convenient to introduce a new parameter

$$\tau = \frac{t - \mathrm{i}\pi}{h} . \tag{7}$$

Then we look for the separatrices in the form

$$x(t,h) = \sum_{k=0}^{\infty} h^{2k} x_k(\tau) , \qquad y(t,h) = \sum_{k=0}^{\infty} h^{2k} y_k(\tau) . \tag{8}$$

Inserting these series into the equation (4) and collecting the terms of the order zero in h we obtain

$$\vec{x}_0(\tau + 1) = F_0(\vec{x}_0(\tau)) . \tag{9}$$

This is a nonlinear finite-difference equation. The higher order terms satisfy linear finite-difference equations. These equation have a lot of solutions and we have to choose the solutions related to the series (5) via a "complex matching" principle, i.e., an expansion of the series (8) as τ goes to infinity should coincide with the expansion of the series (5) as t goes to iπ. This principle defines two solutions, $\vec{x}_0^-$ and $\vec{x}_0^+$, which are different in general. They satisfy the sewing condition when $\mathrm{Re}\,\tau \to -\infty$ and $\mathrm{Re}\,\tau \to +\infty$ respectively. Both of them have the same asymptotic as $\mathrm{Im}\,\tau \to \pm\infty$. We give more details in the next section. One of the solutions approximates the stable separatrix the other one approximates the unstable one.

How one can use these approximations to estimate the splitting of real separatrices? We consider the normal component of the splitting:

$$\tilde{\theta}(t,h) = \det \begin{pmatrix} \dot{x}^- & x^+ - x^- \\ \dot{y}^- & y^+ - y^- \end{pmatrix} (t,h) . \tag{10}$$

Straightforward differentiation shows that the derivative of this function at zero coincides with the homoclinic invariant, $\dot{\tilde{\theta}}(0,h) = \omega$.

The function $\tilde\theta(t,h)$ is very close to a periodic function. A straightforward calculation shows that $\tilde\theta(t+h,h)-\tilde\theta(t,h) = O(\|\vec\delta(t)\|^2)$. The right hand side is an exponentially small value of the second order, if we say that $\vec\delta$ is an exponentially small value of the first order. Indeed, $\tilde\theta(t,h) = \det(\vec\delta^-(t),\vec\delta(t))$, where

$$\vec\delta^-(t) = \dot{\vec x}^-(t)\,, \qquad \vec\delta(t) = \vec x^+(t) - \vec x^-(t)\,.$$

Note, that $\vec\delta^-$ is a solution of the variational equation

$$\vec\delta(t+h) = F'(\vec x^-(t))\,\vec\delta(t)$$

and $\vec\delta$ is an approximate solution for the same equation with the residual quadratic in $\vec\delta$. Then it is necessary to take into account that $\det F' = 1$, because of the area preservation.

In a suitably chosen coordinate system the map F takes the form of a translation in the direction of the first axis: $(T,E)\mapsto(T+h,E)$. It may be shown that the domain of these coordinates is large enough to justify the following arguments. In this coordinate system the splitting is described by an h-periodic function $\theta(t,h) = \tilde\theta(t,h) + O(\|\vec\delta(t)\|^2)$. The existence of the function θ follows from the theorem on analytic integral. In the case of the standard map this theorem was first formulated by V. Lazutkin [10] as a conjecture and recently proved in a form slightly different from the original one [6]. Moreover, for real t

$$\theta(t,h) = 2|\Theta_1(F_0)|h^{-1}\,\mathrm{e}^{-2\pi^2/\mathrm{h}}\sin\frac{2\pi\mathrm{t}}{\mathrm{h}} + \mathrm{O}\left(\mathrm{h}\mathrm{e}^{-2\pi^2/\mathrm{h}}\right)\,. \qquad (11)$$

This estimate may be derive by analyzing Fourier coefficients of the function θ. To do that we consider analytical continuation of the functions. The width of the domain, where we can control the function θ, is related with the positions of the singularities of the unperturbed separatrix (6) at $t=\pm\mathrm{i}\pi$. Since the period h of the function θ is small with respect to the width of its analyticity domain, its Fourier coefficients decay rapidly as function of their number and only the first one should be taken into consideration.

On the segment $\mathrm{Im}\,t = \pi - \frac{\sigma}{2\pi}h\log h^{-1}$, $|\,\mathrm{Re}\,t| < h$, the functions $\vec x^\pm$ in (10) may be replaced by the solutions $\vec x_0^\pm$ of the equation (9). So we have

$$\theta(t,h) \approx= \det\begin{pmatrix} \frac{\mathrm{d}x_0^-}{\mathrm{d}\tau} & x_0^+ - x_0^- \\ \frac{\mathrm{d}y_0^-}{\mathrm{d}\tau} & y_0^+ - y_0^- \end{pmatrix}(\tau) = h^{-1}\Theta_1(F_0)\mathrm{e}^{-2\pi\mathrm{i}\tau} + \mathrm{O}(\mathrm{h}^{1+\sigma})\,,$$

where the factor h^{-1} is due to the change of the parameter (7). The second equality follows from Proposition 3 from the next section. From this estimate we conclude that the first Fourier mode of θ is approximately

$$h^{-1}\Theta_1(F_0)\mathrm{e}^{-2\pi^2/\mathrm{h}}\mathrm{e}^{-2\pi\mathrm{i}t/\mathrm{h}}\,.$$

Taking into account the real analyticity of θ and $\theta(0,h) = 0$ we obtain the exponentially fine estimate (11). The asymptotic formula for the homoclinic invariant is obtained by the differentiation of this equality and the substitution $t=0$.

4 Definition and properties of $\Theta_1(F_0)$

The constant $\Theta_1(F_0)$ is defined from the study of the separatrices associated with the origin, which is a fixed point of the map F_0. Although this point is not hyperbolic, it still has a stable and unstable manifold. It is convenient to represent these manifolds in a parametric form using solutions $\vec{x}_0 = \vec{x}_0^{\pm}$ of the equation

$$\vec{x}_0(\tau + 1) = F_0(\vec{x}_0(\tau)) \tag{12}$$

satisfying the boundary condition

$$\lim_{\tau\to\pm\infty} \vec{x}_0^{\pm}(\tau) = 0\,. \tag{13}$$

The parametrization conjugate the restriction of F_0 to an invariant manifold with the shift $\tau \mapsto \tau + 1$. In the terms of the variable τ the fixed point lies at infinity.

The problem (12), (13) has a solution in the class of formal power series of the form $\vec{x}_0(\tau) = (x_0(\tau), y_0(\tau))$, where

$$x_0(\tau) \sim \sum_{k=2}^{\infty} \frac{x_{0k}}{\tau^k}\,, \qquad y_0(\tau) \sim \sum_{k=3}^{\infty} \frac{y_{0k}}{\tau^k}\,. \tag{14}$$

The coefficient x_{03} may be considered as a free parameter. If it is fixed the series are unique. Note, that if $\vec{x}_0(\tau)$ satisfies the equation (12), then $\vec{x}_0(\tau + \tau_0)$ is a solution too. In the class of formal power series this translation may be replaced by the action of the operator $\exp(\tau_0\partial/\partial\tau)$. This operator leaves unchanged x_{02}, but changes the others. Z Consider the sectors $D_- = \{\, \tau \in \mathbf{C} : |\arg(\tau + A)| > \delta \,\}$ and D_+, which is the reflection of D_- with respect to the imaginary axis. Here A is a sufficiently large positive constant and $\delta \in (0, \pi/2)$ is an arbitrary fixed number.

Lemma 2 *The equation (12) has two analytic solutions, $\vec{x}_0^{\pm}$, defined respectively in the sectors D_- and D_+, such that the series (14) are asymptotic to the solutions on the infinity in the corresponding sectors. These two solutions are unique. On the intersection of the sectors, $\tau \in D_+ \cap D_-$,*

$$\|\vec{x}_0^{+}(\tau) - \vec{x}_0^{-}(\tau)\| = O\left(\mathrm{e}^{-(2\pi-\mu)|\,\mathrm{Im}\,\tau|}\right)\,. \tag{15}$$

The estimate holds for any $\mu > 0$, the constant in the estimate of the error depends on μ, A and δ.

The proof of the lemma is similar to [11].

The functions $\vec{x}_0^-$ and $\vec{x}_0^+$ represent the stable and unstable separatrices associated with the parabolic fixed point of the map F_0. Their restrictions on the real values of τ are shown on Fig. 1 a). In general their analytic continuations do not coincide. To measure the difference between these continuations it is convenient to consider the determinant

$$\hat{\Theta}(\tau) = \det \begin{pmatrix} \frac{\mathrm{d}x_0^-}{\mathrm{d}\tau} & x_0^+ - x_0^- \\ \frac{\mathrm{d}y_0^-}{\mathrm{d}\tau} & y_0^+ - y_0^- \end{pmatrix} (\tau) \,, \tag{16}$$

which represents the splitting component normal to the unstable separatrix. Let define

$$\Theta_1(F_0) \stackrel{\mathrm{def}}{=} \lim_{\operatorname{Im}\tau \to -\infty} \mathrm{e}^{2\pi \mathrm{i}\tau} \hat{\Theta}(\tau) \,. \tag{17}$$

The existence of the limit follows from Proposition 3 below.

The constant $\Theta_1(F_0)$ has the following geometrical meaning. The stable and unstable separatrices of the parabolic fixed point are one dimensional complex curves, which rotate one around the other in the two-dimensional complex phase space. The coefficient $\Theta_1(F_0)$ is a properly normalized amplitude of these rotations. We visualize this by looking on the lines $\operatorname{Im}\tau = \text{const}$, which form invariant foliations of the stable and unstable separatrices, and projecting them on the first coordinate. On Fig. 2a three lines are shown, they correspond to three different values of $\operatorname{Im}\tau$ on the unstable curve of the parabolic Henón map $(x, y) \mapsto (x + y - x^2,\, y - x^2)$. Because of the symmetries the foliation of the stable separatrix may be obtained by a reflection with respect to the real axis and it is not shown on the figure. Figure 2b represents the same picture for an integrable map (time one map of a polynomial hamiltonian flow). Of course, in the last case $\Theta_1 = 0$.

Proposition 3 *The function $\Theta_1(F_0)$ is asymptotically periodic in the sector $-\pi - \delta < \arg\tau < -\delta$, that is for any $\mu > 0$*

$$\hat{\Theta}(\tau) = \Theta_1(F_0)\mathrm{e}^{-2\pi \mathrm{i}\tau} + \mathrm{O}\left(\mathrm{e}^{-(4\pi-\mu)|\operatorname{Im}\tau|}\right) \,.$$

Proof. Let $\vec{\xi}_0(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{x}_0^-(\tau)$ and $\vec{\xi}(\tau) = \vec{x}_0^+(\tau) - \vec{x}_0^-(\tau)$. In these notations $\hat{\Theta} = \det\left(\vec{\xi}_0; \vec{\xi}\right)$. It follows from the equation (12) that

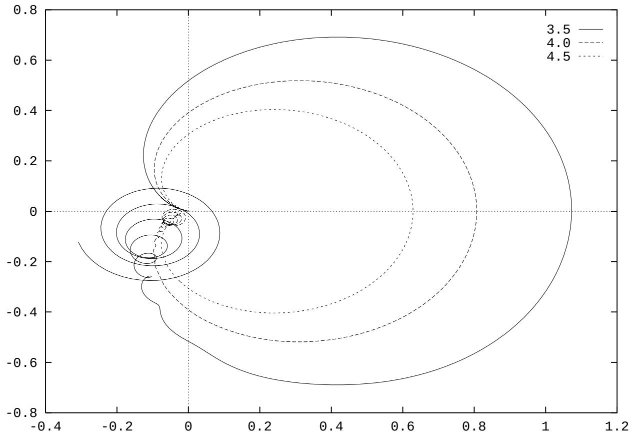
$$\begin{aligned} \vec{\xi}_0(\tau+1) &= F_0'(\vec{x}_0^-)\,\vec{\xi}_0 \,, \\ \vec{\xi}(\tau+1) &= F_0'(\vec{x}_0^-)\,\vec{\xi} + \left(F_0(\vec{x}_0^- + \vec{\xi}) - F_0(\vec{x}_0^-) - F_0'(\vec{x}_0^-)\,\vec{\xi}\right) \,, \end{aligned}$$

where we omitted the argument τ on some of the functions. Since F_0 is area preserving $\det F_0' = 1$ and we have

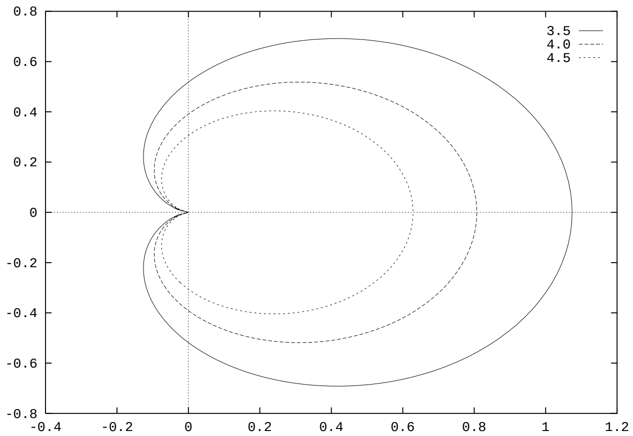
$$\hat{\Theta}(\tau+1) = \hat{\Theta}(\tau) + \det\left(\vec{\xi}_0;\; F_0(\vec{x}_0^- + \vec{\xi}) - F_0(\vec{x}_0^-) - F_0'(\vec{x}_0^-)\,\vec{\xi}\right) \,.$$

This may be considered as a linear nonhomogeneous equation on $\hat{\Theta}$:

$$\hat{\Theta}(\tau+1) - \hat{\Theta}(\tau) = g(\tau) \,.$$



a)



b)

Figure 2: The projections of the invariant foliations of the unstable separatrix: a) for the Henón map, b) for the time one map of an autonomous Hamiltonian flow with $H = \frac{y^2}{2} + \frac{x^3}{3}$

It follows from the estimate (15) that $g(\tau) = O\left(\mathrm{e}^{-(4\pi-2\mu)|\operatorname{Im}\tau|}\right)$. The equation has a particular analytic solution, $G(\tau)$, which affords the estimate of the form $G(\tau) = O\left(\mathrm{e}^{-(4\pi-3\mu)|\operatorname{Im}\tau|}\right)$ (comp. with [11]). The difference $\hat{\Theta}(\tau) - G(\tau)$ satisfies the corresponding homogeneous equation, i.e., it is periodic. A periodic function, which decreases in the sector (the lower connected component of $D_- \cap D_+$), may be replaced by its first Fourier coefficient with the error $O\left(\mathrm{e}^{-4\pi|\operatorname{Im}\tau|}\right)$. We denote this Fourier coefficient by $\Theta_1(F_0)$,

$$\Theta_1(F_0) = \int_{\tau_0}^{\tau_0+1} \mathrm{e}^{2\pi\mathrm{i}\tau}\left(\hat{\Theta}(\tau) - \mathrm{G}(\tau)\right)\,\mathrm{d}\tau\,, \tag{18}$$

where the integral does not depend on the choice of $\tau_0 \in D_- \cap D_+$, $\operatorname{Im}\tau < 0$. We have

$$\begin{aligned}\hat{\Theta}(\tau) &= G(\tau) + \Theta_1(F_0)\mathrm{e}^{-2\pi\mathrm{i}\tau} + \mathrm{O}\left(\mathrm{e}^{-4\pi|\operatorname{Im}\tau|}\right)\\ &= \Theta_1(F_0)\mathrm{e}^{-2\pi\mathrm{i}\tau} + \mathrm{O}\left(\mathrm{e}^{-(4\pi-3\mu)|\operatorname{Im}\tau|}\right)\,.\end{aligned}$$

Since μ is an arbitrary positive constant we may replace μ by $\mu/3$ to obtain the desired representation for $\hat{\Theta}$. □

Proposition 4 *The definition of $\Theta_1(F_0)$ is invariant with respect to analytic symplectic changes of coordinates.*

The proof of this proposition is straightforward.

It is interesting to know how the constant Θ_1 depends on the function F_0. It is easy to see that by a linear (noncanonical) the Taylor series coefficient b_2 of may be turned into 1.

Proposition 5 *Let $F_0(x, y, r)$ be an analytic family of the maps of the form (1) with the value of b_2 independent of r. Then Θ_1 is an analytic function of r.*

Proof. Following the proof of Lemma 2 it is not too difficult to check, that $\vec{x}_0$ depends analytically on the additional parameter r. Then the function g and, consequently, $G(\tau)$, defined in the proof of Lemma 3, also depend on r analytically. Finally, the analyticity of Θ_1 follows from the expression (18) and the analyticity of the functions under the integral. □

It is well known that any map F_0 of the form (1) may be transformed by a formal canonical coordinate change to the standard form

$$GSM_0 : (x, y) \mapsto (x + y + f(x),\ y + f(x))\,, \tag{19}$$

where $f(x) = -bx^2 + \ldots$ is a power series in x with real coefficients, $b = b_2$.

Proposition 6 *Let $x_{03} = 0$. For any analytic standard-like map $\Theta_1(GSM_0)$ is a purely imaginary complex number.*

Proof. We use the reversibility of the standard map. There is an involution, called *reversor*, which conjugates the map with its inverse. One of GSM_0 reversors has the form

$$R : (x, y) \mapsto (x, -y - f(x)) .$$

The reversor maps the stable separatrix into the unstable one. In the case of the standard map $x_{03} = 0$ is a natural choice for (14). Indeed, in this case the system (12) is equivalent to a single equation

$$x_0(\tau + 1) - 2x_0(\tau) + x_0(\tau - 1) = f(x_0(\tau)) .$$

If $x_0(\tau)$ is a solution for this equation then $x_0(-\tau)$ also complies with it. If $x_{03} = 0$ then the expansion (14) contains even power only, because of the uniqueness of the formal solution. The reversor is identical on the first component, therefore the parametrizations of the separatrices are connected by the relation $\vec{x}_0^-(\tau) = R(\vec{x}_0^+(-\tau))$, which implies the following set of equalities

$$\begin{aligned} x_0^-(\tau) &= x_0^+(-\tau) , \\ y_0^-(\tau) &= -y_0^+(-\tau) - f(x_0^+(-\tau)) . \end{aligned}$$

Differentiating these equations we obtain

$$\begin{aligned} \frac{\mathrm{d}x_0^-}{\mathrm{d}\tau}(\tau) &= -\frac{\mathrm{d}x_0^+}{\mathrm{d}\tau}(-\tau) , \\ \frac{\mathrm{d}y_0^-}{\mathrm{d}\tau}(\tau) &= \frac{\mathrm{d}y_0^+}{\mathrm{d}\tau}(-\tau) + f'(x_0^+(-\tau))\frac{\mathrm{d}x_0^+}{\mathrm{d}\tau}(-\tau) . \end{aligned}$$

Substituting these equalities into the definition of $\hat{\Theta}(\tau)$ we have

$$\begin{aligned} \hat{\Theta}(\tau) &= -\det\begin{pmatrix} \frac{dx_0^+}{\mathrm{d}\tau} & x_0^+ - x_0^- \\ \frac{dy_0^+}{\mathrm{d}\tau} & y_0^+ - y_0^- \end{pmatrix}(-\tau) \\ & \quad -\frac{dx_0^+}{\mathrm{d}\tau}(f(x_0^+) - f(x_0^-)) + f'(x_0^+)\frac{dx_0^+}{\mathrm{d}\tau}(x_0^+ - x_0^-) . \end{aligned}$$

Taking into account (15) and real analyticity of the parametrizations we obtain

$$\hat{\Theta}(\tau) = -\overline{\hat{\Theta}}(-\overline{\tau}) + O\left(e^{-2(2\pi-\mu)|\operatorname{Im}\tau|}\right) .$$

Multiplying this equality by $\mathrm{e}^{2\pi\mathrm{i}\tau}$ and going to the limit $\operatorname{Im}\tau \to -\infty$ following the imaginary axis, we obtain

$$\Theta_1 = -\overline{\Theta}_1 ,$$

which is equivalent to $\operatorname{Re}\Theta_1 = 0$. $\square$

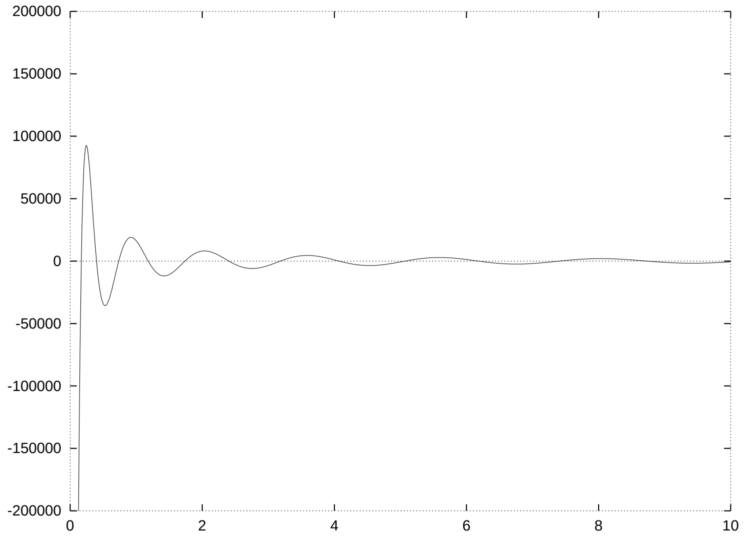


Figure 3: The plot of $|\Theta_1(r)|$ for the family (19) with $f(x) = -x^2 + rx^3$

5 Numerical experiments

For a given map it is possible to compute the constant Θ_1 by a method similar to the methods used in [12, 7]. This method is based on the following observation: $\vec{x}_0(\tau) = F_0^n(\vec{x}_0(\tau - n))$ for any integer n. Taking sufficiently large $|n|$ we can use asymptotic series (14) to evaluate with high accuracy the function $\vec{x}_0(\tau - n)$ and its derivative, then we apply $|n|$ times the map F_0 and its derivative F_0', or, in the case of the negative n, their inverse. We use positive n to get points of the unstable separatrix and negative n in the case of the stable one. Then we evaluate $\hat{\Theta}$ using its definition (16).

According to Proposition 3 the product $\mathrm{e}^{2\pi \mathrm{i}\tau}\hat{\Theta}(\tau)$ approaches the limit exponentially fast as $\operatorname{Im}\tau$ becomes large negative. On the other hand the inevitable errors in the computation of $\hat{\Theta}$ are multiplied by the increasing factor $\mathrm{e}^{2\pi \mathrm{i}\tau}$. The optimal value corresponds to the case, when the contributions of these two sources of errors are of the same order. In this way it is possible to compute 6-8 correct digits using the complex arithmetic with 16 decimals.

As an example we show the dependence of $\operatorname{Im}\Theta_1(r)$ for the generalized standard map (19) with $f(x) = -x^2 + rx^3$ on Fig. 3. For $r = 0$ this map converts to the Henón map with a parabolic fixed point. For the first time the constant $|\Theta_1(0)| = 2.47 \cdot 10^6$ was obtained in [5]. The numerical experiments show, that for negative r there are no oscillations and $\operatorname{Im}\Theta_1$ is a monotone increasing function, which goes very fast to $-\infty$, when r becomes more and more negative.

The dependence of $\operatorname{Im}\Theta_1(r)$ for the map (19) with $f(x) = -x^2 + rx^4$ is shown on Fig. 4.

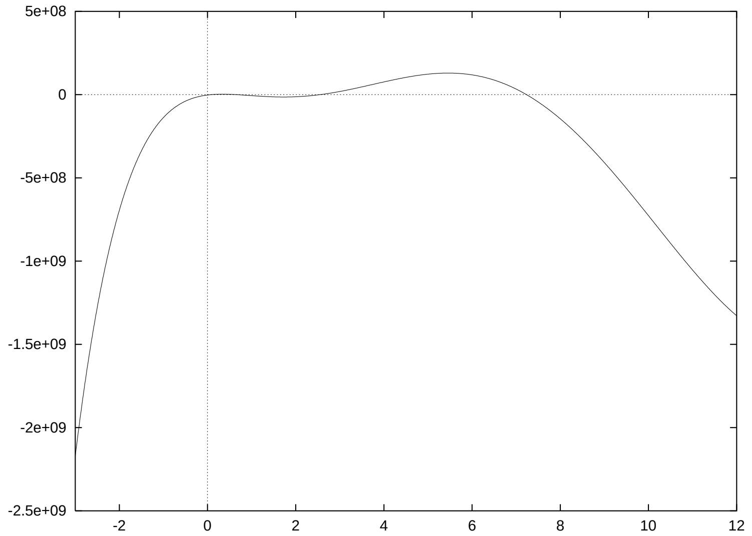


Figure 4: The plot of $|\Theta_1(r)|$ for the family (19) with $f(x) = -x^2 + rx^4$

A remarkable property of these two examples is extremely fast changes of Θ_1 as a function of r (note the scales on the figures).

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