STRICT INEQUALITY FOR CRITICAL PERCOLATION VALUES IN FRUSTRATED RANDOM CLUSTER MODELS

Massimo Campanino[1 2]
Dipartimento di Matematica, Università degli Studi di Bologna,
piazza di Porta S.Donato 5, I-40126 Bologna, Italy
e-mail: campanin@dm.unibo.it

Abstract. We establish an inequality between the critical percolation value of frustraded random cluster models and that of an unfrustrated model where the free occupation probabilities of some bonds are strictly decreased. Using the FK representation this gives an inequality between the critical temperatures of the corresponding spin models. In this way we can prove a strict inequality between critical temperatures for symmetry breaking of some frustrated Ising models and the corresponding ferromagnetic models.

[1] Work supported by Italian G. N. A. F. A and EC grant SC1-CT91-0695.
[2] Investigation supported by University of Bologna. Funds for selected research topics.

§1. Introduction, notation and statement of the results.

Notation.

Let us introduce some notation. In this paper we will work on the lattice Z^d with $d \geq 2$. Elements of $\boldsymbol{Z}^d$ will be called *sites*. On $\boldsymbol{Z}^d$ we shall use the distance $d(x,y) = |x-y| = \sup_i |x_i - y_i|$. Unordered pairs of nearest neighbour sites of $\boldsymbol{Z}^d$ will be called *bonds*. Given a subset $\Lambda \subset \boldsymbol{Z}^d$ we denote by $\hat{\Lambda}$ the set of bonds made up with sites in Λ. Given a set R of bonds $R \subset \hat{\boldsymbol{Z}}^d$ we denote by $\tilde{R}$ the set of sites belonging to some bond $b \in R$. We need also to consider ordered pairs of nearest neighbour sites of $\boldsymbol{Z}^d$: these will be called *ordered bonds*. Given an ordered bond $b = (x,y)$ we denote by $-b$ the ordered bond (y,x). A sequence $p = (b_1, \ldots, b_k)$ of ordered bonds such that for $1 \leq i \leq k-1$ the second site of b_i is equal to the first site of b_{i+1} is called a *path*; the first site of b_1 (respectively the second site of b_k) will be called the *initial site* (respectively the *end site*) of p. A path $(b_1, \ldots, b_k)$ such that the second site of e_k is equal to the first site of b_1 is called a *loop*. Given $\Lambda \subset \boldsymbol{Z}^d$ we say that a site x is *in the k-interior* of Λ where k is a nonnegative integer if $d(x, \boldsymbol{Z}^d \backslash \Lambda) \geq k$. A bond, a path or a loop are in the k-interior of Λ if all their sites are. Given a path $p = (b_1, \ldots, b_k)$, we denote by $-p$ the path $(-b_k, \ldots, -b_1)$. Given two paths $p = (b_1, \ldots, b_k)$, $r = (c_1, \ldots, c_l)$ such that the second site of e_k is equal to the first site of c_1, we denote by (e,r) the path $(b_1, \ldots, b_k, c_1, \ldots, c_l)$. If p, r, s are paths, the symbol (p,r,s) denotes the path $((p,r),s)$ provided the expression is defined, i. e. the paths have the required properties. Similarly for $(p_1, \ldots, p_k)$ if $p_1 \ldots, p_k$ are paths with the required properties.

Given a positive integer L we want to give a precise meaning to the concept of *subdivision* of $\boldsymbol{Z}^d$ or some subset of $\boldsymbol{Z}^d$ into cubes with side L. There is no problem in attributing sites to cubes. For what concerns bonds that connect sites of different cubes we make the convention to attribute them to the cube containing the site with larger coordinates. We say that V is a cube of bonds (with side L) if it is an element of a subdivision as previously defined. As we will be actually interested in configurations of bonds, we will say that a volume seen as a set of bonds can be exactly subdivided into cubes with side L when it is the disjoint union of cubes of bonds with side L.

Given a set R set of bonds the set of *bond configurations* in R is the set $S_R = \{0,1\}^R$. If $e \in S_R$ we say that the bond $b \in R$ is *empty* (respectively *occupied*) in the configuration e if $e_b = 0$ (repectively $e_b = 1$). If p is a path of bonds in R we say that p is *occupied* in the configuration if $e_b = 1$ for all bonds b corresponding to ordered bonds of p. Given a configuration $e \in S_R$ the set of sites $\tilde{R}$ can be partitioned into *clusters*, i. e. connected components, where two sites x y are said to be connected in e if there is an occupied path p with x and y respectively initial and end sites of p.

For $e \in S_R$ $|e|$ denotes number of occupied bonds in e, $\#(s)$ number of clusters in e. Given a finite set of bonds $R \subset \hat{\boldsymbol{Z}}^d$ we say that x is a boundary point of R if there is a bond b such that $x \in b$ and $b \in \hat{\boldsymbol{Z}}^d \backslash R$. Following [4] we define *boundary conditions* on the set of boundary points of a finite set of bonds R. We define actually two kinds of boundary conditions corresponding respectively to "ferromagnetic" and "frustrated" random cluster models that we are going to consider. A boundary condition on R of the first kind is simply a partition π of the boundary sites of R. A boundary condition on R of the second kind is a pair (π, σ) where π is a partition of the set boundary sites of R and $\sigma_{x,y}$ is an assignation of a signature $+1$ or -1 to each unordered pair $\{x,y\}$ of boundary points of R belonging to the same element of the partition π.

If u is a boundary condition on R of the first or the second kind, i. e. $u = \pi$ or $u = (\pi, \sigma)$ and $e \in S_R$ we denote by $\#(e)_u$ the number of clusters in e if declare connected two sites of the boundary of R that belong to the same element of the partition π.

Given a set of bonds $R \subset \hat{\boldsymbol{Z}}^d$, an *interaction* $(J_b)_{b \in R}$ is a map from R to $\mathbb{R}$. A loop $p = (b_1, \ldots, b_k)$ in $R \subset \hat{\boldsymbol{Z}}^d$ is said to be frustrated with respect to the interaction $(J_b)_{b \in R}$ if $\prod_{i=1}^k J_{b_i} <$

0. Given a boundary condition $u = (\pi, \sigma)$ of the second kind a path $p = (b_1, \dots, b_k)$, with initial and end sites x and y boundary sites of R, $x \neq y$, is said to be frustrated with respect to the interaction J if x and y belong to the same element of π and $\sigma_{x,y} \prod_{i=1}^{k} J_{b_i} < 0$.

The Ising spin model.

Given an interaction $(J_b)_{b \in \hat{\boldsymbol{Z}}^d}$, a finite volume $\Lambda \subset \boldsymbol{Z}^d$ and a configuration $\eta \in \{-1,1\}^{\boldsymbol{Z}^d}$ the finite volume Ising Gibbs measure $\mu^{(\Lambda)}$ in Λ corresponding to the interaction (J_b) and boundary condition η is the probability measure on $\{-1,1\}^{\Lambda}$ given by

$$\mu^{\Lambda}(\mathbf{s}) = Z_{\eta}^{(\beta,\Lambda)-1} \exp\left(-\beta\left(\sum_{\langle x,y\rangle} J_{x,y} s_x s_y + \sum_{\langle x,y\rangle} J_{x,y} s_x \eta_y\right)\right), \tag{1.1}$$

where $Z_{\eta}^{(\Lambda)}$ is the normalization constant, the first sum in the exponential is over bonds in $\hat{\Lambda}$, the second sum is over bonds $\{i,j\}$ with $i \in \Lambda$ and $j \in \boldsymbol{Z}^d \backslash \Lambda$.

A *Gibbs state* for the interaction $(J_b)_{b \in \hat{\boldsymbol{Z}}^d}$ at inverse temperature β is any weak limit of finite volume Ising Gibbs measures for sequences of volumes Λ_n and boundary conditions $\eta^{(n)}$ with $\Lambda_n \uparrow \boldsymbol{Z}^d$. Alternatively Gibbs states can be defined through DLR equations for conditional probabilities (see [8]).

Random cluster models.

Given a finite volume of bonds V, an interaction $(J_b)_{b \in V}$ with $J_b \geq 0$ and a boundary condition of the first kind $u = \pi$, we define the corresponding finite volume random cluster measure as the probability measure on S_V given by

$$P_u^{(\beta,V)}(\mathbf{e}) = Z^{(\beta,V)-1}\left(\prod_{e_b=1} p_b\right)\left(\prod_{e_b=0}(1-p_b)\right) q^{\#(\mathbf{e})_u}, \tag{1.2}$$

where q is a real number $q \geq 1$, χ_u is the indicator of the event that there is no occupied frustrated loop and no occupied frustrated path, $p_b = 1 - \exp(-2\beta J_b)$ and $Z^{(\beta,V)}$ is the normalization constant. The superscript (β, V) will be omitted when there is no danger of confusion (see [3]).

Frustrated random cluster models.

Given a finite volume of bonds V, an interaction $(J_b)_{b \in V}$ and a boundary condition of the second kind $u = (\pi, \sigma)$, we define the corresponding finite volume frustrated random cluster measure as the probability measure on the configurations in $\hat{\Lambda}$ given by

$$P_u^{(\beta,V)}(\mathbf{e}) = Z^{(\beta,\Lambda)-1}\left(\prod_{e_b=1} p_b\right)\left(\prod_{e_b=0}(1-p_b)\right) q^{\#(\mathbf{e})_u}\chi_u(e) \tag{1.3}$$

where q is a real number $q \geq 1$, χ_u is the indicator of the event that there is no occupied frustrated loop and no occupied frustrated path, $p_b = 1 - \exp(-2\beta|J_b|)$ and $Z^{(\beta,V)}$ is the normalization constant (see [7]). As before the superscript (β, V) will be omitted when there is no danger of confusion.

In the following the contant q will be fixed once for all so that there will be well definite finite volume (frustrated) random cluster volumes corresponding to an interaction, a boundary condition and a value of β. *Infinite volume (frustrated) random cluster measures* for an interaction J_b, $b \in \hat{\boldsymbol{Z}}^d$, are weak limits of the corresponding finite volume measures. As in the case of Gibbs measures they can be alternatively defined in terms of DLR type equations for conditional probabilities.

Let (J_b) be an interaction defined for $b \in \hat{\boldsymbol{Z}}^d$, the set of bonds of $\boldsymbol{Z}^d$. We say that a cube of bonds C is frustrated with respect to (J_b) if there is a loop $p = (b_1, \dots, b_k)$ in the 3- interior of C which is frustrated with respect to (J_b).

Definition. Let J_b be an interaction on $\hat{\boldsymbol{Z}}^d$. The *critical percolation value* for J_b is defined to be the supremum of those values β such that for any infinite (frustrated) random cluster measure the probability of the existence of an infinite cluster is 0.

Statement of the results.

In [6] Newman proved a weak inequality between the critical point of a frustrated random cluster model with interaction (J_b) and that of a random cluster model with interaction $(|J_b|)$ (see also [7]). Using the FK representation with $q = 2$ this translates into a weak inequality for critical temperatures of Ising models with the same inter actions. In situations where one has a "uniformly distributed frustration" one would expect a strict inequality. Here we prove a result of this kind. We can deal the case when the frustration is uniformly distributed such as a periodic interaction with a frustrated loop. In this case we can get a strict inequality for critical percolation points for random cluster models. By using FK representation in the case $q = 2$ this translates into a strict inequality for critical temperatures for spin flip symmetry breaking of Ising models.

We have the following results:

Theorem 1.1

Let (J_b) be an interaction on $\hat{\boldsymbol{Z}}^d$ such that $0 < K_1 \le |J_b| \le K_2 < \infty\ \forall b \in \hat{\boldsymbol{Z}}^d$, for some constants K_1, K_2. There is a constant $\delta > 0$ such that the critical percolation value β_c for (J_b) is larger than or equal to the critical percolation value for the interaction (I_b) where $I_b = |J_b| - \delta$ if b belongs to the interior of a frustrated cube $C \in G$ and $I_b = |J_b|$ otherwise.

By using the FK representation for $q = 2$ (see [4]), Theorem 1.1 gives the following result for spin models.

Theorem 1.2

Let (J_b) be an interaction on $\hat{\boldsymbol{Z}}^d$ such that $0 < K_1 \le |J_b| \le K_2 < \infty\ \forall b \in \hat{\boldsymbol{Z}}^d$, for some constants K_1, K_2. There exists a constant $\delta > 0$ such that for $\beta < \beta_c$ all Gibbs states correspoding to (J_b) are invariant by spin flip, where β_c is the critical value for the interaction I_b

$$I_b = \begin{cases} |J_b| - \delta & \text{if } b \text{ belongs to the interior of a frustrated cube } C \in G \\ |J_b| & \text{otherwise.} \end{cases}$$

Theorem 1.2 can be applied to give a strict inequality between the critical temperature for spin flip symmetry breaking of some frustrated models and that of the corresponding ferromagnetic models. The following theorem follows immediately from combined application of theorem 1.2 and the results of [1].

Theorem 1.3 Let (J_b) be an interaction on $\hat{\boldsymbol{Z}}^d$ such that $0 < K_1 \le |J_b| \le K_2 < \infty\ \forall b \in \hat{\boldsymbol{Z}}^d$, for some constants K_1, K_2. Assume that for some L there is a subdivision of $\hat{\boldsymbol{Z}}^d$ into frustrated cubes with side L (this is in particular the case of a periodic interaction with some frustrated loop). Then

i) the critical percolation value for the random cluster model corresponding to the interaction (J_b) is strictly larger than that for the interaction $(|J_b|)$;

ii) the critical temperature for symmetry breaking of the Ising model with interaction (J_b) is strictly larger than that for the Ising model corresponding to the interaction $(|J_b|)$.

Remark 2. Recently G. R. Grimmett has proved strict inequalities for random cluster models that allow to extend the results of Theorem 1.3 to frustrated models with random interactions ([4]).

§2. Proofs of the results.

The proof of Theorem 1.1 is based on the following Lemma 2.1.

Let us start with some definitions. Let V be a cube of bonds. We say that E, $E \subset S_V$, is a connection event of V if it refers to connections of boundary sites of V. Formally let $E_{x,y}$ be the event that x and y are connected; E is a connection event of V if it belongs to the algebra generated by the events $E_{x,y}$ as x and y vary on the set of boundary points of V.

Let $V \subset \hat{\boldsymbol{Z}}^d$. An event E, $E \subset S_V$, is said to be *positive* (resp. *negative*) if its indicator χ_E is a nondecreasing (resp. nonincreasing) function on S_V. Given $E \subset S_V$ and $b \in V$ we denote by $\delta_b E$ the event that e is pivotal for E, i. e. the set $\{e \in S_V | \chi_E(T_b e) \neq \chi_E(e)\}$, where $T_b e$ is the configuration with $(T_b e)_{b'} = e_{b'}$ for $b' \neq b$ and $(T_b e)_b \neq e_b$. We decompose $\delta_b E$ as $\delta_b E = \delta_b^{(i)} E \cup \delta_b^{(e)} E$ with $\delta_b^{(i)} E = \delta_b E \cap E$ and $\delta_b^{(e)} E = \delta_b E \cap E^c$.

Lemma 2.1.

Let V be a frustrated cube of bonds for the interaction $(J_b), b \in V$ with $J_b \neq 0 \; \forall b \in V$.

Let (π, σ) be a boundary condition for V and let $\mathbf{P}_{(\pi,\sigma)}$ ($\mathbf{P}^{(F)}_{(\pi)}$) denote the frustrated random cluster model (respectively the random cluster model (here capital F stays for "ferromagnetic")) in V interaction $(J_b), b \in V$ (resp. $(|J_b|), b \in V$), boundary condition (π, σ) (resp. (π). Then for any nontrivial positive connection event E of V we have

$$\mathbf{P}_{(\pi,\sigma)}(E) < \mathbf{P}^{(F)}_{(\pi)}(E) \tag{2.1}$$

.

Proof of Lemma 2.1.

Let $p = (b_1, \ldots, b_k)$ be a frustrated loop in the 3-interior of V. Let U be the event that there is no frustrated loop or path constituted of occupied bonds in V with respect to the boundary conditions (π, σ) ; U is clearly a negative event.

Let now E be any nontrivial positive connection event of V. Since E is nontrivial event there must be a bond $b \in V$ of V such that $\mathbf{P}^{(F)}_{(\pi)}(\delta_b E) > 0$ and hence also $\mathbf{P}^{(F)}_{(\pi)}(\delta_b^{(i)} E) > 0$. Moreover since E is a connection event it is easy to see that b can be taken in $\hat{\Lambda}$ where $\Lambda$is the cube of sites corresponding to V. Let b be chosen with these properties. We want to show that there exists a frustrated loop containing b. If b is one of the bonds of p we have done. Assume that this is not the case. It is easy to check that we can build in the complementary of the support of p two disjoint paths q_1 q_2 joining the extremes of $b = (y_1, y_2)$ with two different sites x_1 and x_2 of p respectively. y_1 and y_2 split p into two paths p_1 and p_2 from y_1 to y_2 and from y_2 to y_1 respectively. We can build two loops containing b : $r_1 = (b, q_2, p_2, -q_1)$ and $r_2 = (e, q_2, -p_1, -q_1)$. It is easy to check that either r_1 or r_2 is a frustrated loop since the products of the J's along p_1 and p_2 have opposite signs. We call r this frustrated loop.

Now let us consider a configuration c of bonds in V with all bonds of r occupied and all other bonds vacant. We have $\mathbf{P}^{(F)}_{(\pi)}(c) > 0$. But c has been built so that $c \subset \delta_e^{(e)} U$. Therefore $\mathbf{P}^{(F)}_{(\pi)}(\delta_e^{(e)} U) > 0$.

The probability measure $\mathbf{P}^{(F)}_{(\pi)}$ satisfies the conditions of FKG inequality. By applying to the events E and F Theorem 1.1 of [2] we get

$$\begin{aligned}
&- \Big(\mathbf{P}^{(F)}_{(\pi)}(E \cap U) - \mathbf{P}^{(F)}_{(\pi)}(E) \mathbf{P}^{(F)}_{(\pi)}(U) \Big) = \mathbf{P}^{(F)}_{(\pi)}(E \cap U^c) - \mathbf{P}^{(F)}_{(\pi)}(E) \mathbf{P}^{(F)}_{(\pi)}(\tilde{U}^c) \geq \\
&\frac{\mathbf{P}^{(F)}_{(\pi)}(e_b = 0)}{\mathbf{P}^{(F)}_{(\pi)}(e_b = 1)} \mathbf{P}^{(F)}_{(\pi)}(\delta_b^{(i)} E) \mathbf{P}^{(F)}_{(\pi)}(\delta_b^{(i)} U^c) = \\
&\frac{\mathbf{P}^{(F)}_{(\pi)}(\omega_e = 0)}{\mathbf{P}^{(F)}_{(\pi)}(\omega_e = 1)} \mathbf{P}^{(F)}_{(\pi)}(\delta_b^{(i)} E) \mathbf{P}^{(F)}_{(\pi)}(\delta_b^{(e)} U) > 0.
\end{aligned} \tag{2.2}$$

We get therefore from (2.2)

$$\mathbf{P}_{(\pi,\sigma)}(E) = \mathbf{P}^{(F)}_{(\pi)}(E|U) = \frac{\mathbf{P}^{(F)}_{(\pi)}(E\cap U)}{\mathbf{P}^{(F)}_{(\pi)}(U)} < \mathbf{P}^{(F)}_{(\pi)}(E) \tag{2.3}$$

Q. E. D.

Proof of Theorem 1.1.

Let V be a frustrated cube of bonds with side L with boundary conditions (π,σ). The probabilities of events with support in V are continuous functions of the interactions and of β . Therefore there exists $\delta > 0$ such that for every nontrivial positive connection event E

$$\mathbf{P}_{(\pi,\sigma)}(E) \leq \mathbf{P}^{'(F)}_{(\pi)}(E), \tag{2.4}$$

where $\mathbf{P}^{'(F)}_{(\pi)}$ is the random cluster Gibbs measure with interaction $I_b = |J_b| - \delta$ for each bond $b \in V$. We have assumed the interaction (J_b) to be bounded. The constant δ can therefore be chosen uniformy for β in a bounded interval $I \subset [0,\infty)$. The inequality holds immediately also for trivial positive events. We can then apply a theorem by Strassen ([9], see also e. g. [5] Theorem 2.4 p. 72) and obtain that there is a coupling between the measures $\mathbf{P}_{(\pi,\sigma)}$ and $\mathbf{P}^{'(F)}_{(\pi)}$ restricted on connection events supported on pairs of connection realizations (c_1, c_2) with $c_1 \leq c_2$ in the natural ordering.

A similar coupling also exists between the measures $\mathbf{P}_{(\pi,\sigma)}$ and $\mathbf{P}^{'(F)}_{(\pi')}$ if π is finer than π' by FKG ordering of the measures $\mathbf{P}_{(\pi,\sigma)}$ with respect to boundary conditions (see [7]). In the case that V is nonfrustrated we have a coupling between the measures $\mathbf{P}_{(\pi,\sigma)}$ and $\mathbf{P}^{(F)}_{(\pi')}$, i. e. with the corresponding model with interactions $|J_b|$, if π is finer than π' (see [7]).

Let now V be a volume of bonds decomposable into cubes $C_1,\ldots,C_k$ with side L. It follows easily from Lemma 2.1 arguments that for any boundary condition (π,σ) of V and for π finer than π' we have that if E is an event that can be expressed in terms of connections between sites on the boundaries of cubes C_i

$$\mathbf{P}_{(\pi,\sigma)}(E) \leq \mathbf{P}^{'(F)}_{(\pi')}(E), \tag{2.5}$$

where $\mathbf{P}^{'(F)}_{(\pi')}$ is the random cluster measure with interaction given by

$$I_b = \begin{cases} |J_b| - \delta & \text{if } b \text{ belongs to some frustrated cube } C_i \\ |J_b| & \text{otherwise.} \end{cases} \tag{2.6}$$

An event E of this kind is the event that the boundary of a cube C_i is connected to the boundary of V. It contains the event that a site in C_i is connected to the boundary of V.

We remark that on random cluster models the boundary conditions on a volume of bonds $C \subset V$ are determined just by the connections of boundary sites of C in $V\backslash C$. Inequality (2.5) can be established by constructing a coupling between connection realizations on the sites of the boundaries of the cubes C_i. One way to check it is to use the couplings between conditional probabilities to couple two "Glauber type" dynamics on connection realizations on the boundary sites that have as invariant measures $\mathbf{P}_{(\pi,\sigma)}$ and $\mathbf{P}^{'(F)}_{(\pi')}$ respectively and such that in the dynamics the first component is always less than or equal to the second. This can be achieved by choosing at random one of the cubes C_i and ajourning the two connection configurations on the boundary of C_i while leaving invariant the connection configurations on the boundaries of other cubes. The transition probabilities are just the coupling between conditional probabilities that have been shown to exist if the partition π corresponding to the first component is finer than the partition π' corresponding to the second. Last condition is always verified since it is implied by the ordering between the two components.

Let now V be cube centered at the origin and let E be the event that the boundary of the cube C_i that contains the origin is connected to the boundary of V. By taking the volume V tending to $\hat{\boldsymbol{Z}}^d$ we get that if the r. h. s. of (2.5) tends to 0 so does the l. h. s.. This can be easily extended to any sequence of volumes (not necessarily decomposable into cubes with side L) tending to $\hat{\boldsymbol{Z}}^d$. This implies the inequality between critical percolation values. Q. E. D.

References:

[1] C. E. Bezuidenhout, G. R. Grimmett and H. Kesten "Strict inequality for critical values of Potts models and random-cluster processes" Commun. Math. Phys. **158**1-16 (1993).

[2] M. Campanino "A lower bound for the covariance of positive events" Preprint of the Department of Mathematics of the University of Bologna n.15 (1997) mparc 97-529 to appear in Forum Mathematicum.

[3] G. R. Grimmett "The stochastic random-cluster process and the uniqueness of random-cluster measures" Annals of Probability **23** 1461-1510 (1997).

[4] G. R. Grimmett "Strict inequalities for quenched random-cluster models" preliminary draft, private communication (1997).

[5] T. M. Liggett. Interacting Particle Systems. Springer-Verlag, Berlin (1985).

[6] C. M. Newman "Disordered Ising systems and random cluster representations" in *Probability and Phase Transitions*, Geoffrey Grimmett ed., NATO ASI Series C: Mathematical and Physical Sciences - Vol. **420** 247-260 (1994), Kluwer Academic Publishers, Dordrecht.

[7] C. M. Newman. "Topics in Disordered Systems" Birkhäuser. Basel (1997).

[8] D. Ruelle. Thermodynamic formalism. Addison-Wesley, Reading Mass (1978).

[9] V. Strassen "The existence of probability measures with given marginals" Ann. Math. Statist., **36** 423-439 (1965).