RENEWAL SEQUENCES AND INTERMITTENCY

Stefano Isola

Dipartimento di Matematica, Università degli Studi di Bologna, piazza di Porta S.Donato 5, I-40127 Bologna, Italy.

e-mail: isola@dm.unibo.it

Abstract. In this paper we examine the generating function $\Phi(z)$ of a renewal sequence arising from the distribution of return times in the 'turbulent' region for a class of piecewise affine interval maps introduced by Gaspard and Wang[(1)] and studied by several authors[(2−8)]. We prove that it admits a meromorphic continuation to the entire complex z-plane with a branch cut along the ray $(1, +\infty)$. Moreover we compute the asymptotic behaviour of the coefficients of its Taylor expansion at $z = 0$. From this, the exact polynomial asympotics for the rate of mixing when the invariant measure is finite and of the scaling rate when it is infinite are obtained.

0. INTRODUCTION.

The Pomeau-Manneville[9] type 1 intermittency model (at the tangent bifurcation point) consists of a class of smooth transformations $f:[0,1]\to[0,1]$ which are expanding everywhere but at a neutral fixed point at the origin. Such *intermittent interval maps* provide with no doubt the simplest examples of 'chaotic' dynamical systems with anomalous statistical behaviour. For instance they may possess but a σ-finite non-normalizable invariant measure[10−12]. Or else, they may leave invariant a probability measure with slow (e.g. polynomial) speed of mixing[14−16]. Finally, in the framework of thermodynamic formalism, they may exhibit phase transitions[6−8] and dynamical ζ-functions with non-polar singularities[14,18].

Here we consider a one parameter family of linearized intermittent interval maps (see eq.(1.2) below) introduced by Gaspard and Wang[1,2]. As far as its statistical properties are concerned, it is equivalent to a Markov chain with countable state space[13]. It is also related to the statistical mechanics model introduced by Fisher[17] and successively studied by Gallavotti[18] (see also Hofbauer[19]). Without pretending to be a typical example, the main advantage of this approximation scheme is that it partially allows for exact calculations.

The main concern of this paper is the study of the generating function $\Phi(z)$ of a renewal sequence arising from the distribution of return times in a region where the map is uniformly expanding. This function coincides, up to a factor $(1-z)^{-1}$, with the Ruelle dynamical ζ-function[30], and is shown to admit a meromorphic continuation to the the entire complex z-plane with a branch cut along the ray $(1,+\infty)$. It appears that finding analytic continuation of dynamically defined functions, which are holomorphic in a domain given *a priori*, can be a rewarding mathematical achievement in itself. Moreover, it will be shown that the main statistical features of this dynamical system such as the rate of mixing or the scaling rate, are embodied in the behaviour of its Taylor coefficients.

1. PRELIMINARIES.

Let q, r, s be three real parameters satisfying $0 < q < 1$, $s > 0$ and $r + 1 = q^{-s}$. Let moreover d_n, $n \geq 0$, be defined by

$$d_0 = 1, \;\; d_n = (1 + nr)^{-1/s}, \quad n \geq 1. \tag{1.1}$$

In particular $d_1 = q$. The numbers d_n form a Kaluza sequence[20], i.e. they satisfy:

$$0 < d_n \leq d_0 = 1, \qquad d_n^2 \leq d_{n-1} d_{n+1}, \tag{1.2}$$

which is equivalent to the assertion that $d_{n+1} - 2\lambda d_n + \lambda^2 d_{n-1} \geq 0$ for all $\lambda > 0$. Moreover they generate a countable partition of $[0, 1]$ into the intervals $A_n = [d_n, d_{n-1}]$, $n \geq 1$, and the length ρ_n of the interval A_n has the asymptotic behaviour

$$\rho_n = d_{n-1} - d_n = \; (r/s)\,(1 + nr)^{-1-1/s} + \mathcal{O}((1 + nr)^{-2-1/s})\,. \tag{1.3}$$

Set $\alpha_n = \rho_n / \rho_{n-1}$, $n \geq 1$ (with $A_0 = [0, 1]$). One then readily verifies that $1 \geq \alpha_n > \alpha_{n-1}$ and $\lim \alpha_n = \sup \alpha_n = 1$. We shall consider the piecewise affine map $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} (x - d_1)/\alpha_1, & \text{if } x \in A_1 \\ d_{k-1} + (x - d_k)/\alpha_k, & \text{if } x \in A_k, \; k \geq 2. \end{cases} \tag{1.4}$$

This map has been introduced as as a simplified model for an intermittent interval map whose behaviour when $x \to 0^+$ is given by

$$f(x) = x + u\, x^{1+s} + \mathcal{O}(x^{1+s+\epsilon})$$

where $u = r/s$ and $\epsilon > 0$. Observe that the fixed point at the origin is neutral: $f'(0) = 1$, and $f'(x)$ is only Hölder continuous at $x = 0$, with exponent s.
We finally notice that $f(A_n) = A_{n-1}$ for any $n \geq 1$, so that $\{A_n\}$ is a Markov partition for the map f.

A countable Markov chain. One can say more: the iteration process $x_n = f^n(x)$, with f as above, is actually isomorphic (mod 0) to a Markov chain with state space $I\!N$ and transition matrix $P = (p_{ij})$ given by

$$P = \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.5}$$

To see this, let X be the residual set of points in $(0,1]$ which are not preimages of 1 with respect to the map f, namely $X = (0,1] \setminus \{d_n\}_{n\geq 0}$. Let moreover Ω be the set of all one-sided sequences $\omega = (\omega_0\omega_1 \dots)$, $\omega_i \in I\!N$ s.t. given ω_i then $\omega_{i-1} = \omega_i + 1$ or $\omega_{i-1} = 1$. Then the map $\varphi : \Omega \to [0,1]$ defined by

$$\varphi(\omega) = x \quad \text{according to} \quad f^j(x) \in A_{\omega_j}, \;\; j \geq 0$$

is a bijection between Ω and X and conjugates the map f with the shift T on Ω. It is then immediate to check that the stochastic process on Ω given by $x_i(\omega) = \omega_i$, $i \geq 0$, is a Markov chain with conditional probabilities $p_{ij} = P(x_n(\omega) = j \,|\, x_{n-1}(\omega) = i) = m(f^{-1}(A_j) \bigcap A_i)/m(A_i)$, which coincide with those in (1.5). Since g.c.d.$\{n : \rho_n > 0\} = 1$ the chain is aperiodic and recurrent. Consider the infinite sequence $t_1, t_2, \dots$ of successive entrance times in the state 1: $t_1 = \inf\{i \geq 0 \;:\; \omega_i = 1\}$ and, for $j \geq 2$, $t_j = \inf\{i > t_{j-1} \;:\; \omega_i = 1\}$. Let moreover $r_j = t_{j+1} - t_j$ be the sequence of times between returns. The state 1 being recurrent, the numbers r_j are i.i.d.r.v. under the probability $P_1(\cdot) = P(\;\cdot\;|x_0(\omega) = 1)$. Their common distribution is $P_1(r_j = n) = \rho_n$ and their expectation value is given by $E_1(r_j) = \sum n\,\rho_n = \sum d_n$, which may be finite (positive-recurrent chain) or infinite (null-recurrent chain) according whether $s < 1$ or $s \geq 1$. More specifically, we can define a family of moments

$$M^{(\ell)} = E_1(r_j^\ell) \equiv \sum n^\ell\, \rho_n, \qquad \ell \geq 0, \tag{1.6}$$

and say that the chain has *ergodic degree* ℓ if $M^{(\ell)} < \infty$ but $M^{(\ell+1)} = \infty$. Notice that $M^{(0)} = 1$, so that the chain has degree at least zero (null-recurrent case).

Finally, the steady-state equation is $\pi_n = \sum_{i\in S} \pi_i\, p_{in}$ and is formally solved by $\pi_n = \pi_1\, d_{n-1}$, $n \geq 1$. In the positive-recurrent case one finds $\pi_1 = (\sum d_n)^{-1}$. For more details on this Markov chain we refer to Isola[13].

Invariant measure and return times. We now return to our interval map f. An easy consequence of the previous discussion is that it preserves an absolutely continuous σ-finite measure ν, whose density e is given by

$$e(x) = d_{n-1}/\rho_n, \qquad d_n < x \leq d_{n-1}. \tag{1.7}$$

It may be noted that

$$\nu(A_n) = d_{n-1} = \sum_{l\geq n} m(A_l), \tag{1.8}$$

where m denotes the Lebesgue measure. More specifically, if $E \subseteq A_n$, one readily sees that

$$\nu(E) = \sum_{k\geq 0} m(A_{n+k} \cap f^{-k}E) = \sum_{k\geq 0} \frac{m(E)\cdot m(A_{n+k})}{m(A_n)} = \frac{m(E)}{m(A_n)}\, d_{n-1}. \tag{1.9}$$

Let $\tau : X \to I\!N$ be the *first passage time* in the interval A_1, that is

$$\tau(x) = 1 + \min\{n \geq 0 \ : \ f^n(x) \in A_1 \ \}, \tag{1.10}$$

so that A_n is the closure of the set $\{x \in X \ : \ \tau(x) = n\}$. On the other hand, the *return time* function $r : X \to I\!N$ in the interval A_1 is given by

$$r(x) = \min\{n \geq 1 \ : \ f^n(x) \in A_1 \ \} = \tau \circ f(x). \tag{1.11}$$

Let B_n = closure of $\{x \in A_1 \ : \ r(x) = n\}$. Clearly $B_n = A_1 \cap f^{-1}A_n \subset A_1$, and therefore, using (1.9), we get

$$\nu(B_n) = \frac{m(A_1 \cap f^{-1}A_n)}{m(A_1)} = m(A_n). \tag{1.12}$$

Putting togheter (1.8) and (1.12) we have the following chain of identities:

$$\begin{aligned} \nu([0,1]) &= \sum \nu(A_n) = \sum n\cdot m(A_n) = \int_0^1 \tau(x)\, m(dx) \\ &= \sum_n n\cdot \nu(B_n) = M^{(1)} = \int_{A_1} r(x)\, \nu(dx), \end{aligned} \tag{1.13}$$

which is a version of Kac's formula. Using the shorthand $M \equiv M^{(1)}$ we then have the following dichotomy: either $M < \infty$, and then there exists an f-invariant a.c. probability measure $\mu = \nu/M$; or $M = \infty$ so that ν is not normalizable and no invariant a.c. probability measure exists. In the latter case, the ergodic means $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}$ converge weakly to the Dirac delta at $0^{(21,11)}$.

For later use, we now define recursively a family of formal 'tail sequences' $d_n^{(\ell)}$, with $\ell \geq 0$, derived from d_n as follows:

$$d_n^{(0)} = d_n \quad \text{and} \quad d_n^{(\ell)} = \sum_{l>n} d_l^{(\ell-1)} \quad \text{for} \quad \ell > 0. \tag{1.14}$$

Moreover we say that a_n and b_n are asymptotically equivalent as n approaches ∞, denoted as $a_n \sim b_n \quad (n \to \infty)$, if the quotient a_n/b_n tends to unity. From (1.1) (see also (1.3)) we have that if $s < 1/\ell$, with $\ell \geq 1$, then the terms $d_n^{(k)}$ are finite for $0 \leq k \leq \ell$ and satisfy

$$d_n^{(k)} \sim (1 + (n+k)r)^{k-\frac{1}{s}}. \tag{1.15}$$

It is also easy to check that $M^{(\ell)}$ is finite if and only if $d_n^{(\ell)}$ is. Of special importance will be the asymptotic behaviour of $d_n^{(1)}$, when $s < 1$:

$$d_{n-1}^{(1)} = \nu\,(x \in X \,:\, \tau(x) > n) \sim n^{1-\frac{1}{s}}. \tag{1.16}$$

2. THE GENERATING FUNCTION OF THE RETURN TIMES DISTRIBUTION.

If we view the element A_n of the countable Markov partition introduced in Section 1 as the n-th 'state' for our dynamical system, the number ρ_n can be interpreted as the m-probability that a first passage in the state 1 occurs after n iterates. Let us now consider the quantity $u_n := m(f^{-(n-1)}A_1)$ which, for $n \geq 1$, gives the m-probability to observe a *passage* in the state 1 after $n-1$ iterates (for the first time or not). We shall see in the next Section that u_n is also equal to $\nu(A_1 \cap f^{-n}A_1)$, and can thus be interpreted as the ν-probability to observe a *return* in the state 1 after n iterates. According to the discussion given in the previous Section, the iteration process $x_n = f^n(x)$ 'starts afresh' at each passage in the state 1, and the sequence $u_0, u_1, \dots$ satisfies the recurrence relation

$$u_0 = 1 \quad \text{and} \quad u_n = \rho_n + u_1\,\rho_{n-1} \cdots + u_{n-1}\,\rho_1 \quad \text{for} \quad n \geq 1. \tag{2.1}$$

In other words, $u_0, u_1, \dots$ is the *renewal sequence*(20) associated with the sequence $\rho_1, \rho_2, \dots$. We now turn to the study of the generating function $\Phi(z)$ of the sequence u_n, which is given by the following formal power series

$$\Phi(z) = \sum_{n=0}^{\infty} u_n z^n = \left(1 - \sum_{n=1}^{\infty} \rho_n\, z^n\right)^{-1} = \left((1-z)\sum_{n=0}^{\infty} d_n z^n\right)^{-1}. \tag{2.2}$$

The next result can be viewed as a sharpening of a renewal theorem proved by Erdös, Feller and Pollard(22).

Theorem 2.1 (Part one). *The power series defined in (2.2) defines a holomorphic function $\Phi(z)$ in the open unit disk and converges at every point of the unit circle with the exception of $z = 1$, where it has a non-polar singular point. Moreover, one has the following asymptotic behaviour of the coefficients u_n:*

a) for $s < 1$ we have $u_n \sim (1 + n^{1-1/s})/M$;

b) for $s \geq 1$ we have

$$u_n \sim \begin{cases} n^{-1+1/s}, & \text{if } s > 1 \\ 1/\log n & \text{if } s = 1. \end{cases}$$

Proof. We first notice that

$$0 \leq \frac{1}{\sum_{n=0}^{\infty} d_n} = \frac{1}{M} < 1.$$

It is then easy to see that the function $D(z) := \sum_{n=0}^{\infty} d_n z^n$ has no zeros for $|z| \leq 1$. Indeed, for $|z| < 1$ this follows from (2.2), since $\rho_n > 0$ and therefore $|\sum_{n=1}^{\infty} \rho_n z^n| < 1$ for $|z| < 1$. Furthermore, from the above identity it follows that any zeros of $D(z)$ must be of the form $e^{i\phi}$, $0 < \phi < 2\pi$. Now, if $D(e^{i\phi}) = 0$ then (2.2) implies $\sum_{n=1}^{\infty} \rho_n\, e^{in\phi} = 1$, that is $\cos(n\phi) = 1$, $\forall n \geq 1$, which is impossible. Then the function $1/D(z)$ has no singularities in $|z| < 1$ and we can expand it in a power series $1/D(z) = \sum_{n=0}^{\infty} \gamma_n z^n$. Notice that $\gamma_0 = 1$. Set $h_n = -\gamma_n$ $(n \geq 1)$. We can then say more. By the property (1.2) of the sequence d_n, we can apply Hardy[(23)], Theorem 22, and obtain

$$h_n \geq 0, \qquad \sum_{n=1}^{\infty} h_n \leq 1. \tag{2.3}$$

In addition, if $M = \infty$, then $\sum_{n=1}^{\infty} h_n = 1$. In particular, it appears that $1/\sum_{n=0}^{\infty} d_n z^n$ is absolutely convergent for $|z| \leq 1$. This yields the announced analytic properties of $\Phi(z)$ (the nature of the singularity at $z = 1$ will be clarified at the end of the proof).

To show statement (b), we start noticing that $u_n = 1 - h_1 - \ldots - h_n$ so that, if $M = \infty$, the sequence a_n decreases monotonically to 0. Assertion (b) then follows from (1.14) and a repeated application of a Tauberian theorem for power series (see, e.g., Feller[(24)], chap XIII.5, Theorem 5).

Next, we are going to prove statement (a), for $s < 1$. In this case, we have $u_n \to 1/M$ as $n \to \infty$. To obtain more information we first note that the relation

$$\sum_{n=0}^{\infty} u_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} z^n$$

implies

$$\sum_{n=0}^{\infty} v_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} d_n^{(1)} z^n \tag{2.4}$$

where $d_n^{(1)}$ is defined in (1.14) (see also (1.16)) and

$$v_n := M\, u_n - 1 \quad (n \geq 0). \tag{2.5}$$

Moreover we have $v_n = M \sum_{l>n} h_l$, so that the sequence v_n is positive and decreases monotonically to 0.

Put first $1/2 \leq s < 1$. Then, according to (1.15), the term $d_0^{(1)}$ is finite and the power series $\sum_{n=0}^{\infty} d_n^{(1)} z^n$ is divergent at $z = 1$. Thus, for these values of s, a direct application of (1.15) and the same Tauberian theorem for power series used above give $v_n \sim n^{1-1/s}$ and hence (a). Furthermore, using again (1.15), we have that if $1/(\ell+1) \leq s < 1/\ell$, with $\ell > 1$, then for $k \leq \ell$ the terms $d_0^{(k)}$ are finite and the power series $\sum_{n=0}^{\infty} d_n^{(\ell)} z^n$ is divergent at $z = 1$. On the other hand, it is easy to check that under these circumstances (2.4) can be rewritten in the following way:

$$\begin{aligned} \sum_{n=0}^{\infty} v_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n \; = \; & (z-1)^{\ell-1} \sum_{n=0}^{\infty} d_n^{(\ell)} z^n + \\ & + \sum_{k=2}^{\ell} (z-1)^{k-2} \left(d_0^{(k-1)} + d_0^{(k)} \right) \end{aligned} \tag{2.6}$$

so that the claimed result follows using the same reasoning as above, along with the positivity and monotonicity of the sequences $d_n^{(\ell)}$.

It remains to show that $z = 1$ is a non-polar singularity for $\Phi(z)$. Now, from (1.15) we have that if $s \geq 1$, then $(1-z)\Phi(z) \to 0$ even though $\Phi(z) \to \infty$ as $z \to 1_-$. Moreover, if $1/(\ell+1) \leq s < 1/\ell$, then, denoting by $H_\ell(z)$ the expression in the r.h.s. of (2.6) and using again (1.15), we have $(z-1)^\ell H_\ell(z) \to 0$ but $(z-1)^{\ell-1} H_\ell(z) \to \infty$ as $z \to 1_-$. The assertion then follows for each of these functions, and in particular for $\Phi(z)$. q.e.d.

Let us now observe that the coefficients d_n can be considered as values of a function $d(x)$ when x ranges over the natural numbers. One may then examine the relation between the analytic properties of the function $d(x)$ determining the coefficients and those of the function defined by $D(z) = \sum_n d_n z^n$ (see for instance Dienes[(26)], p.335). Along these lines we now prove the following theorem.

Theorem 2.1 (Part two). *The function $\Phi(z)$ can be continued meromorphically to the entire z-plane with a branch cut along the ray $(1, +\infty)$. The meromorphic continuation is given by the formula, valid for any $\delta > 0$,*

$$\Phi(z) = \frac{1}{(1-z)} \left(\frac{1}{2\pi i} \int_1^{+\infty} \int_{\mathrm{Re}\, x = \delta} d(x) \frac{t^{-x}}{t-z} dx dt \right)^{-1}$$

where $d(x) = (1 + rx)^{-1/s}$.

Proof. The following proof relies on standard techniques of analytic continuation of power series based on the use of the Mellin transform. The first step in this approach is the construction of a function $d(x)$ defined on $I\!R_+$, which reproduces the numbers d_n at $x = n$ and extends to a function regular in the half-plane $\mathrm{Re}\, x > 0$. For our example this construction is effortless: $d(x) = (1 + rx)^{-1/s}$. Nevertheless we shall sketch below a procedure which may be applied in more general situations, e.g. when the d_n's are not explicitly known. To this end, we first notice that if $\psi_0(x)$ denotes the inverse branch of f which maps $[0, 1]$ onto $[0, q]$, then we have $d_n = \psi_0^n(1)$. Let moreover $\psi : [0, 1] \to [0, q]$ be a suitable smooth function which interpolates ψ_0 at those points: $\psi(d_n) = d_{n+1}$, so that $d_n = \psi^n(1)$ as well. Now, a standard method for dealing with the asymptotic behaviour of iterated functions starts considering the Abel equation (see, e.g., de Bruijn[(27)], p.160): $G(\psi(x)) = G(x) + 1$. If G is known, up to an additive constant, and ψ satisfies the above equation, one finds ψ^n by solving $G(\psi^n(x)) = G(x)+n$ for $\psi^n(x)$. Suppose then to be able to determine a solution $G : [q, 1] \to [0, 1]$ of the Abel equation[1], satisfying $G(1) = 0$ and $G(q) = 1$. Let $F(x) = G^{-1}(x) : [0, 1] \to [q, 1]$. A candidate for the function $d(x)$ is then obtained by extending $F(x)$ to $I\!R_+$ as follows:

$$d(x) = \psi^n\big(F(x - n)\big), \quad n \le x \le n + 1, \quad n \ge 0.$$

It is easy to check that in our case the function

$$\psi(x) := x(1 + rx^s)^{-1/s} \tag{2.7}$$

satisfies the above requirement[2] and a real analytic solution $G : [q, 1] \to [0, 1]$ of the Abel equation (with ψ as in (2.7)) which satisfies $G(1) = 0$ and $G(q) = 1$ is

[1] The problem of the existence of solutions of the Abel equation for a broad class of intermittent maps has been investigated by Prellberg[(6)].

[2] By the way, (2.7) is but the exact solution of the fixed point equation for the renormalization transformation with intermittency boundary conditions[(28)].

given by $G(x) = (x^{-s} - 1)/r$, and its inverse is $F(x) = G^{-1}(x) = (1 + rx)^{-1/s}$. We therefore get $\psi^n(x) = F(G(x)+n) = x(1+nrx^s)^{-1/s}$ and $d(x)$ as announced above. Accordingly, the function $d(x)$ extends to a function regular in the half-plane $\operatorname{Re} x > 0$ and, for any $\delta > 0$,

$$d(x) \to 0, \quad d'(x) = \mathcal{O}(x^{-1-\frac{1}{s}}), \quad x \to \infty, \quad \operatorname{Re} x \geq \delta$$

uniformly in $\arg x$. We can then proceed as in Evgrafov[29], Section VII, Theorem 6.1. First, we take the Mellin transform of $d(-x)$,

$$d(-x) = \int_1^\infty w(t)\, t^x\, \frac{dt}{t}, \qquad w(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} x = -\delta} d(-x)\, t^{-x}\, dx.$$

In the first expression we put $x = -n$ and multiply by z^n. Taking $|z|$ smaller than the distance from the origin to the contour $(1, \infty)$, that is $|z| < 1$, we sum over $n \geq 0$ and, as we may interchange the order of summation and integration, we get

$$\sum_{n=0}^{\infty} d_n z^n = \int_1^\infty \sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}}\, w(t)\, dt = \int_1^\infty \frac{w(t)}{t - z} dt.$$

The last integral converges uniformly in any closed region not containing points of the ray $(1, \infty)$. We finish the proof by inserting the representation of $w(t)$ in the above integral q.e.d.

Remarks.

1. If one wishes, one can investigate the behaviour of $\Phi(z)$ in a neighborhood of the branch point $z = 1$ with the help of the above formula. For example, taking $s = 1$ one finds that $\Phi(z)$ has a logarithmic branch point at $z = 1$.

2. Consider the dynamical zeta function $\zeta(z)$ defined by the following formal series[30]:

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n, \qquad Z_n = \sum_{x = f^n(x)} \prod_{k=0}^{n-1} \frac{1}{|f'(f^k(x))|}.$$

It is an easy task to realize that $Z_n = 1 + \operatorname{tr}(P_N)^n$ provided $N > n$, where P_N is the $N \times N$ truncation of the transition matrix (1.5) and the 1 comes from the neutral fixed point. A staightforward algebraic calculation then gives

$$
\begin{aligned}
(1-z)/\zeta(z) &= \lim_{N\to\infty} \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(P_N)^n \\
&= \lim_{N\to\infty} \det(I - zP_N) = 1 - \sum_{n=1}^{\infty} \rho_n\, z^n
\end{aligned}
$$

and therefore

$$\zeta(z) = (1-z)^{-1}\,\Phi(z).$$

The above identity and Theorem 2.1 (Part two) answer a question raised by Dalqvist[(31)] for this particular model (see also Rugh[(32)] for related results on Fredholm determinants).

3. SCALING AND MIXING RATES.

Given $U, V \in L^2([0,1], \mathcal{B}, \nu)$ one may consider the formal power series $S_{UV}(z)$ given by

$$S_{UV}(z) := \sum_{n=0}^{\infty} z^n \nu(U \cdot V \circ f^n).$$

Take first $U = V = \chi_1$, the indicator function of the interval A_1. We have $\nu(\chi_1 \cdot \chi_1 \circ f^n) = \nu(A_1 \cap f^{-n} A_1) =: u_n^{(1)}$, which is the ν-probability to observe a return in the state 1 after n iterates (recall that $\nu(A_1) = 1$). Clearly, it satisfies the recurrence relation:

$$u_0^{(1)} = 1 \quad \text{and} \quad u_n^{(1)} = u_0^{(1)}\,\nu(B_n) + \cdots + u_{n-1}^{(1)}\,\nu(B_1) \quad \text{for} \quad n \geq 1, \tag{3.1}$$

where $B_n = \{x \in A_1 \,:\, r(x) = n\}$ and $r(x)$ is defined in (1.11). On the other hand we know that $\nu(B_n) = m(A_n) \equiv \rho_n$ and, comparing with (2.1), we get $u_n^{(1)} \equiv u_n$, $\forall n$. We then have the following

Proposition 3.1. $S_{\chi_1\chi_1}(z) = \Phi(z)$.

Furthermore, if $M < \infty$, we can consider the generating function of the correlation function of probability measure $\mu = \nu/M$ for the observable χ_1. An easy calculation shows that

$$\sum_{n=0}^{\infty} z^n \left(\mu(A_1 \cap f^{-n} A_1) - (\mu(A_1))^2 \right) = (\mu(A_1))^2 \cdot \sum_{n=0}^{\infty} v_n z^n \tag{3.2}$$

where $\mu(A_1) = 1/M$ and the v_n's are defined in (2.5). Putting together Theorem 2.1 (Part one), Proposition (3.1) and equation (3.2) we obtain the following

Theorem 3.2.

a) if $s < 1$, then

$$\mu(A_1 \cap f^{-n} A_1) - (\mu(A_1))^2 \sim (\mu(A_1))^2 \; n^{1-\frac{1}{s}};$$

b) if $s \geq 1$, then

$$\lim_{n\to\infty} \nu(A_1 \cap f^{-n} A_1) = 0$$

but

$$\lim_{n\to\infty} \frac{1}{u_n} \nu(A_1 \cap f^{-n} A_1) = (\nu(A_1))^2 = 1$$

with u_n given in (2.1).

We now briefly dwell upon some consequences of the above result.

Mixing rate when the invariant measure is finite. Statement (a) of Theorem 3.2 can be generalized to a certain extent. Given $k \in \mathbb{Z}^+$, $k > 1$, the analogous of relation (3.1) for $u_n^{(k)} := \nu(A_k \cap f^{-n} A_k)$ reads $u_n^{(k)} = 0$ for $n < k$, and

$$\frac{u_n^{(k)}}{\nu(A_k)} = u_0 \, \rho_n + \cdots + u_l \, \rho_{n-l}, \quad n = k + l, \quad l \geq 0.$$

Therefore we have

$$\frac{1}{\nu(A_k)} \sum_{n=0}^{\infty} z^n \nu(A_k \cap f^{-n} A_k) = \Phi(z) \cdot \sum_{n \geq k} z^n \, \rho_n \tag{3.3}$$

We now use the following result, whose proof can be found in Chung[(33)], Chap. I.5, Lemma A.

Lemma 3.3. *Let $\{s_n\}_{n\geq 0}$ be a sequence of nonnegative numbers not all vanishing. If*

$$\lim_{n\to\infty} \frac{s_n}{\sum_{m=0}^{n} s_m} = 0$$

then, whenever the sequence $\{t_n\}_{n\geq 0}$ of real numbers has a limit, we have

$$\lim_{n\to\infty} \frac{\sum_{m=0}^n s_m t_{n-m}}{\sum_{m=0}^n s_m} = \lim_{n\to\infty} t_n.$$

Using this result with $t_m = u_m$ and $s_m = \rho_{k+m}$ along with (3.3), we see that

$$\nu(A_k \cap f^{-n} A_k) \sim \nu(A_k) \cdot \left(\sum_{m=0}^{n-k} \rho_{k+m} \right) \cdot u_{n-k} \sim (\nu(A_k))^2 \cdot u_{n-k} \sim (\nu(A_k))^2 \cdot u_n \tag{3.4}$$

where the last asymptotic equivalence holds for each fixed $k \in \mathbb{Z}^+$. This gives

$$\mu(A_k \cap f^{-n} A_k) - (\mu(A_k))^2 \sim (\mu(A_k))^2 \; n^{1-\frac{1}{s}}. \tag{3.5}$$

The same reasoning gives for a subset $E \subseteq A_k$, with $m(E) > 0$,

$$\nu(E \cap f^{-k-l}(E)) \sim \nu(E) \cdot \left(\sum_{m=0}^{l} m(f^{-m}(E) \cap A_{k+m}) \right) \cdot u_l \sim (\nu(E))^2 \cdot u_{l+k}$$

where (1.9) has been used. We then get again

$$\mu(E \cap f^{-n} E) - (\mu(E))^2 \sim (\mu(E))^2 \; n^{1-\frac{1}{s}}. \tag{3.6}$$

In an entirely analogous way one shows that (3.6) holds true for $E \subseteq \cup_{l\in J} A_l$ where $J \subset \mathbb{Z}^+$ is any given finite set.

Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$ and consider the quadruple $([0,1], \mathcal{B}, \nu, f)$. Clearly $\mathcal{A}$ generates $\mathcal{B}$. Given $E \in \mathcal{B}$, we define the *mixing rate* $\mu_n(E)$ of E as

$$\mu_n(E) := \frac{\mu(E \cap f^{-n} E) - (\mu(E))^2}{(\mu(E))^2}. \tag{3.7}$$

According to the above (see e.g. (3.4)) the mixing rate is not uniform in $E \in \mathcal{B}$. To recover uniformity we define

$$B_+ := \cup_\epsilon \{ E \in \mathcal{B} : \; m(E) > 0, \; E \subseteq [0,1] \setminus (0,\epsilon) \}, \tag{3.8}$$

where the union can be restricted to rational (positive) ϵ because the set in the union increases as ϵ decreases. An easy consequence of the above discussion is the following result

Lemma 3.4. *Let $E, F \in B_+$. Then $\mu_n(E) \sim \mu_n(F)$.*

Therefore, one can define the (self-) mixing rate $\mu_n(f)$ of the map f as the rate of asymptotic decay of the sequences $\{\mu_n(E)\}$, with $E \in B_+$. We summarize the above results in the following

Theorem 3.5. *If $M < \infty$ then $\mu_n(f) = n^{1-\frac{1}{s}}$.*

Remarks.

1. We point out that Theorems 3.2 and 3.5 give the exact rate of mixing of the map f, not just a bound for it. In particular they improve all previously known bounds[4,5]. The above results can be viewed as statements about the decay of correlations for test functions as simple as indicators of sets in B_+. This makes the mixing rate (as defined above) determined by nothing but the distribution of return times: $\nu\{x \in X \,:\, \tau(x) > n\}$ (compare (1.16)). On the other hand, when dealing with correlation functions of a broader class of observables, one expects a richer behaviour depending also of the smoothness properties of the functions involved. In particular one may obtain faster decays. We refer to Isola[14], Liverani et al[15] and Young[16] for different approaches yielding more general results.

2. It turns out that the mixing rate is the same as above also for the set $D_k = \cup_{l>k} A_l$ for some fixed $k \in \mathbb{Z}^+$. Indeed, we get

$$\frac{1}{\nu(D_k)} \sum_{n=0}^{\infty} z^n \nu(D_k \cap f^{-n} D_k) = \Phi(z) \cdot \sum_{n \geq k} z^n \, d_n.$$

Since $\nu(D_k) = \sum_{n\geq k} d_n$, Lemma 3.3 gives

$$\nu(D_k \cap f^{-n} D_k) \sim (\nu(D_k)) \cdot \left(\sum_{m=0}^{n-k} d_{k+m} \right) \cdot u_{n-k} \sim (\nu(D_k))^2 \cdot u_n,$$

and therefore $\mu(D_k \cap f^{-n} D_k) - (\mu(D_k))^2 \sim (\mu(D_k))^2 \; n^{1-\frac{1}{s}}$.

Scaling rate, wandering rate and return sequence when the invariant measure is infinite. When $M = \infty$, given $E \in \mathcal{B}$, with $\nu(E) > 0$, we can define the *scaling rate* $\sigma_n(E)$ of E as

$$\sigma_n(E) := \frac{\nu(E \cap f^{-n}E)}{(\nu(E))^2}. \tag{3.9}$$

From Theorem 3.2 we have that $\sigma_n(A_1) \sim u_n$. Moreover, reasoning as above, one shows that $\sigma_n(E) \sim u_n$ for all $E \in B_+$. We then define the scaling rate $\sigma_n(f)$ of the map f as the rate of asymptotic decay of the sequences $\{\sigma_n(E)\}$, $E \in B_+$. By Theorem 2.1 (Part one) we obtain the following,

Theorem 3.6. *If $M = \infty$ then*

$$\sigma_n(f) = \begin{cases} n^{-1+1/s}, & \text{if } s > 1 \\ 1/\log n & \text{if } s = 1. \end{cases}$$

From the scaling rate, defined above, one can compute some other natural objects arising in the ergodic theory of transformations preserving infinite measures, notably the *wandering rate* $w_n(f)$ and the *return sequence* $r_n(f)$. We refer to Thaler[12], Theorem 3, and also Aaronson[10], Theorem 3, for precise definitions and a discussion of their relevant properties. Intuitively, $w_n(f)$ measures the amount (measured with the measure ν) of $[0,1]$ visited by f-iterates of points of B_+ up to time n. In our case it is simply given by the partial sums $\sum_{k=0}^n d_k$. On the other hand, the existence of $r_n(f)$ is what makes the transformation f *weakly ergodic*[11], namely such that $\frac{1}{r_n}\sum_{k=0}^{n-1} U(f^k x)$ converges in measure (but not m-almost surely) to $\nu(U)$ for any $U \in L^1([0,1], \mathcal{B}, \nu)$. These quantities can be readily obtained putting together Theorem 3.6 and the asymptotic equivalences[10]:

$$w_n(f) \sim \frac{n}{\sum_{k=0}^n \sigma_k}, \qquad r_n(f) \sim \sum_{k=0}^n \sigma_k.$$

Concluding remarks. We finally point out that using the above and results from Feller[25] one can obtain several limit theorems, at least for observables such as indicator functions of sets in B_+. To give an example where Feller results are directly applicable, consider the test function $U = \chi_1$. Then $N_n(x) :=$

$U(x) + \cdots + U(f^{n-1}(x))$ gives the number of passages in the state 1 up to the n-th iterate of the map f. Let moreover $g : X \to X$ be the induced map defined by $g(x) = f^{\tau(x)}(x)$. Then $S_n(x) := \tau(x) + \tau(g(x)) \cdots + \tau(g^{n-1}(x))$ is the total number of iterates of f needed to observe n passages in the state 1. Take first $s < 1/2$. Then, using the notation of Section 1, we have $\sigma^2 := M^{(2)} - M^2 < \infty$. This implies that the associated Markov chain has ergodic degree at least two. One then shows[25] that the mean and the variance of the random variable N_n are asymptotically equal to n/M and $\sigma/M^{3/2}$. Moreover, observing that $m(x \in X : N_n(x) \geq k) = m(x \in X : S_k(x) \leq n)$, one obtain the following (central) limit theorem:

$$m\left(x \in X \,:\, N_n(x) \geq \frac{n}{M} - \frac{\sigma\sqrt{n}}{M^{3/2}}\,\alpha\right) \to \frac{1}{2\pi}\int_{-\infty}^{\alpha} e^{-y^2/2}dy.$$

In the case $1/2 \leq s < 1$, in which the associated Markov chain has ergodic degree one, as well as in the null recurrent case ($s \geq 1$), one obtains different, non-normal, limiting distributions for which we refer to Feller's paper.

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