From the adiabatic piston to macroscopic motion induced by fluctuations

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Abstract

The controversial problem of the evolution of an isolated system with an internal adiabatic wall is investigated with the use of a simple microscopic model and the Boltzmann equation. In the case of two infinite volume one-dimensional ideal fluids separated by a piston whose mass is equal to the mass of the fluid particles we obtain a rigorous explicit stationary non-equilibrium solution of the Boltzmann equation. It is shown that at equal pressures on both sides of the piston, the temperature difference induces a non-zero average velocity, oriented toward the region of higher temperature. It thus turns out that despite the absence of macroscopic forces the asymmetry of fluctuations results in a systematic macroscopic motion. This remarkable effect is analogous to the dynamics of stochastic ratchets, where fluctuations conspire with spatial anisotropy to generate directed motion. However, a different mechanism is involved here. The relevance of the discovered motion to the adiabatic piston problem is discussed.

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I. INTRODUCTION

The recent work of E.Lieb and J.Yngvason [?] on the foundations of thermostatics has given a new interest to an old but still controversial problem of thermodynamics. This is the so called "adiabatic piston problem" described as follows [?] (see also recent references in [?]). The system is composed of a finite volume cylinder containing two fluids separated by an adiabatic movable piston, as shown in Fig.1. A brake maintains initially the piston at rest, and the two fluids are in equilibrium with temperature, pressure and volume given by $(T_1^{(0)}, p_1^{(0)}, V_1^{(0)} = A(L_1+X^{(0)}))$, and $(T_2^{(0)}, p_2^{(0)}, V_2^{(0)} = A(L_2-X^{(0)}))$, respectively. A denotes here the cross section area of the cylinder, and $X^{(0)}$ the initial position of the piston. At a certain time the brake is released, and the problem is to find the final equilibrium state.

Figure 1: Adiabatic piston problem

The laws of thermostatics give immediately the condition for the mechanical equilibrium, namely that in the final equilibrium state the pressures must be equal, i.e. $p_1^f = p_2^f$ (with $E_1^f + E_2^f = E_1^0 + E_2^0$). However, they give no information concerning the final temperatures, except for the condition that they must be such that for both fluids the final entropy cannot be smaller than its initial value, i.e. $S_1^f \geq S_1^0$, and $S_2^f \geq S_2^0$.

On the other hand, from the laws of thermodynamics one can derive ordinary differential equations for the time evolution of the system. Solving these equations would then give the unique solution of the problem, and in particular the final temperatures. This idea has been recently followed in [?], where a system of thermodynamic evolution equations has been derived within a very simple model. One had to distinguish then between the case where the two fluids have vanishing viscosities and the case where at least one of the fluids has a non-zero viscosity. For non-viscous fluids the thermodynamic equations predicts a periodic motion subject to the energy conservation law, with constant subsystem entropies: $S_1(t) = S_1^{(0)}$, $S_2(t) = S_2^{(0)}$. When at least one of the fluids is viscous, the system will attain the final equilibrium state with equal pressures $p_1 = p_2$. However, to find the final entropies (or temperatures) one has to solve the evolution equations which so far could not be achieved explicitly. These same evolution equations were also obtained in the framework of the kinetic theory of gases, assuming the fluctuations of the piston to be negligibly small and considering perfect gases, and several numerical solutions have been obtained [**?**].

But even having obtained the macroscopic description of the evolution of the system the controversial question remains whether the stochastic frictionless motion of the piston induced by collisions with the fluid particles could eventually lead (on a much longer time scale) to a final state where the temperatures of the fluids are equal, and whether such a microscopic fluctuation mechanism would not violate the second law of thermodynamics. Clearly, the analysis of such problem requires the microscopic description of the dynamical evolution.

In this article we thus present a microscopic study of the motion of the adiabatic piston in contact with two infinite volume reservoirs. Our analysis, based on the Boltzmann equation, has been inspired by the above fundamental adiabatic piston question.

In Section 2 we define the microscopic model and we discuss the applicability of the Boltzmann equation to the description of the motion of the piston. The rigorous determination of the stationary state in one dimension for the case where the mass of the piston is equal to the fluid particle mass is presented in Section 3. The derived solution at equal temperatures reduces simply to the Maxwell equilibrium state. But when the temperatures are different a non-equilibrium stationary state appears characterized by an average velocity of the piston oriented toward the higher temperature region, although no macroscopic force is present (the pressures on both sides of the piston are supposed to be equal). This situation is thus precisely that of the "molecular motor problem". The rigorous derivation (within Boltzmann's theory) of the fluctuation induced macroscopic motion is the main result of our work. The relevance of this prediction to a much more realistic problem of a massive piston is discussed in the closing Section 4. The case where the mass of the piston is much larger than that of a fluid particle will be discussed in the forthcoming paper [**?**].

II. ADIABATIC PISTON SEPARATING TWO THERMOSTATS: THE BOLTZMANN EQUATION

In order to simplify the analysis as much as possible we consider in the following the one-dimensional system occupying the whole infinite line. The two fluids are assumed to be perfect gases composed of identical point particles of mass m. They are separated by a structureless particle of mass M which plays the role of the piston. The perfectly elastic collisions between the fluid particles result in instantaneous exchanges of velocities. So, from the point of view of the dynamics of the piston the fluid particles can be considered as non-interacting, penetrating each other in their unperturbed free motion. The only perturbation of the fluid states is due to collisions with the piston. The similar model has been already investigated almost forty years ago by Lebowitz [**?**] and few years later by Alkemade, van Kampen and Mac Donald [**?**].

When a collision occurs, the pre-collisional velocities V and v of the piston and the fluid particle, respectively, suffer a change according to the elastic collision law

$$\begin{aligned} V \to V' &= V + \frac{2m}{M+m}(v-V) \\ v \to v' &= v - \frac{2M}{M+m}(v-V) \end{aligned} \tag{1}$$

Let us suppose that at the initial moment the fluids to the left and to the right of the piston are at equilibrium with temperatures T_1 and T_2, and number densities n_1 and n_2, respectively. Moreover, the macroscopic mechanical equilibrium will be assumed: the pressures on both sides of the piston are equal

$$p_1 = n_1 k_B T_1 \ = \ p_2 = n_2 k_B T_2 \tag{2}$$

(k_B is Boltzmann's constant).

When $M \gg m$, the probability that the piston interacts back with the fluid particle it already collided with becomes negligible. In the absence of recollisions the piston always "sees" the unperturbed Maxwell distributions of the arriving fluid particles, and no correlations can occur between its velocity and the precollisional velocity of the fluid particles. The perturbations caused by collisions with the piston go away to infinity. So, in the large mass limit $m/M \to 0$, we can expect the linear Boltzmann equation to give a correct description of the dynamical evolution of the piston. In order to write down this equation we denote by $\phi_{T_1}^m$ and $\phi_{T_2}^m$ the Maxwell velocity distributions of the two surrounding fluids, where

$$\phi_T^m = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{3}$$

Let $F(X, V; t)$ denote the probability density for finding the piston at point X with velocity V at time t. When $m/M \ll 1$, $F(X, V; t)$ satisfies the linear Boltzmann equation of the form

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial X}\right) F(X, V; t) = \int dv |V - v| \times \qquad (4)\\
&[\theta(V - v) n_1 \phi_{T_1}^m \left(v - \frac{2M}{M+m}(v - V)\right) F\left(X, V + \frac{2m}{M+m}(v - V); t\right)\\
&-\theta(v - V) n_1 \phi_{T_1}^m(v) F(X, V; t)\\
&+\theta(v - V) n_2 \phi_{T_2}^m \left(v - \frac{2M}{M+m}(v - V)\right) F\left(X, V + \frac{2m}{M+m}(v - V); t\right)\\
&-\theta(V - v) n_2 \phi_{T_2}^m(v) F(X, V; t)]
\end{aligned}$$

where θ is the unit Heaviside step function.

Our aim is to determine the velocity distribution of the piston in the asymptotic stationary state attained for $t \to \infty$. It turns out that this problem can be given a rigorous answer in the special case of $M = m$. But when the mass M of the piston is equal to the fluid particle mass m, the recollisions can take place, and the validity of the Boltzmann equation becomes questionable. However, what we are really interested in is the case of $M \gg m$. We thus expect that neglecting the effect of recollisions in the equal mass case will lead to the evolution qualitatively similar to that occuring for $M \gg m$, simply because the same class of dynamical events will be taken into account. This should apply in particular to the nature of the stationary state. It should be noticed here that even when compared with the exact results the linear Boltzmann equation for $M = m$ represented a very reasonable approximation to the velocity correlation function and to the Van Hove self-correlation function of the one-dimensional hard rod fluid [**?**].

Having adopted the above point of view we are going to study in the next section the equation satisfied by the stationary probability density $\Phi(V)$ for finding the piston with velocity V in the equal pressure case (see eq.(2)). Denoting by $F(X,V)$ the stationary solution of equation (4), we have

$$\Phi(V) = \int dX F(X,V) \tag{5}$$

III. SOLVING THE BOLTZMANN EQUATION: APPEARANCE OF A MOLECULAR MOTOR

According to the linear Boltzmann equation (4) the stationary velocity distribution $\Phi(V)$ in the equal pressure case (2) satisfies the equation

$$\int dv|V-v|\left[\theta(v-V)\frac{1}{T_1}\phi^m_{T_1}(v) + \theta(V-v)\frac{1}{T_2}\phi^m_{T_2}(v)\right]\Phi(V) \tag{6}$$

$$= \int dv|V-v|\left[\theta(V-v)\frac{1}{T_1}\phi^m_{T_1}(V) + \theta(v-V)\frac{1}{T_2}\phi^m_{T_2}(V)\right]\Phi(v)$$

In the following we shall assume $T_2 > T_1$.

It is convenient to express eq.(6) in terms of the following two functions

$$I(V) = \int_V^\infty dv(v-V)\frac{1}{T_1}\phi^m_{T_1}(v) + \int_{-\infty}^V dv(V-v)\frac{1}{T_2}\phi^m_{T_2}(v) \tag{7}$$

$$G(V) = \int_V^\infty dv(v-V)\Phi(v) + \int_{-\infty}^V dv(V-v)\Phi(v) \tag{8}$$

Using the identities

$$|V-v|\theta(V-v) = \frac{1}{2}[(V-v) + |V-v|]$$