A note on the construction of Hamiltonian trajectories along heteroclinic chains

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Abstract

We revisit chapter 8 of [3] (L. Chierchia, G. Gallavotti, *Drift and diffusion in phase space*, Ann. Inst. Henri Poincaré, **60**, 1, 1994) with the purpose of providing a short, simple proof of the existence of Hamiltonian trajectories arbitrarily close to a chain of heteroclinic orbits connecting whiskered tori. We also discuss a characterization of transversality for whiskers that are graphs "over the angles". In Appendix we consider a model problem (a "standard chain of transition tori") and prove that the "drifting times" are proportional, for such a model, to a power of the number of tori forming the chain (in the perturbative case, this would correspond to drifting times which are polynomial in the inverse of the perturbation parameter).

Contents

1 Introduction

"Arnold diffusion" (i.e. order–one drift of action variables in general nearly–integrable Hamiltonian systems) takes place near long chains of heteroclinic orbits connecting lower–dimensional, invariant, whiskered tori: for a general theory of "Arnold diffusion" for perturbations of "a–priori unstable" nearly–integrable Hamiltonian systems[1], see [3]. This note is devoted to give a short (albeit complete) proof of the existence of drifting (or "shadowing") orbits along a *given* chain of whiskered tori as they arise, for example, in general perturbations of "a–priori unstable" integrable systems.

[1] Roughly speaking, "a–priori unstable" nearly–integrable Hamiltonian systems are perturbations of integrable Hamiltonian systems which possess a (one–dimensional) separatrix, like for example $(\ell-1)$–rotators plus a pendulum, i.e. a system governed by the Hamiltonian $\sum_{j=1}^{\ell-1} A_j^2/2 + I^2/2 + \cos\varphi - 1$, (here (A_j, α_j) and (I, φ) are canonically conjugated "action–angle" coordinates; see below for precise notations).

The proof presented here (in § 2) follows essentially the scheme given in [3] and corrects a minor error in § 8 of [3] (see [4]). In [3] certain parameters (measuring the expansion rates in the "hyperbolic variables" and in the "quasi–periodic variables") are taken to be different and this, in general, might not be possible (or needs at least a justification). However if one takes such parameters *equal*, the proof in § 8 of [3] goes through word–by–word (leading to extremely pessimistic estimates on "drifting times", i.e., the times needed for Arnold diffusion to take place). As mentioned above, we present here a slight modification of the argument used in [3] obtaining a short proof of the existence of Arnold diffusion *given* a long "transition chain" (see the beginning of the next section for precise definitions).

Section 2 ends with a characterization of transversality for intersections of whiskers that are graphs "over the angle–variables" (in a sense to be specified below): such a situation is often encountered in applications (see, again, [3]).

We include in this note (in the Appendix) some upper bounds on the drifting–times in a *model problem*, which we call a "standard chain of transition tori". Such a model is motivated by the "standard whiskered tori" of [1], which we assumed to be connected by a simple dynamical rule. We prove that, for the standard chain, the time needed for drifting orbits to go from one end of the chain to the other is *polynomial* in the number of the tori forming the chain. In the nearly–integrable setting, such an estimate would lead to drifting times bounded by an inverse power of the perturbation parameter. For other estimates on drifting times see, also, [7], [8], [5] and [6].

2 Construction of shadowing orbits

We begin by making precise the notion of "transition chain" in the context of Arnold diffusion. The main result is the "local" construction presented in Proposition 1, which implies immediately the existence of Arnold diffusion (see the Corollary below).

Let ϕ^t denote the flow at time t generated by a (real–analytic) Hamiltonian H on a fixed energy level $\mathcal{S}_E \equiv \{H = E\}$ contained in the phase space $V \times \mathbb{T}^\ell$ endowed with standard canonical "action–angle variables"; here V is some bounded domain in $\mathbb{R}^\ell$ and $\mathbb{T}^\ell$ denotes the standard flat ℓ–dimensional torus. *We assume that, in $\mathcal{S}_E$, there exist a "transverse, heteroclinic chain of whiskered, codimension–one KAM tori"*, i.e., there exist $(\ell-1)$–dimensional (different) tori $\mathcal{T}_1, \dots, \mathcal{T}_N$ such that each torus $\mathcal{T}_i$ is ϕ^t–invariant and has two asymptotic, invariant, Lagrangian manifolds ("whiskers") W_i^s and[2] W_i^u such that:

(i) For $1 \le i \le N-1$, W_i^u intersects W_{i+1}^s transversally in $\mathcal{S}_E$ at a point[3] z_i;

(ii) the dynamics in a neighbourhood U_i of $\mathcal{T}_i$ is described by standard "KAM normal" coordinates[4] $(\vec{A}', \vec{\psi}, p, q) \in \mathbb{R}^{\ell-1} \times \mathbb{T}^{\ell-1} \times \mathbb{R} \times \mathbb{R}$: there exists $r, \hat{r} > 0$, a canonical change of variables $\mathcal{C}$, and (real–analytic) functions of one variable $\vec{A}'_i$, $\vec{\omega}_i$, g_i, with $|\vec{A}'_i(J) - \vec{A}'_i(0)| < r$ for every $|J| < \hat{r}^2$, satisfying the following properties:

$$
\begin{aligned}
&\mathcal{C} : \widehat{U}_i \equiv \{|\vec{A}' - \vec{A}'_i(0)| < r, \vec{\psi} \in \mathbb{T}^{\ell-1}, |p| < \hat{r}, |q| < \hat{r}\} \to U_i \\
&H \circ \mathcal{C}(\vec{A}'_i(pq), \vec{\psi}, p, q) \equiv E \ , \ \forall\, \vec{\psi} \in \mathbb{T}^{\ell-1}, |p| < \hat{r}, |q| < \hat{r} \ , \\
&\partial_p\big(H \circ \mathcal{C}\big)(z) \ne 0 \ , \ \forall\, z \in \widehat{U}_i \cap \{q \ne 0\} \ , \quad \partial_q\big(H \circ \mathcal{C}\big)(z) \ne 0 \ , \ \forall\, z \in \widehat{U}_i \cap \{p \ne 0\} \ , \qquad (2.1)
\end{aligned}
$$

[2]As customary, the superscripts s and u stand for "stable" and "unstable".

[3]We recall that if S is a differentiable manifold and if M and N are two submanifolds of S one says that M intersects transversally N in S at a point $p \in M \cap N$ if $TM_p + TN_p = TS_p$ ("T" denoting, here, "tangent space of").

[4]The notations we use here follow quite closely those of [3]: in [3] the canonical variables on the phase space $V \times \mathbb{T}^\ell$ are denoted $(\vec{A}, I, \vec{\alpha}, \varphi)$ and the "KAM normal" coordinates $(\vec{A}', \vec{\psi}, p, q)$; as in [3], the vector symbol $\vec{\ }$ will be attached to quantities in $\mathbb{R}^{\ell-1}$ or in $\mathbb{T}^{\ell-1}$; ϕ^t and $\mathcal{C}$ are denoted in [3], respectively, S_t and $\mathcal{C}_\infty$ (see, in particular, [3], page 33, fifth paragraph: $\mathcal{C}$ is the transformation generated by $\widetilde{\phi_\infty}$); $\vec{\omega}_i$ and g_i in (2.2) correspond, respectively, to $(1+\gamma(J,\sigma_i,\mu))\vec{\omega}_{\sigma_i}$ and $g_{\sigma_i}(1+\gamma'(pq,\sigma_i,\mu))$ in [3].

and, for p, q, t satisfying $|pe^{-g_i(pq)t}| < \hat{r}$, $|qe^{g_i(pq)t}| < \hat{r}$,

$$\phi^t \circ \mathcal{C}(\vec{A}'_i(pq), \vec{\psi}, p, q)) = \mathcal{C}(\vec{A}'_i(pq), \vec{\psi} + \vec{\omega}_i(pq)\, t, pe^{-g_i(pq)t}, qe^{g_i(pq)t}) \ . \tag{2.2}$$

The dependence of $\vec{\omega}_i$ upon the variable $J = pq$ is scalar: $\vec{\omega}_i(J) = \tau_i(J)\vec{\omega}_i^0$ and the functions τ_i and g_i are uniformly bounded and bounded away from zero (and are, say, positive); the constant vectors $\vec{\omega}_i^0$ are rationally independent (diophantine).

In particular, (ii) implies that the torus $\mathcal{T}_i$, the *local* stable and unstable whiskers around $\mathcal{T}_i$ (that we denote $W^s_{i,\text{loc}}$ and $W^u_{i,\text{loc}}$) are given, in normal coordinates, respectively, by

$$\begin{aligned} &\mathcal{T}_i = \{(\vec{A}'_i(0), \vec{\psi}, 0, 0) : \ \vec{\psi} \in \mathbb{T}^{\ell-1}\} \ , \\ &W^s_{i,\text{loc}} = \{(\vec{A}'_i(0), \vec{\psi}, p, 0) : \ \vec{\psi} \in \mathbb{T}^{\ell-1} \ , |p| < \hat{r}\} \ , \quad W^u_{i,\text{loc}} = \{(\vec{A}'_i(0), \vec{\psi}, 0, q) : \ \vec{\psi} \in \mathbb{T}^{\ell-1} \ , |q| < \hat{r}\} \ . \end{aligned} \tag{2.3}$$

Assumptions (i) and (ii) are satisfied in general perturbation problems, at least in the case of "a priori–unstable" nearly–integrable Hamiltonian systems (compare[5] [3]); in the following only one of the conditions in the third line of (2.1) is needed: the requirement on both derivatives is made for simmetry.

We now consider a piece of the stable manifold W^s_{i+1} lying in U_i and parameterize it, in normal coordinates, as

$$\Delta^s_i \equiv \mathcal{C}\Big(\{(\vec{A}^\star_i(u), \vec{\psi}^\star_i(u), p^\star_i(u), q^\star_i(u)), \ u \in U \subset \mathbb{R}^\ell\}\Big) \ .$$

Let w_i be a point in the transverse intersection between W^u_i and W^s_{i+1}, lying in U_i. Without loss of generality, we can write

$$w_i = \mathcal{C}(\hat{w}_i) \quad \text{with} \quad \hat{w}_i = (\vec{A}'_i(0), \vec{\psi}_i, 0, \hat{r}/2) = (\vec{A}^\star_i(0), \vec{\psi}^\star_i(0), p^\star_i(0), q^\star_i(0)) \ ,$$

(to get the q–value equal, for all i, to $\hat{r}/2$, one can move w_i with the flow). We now build a curve in Δ^s_i, passing through w_i, parameterized by the hyperbolic variable p and with q–coordinate fixed. More precisely we have the following

Lemma 1. *With the above assumptions and notations, there exist a suitable $\tilde{r} > 0$ and a suitable (smooth) function $\vec{\psi}_i(p)$ such that*

$$\Delta^s_i \supseteq \mathcal{C}\Big(\{\Gamma_i(p) \equiv (\vec{\mathcal{A}}'_i(p), \vec{\psi}_i(p), p, \hat{r}/2) \ , \ |p| < \tilde{r}\}\Big) \equiv \mathcal{C}(\Gamma_i) \ ,$$

with $\vec{\mathcal{A}}'_i(p) \equiv \vec{A}'_i(p\hat{r}/2)$ and $\vec{\psi}_i(0) = \vec{\psi}_i$.

Proof. From the transversality (in the energy level) between Δ^s_i and W^u_i at w_i we have

$$2\ell - 1 = \text{rank} \begin{pmatrix} \partial_u \vec{A}^\star_i(0) & 0 & 0 \\ \partial_u \vec{\psi}^\star_i(0) & I & 0 \\ \partial_u p^\star_i(0) & 0 & 0 \\ \partial_u q^\star_i(0) & 0 & 1 \end{pmatrix} = \text{rank} \begin{pmatrix} \partial_u \vec{A}^\star_i(0) & 0 \\ \partial_u p^\star_i(0) & 0 \\ \partial_u q^\star_i(0) & 1 \end{pmatrix} + \ell - 1 \tag{2.4}$$

where I denotes here the $(\ell-1)\times(\ell-1)$ identity matrix. By energy conservation ($\Delta^s_i \subset \mathcal{S}_E$ and second line of (2.1))

$$H \circ \mathcal{C}(\vec{A}^\star_i(u), \vec{\psi}^\star_i(u), p^\star_i(u), q^\star_i(u)) = E = H \circ \mathcal{C}(\vec{A}_i(0), \vec{\psi}, 0, q) \tag{2.5}$$

so that, differentiating with respect to u the first equality and with respect to $\vec{\psi}$ and q the second one, one obtains

$$\partial_u p^\star_i(0) = \vec{a} \cdot \partial_u \vec{A}^\star_i(0) \tag{2.6}$$

[5] See, in particular, Lemma 1, which provides, in general settings, the dynamics described by (ii), and § 6÷§ 8, which provide general conditions ensuring (i). For a thorough discussion of Lemma 1 of [3] see also [9].

with[6] $\vec{a} \equiv -\partial_{\vec{A}'}(H\circ\mathcal{C})(\hat{w}_i)/\partial_p(H\circ\mathcal{C})(\hat{w}_i)$.

Furthermore, since $w_i \in W_i^u \cap W_{i+1}^s$ the trajectory $w(t) \equiv \phi^t(w_i)$ is a *heteroclinic* orbit belonging to $W_i^u \cap W_{i+1}^s$ for all times. In particular (since $w(t) \in W_i^u$) $w(t) = \mathcal{C}(\vec{A}_i'(0), \vec{\psi}_i^h(t), 0, q_i^h(t))$ for suitable functions $\vec{\psi}_i^h$, $q_i^h(t)$ while (since $w(t)$ lies, at least for small $|t|$, in $\Delta_i^s \subset W_{i+1}^s$) there exists a (smooth) function $u(t)$ such that $u(0)=0$ and $w(t) = \mathcal{C}\circ\left(\vec{A}_i^\star(u(t)),\, \vec{\psi}_i^\star(u(t)),\, p_i^\star(u(t)),\, q_i^\star(u(t))\right)$. Thus $\vec{A}_i^\star(u(t)) = \vec{A}_i'(0)$, $p_i^\star(u(t)) = 0$ and $q_i^\star(u(t)) = q_i^h(t)$ (for $|t|$ small enough). Differentiating with respect to t the relations $\vec{A}_i^\star(u(t)) = \vec{A}_i'(0)$ and $q_i^\star(u(t)) = q_i^h(t)$ one sees that

$$\begin{pmatrix} \partial_u \vec{A}_i^\star(0) \\ \partial_u q_i^\star(0) \end{pmatrix} \bar{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{2.7}$$

with $\bar{a} \equiv \dot{u}(0)/\partial_p(H\circ\mathcal{C})(\hat{w}_i)$. So, from (2.4), (2.6), (2.7) it follows that

$$\ell = \text{rank}\begin{pmatrix} \partial_u \vec{A}_i^\star(0) & 0 \\ \partial_u p_i^\star(0) & 0 \\ \partial_u q_i^\star(0) & 1 \end{pmatrix} = \text{rank}\begin{pmatrix} \partial_u \vec{A}_i^\star(0) & 0 \\ \partial_u q_i^\star(0) & 1 \end{pmatrix} = \text{rank}\begin{pmatrix} \partial_u \vec{A}_i^\star(0) \\ \partial_u q_i^\star(0) \end{pmatrix}$$

thus the matrix $\partial_u(\vec{A}_i^\star, q_i^\star)\Big|_{u=0}$ is nonsingular. By the Inverse Function Theorem, there exists a function $v(p)$ such that $v(0)=0$ and, for small p (say $|p|<\tilde{r}$),

$$\vec{A}_i^\star\,(v(p)) = \vec{A}_i'(p\hat{r}/2)\,, \qquad q_i^\star\,(v(p)) = \hat{r}/2\ . \tag{2.8}$$

Moreover, from (2.5) evaluated at $u=v(p)$ and from the second line in (2.1) evaluated at $q=\hat{r}/2$, $\vec{\psi} = \vec{\psi}_i^\star(v(p))$, there follows

$$\begin{aligned} H\circ\mathcal{C}\left(\vec{A}_i^\star(v(p)), \vec{\psi}_i^\star(v(p)), p_i^\star(v(p)), q_i^\star(v(p))\right) &= H\circ\mathcal{C}\left(\vec{A}_i'(p\hat{r}/2), \vec{\psi}_i^\star(v(p)), p_i^\star(v(p)), \hat{r}/2\right) \\ &= E = H\circ\mathcal{C}\left(\vec{A}_i'(p\hat{r}/2), \vec{\psi}_i^\star(v(p)), p, \hat{r}/2\right)\ , \end{aligned}$$

which implies[7] that $p_i^\star(v(p)) = p$.

So $\Delta_i^s \supseteq \left\{\left(\vec{A}_i^\star(v(p)), \vec{\psi}_i^\star(v(p)), p_i^\star(v(p)), q_i^\star(v(p))\right),\ |p|<\tilde{r}\right\} = \left\{(\vec{A}_i'(p\hat{r}/2), \vec{\psi}_i^\star(v(p)), p, \hat{r}/2\right),\ |p|<\tilde{r}\right\}$. ∎

Proposition 1. *Assume* (i) *and* (ii) *above and denote, for* $1\le i\le N-1$, *by* Δ_i^s *a submanifold of* W_{i+1}^s *contained in* U_i *and intersecting transversally* $W_{i,\text{loc}}^u$ *in* $\mathcal{S}_E$ *at* $w_i \in U_i$. *Given a neighbourhood* B_{i-1} *of some*[8] $\xi_{i-1} \in \Delta_{i-1}^s \cap (W_{i-1,\text{loc}}^u)^c$ *one can find* $\xi_i \in \Delta_i^s \cap (W_{i,\text{loc}}^u)^c$, *a neighbourhood* B_i *of* ξ_i *and a time* $T_i>0$ *such that* $\phi^{-T_i}B_i \subset B_{i-1}$.

Proof. Since $\xi_{i-1} \in \Delta_{i-1}^s \subset W_i^s$, ξ_{i-1} will evolve, in a suitable time $T_i^\star>0$, into a point $\xi_i' \in U_i$ having normal coordinates given by $(\vec{A}_i'(0), \vec{\chi}_i', \hat{r}/2, 0)$ for a suitable $\vec{\chi}_i' \in \mathbb{T}^{\ell-1}$. Set $B_i' \equiv \phi^{T_i^\star}(B_{i-1})$. It remains to show that in $\Delta_i^s \cap (W_{i,\text{loc}}^u)^c$ one can find a point ξ_i and a time $t_i^\star>0$ such that $\phi^{-t_i^\star}(\xi_i) \in B_i'$ (from this the existence of B_i with the desired properties follows at once). The point ξ_i will be selected on the curve Γ_i lying in Δ_i^s constructed in the above Lemma 1. Let $\widehat{B}_i$ be a small sphere of radius $s>0$ (in normal coordinates) centered at $\mathcal{C}^{-1}(\xi_i')$ and such that $\mathcal{C}(\widehat{B}_i) \subset B_i'$. By the construction of Γ_i and (2.2) one has, for p and t satisfying $|pe^{-g_i(p\hat{r}/2)t}|<\hat{r}$ and $\hat{r}e^{g_i(p\hat{r}/2)t}/2<\hat{r}$,

$$\phi^t\circ\mathcal{C}(\Gamma_i(p)) = \mathcal{C}(\vec{A}_i'(p), \vec{\psi}_i(p) + \vec{\omega}_i(p\hat{r}/2)\,t, pe^{-g_i(p\hat{r}/2)t}, \hat{r}e^{g_i(p\hat{r}/2)t}/2)\ . \tag{2.9}$$

Define, for $p>0$, $t_i(p)$ by setting $pe^{g_i(p\hat{r}/2)t_i(p)} = \hat{r}/2$ and notice that[9] $\tau_i(p\hat{r}/2)t_i(p)\to\infty$ as $p\to 0$. Since the flow $t\to\vec{\omega}_i^0 t$ is dense on $\mathbb{T}^{\ell-1}$, for any $r_0>0$ we can find $0<p_i^\star<r_0$ such that $\vec{\psi}_i - \vec{\omega}_i(p_i^\star\hat{r}/2)t_i(p_i^\star) \equiv \vec{\psi}_i -$

[6] Recall that, since $\hat{w}_i = (\vec{A}_i'(0), \vec{\psi}_i, 0, \hat{r}/2)$, (ii) implies that $\partial_p(H\circ\mathcal{C})(\hat{w}_i) \neq 0$.
[7] Use (2.8) and recall again that $\partial_p(H\circ\mathcal{C})(\hat{w}_i) \neq 0$
[8] The superscript ^{c} denotes complementary set.
[9] Recall the definitions of and the assumptions on g_i and τ_i given in (ii).

$\vec{\omega}_i^0 \tau_i(p_i^\star \hat{r}/2) t_i(p_i^\star)$ is arbitrarily close to $\vec{\chi}_i'$. Now, choose $0 < r_0 < \tilde{r}$ be such that $|\vec{A}_i'(p) - \vec{A}_i'(0)| < s$ and $|\vec{\psi}_i(p) - \vec{\psi}_i| < s/2$ for $|p| < r_0$ and let $t_0 > 0$ be such that $e^{-g_i(p\hat{r}/2)t}\hat{r}/2 < s$ for all $t > t_0$ and $|p| < r_0$. Then one can find $0 < p_i^\star < r_0$ so that $t_i(p_i^\star) > t_0$ and $|\vec{\psi}_i - \vec{\omega}_i(p_i^\star \hat{r}/2) t_i(p_i^\star) - \vec{\chi}_i'| < s/2$, which, in view of the above choices, implies that $\phi^{-t_i(p_i^\star)} \circ \mathcal{C}(\Gamma_i(p_i^\star)) \in \mathcal{C}(\widehat{B}_i) \subset B_i'$. ∎

Since ϕ^{-t} is a diffeomorphism (and therefore preserves transversality), one can take as Δ_{i-1}^s the image under ϕ^{-T} (for a suitable $T > 0$) of a connected component of W_i^s in a neighbourhood of z_{i-1} (compare assumption (i)). An iteration of Proposition 1 leads then immediately to the following

Corollary. *With the same assumptions made in Proposition 1, there exist a positive $T^\star$ and a positive measure set of initial data in a neighbourhood of $\mathcal{T}_1$ whose $\phi^{T^\star}$–image is contained into a neighbourhood of $\mathcal{T}_N$.*

Often, in applications, the whiskers appear naturally as *graphs* over the angles. In such a case trasversality takes a simple form:

Lemma 2. *Let $\Delta_i^s \subset U_i$ be a submanifold of W_{i+1}^s which may be expressed as a graph over $(\vec{\psi}, q)$, i.e., there exist functions $\vec{A}_i(\vec{\psi}, q)$ and $p_i(\vec{\psi}, q)$ defined (for suitable $d > 0$, $\vec{\psi}_i \in \mathbb{T}^{\ell-1}$, $0 < |q_i| < \hat{r} - d$) for $|\vec{\psi} - \vec{\psi}_i| < d$ and $|q - q_i| < d$ such that*

$$\Delta_i^s = \mathcal{C}\Big(\{(\vec{A}_i(\vec{\psi}, q), \vec{\psi}, p_i(\vec{\psi}, q), q) : \ |\vec{\psi} - \vec{\psi}_i| < d \ , |q - q_i| < d\}\Big) \ , \quad \vec{A}_i(\vec{\psi}_i, q_i) = \vec{A}_i'(0) \ , \quad p_i(\vec{\psi}_i, q_i) = 0 \ . \tag{2.10}$$

Then Δ_i^s intersects transversally $W_{i,\mathrm{loc}}^u$ in $\mathcal{S}_E$ at a point $w_i \equiv \mathcal{C}(\vec{A}_i(\vec{\psi}_i, q_i), \vec{\psi}_i, p_i(\vec{\psi}_i, q_i), q_i)$ if and only if

$$\det \frac{\partial \vec{A}_i}{\partial \vec{\psi}}(\vec{\psi}_i, q_i) \neq 0 \ . \tag{2.11}$$

Proof. As above, using normal coordinates we see that the span of the tangent spaces of Δ_i^s and $W_{i,\mathrm{loc}}^u$ is given by the rank of the following matrix

$$M \equiv \left.\begin{pmatrix} \partial_{\vec{\psi}} \vec{A}_i & \partial_q \vec{A}_i & 0 & 0 \\ I & 0 & I & 0 \\ \partial_{\vec{\psi}} p_i & \partial_q p_i & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right|_{(\vec{\psi}, q) = (\vec{\psi}_i, q_i)} \ . \tag{2.12}$$

Thus, Δ_i^s and $W_{i,\mathrm{loc}}^u$ intersects transversally in $\mathcal{S}_E$ at w_i if and only if $\operatorname{rank} M = 2\ell - 1$. But, because of the structure of M,

$$\operatorname{rank} M = \ell + \operatorname{rank} \left.\begin{pmatrix} \partial_{\vec{\psi}} \vec{A}_i & \partial_q \vec{A}_i \\ \partial_{\vec{\psi}} p_i & \partial_q p_i \end{pmatrix}\right|_{(\vec{\psi}_i, q_i)} \ . \tag{2.13}$$

Since $w_i \in W_i^u \cap W_{i+1}^s$, the trajectory $\phi^t(w_i)$ is a *heteroclinic trajectory* belonging to $W_i^u \cap W_{i+1}^s$ for all t. Thus, if we call $(\vec{A}_i^h(t), \vec{\psi}_i^h(t), p_i^h(t), q_i^h(t))$ the normal coordinates of $\phi^t(w_i)$ we see that

$$\vec{A}_i^h(t) \equiv \vec{A}_i'(0) \equiv \vec{A}_i(\vec{\psi}_i^h(t), q_i^h(t)) \ , \quad p_i^h(t) \equiv 0 \equiv p_i(\vec{\psi}_i^h(t), q_i^h(t)) \ . \tag{2.14}$$

Differentiating with respect to t such relation we find $\partial_q \vec{A}_i = \vec{a} \cdot \partial_{\vec{\psi}} \vec{A}_i$ and $\partial_q p_i = \vec{a} \cdot \partial_{\vec{\psi}} p_i$ with[10] $\vec{a} \equiv -\dot{\vec{\psi}}_i^h(t)/\dot{q}_i^h(t)$. Furthermore, differentiating with respect to $\vec{\psi}$ the energy relation $H \circ \mathcal{C}(\vec{A}_i(\vec{\psi}, q), \vec{\psi}, p_i(\vec{\psi}, q), q) = E$, we find $\partial_{\vec{\psi}} p_i = \vec{b} \cdot \partial_{\vec{\psi}} \vec{A}_i$ with $\vec{b} \equiv -\partial_{\vec{A}'}(H \circ \mathcal{C})/\partial_p(H \circ \mathcal{C})$ (see footnote 10). By these observations and (2.13) there follows

$$\operatorname{rank} M = \ell + \operatorname{rank} \begin{pmatrix} \partial_{\vec{\psi}} \vec{A}_i \\ \partial_{\vec{\psi}} p_i \end{pmatrix} = \ell + \operatorname{rank} \partial_{\vec{\psi}} \vec{A}_i \ .$$

[10] Recall that $\dot{q}_i^h(t) \neq 0$ by (the third relation in) (2.1) and by the equations of motion.

This implies that Δ_i^s and $W_{i,\mathrm{loc}}^u$ intersects transversally in $\mathcal{S}_E$ at w_i if and only if $\mathrm{rank}\,\partial_{\vec\psi}\vec A_i = \ell - 1$, which is (2.11). ∎

A Polynomial drifting times for standard chains

In this appendix we discuss some estimates on drifting–times for a model inspired by [1]. Such a model, which we shall call a "standard chain of transition tori", (besides satisfying (i) of § 2) is characterized by having a very strong normal form in a neighborhood of the tori (see (ii*) below, which will replace (ii) of § 2) and by having a simple rule describing the dynamics of the connection between "neighbouring tori" (see (iii) below).

The strong normal form is formulated as follows[11]:

(ii*) The dynamics in a neighbourhood U_i of $\mathcal{T}_i$ is described by coordinates $(\vec A', \vec\psi, p, q) \in \mathbb{R}^{\ell-1}\times\mathbb{T}^{\ell-1}\times\mathbb{R}\times\mathbb{R}$ as follows: there exists $r, \hat r, \tilde r > 0$, a canonical change of variables $\mathcal{C}$, a fixed number $g_i > 0$ and (real–analytic) functions of one variable $\vec A_i'$, $\vec\omega_i$, with $|\vec A_i'(J) - \vec A_i'(0)| < r$ for every $|J| < \hat r^2$, verifying (2.1). Moreover, if $\vec A', p, q, t$ satisfy $|pe^{-g_i t}| < \hat r$, $|qe^{g_i t}| < \hat r$, $\inf_{|J|<\hat r^2} |\vec A' - \vec A_i'(J)| = \mathrm{dist}\,(\vec A', \vec A_i'(J)) < r'$, then

$$\phi^t \circ \mathcal{C}(\vec A', \vec\psi, p, q)) = \mathcal{C}(\vec A', \vec\psi + \vec\omega_i(pq)\,t, pe^{-g_i t}, qe^{g_i t}) \ . \tag{A.1}$$

Furthermore, as in (ii), the dependence of $\vec\omega_i$ upon the variable $J = pq$ is scalar: $\vec\omega_i(J) = \tau_i(J)\vec\omega_i^0$ and the function τ_i is uniformly bounded and bounded away from zero (say positive); the constant vectors $\vec\omega_i^0$ are diophantine.

To describe the dynamics connecting neighbouring tori we introduce the following notation[12]:

$$B_{(r^{\vec A'}, r^{\vec\psi}, r^p, r^q)}(\vec A'_\star, \vec\psi_\star, p_\star, q_\star) \equiv \{|\vec A' - \vec A'_\star| \le r^{\vec A'}, \quad |\vec\psi - \vec\psi_\star| \le r^{\vec\psi}, \quad |p - p_\star| \le r^p, \quad |q - q_\star| \le r^q\} \ ,$$

and if Γ_i is the curve in $\Delta_i^s \subset W_{i+1}^s$ constructed in Lemma 1 we define, for any $|p| < \tilde r$, (smooth) functions $\vec\psi_i^s(p)$ and $T_i^\star(p)$ by the relation $\phi^{T_i^\star(p)} \circ \mathcal{C}(\Gamma_i(p)) = \mathcal{C}(\vec A_{i+1}'(0), \vec\psi_{i+1}^s(p), \hat r/2, 0)$. Then the connecting dynamics is described as follows:

(iii) There exists a constant $0 < \theta \le 1$ such that[13]

$$\phi^{T_i^\star(p)} \circ \mathcal{C}\left(B_{(r^{\vec A'}, r^{\vec\psi}, r^p, r^q)}(\vec{\mathcal{A}}_i'(p), \vec\psi_i(p), p, \hat r/2)\right) \supseteq \mathcal{C}\left(B_{(\theta r^{\vec A'}, r^{\vec\psi}, \theta r^q, \theta r^p)}(\vec A_{i+1}'(0), \vec\psi_{i+1}^s(p), \hat r/2, 0)\right) \ , \tag{A.2}$$

for any (suitably small) choice of the radii $r^{\vec A'}, r^{\vec\psi}, r^p, r^q$.

For definiteness, we assume also that τ_i is a *non increasing* function of[14] $J = pq$ and that $|\partial_p \vec\psi_i| < D$, $|\partial_p \vec{\mathcal{A}}_i'| < D$ for[15] some $D > 0$. Finally, we shall call a and b the constants of ergodization of the tori: $\forall\delta > 0$, $\forall i = 1, \dots, N$, $\forall\vec\psi \in \mathbb{T}^{\ell-1}$ there exists a time $t = t(i, \vec\psi, \delta)$ such that $|\vec\psi - \vec\omega_i^0\,t| \le \delta$, with $0 \le t \le a/\delta^b$ (compare [2] and [6]).

A chain of whiskered tori satisfying (i), (ii*) and (iii) above will be called a "standard transition chain".

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[11] In practice, we assume that (2.2) is valid not only on the curve $\vec A_i'(pq)$ but in a full (i–independent) neighbourhood of it: such assumption is verified, for example, in the "isocronous case" considered in [6]. Also, the Lyapunov exponent g_i is assumed to be *constant* (i.e. not depending on $J = pq$).

[12] Beware that p and q, here, are *never* exponents but just *suffixes*. The norm used on the $(\ell-1)$–vectors is, for fixing ideas, the max–norm: $|(v_1, \dots, v_{\ell-1})| \equiv \max_{1\le i\le \ell-1} |v_i|$.

[13] Notice the exchange between the p and q–radii: this exchange is by no means important and is done only to make the model more "realistic" (think to the dynamics of a perturbed pendulum).

[14] If τ is increasing, just replace p with $-p$.

[15] The functions $\vec{\mathcal{A}}_i'$ and $\vec\psi_i$ have been introduced in Lemma 1.

Proposition 2. *There are points near the first torus of a standard transition chain arriving near the N^{th} one in a time bounded above by a power of N.*

Proof. The argument is by induction. Fix a point ξ_1' of $W^s_{1,\text{loc}}$, and set $r_1^{\vec{A}'} = r_1^{\vec{\psi}} = r_1^p = r^\star$ and $r_1^q = cr^\star$ (with suitably small positive c and $r^\star$). Now, given, for $1 \le i \le N-1$, a point $\xi_i' \in W^s_{i,\text{loc}}$ and positive radii $r_i^{\vec{A}'}$, $r_i^{\vec{\psi}}$, r_i^p, r_i^q we construct a new point $\xi_{i+1}' \in W^s_{i+1,\text{loc}}$ and new radii $r_{i+1}^{\vec{A}'}$, $r_{i+1}^{\vec{\psi}}$, r_{i+1}^p, r_{i+1}^q such that

$$\phi^{\bar{T}_i} \circ \mathcal{C}\left(B_{(r_i^{\vec{A}'}, r_i^{\vec{\psi}}, r_i^p, r_i^q)}(\xi_i')\right) \supseteq \mathcal{C}\big(B_{(r_{i+1}^{\vec{A}'}, r_{i+1}^{\vec{\psi}}, r_{i+1}^p, r_{i+1}^q)}(\xi_{i+1}')\big) \ , \tag{A.3}$$

for a suitable time $0 \le \bar{T}_i \le i^{\text{const}}$.

We assume (as in § 2) that the normal coordinates of ξ_i' are given by $(\vec{A}_i'(0), \vec{\chi}_i', \hat{r}/2, 0)$. Set

$$\rho_i^\star \equiv r_i^{\vec{\psi}}(1 - e^{-1/(2i)^2}) \ , \quad p_i^{\max} \equiv r_i^q/2 \ , \quad \bar{g}_i \equiv g_i/\tau_i(p_i^{\max}\hat{r}/2) \ , \quad p_i^{\min} \equiv p_i^{\max}\, e^{-\bar{g}_i a/(\rho_i^\star)^b} \ ,$$

and define, $\forall p \in [p_i^{\min}, p_i^{\max}]$, $t_i(p) \equiv \big(1/g_i\big)\, \log(\hat{r}/(2p))$. Define also the new radii as follows:

$$\begin{array}{llll} \rho_{i+1}^{\vec{A}'} \equiv r_i^{\vec{A}'}/2\,, & \rho_{i+1}^{\vec{\psi}} \equiv r_i^{\vec{\psi}}\, e^{-1/i^2}\,, & \rho_{i+1}^{p} \equiv 2 r_i^p\, p_i^{\min}/\hat{r}\,, & \rho_{i+1}^{q} \equiv \hat{r}/2\,, \\ r_{i+1}^{\vec{A}'} \equiv \theta \rho_{i+1}^{\vec{A}'}\,, & r_{i+1}^{\vec{\psi}} \equiv \rho_{i+1}^{\vec{\psi}}\,, & r_{i+1}^{p} \equiv \theta\rho_{i+1}^{q}\,, & r_{i+1}^{q} \equiv \theta\rho_{i+1}^{p}\,. \end{array}$$

From these definitions it follows immediately, for $i \ge 1$, that

$$r_i^{\vec{A}'} = (\theta/2)^{i-1} r_1^{\vec{A}'} \ , \quad r_i^{\vec{\psi}} = r_1^{\vec{\psi}} \exp\Big(-\sum_{j=1}^{i-1}\frac{1}{j^2}\Big) \ , \quad r_i^p = \theta\hat{r}/2 \ ,$$

$$r_{i+1}^q = \frac{\theta^2}{2}\, r_i^q\, e^{-\bar{g}_i a/(\rho_i^\star)^b} = \ldots = \Big(\frac{\theta^2}{2}\Big)^{i-1} r_2^q \,\exp\Big(-\sum_{j=2}^{i}\bar{g}_j a/(\rho_j^\star)^b\Big) \ . \tag{A.4}$$

Choosing $r^\star$ suitably one sees that $r_2^q \le r_1^q\theta^2/2$: then (A.4) implies that, $\forall 1 \le i \le N$, $r_i^q \le (\theta^2/2)^{i-1}\cdot r_1^q$ and so, if we choose c suitably,

$$p_i^{\max} \le \min\Big\{ r_i^{\vec{A}'}/(2D),\, r_i^{\vec{\psi}}(e^{-1/(2i)^2} - e^{-1/i^2})/D \Big\} \ , \qquad \forall 1 \le i \le N \ . \tag{A.5}$$

Now (recalling the definition of t_i and that τ_i is not increasing) $\tau_i(p_i^{\min}\hat{r}/2)\, t_i(p_i^{\min}) - \tau_i(p_i^{\max}\hat{r}/2)\, t_i(p_i^{\max}) \ge (1/\bar{g}_i)\Big(\log(\hat{r}/(2p_i^{\min})) - \log(\hat{r}/(2p_i^{\max}))\Big) = a/(\rho_i^\star)^b$. Hence, there exists $p_i^\star \in [p_i^{\min}, p_i^{\max}]$ such that

$$|\vec{\psi}_i(0) - \vec{\omega}_i(p_i^\star\hat{r}/2)\, t_i(p_i^\star) - \vec{\chi}_i'| = |\vec{\psi}_i(0) - \vec{\omega}_i^0\, \tau_i(p_i^\star\hat{r}/2)\, t_i(p_i^\star) - \vec{\chi}_i'| \le \rho_i^\star \ . \tag{A.6}$$

Setting $\xi_i \equiv \Big(\vec{A}_i'(p_i^\star), \vec{\psi}_i(p_i^\star), p_i^\star, \hat{r}/2\Big)$ and $t_i^\star \equiv t_i(p_i^\star)$, one can check (using the definition of $t_i(p)$, (A.5), (A.6) and (A.1)) that

$$\mathcal{C}\big(B_{(r_i^{\vec{A}'}, r_i^{\vec{\psi}}, r_i^p, r_i^q)}(\xi_i')\big) \supseteq \phi^{-t_i^\star} \circ \mathcal{C}\left(B_{(\rho_{i+1}^{\vec{A}'}, \rho_{i+1}^{\vec{\psi}}, \rho_{i+1}^p, \rho_{i+1}^q)}(\xi_i)\right) \ . \tag{A.7}$$

Furthermore, (A.4) implies that $r_i^q \ge e^{-\text{const}\, i^{2b+1}}$. Thus

$$t_i^\star \le \text{const}\log(\hat{r}/(2p_i^{\min})) \le \text{const}\cdot a/(\rho_i^\star)^b + \text{const}\log(\hat{r}/r_i^q) \le \text{const}\, i^{2b+1} \ .$$

Let $T_i^\star$ be the time (of order 1) needed to carry ξ_i in U_{i+1} with normal coordinates q equal to $\hat{r}/2$, i.e. $\phi^{T_i^\star}(\xi_i) = \mathcal{C}(\vec{A}_{i+1}'(0), \vec{\chi}_{i+1}', \hat{r}/2, 0)$ for a suitable $\vec{\chi}_{i+1}' \in \mathbb{T}^{\ell-1}$. Setting $\xi_{i+1}' \equiv \phi^{T_i^\star}(\xi_i)$ and $\bar{T}_i \equiv t_i^\star + T_i^\star$, one obtains (A.3), thanks to (A.2) and to the definitions of the r_{i+1}'s in terms of the ρ_{i+1}'s.

Iterating (A.3), we have $\phi^{T^\star} \circ \mathcal{C}\left(B_{(r_1^{\vec{A}'}, r_1^{\vec{\psi}}, r_1^p, r_1^q)}(\xi_1')\right) \supseteq \mathcal{C}\big(B_{(r_N^{\vec{A}'}, r_N^{\vec{\psi}}, r_N^p, r_N^q)}(\xi_N')\big)$, with $0 \le T^\star \equiv \bar{T}_1 + \ldots + \bar{T}_N \le N^{\text{const}}$. ∎

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References

[1] V.I. Arnold, *Instabilty of dynamical systems with several degrees of freedom*, Soviet Math. Dokl., **5**, 1964, p. 581.

[2] J. Bourgain, F. Golse, S. Wennberg, *The ergodisation time for linear flows on tori: application to kinetic theory*, preprint, 1995.

[3] L. Chierchia, G. Gallavotti, *Drift and diffusion in phase space*, Ann. Inst. Henri Poincaré, **60**, **1**, 1994, p. 1.

[4] L. Chierchia, G. Gallavotti, *Erratum for "Drift and diffusion in phase space", Ann. Inst. Henri Poincaré, 60, 1, 1994*, Ann. Inst. Henri Poincaré **68**, **1**, 1998, p. 135.

[5] J. Cresson, *Propriétés d'instabilité des systèmes Hamiltoniens proches de systèmes intégrabiles*, Doctoral dissertation, Observatoire de Paris, 1997.

[6] G. Gallavotti, *Fast Arnold's diffusion in isochronous systems*, preprint, 1997.

[7] J.P. Marco, *Transition le long des chaînes de tores invariants pur les systèmes hamiltoniens analytiques*, Ann. Inst. Henri Poincaré, **64**, **2**, 1996, p. 205.

[8] L. Niederman, *Dynamic around a chain of simple resonant tori in nearly integrable Hamiltonian systems*, preprint, 1997.

[9] E. Valdinoci, *Whiskered transition tori in "a priori" stable & unstable hamiltonian systems*, preprint, 1998.