BRYUNO FUNCTION AND THE STANDARD MAP

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Abstract. For the standard map the homotopically non-trivial invariant curves of rotation number ω satisfying the Bryuno condition are shown to be analytic in the perturbative parameter ε, provided $|\varepsilon|$ is small enough. The radius of convergence $\rho(\omega)$ of the Lindstedt series – sometimes called *critical function* of the standard map – is studied and the relation with the Bryuno function $B(\omega)$ is derived: the quantity $|\log \rho(\omega) + 2B(\omega)|$ is proved to be bounded uniformily in ω.

1. Introduction

We continue the study, started in [1], of the radius of convergence of the Lindstedt series for the standard map, for rotation numbers close to rational values. We consider real rotation numbers ω satisfying the Bryuno condition (see below), and study how the corresponding radius of convergence depends on the Bryuno function $B(\omega)$, introduced by Yoccoz in [2].

The standard map is a discrete time, one-dimensional dynamical system generated by the iteration of the area-preserving – symplectic – map of the cylinder into itself $T_\varepsilon : \mathbb{T} \times \mathbb{R} \mapsto \mathbb{T} \times \mathbb{R}$, given by:

$$T_\varepsilon : \begin{cases} x' = x + y + \varepsilon \sin x, \\ y' = y + \varepsilon \sin x. \end{cases} \tag{1.1}$$

Given a real *rotation number* $\omega \in [0, 1)$, we can look for (homotopically non-trivial) invariant curves described parametrically by:

$$\begin{cases} x = \alpha + u(\alpha, \varepsilon; \omega), \\ y = \alpha + u(\alpha, \varepsilon; \omega) - u(\alpha - 2\pi\omega, \varepsilon; \omega), \end{cases} \tag{1.2}$$

such that the dynamics induced in the variable α is given by rotations by ω:

$$\alpha' = \alpha + 2\pi\omega. \tag{1.3}$$

For irrational rotation numbers ω, by imposing that the average of u over α is 0, the (formal) conjugating function u is unique and odd in α, and has a formal expansion – known as Lindstedt series – of the form:

$$u(\alpha, \varepsilon) = \sum_{\nu \in \mathbb{Z}} u_\nu(\varepsilon) e^{i\nu\alpha} = \sum_{k \geq 1} u^{(k)}(\alpha) \varepsilon^k = \sum_{k \geq 1} \sum_{\nu \in \mathbb{Z}} u_\nu^{(k)} e^{i\nu\alpha} \varepsilon^k; \tag{1.4}$$

the coefficients $u_\nu^{(k)}$ can be expressed graphically in terms of sums over *trees* as explained shortly (see also [1] and references quoted therein). The *radius of convergence* of the series (1.4), called sometimes the *critical function* of the standard map, is defined as:

$$\rho(\omega) = \inf_{\alpha\in\mathbb{T}} \Big(\limsup_{k\to\infty} \big| u^{(k)}(\alpha) \big|^{1/k} \Big)^{-1} . \tag{1.5}$$

Given ω, let $\{p_n/q_n\}$ be the sequence of *convergents* defined by the standard continued fraction expansion of ω, and let:

$$B_1(\omega) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} . \tag{1.6}$$

The irrational number $\omega \in [0,1)$ *satisfies the Bryuno condition* if $B_1(\omega) < \infty$; we also say that in this case ω is a *Bryuno number.* After Yoccoz [2], we define on the irrational numbers the Bryuno function $B(\omega)$ by the functional equation:

$$\begin{cases} B(\omega) = -\log\omega + \omega B(\omega^{-1}) & \text{for } \omega\in(0,\frac{1}{2}) \text{ and irrational,} \\ B(\omega+1) = B(-\omega) = B(\omega). \end{cases} \tag{1.7}$$

It can be proved that such functional equation has a unique solution in L_p, $p \ge 1$; moreover $B(\omega)$ is related to the series $B_1(\omega)$ by the inequality:

$$\big| B(\omega) - B_1(\omega) \big| < C_1 , \tag{1.8}$$

for some constant C_1. See [2] and [3] for the proofs of these statements.

We prove the following theorem.

Theorem. *Consider the standard map* (1.1) *and let ω be an irrational number, $\omega \in [0,1)$, satisfying the Bryuno condition. Then the radius of convergence* (1.5) *satisfies the bound:*

$$|\log\rho(\omega) + 2B(\omega)| \le C_0 , \tag{1.9}$$

where C_0 is a constant independend on ω.

An analogous result was proved by Davie [4] for the semistandard map (where the nonlinear term $\sin x$ in (1.1) is replaced by e^{ix}); in the same paper it was also shown that the upper bound in (1.9) holds:

$$\log\rho(\omega) + 2B(\omega) < C_2 , \tag{1.10}$$

for some constant C_2. In ref. [5] it was proved, by "phase space renormalization" arguments, that $\forall\eta > 0$ $\exists C_3$, depending on η, such that:

$$\log\rho(\omega) + (2+\eta)B(\omega) > C_3 . \tag{1.11}$$

So our theorem improves the result of [5] (using also a different, direct technique, taken from [6] – and inspired to the works [7] and [8] –, in some sense more elementary than the one of [5]) and proves for the standard map the conjecture first stated in [9].

The paper is organized as follows. In sect. 2 we introduce the formalism and give the scheme of the proof of the theorem, elucidating the major difficulties, due to the accumulation of small divisors in the Lindstedt series, and showing that, in absence of such a phenomenon, the proof could be carried out by a detailed analysis of the single terms of the series. In sect. 3 and 4, we shall see how to handle the small divisors problem, by showing that there are cancellation mechanisms, operating to all perturbative orders between different terms of the Lindstedt series, which assure its convergence. Finally sect. 5 and 6 deal with the proof of the main technical lemmata used in the proof of the theorem.

2. Formalism: trees, clusters and resonances

As in [1], we can express graphically the coefficients $u_\nu^{(k)}$ in (1.4) in terms of *trees*. We shall only recall the definitions used in this paper and set up the notations, leaving the full details of the tree expansion for our problem to [1] and the references quoted therein.

A tree ϑ consists of a family of lines arranged to connect a partially ordered set of points – nodes –, with the lower nodes to the right. All the lines have two nodes at their extremes, except the highest which has only one node, the *last node* u_0 of the tree; the other extreme r will be called the *root* of the tree and it will not be regarded as a node.

We denote by $\preccurlyeq$ the partial ordering relation between nodes defined as follows: given two nodes u, v, we say that $u \preccurlyeq v$ if u is along the path of lines connecting v to the root r of the tree – they could coincide: we say that $u \prec w$ if they do not. So our trees are "rooted trees", following the terminology of [10].

We assign to each line ℓ joining two nodes u and u' an "arrow" pointing from the highest to the lowest node according to the order relation just defined; if $u' \prec u$, we say that the line ℓ exists from u and enters u'. We write $u_0' = r$ even if, strictly speaking, r is not considered a node. For each node u there is a unique exiting line, and $m_u \geq 0$ entering lines; as there is a one-to-one correspondence between lines and nodes, we can associate to each node u the line ℓ_u exiting from it. The line ℓ_{u_0} exiting the last node u_0 will be called the *root line*. Note that each line ℓ can be considered the root line of the subtree consisting of the nodes satisfying $v \preccurlyeq u$, and u' will be the root of such tree. The *order* k of the tree is defined as the number of its nodes.

To each node $u \in \vartheta$ we associate a *mode label* $\nu_u = \pm 1$, and define the *momentum* flowing through the line ℓ_u as:

$$\nu_{\ell_u} = \sum_{w \preccurlyeq u} \nu_w, \quad \nu_w = \pm 1; \tag{2.1}$$

note that no line can have zero momentum, as $u_0^{(k)} = 0$.

While in [1] we could get along considering only two "scales", we need a full multiscale decomposition of the momenta associated to each line.

Given a rotation number $\omega \in [0,1)\backslash\mathbb{Q}$, let $\{p_n/q_n\}$ be the sequence of convergents coming from the standard continued fraction expansion of ω. For $x \in \mathbb{R}$, let:

$$||x|| = \inf_{\nu \in \mathbb{Z}} |x - p| \tag{2.2}$$

be the distance of x from the nearest integer. Let now:

$$\gamma(\nu) = 2(\cos 2\pi\omega\nu - 1); \tag{2.3}$$

then we have the estimate:

$$|\gamma(\nu)| = 2|\cos 2\pi\omega\nu - 1| \geq \Gamma ||\omega\nu||^2, \tag{2.4}$$

for some constant Γ.

We introduce a C^∞ partition of unity in the following way. Let $\chi(x)$ a C^∞, non-increasing, compact-support function defined on $\mathbb{R}^+$, such that:

$$\chi(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases} \tag{2.5}$$

and define for each $n \in \mathbb{N}$:

$$\begin{cases} \chi_0(x) = 1 - \chi(96 q_0 x), \\ \chi_n(x) = \chi(96 q_n x) - \chi(96 q_{n+1} x), \quad \text{for } n \geq 1. \end{cases} \tag{2.6}$$

Then for each line ℓ set:

$$g(\nu_\ell) \equiv \frac{1}{\gamma(\nu_\ell)} = \sum_{n=0}^{\infty} \frac{\chi_n(||\omega\nu_\ell||)}{\gamma(\nu_\ell)} \equiv \sum_{n=0}^{\infty} g_n(\nu_\ell), \tag{2.7}$$

and call $g_n(\nu_\ell)$ the *propagator on scale* n.

Given a tree ϑ, we can associate to each line ℓ of ϑ a scale label n_ℓ, using the multiscale decomposition (2.7) and singling out the summands with $n = n_\ell$. We shall call n_ℓ the *scale label* of the line ℓ, and we shall say also that the line ℓ is *on scale* n_ℓ.

Remark 1. Given a value ν_ℓ there can be at most two possible – consecutive – values of n such that the corresponding $\chi_n(||\omega\nu_\ell||)$ are not vanishing. This means that at most only two summands of the infinite series (2.7) really appear; nevertheless keeping all terms is more convenient, in order to have a label to characterize the "size" of the "propagators" $g(\nu_\ell)$.

Remark 2. Note that if a line ℓ has momentum ν_ℓ and scale n_ℓ, then:

$$\frac{1}{96 q_{n_\ell+1}} \leq ||\omega\nu_\ell|| \leq \frac{1}{48 q_{n_\ell}}, \tag{2.8}$$

provided that one has $\chi_{n_\ell}(||\omega\nu_\ell||) \neq 0$.

A group $\mathcal{G}$ of tranformations acts on the trees, generated by the permutations of all the subtrees emerging from each node with at least one entering line: $\mathcal{G}$ is therefore a cartesian product of copies of the symmetric groups of various orders. Two trees that can be transformed into each other by the action of the group $\mathcal{G}$ are considered identical.

Denote by $\mathcal{T}_{\nu,k}$ the set of trees, with nonvanishing value, of order k and total momentum $\nu_{\ell_{u_0}} = \nu$, if u_0 is the last node of the tree. The number of elements in $\mathcal{T}_{\nu,k}$ is bounded by $2^k \cdot 2^k \cdot 2^{2k} = 2^{4k}$: the number of semitopological trees (see [1]) of order k is bounded by 2^{2k},[1] and there are two possible values for the mode label of each node and two possible values for the scale label of each line.

[1] The number of semitopological trees can be bounded by the number of one-dimensional random walks with $2k - 1$ steps.

Then, as in [1] – to which we refer for more details and figures – one finds:

$$u_\nu^{(k)} = \frac{1}{2^k} \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta), \quad \mathrm{Val}(\vartheta) = -i \left[\prod_{u \in \vartheta} \frac{\nu_u^{m_u+1}}{m_u!} \right] \left[\prod_{\ell \in \vartheta} g_{n_\ell}(\nu_\ell) \right]; \tag{2.9}$$

the factors $g_{n_\ell}(\nu_\ell)$ above are called *propagators* of *small divisors* on scale n_ℓ, and the quantity $\mathrm{Val}(\vartheta)$ will be called the *value* of the tree ϑ.

We define now the main combinatorial tools.

Definition (Cluster). Given a tree ϑ, a cluster T of ϑ on scale n is a maximal connected set of lines of lines on scale $\le n$ with at least one line on scale n. We shall say that such lines are *internal* to T, and write $\ell \in T$ for an internal line T. A node u is called *internal* to T, and we write $u \in T$, if at least one of its entering lines or exiting line is in T. Each cluster has an arbitrary number $m_T \ge 0$ of entering lines but only one exiting line; we shall call *external* to T the lines entering or exiting T (which are all on scale $> n$). We shall denote with n_T the scale of the cluster T, with n_T^i the minimum of the scales of the lines entering T, with n_T^o the scale of the line exiting T and with k_T the number of nodes in T.

Note that, despite the name, not all lines outside T are "external" to it: only those lines outside T which enter or exit T are external to it. On the contrary a line inside T is said to be "internal" to it. The use of such a terminology is inherited from Quantum Field Theory.

Definition (Resonance). Given a tree ϑ, a cluster V of ϑ will be called a *resonance* with *resonance-scale* $n = n_V^R \equiv \min\{n_V^i, n_V^o\}$, if:

1. the sum of the mode labels of its nodes is 0:
$$\nu_V \equiv \sum_{u \in V} \nu_u = 0; \tag{2.10}$$
2. all the lines entering V are on the same scale except at most one, which can be on a higher scale;
3. $n_V^i \le n_V^o$ if $m_V \ge 2$, and $|n_V^i - n_V^o| \le 1$ for $m_V = 1$;
4. $k_T < q_n$;
5. $m_V = 1$ if $q_{n+1} \le 4q_n$;
6. if $q_{n+1} > 4q_n$ and $m_V \ge 2$, denoting by k_0 the sum of the orders of the subtrees of order $< q_{n+1}/4$ entering V, either
   (a) there is a only one subtree of order $k_1 \ge q_{n+1}/4$ entering V and $k_1 + k_0 + k_T \ge q_{n+1}/4$, $k_0 < q_{n+1}/8$, or
   (b) there is no such subtree and $k_0 + k_T < q_{n+1}/4$.

Remark 3. Note that for any resonance V one has $n_V^R \ge n_V + 1$, if n_V is the scale of the resonance V as a cluster. As in [11] we use the notation with a hyphen for the resonance-scale to avoid confusion between n_V^R and n_V.

Remark 4. One would be tempted to give a simpler definition of resonance (for instance, by imposing only condition 1 to the cluster V). This temptation should be resisted, as it would make impossible to exploit the cancellations leading to the improvement of the bound discussed at the end of this section (in fact, no relation would continue to subsist between momenta and scale labels and factorials would arise from counting the summands generated by the renormalization procedure described in sect. 4). On the other hand we shall see in sect. 5 that no problems should arise if no resonances – exactly as they defined above – could appear.

In the following we shall need to introduce trees in which it can happen that a line ℓ is on a scale n_ℓ and yet its momentum does not satisfy (2.8). The value of any such tree ϑ is vanishing as $\chi_{n_\ell}(||\omega\nu_\ell||)=0$; nevertheless it will be useful to write $\mathrm{Val}(\vartheta)$ as sum of two (possibly) nonvanishing terms: one of them will be used to cancel terms arising from other tree values, so it will disappears, while the other one is left and has to be bounded. This means that we shall have to deal with trees in which there are lines ℓ with momentum ν_ℓ and scale n_ℓ which do not satisfy (2.8). What will be shown to hold is that for such lines a bound similar to (2.8), though weaker, still holds; more precisely, a line ℓ with momentum ν_ℓ will have only scales n_ℓ such that

$$\frac{1}{768 q_{n_\ell+1}} \le ||\omega\nu_\ell|| \le \frac{1}{8 q_{n_\ell}}, \tag{2.11}$$

and, for fixed ν_ℓ, the number of possible scales to associate to ℓ is bounded by an absolute constant.

As (2.11) is implied by (2.8), even for trees with nonvanishing value we shall use that if a line is on scale n_ℓ then (2.11) holds.

Then, if $N_n(\vartheta)$, $n\in\mathbb{N}$, denotes the number of lines on scale n in ϑ, we have trivially for a given tree ϑ the bound:

$$|\,\mathrm{Val}(\vartheta)| \le D_1^k \prod_{n=0}^{\infty} \left(768 q_{n+1}\right)^{2N_n(\vartheta)}, \tag{2.12}$$

for some constant D_1 (actually $D_1 = 1/\Gamma$; see (2.4), (2.9) and (2.11)).

Given a tree ϑ, let us denote with $N_n^R(\vartheta)$ the number of resonances with resonance-scale n and by $P_n(\vartheta)$ the number of

resonances on scale n. Of course $N_0^R=0$.

Remark 5. Note that the number $N_n^R(\vartheta)$ of resonances with resonance-scale n can be counted by counting *the number of lines exiting resonances with resonance-scale n*; analogously $P_n(\vartheta)$ can be counted by counting the number of lines exiting resonances on scale n. Such counts will be performed in sect. 5.

The following simple lemmata contain all the arithmetic we shall need, and are basically adapted from [4].

Lemma 1 (Davie's lemma). *Given $\nu\in\mathbb{Z}$ such that $||\omega\nu||\le 1/4q_n$, then*

1. *either $\nu=0$ or $|\nu|\ge q_n$,*
2. *either $|\nu|\ge q_{n+1}/4$ or $\nu = s q_n$ for some integer s.*

Lemma 2. *If a tree ϑ has $k<q_n$ nodes, then $N_n(\vartheta)=0$ and $P_{n-1}(\vartheta)=0$.*

Lemma 3. *For any irrational number $\omega\in[0,1)$:*

$$\sum_{n=0}^{\infty} \frac{\log q_n}{q_n} \le D_2, \tag{2.13}$$

for a constant D_2; here q_n are the denominators of the convergents of ω.

Lemma 4. *Given a momentum ν such that*

$$\frac{1}{768 q_{n+1}} \leq ||\omega\nu|| \leq \frac{1}{8q_n}, \tag{2.14}$$

then one can have $\chi_{n'}(||\omega\nu||) \neq 0$ only for n' such that $n-8 \leq n' \leq n+8$.

Proof of lemma 1. If $\{q_n\}$ are the denominators of the convergents of ω, then (see *e.g.* [12], Ch. 1, §3):

$$\frac{1}{2q_{n+1}} < ||\omega q_n|| < \frac{1}{q_{n+1}}, \tag{2.15}$$

and:

$$\forall |\nu| < q_{n+1}, |\nu| \neq q_n : \quad ||\omega\nu|| > ||\omega q_n||. \tag{2.16}$$

To prove 1 note that if $\nu = 0$ nothing has to be proved: so we assume $\nu \neq 0$. If $|\nu| < q_n$, by (2.16) and (2.15), $||\omega\nu|| \geq ||\omega q_{n-1}|| > 1/2q_n$, so that $||\omega\nu|| < 1/4q_n$ implies $|\nu| \geq q_n$, proving the first assertion of lemma 1.

To prove 2, again if $\nu = 0$ nothing has to be proved (and $s = 0$): so we assume $\nu \neq 0$, and proceed by *reductio ad absurdum.* If $0 < \nu < q_{n+1}/4$ and there does not exist any $s \in \mathbb{Z}$ such that $\nu = sq_n$, then one has $\nu = mq_n + r$, with $0 < r < q_n$ and $m < q_{n+1}/4q_n$; then, by (2.15), $||\omega m q_n|| \leq m||\omega q_n|| < m/q_{n+1} < 1/4q_n$, and, by (2.16), $||\omega r|| \geq ||\omega q_{n-1}|| > 1/2q_n$, as $r \neq 0$; so $||\omega\nu|| \geq ||\omega r|| - ||\omega m q_n|| > 1/4q_n$. The case $0 > \nu > -q_{n+1}/4$ is identical as $||\cdot||$ is even. □

Proof of lemma 2. If $k < q_n$, then for any $\ell \in \vartheta$ one has $|\nu_\ell| \leq k < q_n$, so that, by (2.15) and (2.16), $||\omega\nu_\ell|| \geq ||\omega q_{n-1}|| > 1/2q_n$, hence $n_\ell < n$ and so $N_n(\vartheta) = 0$. If there are no lines on scale n, it is impossible to form a cluster on scale $n-1$, *a fortiori* a resonance. □

Proof of lemma 3. The denominators of the convergents $\{q_n\}$ of ω satisfy $q_0 = 1$, $q_1 \geq 1$ and $q_n \geq 2q_{n-2}$ for any $n \geq 2$. So we can write:

$$\sum_{n=0}^{\infty} \frac{\log q_n}{q_n} = \sum_{n=0}^{\infty} \frac{\log q_{2n}}{q_{2n}} + \sum_{n=0}^{\infty} \frac{\log q_{2n+1}}{q_{2n+1}}; \tag{2.17}$$

using the fact that, for $x \geq e$, $x^{-1}\log x$ is decreasing, we obtain easily:

$$\sum_{n=0}^{\infty} \frac{\log q_n}{q_n} \leq 3 \max_{x \geq 1}\left\{\frac{\log x}{x}\right\} + 2\log 2 \sum_{k=2}^{\infty} \frac{k}{2^k} = 3(e^{-1} + \log 2) \equiv D_2, \tag{2.18}$$

which also gives an explicit value for the constant D_2. □

Proof of lemma 4. Simply use that $q_{n+1} \geq q_n$ and $q_{n+2} \geq 2q_n$ for all $n \geq 0$, to deduce that $1/48q_{n+9} < 1/768q_{n+1}$ and $1/96q_{n-8} > 1/8q_n$. □

The following "counting" lemma is the main result stated in this section, and it can be considered an adaption and extension of lemma 2.3 in [4]. We postpone its proof to sect. 5.

Lemma 5. *Given a tree ϑ, let $M_n(\vartheta) = N_n(\vartheta) + P_n(\vartheta)$. Then:*

$$M_n(\vartheta) \le \frac{k}{q_n} + \frac{8k}{q_{n+1}} + N_n^R(\vartheta), \tag{2.19}$$

where k is the order of ϑ.

Therefore we can rewrite the bound (2.12) on the tree value as:

$$\begin{aligned} |\operatorname{Val}(\vartheta)| &\le D_1^k \prod_{n=0}^{\infty} \big(768 q_{n+1}\big)^{2(M_n(\vartheta)-P_n(\vartheta))} \\ &\le D_1^k \prod_{n=0}^{\infty} \big(768 q_{n+1}\big)^{2(k/q_n + 8k/q_{n+1} + N_n^R(\vartheta) - P_n(\vartheta))}. \end{aligned} \tag{2.20}$$

Note that at this point it would be very easy to prove the lower bound in (1.9) *for the semistandard map* and, by simple modifications of the same scheme, *for Siegel problem*, since in these cases no resonances appear. On the contrary, in the more difficult case of the standard map we lack, for the moment, a control on the number $N_n^R(\vartheta)$ of resonances in ϑ with resonance-scale n.

In sect. 3 and 4 we shall see how to improve the bound *on the sum* over the trees of fixed order and total momentum, in order to prove the theorem stated in sect. 1. We postpone to forthcoming sections the proofs, limiting ourselves here to a heuristic discussion in order to give an idea of the structure of the proof.

We perform a suitable resummation – described in sect. 3 and 4 – whose consequence is that, for each resonance V, *it is as if one of the external lines on scale n_V^R contributed $\big(768 q_{n_V+1}\big)^2$ instead of $\big(768 q_{n_V^R+1}\big)^2$*. To obtain such a result, we shall perform on trees transformations which will lead to the introduction of new trees: so we extend $\mathcal{T}_{\nu,k}$ to a larger set $\mathcal{T}^*_{\nu,k}$. However we shall prove that the value of each single tree in $\mathcal{T}^*_{\nu,k}$ still admits the bound (2.20) – even if, unlike the values of the trees in $\mathcal{T}_{\nu,k}$, it fails to satisfy the same bound with 768 replaced with 96 – and the number of elements in $\mathcal{T}^*_{\nu,k}$ is bounded by a constant to the power k (*i.e.* no bad counting factors, like factorials, appear). Then we obtain, *for the sum of the resummed trees*, a bound of the form (2.20) with:

$$\prod_{n=0}^{\infty} \big(768 q_{n+1}\big)^{2N_n^R(\vartheta)}$$

replaced with:

$$D_3^k \prod_{n=0}^{\infty} \big(768 q_{n+1}\big)^{2P_n(\vartheta)},$$

for some constant D_3. By using that the number of trees in $\mathcal{T}^*_{\nu,k}$ will be shown to be bounded by a constant to the power k, we obtain, for some constants D_4, D_5:

$$\begin{aligned} |u^{(k)}(\alpha)| &\le \Big| \sum_{|\nu|\le k} \sum_{\vartheta\in\mathcal{T}_{\nu,k}} \operatorname{Val}(\vartheta) \Big| \le \Big| \sum_{|\nu|\le k} \sum_{\vartheta\in\mathcal{T}^*_{\nu,k}} \operatorname{Val}(\vartheta) \Big| \\ &\le D_4^k \prod_{n=0}^{\infty} \big(768 q_{n+1}\big)^{2k/q_n + 16k/q_{n+1}} \\ &\le D_5^k \exp\Big[2k \sum_{n=0}^{\infty} \Big(\frac{\log q_{n+1}}{q_n} + \frac{8 \log q_{n+1}}{q_{n+1}} \Big) \Big], \end{aligned} \tag{2.21}$$

which, by making use of lemma 3, gives:

$$\log \rho(\omega) + 2B_1(\omega) \geq -16D_2 - \log D_5. \tag{2.22}$$

By making rigorous the above discussion in sect. 3 and 4, we shall complete the proof of the theorem, since the bound from above was already proved in [4].

3. Renormalization of resonances: set-up and the first step

Given a tree ϑ, let us consider maximal resonances, *i.e.* resonances *not* contained in any larger resonance; let us call them *first generation resonances.* Inside the first generation resonances let us consider the "next maximal" resonances, *i.e.* the resonances not contained in any larger resonance except first generation resonances, and let us call them *second generation resonances.* We can define in this way *j-th generation resonances*, for $j \geq 2$, as resonances which are maximal within $(j-1)$-th generation resonances.

Let $\mathbf{V}$ be the set of all resonances of a tree ϑ, and $\mathbf{V}_j$ the set of all resonances of j-th generation, with $j = 1, \ldots, G$, for some integer G, depending on ϑ.

Given a tree ϑ and a resonance $V \in \mathbf{V}_j$ with m_V entering lines, define V_0 as the set of nodes and lines internal to V and outside any resonances contained in V. Let $L_V = \{\ell_1, \ldots, \ell_{m_V}\}$ be the set of entering lines of V; we define L_V^R as the subset of the lines in L_V which enter some resonances of higher generation contained inside V and $L_V^0 = L_V \setminus L_V^R$ as the subset of lines in L_V which enter nodes in V_0.

For any line $\ell_m \in L_V^R$, let $V(\ell_m)$ be the minimal resonance containing the node which the line ℓ_m enters (*i.e.* the highest generation resonance containing such a node) and $V_0(\ell_m)$ the set of nodes and lines internal to $V(\ell_m)$ and outside resonances contained in $V(\ell_m)$. Define:

$$\tilde{\mathbf{V}}(V) = \{\tilde{V} \subset V \,:\, \tilde{V} = V(\ell_m) \text{ for some } \ell_m \in L_V^R\}. \tag{3.1}$$

Call m_{V_0} the number of lines in L_V^0. The number of lines in L_V^R entering the same resonance $\tilde{V} \in \tilde{\mathbf{V}}(V)$ is not arbitrary: it is always 1, as it is shown by the following lemma.

Lemma 6. *For $j \geq 1$, given a resonance $W \in \mathbf{V}_{j+1}$ contained inside a resonance $V \in \mathbf{V}_j$, only one among the entering lines W can also enter V.*

Proof. By item 3 of the definition of resonance one has $n_W^R \leq n_V$, otherwise V would be a cluster on scale $< n_W^R$, so that all the lines external to W would be also external to V and $V = W$, while we assumed that $V \subsetneq W$. Then if a line ℓ enter both V and W, one must have $n_\ell > n_W^R$. But, by item 2 in the definition of resonance, all lines external to W have the same scale n_W^R except at most one. $\square$

We define the *resonance family* $\mathcal{F}_V(\vartheta)$ of $V \in \mathbf{V}$ in ϑ as the set of trees obtained from ϑ by the action of a group of transformations $\mathcal{P}_V$ on ϑ, generated by the following operations:

1. Detach the line ℓ_1, then if $\ell_1 \in L_V^R$ reattach it to all nodes internal to $V_0(\ell_1)$, while if $\ell_1 \in L_V^0$ reattach it to all nodes in V_0; for each tree so obtained, do the same operations with the line ℓ_2 and so forth for each line entering the resonance.
2. In a given tree, each node $u \in V$ will have m_u entering lines, of which s_u are inside V and $r_u = m_u - s_u$ are outside V (*i.e.* are entering lines of V); then we can apply to the set of lines entering u a transformation in the group obtained as the quotient of the group of permutations of the m_u lines entering u by the groups of permutations of the s_u internal entering lines and of permutations of the r_u entering lines outside V; in this way for each node $u \in V$ a number of trees equal to:
$$\binom{m_u}{s_u} = \frac{m_u!}{s_u! r_u!}$$
is obtained.
3. Flip simultaneously all the mode labels of the nodes internal to V.

We shall call *renormalization transformations* (of type 1, 2, 3) the operations described above.

Remark 6. Note that in all such transformations the scales are not changed (by definition) and the set of resonance $\mathbf{V}$ remains the same (by construction). On the contrary the momenta flowing through the lines can change (because of the shift of the lines entering resonances) and in particular one can have for some lines ℓ, $\chi_{n_\ell}(||\omega\nu_\ell||) = 0$, if ν_ℓ is the modified momentum flowing through ℓ.

Remark 7. The definition of resonance families is aimed at grouping together the trees between which one will look for compensations, but in doing so one has to avoid overcountings. In fact, to each tree ϑ we associate a value $\mathrm{Val}(\vartheta)$ according to (2.9); when applying the transformations of the group $\mathcal{P}_V$ on the tree ϑ, the same tree ϑ' can be obtained, in general, in several ways; however, it has to be counted once. This means that $\mathcal{P}_V$, as a group, defines an equivalence class, and only inequivalent elements obtained through the transformations defining $\mathcal{P}_V$ have to be retained.

Let us call $\mathcal{F}_{\mathbf{V}_1}(\vartheta)$ the family obtained by the composition of all transformations defining the resonance families $\mathcal{F}_{V_1}(\vartheta)$, $V_1 \in \mathbf{V}_1$.

For any tree $\vartheta_1 \in \mathcal{F}_{\mathbf{V}_1}(\vartheta)$, let V_2 be a resonance in $\mathbf{V}_2$ and let us define the resonance family $\mathcal{F}_{V_2}(\vartheta_1)$ of V_2 in ϑ_1 as the set of trees obtained from ϑ_1 by the action of the group of transformations $\mathcal{P}_{V_2}$. The composition of all transformations defining the resonance families $\mathcal{F}_{V_2}(\vartheta_1)$, for all $\vartheta_1 \in \mathcal{F}_{\mathbf{V}_1}(\vartheta)$ and all $V_2 \in \mathbf{V}_2$, gives a family that we shall denote by $\mathcal{F}_{\mathbf{V}_2}(\vartheta)$.

We continue by considering resonances of 3-rd generation, and so on until the G-th generation resonances are reached. At the end we shall have a family $\mathcal{F}(\vartheta)$ of trees obtained by the composition of all transformations of the groups $\mathcal{P}_V$, $V \in \mathbf{V}$, defined recursively through the application of the renormalization transformations corresponding to resonances $V \in \mathbf{V}_j$ to all trees ϑ' belonging to the family $\mathcal{F}_{\mathbf{V}_{j-1}}(\vartheta)$.

Remark 8. Given a tree $\vartheta \in \mathcal{T}_{\nu,k}$ and a family $\mathcal{F}(\vartheta)$, when considering another tree $\vartheta' \in \mathcal{F}(\vartheta)$ with nonvanishing value $\mathrm{Val}(\vartheta')$, the same family $\mathcal{F}(\vartheta') = \mathcal{F}(\vartheta)$ is obtained (by construction). Note however that $\mathcal{F}(\vartheta)$ can contain also trees with vanishing values, as they can have lines ℓ such that $\chi_{n_\ell}(||\omega\nu_\ell||) = 0$ (see remark 6).

Define also $\mathcal{N}_{\mathcal{F}(\vartheta)}$ the number of trees in $\mathcal{F}(\vartheta)$ whose value is not vanishing; of course $\mathcal{N}_{\mathcal{F}(\vartheta)} \le |\mathcal{F}(\vartheta)|$, if $|\mathcal{F}(\vartheta)|$ is the number of elements in $\mathcal{F}(\vartheta)$.

Write:

$$\sum_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta) = \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \frac{1}{\mathcal{N}_{\mathcal{F}(\vartheta)}} \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta') = \sum_{\vartheta \in \mathcal{T}^*_{\nu,k}} \frac{1}{|\mathcal{F}(\vartheta)|} \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta'), \quad (3.2)$$

where the factors $\mathcal{N}_{\mathcal{F}(\vartheta)}$ and $|\mathcal{F}(\vartheta)|$ have been intoduced in order to avoid overcountings (see remark 8) and the last sum implicitly defines the set $\mathcal{T}^*_{\nu,k}$: so $\mathcal{T}^*_{\nu,k}$ is the set of inequivalent trees in $\cup_{\vartheta \in \mathcal{T}_{\nu,k}} \mathcal{F}(\vartheta)$.

Consider a tree $\vartheta \in \mathcal{T}^*_{\nu,k}$. Then $\vartheta \in \mathcal{F}(\vartheta_0)$, for some tree $\vartheta_0 \in \mathcal{T}_{\nu,k}$; however one has to bear in mind that ϑ, unlike ϑ_0, could vanish.

Given a tree $\vartheta \in \mathcal{T}^*_{\nu,k}$, if V is a first generation resonance, we define its *resonance factor* $\mathcal{V}_V(\vartheta)$ as its contribution to the value of the tree ϑ, namely:

$$\mathcal{V}_V(\vartheta) = \left[\prod_{u \in V} \frac{\nu_u^{m_u+1}}{m_u!} \right] \left[\prod_{\ell \in V} g_{n_\ell}(\nu_\ell) \right], \quad (3.3)$$

which of course depends on the subset of ϑ outside the resonance V only through the momenta of the entering lines of V. Given a node $u \in V$, let us denote with $\mathcal{E}_u$ the set of lines entering V such that they end into nodes preceding u.

For future notational convenience, we rewrite (3.3) as:

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta) L_V(\vartheta), \quad U_V(\vartheta) = \prod_{u \in V} \frac{\nu_u^{m_u+1}}{m_u!}, \quad L_V(\vartheta) = \prod_{\ell \in V} g_{n_\ell}(\nu_\ell). \quad (3.4)$$

In the following, we shall consider the quantities $\omega\nu$, $\nu \in \mathbb{Z}$, modulo 1, and shall continue to use the symbol $\omega\nu$ to denote the representative of the equivalence class within the interval $(-1/2, 1/2]$.

For any node u contained in a resonance V, we shall write:

$$\nu_{\ell_u} = \nu^0_{\ell_u} + \sum_{\ell' \in \mathcal{E}_u} \nu_{\ell'}, \quad \nu^0_{\ell_u} = \sum_{\substack{w \in V \\ w \preccurlyeq u}} \nu_w, \quad (3.5)$$

where the set $\mathcal{E}_u$ was defined after (3.3).

We shall consider the resonance factor (3.3) as a function of the quantities $\mu_1 = \omega\nu_{\ell_1}, \ldots, \mu_{m_V} = \omega\nu_{\ell_{m_V}}$, where $\nu_{\ell_1}, \ldots, \nu_{\ell_{m_V}}$ are the momenta flowing through the lines $\ell_1, \ldots, \ell_{m_V}$ entering V. More precisely, we let:

$$\mathcal{V}(\vartheta) \equiv \mathcal{V}_V(\vartheta; \omega\nu_{\ell_1}, \ldots, \omega\nu_{\ell_{m_V}}), \quad (3.6)$$

and we write:

$$\begin{aligned} &\mathcal{V}_V(\vartheta; \omega\nu_{\ell_1}, \ldots, \omega\nu_{\ell_{m_V}}) = \\ &\quad = \mathcal{L}\mathcal{V}_V(\vartheta; \omega\nu_{\ell_1}, \ldots, \omega\nu_{m_V}) + \mathcal{R}\mathcal{V}_V(\vartheta; \omega\nu_{\ell_1}, \ldots, \omega\nu_{m_V}), \end{aligned} \quad (3.7)$$

where:

$$\begin{aligned}\mathcal{L}\mathcal{V}_V(\vartheta;\omega\nu_{\ell_1},\ldots,\omega\nu_{\ell_{m_V}}) &= \\ &= \mathcal{V}_V(\vartheta;0,\ldots,0)+\sum_{m=1}^{m_V}\omega\nu_{\ell_m}\frac{\partial}{\partial\mu_m}\mathcal{V}_V(\vartheta;0,\ldots,0)\end{aligned}\tag{3.8}$$

is the *localized part* of the resonance factor, or *localized resonance factor*, while:

$$\begin{aligned}\mathcal{R}\mathcal{V}_V(\vartheta;\omega\nu_{\ell_1},\ldots,\omega\nu_{\ell_{m_V}}) = \sum_{m,m'=1}^{m_V}\omega\nu_{\ell_m}\,\omega\nu_{\ell_{m'}}\,\cdot& \\ \cdot\int_0^1 \mathrm{d}t\,(1-t)\frac{\partial^2}{\partial\mu_m\partial\mu_{m'}}\mathcal{V}_V(\vartheta;t\omega\nu_{\ell_1},\ldots,t\omega\nu_{\ell_{m_V}})&\end{aligned}\tag{3.9}$$

is the *renormalized part* of the resonance factor, or *renormalized resonance factor*. In (3.7) $\mathcal{L}$ is called the *localization operator* and $\mathcal{R}=1-\mathcal{L}$ is called the *renormalization operator*. Using the notations (3.4), we can write:

$$\mathcal{L}\mathcal{V}_V(\vartheta)=U_V(\vartheta)\mathcal{L}L_V(\vartheta),\quad \mathcal{R}\mathcal{V}_V(\vartheta)=U_V(\vartheta)\mathcal{R}L_V(\vartheta),\tag{3.10}$$

as only the factors in $L_V(\vartheta)$ depend on the momenta flowing through the lines entering the resonance V.

Remark 9. Note that in the localized part (3.8) the momentum ν_ℓ flowing through any line ℓ internal to V is changed into ν_ℓ^0 (see (3.5)).

Then we perform the renormalization transformations in $\mathcal{P}_V$ described above. By remark 9, for all trees obtained by applying the group $\mathcal{P}_V$ the contribution to the localized resonance factor arising from the $L_V(\vartheta)$ term in (3.4) is the same, *i.e.* :

$$\mathcal{L}L_V(\vartheta)=\mathcal{L}L_V(\vartheta'),\quad \forall\vartheta'\in\mathcal{F}_V(\vartheta),\tag{3.11}$$

so that we can consider:

$$\sum_{\vartheta'\in\mathcal{F}_V(\vartheta)}\mathcal{L}\mathcal{V}_V(\vartheta').\tag{3.12}$$

The sum over all the trees in the resonance family $\mathcal{F}_V(\vartheta)$ of the localized resonance factors produces zero, so that only the renormalized part has to be taken into account. The proof of this assertion is similar to the proof of the analogous statement in [1], and it is given in sect. 6 as a particular case of the proof of the more general statement in lemma 8 below.

Then only the second order terms have to be taken into account in (3.7). This leads to the following expression for the renormalized resonance factor:

$$\begin{aligned}\mathcal{R}\mathcal{V}_V(\vartheta)=U_V(\vartheta)\sum_{m,m'=1}^{m_V}\omega\nu_{\ell_m}\,\omega\nu_{\ell_{m'}}\,\cdot&\\ \cdot\Bigg[\sum_{\substack{\ell_V^1,\ell_V^2\in V\\ \ell_V^1\neq\ell_V^2}}\left(\frac{\partial}{\partial\mu_m}g_{n_{\ell_V^1}}(\nu_{\ell_V^1})\right)\left(\frac{\partial}{\partial\mu_{m'}}g_{n_{\ell_V^2}}(\nu_{\ell_V^2})\right)\Bigg(\prod_{\substack{\ell\in V\\ \ell\neq\ell_V^1,\ell_V^2}}g_{n_\ell}(\nu_\ell)\Bigg)+&\\ +\sum_{\ell_V\in V}\left(\frac{\partial}{\partial\mu_m}\frac{\partial}{\partial\mu_{m'}}g_{n_{\ell_V}}(\nu_{\ell_V})\right)\Bigg(\prod_{\substack{\ell\in V\\ \ell\neq\ell_V}}g_{n_\ell}(\nu_\ell)\Bigg)\Bigg],&\end{aligned}\tag{3.13}$$

from the very definition of the renormalized resonance factor (3.9), by noting that the two derivatives in (3.9) act either on two distinct propagators (the sum with $\ell_V^1 \neq \ell_V^2$ in (3.13)) or on the same propagator (the sum with only one line ℓ_V in (3.13)).

Note that it can happen that $\vartheta \in \mathcal{F}_V(\vartheta_0)$, for some tree $\vartheta_0 \in \mathcal{T}_{\nu,k}$, *i.e.* for some tree ϑ_0 with nonvanishing value, while $\mathcal{V}_V(\vartheta) = 0$ (correspondingly there does not exist any tree in $\mathcal{T}_{\nu,k}$ of that shape associated with the given choice of mode and scale labels). The tree ϑ is obtained from ϑ_0 through a transformation of $\mathcal{P}_V$, so that there is a correspondence between the lines of ϑ_0 and the lines of ϑ: we shall say that the lines are *conjugate*. The tree ϑ inherits the scale labels of the tree ϑ_0, *i.e* the lines in ϑ have the same scales of the conjugate lines of ϑ_0. So it can happen that in ϑ_0 some line internal to V has a scale n_ℓ and a momentum $\tilde{\nu}_\ell$ such that $\chi_{n_\ell}(||\omega\tilde{\nu}_\ell||) \neq 0$, while the momentum ν_ℓ of the line ℓ seen as a line of ϑ (*i.e.* of the line of ϑ conjugate to the line ℓ of ϑ_0) is such that $\chi_{n_\ell}(||\omega\nu_\ell||) = 0$ (see remark 8). This means that for such a line (2.8) does not hold. Nevertheless, as anticipated in remark 6, one finds that the momentum ν_ℓ can not change "too much" with respect to $\tilde{\nu}_\ell$; more precisely:

$$\frac{1}{768 q_{n_\ell+1}} \leq ||\omega\nu_\ell|| \leq \frac{1}{24 q_{n_\ell}}, \tag{3.14}$$

as we shall prove, using the following result.

Lemma 7. *Given a tree $\vartheta_0 \in \mathcal{T}_{\nu,k}$ and a resonance V, let $\vartheta \in \mathcal{T}^*_{\nu,k}$ be a tree obtained by the action of the group $\mathcal{P}_V$,* i.e. *$\vartheta \in \mathcal{F}_V(\vartheta_0)$. If $||\omega\nu_{\ell_m}|| \leq 1/8q_{n_V^R}$ for any entering line ℓ_m of V, $m = 1, \dots, m_V$, then, for any line $\ell \in V$, one has*

$$\big| ||\omega\nu_\ell|| - ||\omega\tilde{\nu}_\ell|| \big| \leq \frac{1}{4q_{n_V^R}}, \qquad ||\omega\nu_\ell|| \geq \frac{1}{4q_{n_V^R}}, \qquad ||\omega\tilde{\nu}_\ell|| \geq \frac{1}{4q_{n_V^R}}, \tag{3.15}$$

if ν_ℓ and $\tilde{\nu}_\ell$ are the momenta flowing through ℓ in ϑ and ϑ_0, respectively.

Proof. As V is a resonance, then for each line $\ell \in V$ one has $|\nu_\ell^0| \leq k_V < q_{n_V^R}$ (see item 4 in the definition of resonance), so that

$$||\omega\nu_\ell^0|| \geq ||\omega q_{n_V^R-1}|| > \frac{1}{2q_{n_V^R}}, \tag{3.16}$$

by (2.15) and (2.16). On the other hand

$$||\omega\nu_\ell - \omega\nu_\ell^0|| \leq \sum_{m=1}^{m_V} ||\omega\nu_m||, \tag{3.17}$$

if $\nu_1, \dots, \nu_{m_V}$ are the momenta flowing through the lines $\ell_1, \dots, \ell_{m_V}$ entering V. By hypothesis

$$||\omega\nu_{\ell_m}|| \leq \frac{1}{8q_{n_V^R}}, \quad \forall m = 1, \dots, m_V. \tag{3.18}$$

If $m_V \geq 2$ then one must have $q_{n_V^R+1} > 4q_{n_V^R}$ (see item 5 in the definition of resonance). In such a case if there is an entering line (say ℓ_1) which is the root line of a tree of order $\geq q_{n_V^R+1}/4$, then all the other lines are the root lines of subtrees of orders $k_2, \dots, k_{m_V}$ such that $k_0 \equiv k_2 + \ldots + k_{m_V} < q_{n_V^R+1}/8$ (see item 6a in the definition of resonance). Moreover, for each $m = 2, \dots, m_V$, $k_m \geq q_{n_V^R}$, otherwise the line ℓ_m would not be on scale $\geq n_V^R$. By lemma 1, $\nu_m = s_m q_{n_V^R}$ for all $m = 2, \ldots, m_V$, with $s_m \in \mathbb{Z}$, and

$$|s_2| + \ldots + |s_{m_V}| \leq \frac{k_0}{q_{n_V^R}} \leq \frac{q_{n_V^R+1}}{8q_{n_V^R}}, \tag{3.19}$$

so that

$$\sum_{m=1}^{m_V} ||\omega\nu_m|| \leq \frac{1}{8q_{n_V^R}} + \sum_{m=2}^{m_V} |s_m|\, ||\omega q_{n_V^R}|| \leq \frac{1}{8q_{n_V^R}} + \frac{1}{8q_{n_V^R}} \leq \frac{1}{4q_{n_V^R}}, \tag{3.20}$$

where use was made of (2.15). Therefore, when replacing ϑ_0 with ϑ, (3.15) follows.

If there is no entering line of V which is the root line of a tree of order $\geq q_{n_V^R+1}/4$ and the tree having as root line the exiting line of V is of order $k < q_{n_V^R+1}/4$ (see item 6b in the definition of resonance), then

$$\sum_{m=1}^{m_V} |s_m| q_{n_V^R} \leq k_1 + \ldots + k_{m_V} \equiv k - k_T < k \leq \frac{q_{n_V^R+1}}{4}\ , \tag{3.21}$$

so that

$$\sum_{m=1}^{m_V} ||\omega\nu_m|| \leq \sum_{m=1}^{m_V} |s_m|\, ||\omega q_{n_V^R}|| \leq \frac{q_{n_V^R+1}}{4q_{n_V^R}} \frac{1}{q_{n_V^R+1}} = \frac{1}{4q_{n_V^R}}. \tag{3.22}$$

which implies again (3.15). If $m_V = 1$, then (3.15) follows immediately from (3.17) and (3.18). $\square$

We come back to the proof of (3.14). Note that inside V in ϑ_0 (hence also in ϑ, see remark 6) only lines on scale n_ℓ such that $1/48q_{n_\ell} > 1/4q_{n_V^R}$ are possible, by the second inequality in (3.15) and the definition of scale (see (2.8)).

As the entering lines of V satisfy (2.8), hence (2.11), lemma 7 applies. Then, given a line ℓ internal to V on scale n_ℓ, one has

$$||\omega\nu_\ell|| \leq \frac{1}{48q_{n_\ell}} + \frac{1}{4q_{n_V^R}} \leq \frac{1}{48q_{n_\ell}} + \frac{1}{48q_{n_\ell}} \leq \frac{1}{24q_{n_\ell}}. \tag{3.23}$$

Likewise, if $1/96q_{n_\ell+1} > 2/q_{n_V^R}$, one has

$$||\omega\nu_\ell|| \geq \frac{1}{96q_{n_\ell+1}} - \frac{1}{4q_{n_V^R}} \geq \frac{1}{96q_{n_\ell+1}} - \frac{1}{768q_{n_\ell+1}} \geq \frac{1}{96q_{n_\ell+1}}\left(1 - \frac{1}{8}\right), \tag{3.24}$$

while, if $1/96q_{n_\ell+1} < 2/q_{n_V^R}$, one has

$$||\omega\nu_\ell|| \geq \frac{1}{4q_{n_V^R}} \geq \frac{1}{768q_{n_\ell+1}}\ . \tag{3.25}$$

by the third inequality in (3.15). Then (3.14) follows: so in particular the momentum ν_ℓ of the line $\ell \in \vartheta$ still fulfills (2.11).

Note that (3.13) and (2.11) imply the following bound for the renormalized resonance factor of a first generation resonance:

$$\begin{aligned} |\mathcal{R}\mathcal{V}_V(\vartheta)| \leq D_6 D_7^{k_V} & \sum_{m,m'=1}^{m_V} ||\omega\nu_{\ell_m}||\, ||\omega\nu_{\ell_{m'}}|| \cdot \\ & \cdot \left(768q_{n_V+1}\right)^2 \left(\prod_{\ell\in V} \left(768q_{n_\ell+1}\right)^2\right), \end{aligned} \tag{3.26}$$

(for some constants D_6 and D_7), where the last product (times Γ^{-k}) represents a bound on the resonance factor (3.3). The proof of such an assertion again is as in [1] (see the proof of the Corollary in [1], §3), and follows immediately by noting that for any line $\ell \in V$ one has $n_\ell \geq n_V$. The only difference with respect to [1] is that now the derivatives can act also on the compact support functions: they were just missing in [1]; it is nevertheless straightforward to see that:

$$\left| \frac{\partial^p}{\partial^p \mu} \chi_n(||\omega\nu_\ell||) \right| \leq D_8 \big(768 q_{n+1}\big)^p, \tag{3.27}$$

with $p = 1, 2$, for some constant D_8, so that:

$$\left| \frac{\partial^p}{\partial^p \mu} g_n(\nu_\ell) \right| \leq D_9 \big(768 q_{n+1}\big)^{p+2}, \tag{3.28}$$

with $p = 0, 1, 2$, for some constant D_9.

For any tree in $\mathcal{F}_V(\vartheta)$ the bound (2.11) holds, so that lemma 5 applies (see remark 15 in sect. 5).

Note that the two factors $||\omega\nu_{\ell_m}||$, $||\omega\nu_{\ell_{m'}}||$ in (3.26) allow us to neglect the propagator corresponding to a line entering a resonance with resonance scale n_V^R, provided such a propagator is replaced by a factor $(768 q_{n_V+1})^2$, where n_V is the scale of the resonance as a cluster. Such a mechanism corresponds to the discussion leading to (2.21), as far as only the first generation resonances are considered.

In general a tree will contain more resonances, and the resonances can be contained into each other. Then the above discussion has to be extended to cover the more general case: which will be done in the next section.

4. Renormalization of resonances: the general step

We proceed following strictly the techniques of [6] and [13].

Consider a tree $\vartheta \in \mathcal{T}^*_{\nu,k}$ in (3.2). For each resonance V of any generation, let us define a pair of *derived lines* ℓ^1_V, ℓ^2_V internal to V – possibly coinciding – with the following "compatibility" condition: if V is inside some other resonance W, the set $\{\ell^1_V, \ell^2_V\}$ must contain those lines of $\{\ell^1_W, \ell^2_W\}$ which are inside V. Clearly there can be 0, 1 or 2 such lines, and correspondingly we shall say that the resonance V is of type 2 if none of its derived lines is a derived line for one of the resonances containing it, of type 1 if just one of its two derived lines is a derived line for one of the resonances containing it, and of type 0 if both derived lines are derived lines for some resonances W, W' – possibly coinciding – containing V; we shall use a label $z_V = 0, 1, 2$ to take note of the type of the resonance V. One associates also to each resonance V a pair of entering lines ℓ^V_m, $\ell^V_{m'}$ if $z_V = 2$ and a single line ℓ^V_m if $z_V = 1$, with $m, m' = 1, \ldots, m_V$. Moreover for each resonance we shall introduce an interpolation parameter t_V and a measure $\pi_{z_V}(t_V)\,\mathrm{d}t_V$ such that:

$$\pi_z(t) = \begin{cases} (1-t), & z=2 \\ 1, & z=1 \\ \delta(t-1), & z=0; \end{cases} \tag{4.1}$$

we shall denote with $\mathbf{t} = \{t_V\}_{V\in\mathbf{V}}$ the set of all interpolation parameters.

The momentum flowing through a line ℓ_u internal to any resonance V will be defined recursively as:

$$\nu_{\ell_u}(\mathbf{t}) = \nu^0_{\ell_u} + t_V \sum_{\ell\in\mathcal{E}_u} \nu_\ell(\mathbf{t}), \quad \nu^0_{\ell_u} = \sum_{\substack{w\in V \\ w\preccurlyeq u}} \nu_w; \tag{4.2}$$

of course $\nu_{\ell_u}(\mathbf{t})$ will depend only on the interpolation parameters corresponding to the resonances containing the line ℓ_u (by construction).

For any resonance V the resonance factor is defined as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta)\left[\prod_{\ell\in V} g_{n_\ell}(\nu_\ell(\mathbf{t}))\right], \tag{4.3}$$

when $z_V = 2$, as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta)\left[\Big(\frac{\partial}{\partial\mu} g_{n_{\ell^1_V}}(\nu_{\ell^1_V}(\mathbf{t}))\Big)\Big(\prod_{\substack{\ell\in V,\\ \ell\neq\ell^1_V}} g_{n_\ell}(\nu_\ell(\mathbf{t}))\Big)\right], \tag{4.4}$$

when $z_V = 1$ (and we have called ℓ^1_V the line in $\{\ell^1_V, \ell^2_V\}$ which belongs to the set $\{\ell^1_W, \ell^2_W\}$ for some resonance W containing V), as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta)\left[\Big(\frac{\partial^2}{\partial\mu\partial\mu'} g_{n_{\ell^1_V}}(\nu_{\ell^1_V}(\mathbf{t}))\Big)\Big(\prod_{\substack{\ell\in V,\\ \ell\neq\ell^1_V}} g_{n_\ell}(\nu_\ell(\mathbf{t}))\Big)\right], \tag{4.5}$$

when $z_V = 0$ and $\ell^1_V = \ell^2_V$, and as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta)\left[\Big(\frac{\partial}{\partial\mu} g_{n_{\ell^1_V}}(\nu_{\ell^1_V}(\mathbf{t}))\Big)\Big(\frac{\partial}{\partial\mu'} g_{n_{\ell^2_V}}(\nu_{\ell^2_V}(\mathbf{t}))\Big)\cdot\right.$$
$$\left.\cdot\Big(\prod_{\substack{\ell\in V,\\ \ell\neq\ell^1_V,\ell^2_V}} g_{n_\ell}(\nu_\ell(\mathbf{t}))\Big)\right], \tag{4.6}$$

when $z_V = 0$ and $\ell^1_V \neq \ell^2_V$.

In (4.4)$\div$(4.6) one has $\mu = \omega\nu_{\ell^W_m}$ and $\mu' = \omega\nu_{\ell^{W'}_{m'}}$, for some lines ℓ^W_m and $\ell^{W'}_{m'}$ (possibly coinciding) entering, respectively, some resonances W and W' (possibly coinciding) containing V.

We define the renormalization operator according to the type of the resonance; namely, if $z_V = 2$, then:

$$\mathcal{R}\mathcal{V}_V(\vartheta;\omega\nu_{\ell_1}(\mathbf{t}),\ldots,\omega\nu_{\ell_{m_V}}(\mathbf{t})) = \sum_{m,m'=1}^{m_V} \omega\nu_{\ell_m}(\mathbf{t})\omega\nu_{\ell_{m'}}(\mathbf{t})\cdot$$
$$\cdot\int_0^1 \mathrm{d}t_V\,(1-t_V)\frac{\partial^2}{\partial\mu_m\partial\mu_{m'}}\mathcal{V}_V(\vartheta, t_V\omega\nu_{\ell_1}(\mathbf{t}),\ldots,t_V\omega\nu_{\ell_{m_V}}(\mathbf{t})); \tag{4.7}$$

if $z_V = 1$, then:

$$\mathcal{R}\mathcal{V}_V(\vartheta;\omega\nu_{\ell_1}(\mathbf{t}),\ldots,\omega\nu_{\ell_{m_V}}(\mathbf{t})) = \sum_{m=1}^{m_V} \omega\nu_{\ell_m}(\mathbf{t}) \cdot \\ \cdot \int_0^1 \mathrm{d}t_V \, \frac{\partial}{\partial\mu_m}\mathcal{V}_V(\vartheta, t_V\omega\nu_{\ell_1}(\mathbf{t}),\ldots,t_V\omega\nu_{\ell_{m_V}}(\mathbf{t})); \quad (4.8)$$

finally if $z_V = 0$, then:

$$\mathcal{R}\mathcal{V}_V(\vartheta)(\vartheta;\omega\nu_{\ell_1}(\mathbf{t}),\ldots,\omega\nu_{\ell_{m_V}}(\mathbf{t})) = \mathcal{V}_V(\vartheta)(\vartheta;\omega\nu_{\ell_1}(\mathbf{t}),\ldots,\omega\nu_{\ell_{m_V}}(\mathbf{t})). \quad (4.9)$$

In all cases set $\mathcal{L} = 1 - \mathcal{R}$.

Remark 10. Note that z_V equals the order of the renormalization performed on the resonance V.

Remark 11. If a resonance V has a resonance scale n_V^R, then there is a line ℓ_V^0 on scale n_V^R entering V such that $||\omega\nu_\ell|| \le ||\omega\nu_{\ell_V^0}||$ for each ℓ entering V. If there is ambiguity, ℓ_V^0 can be chosen arbitrarily. For any resonance V one has a factor bounded by $||\omega\nu_{\ell_V^0}||^{z_V}$, from (4.7), (4.8) and (4.9) and by the definition of ℓ_V^0.

To each line ℓ derived once one can associate the line $\ell_m(\ell)$ corresponding to the quantity $\mu_m = \omega\nu_{\ell_m(\ell)}$ with respect to which the propagator $g_{n_\ell}(\nu_\ell(\mathbf{t}))$ is derived. If the line ℓ is derived twice one associates to it the two lines $\ell_m(\ell)$ and $\ell_{m'}(\ell)$ such that $\mu_m = \omega\nu_{\ell_m(\ell)}$ and $\mu_{m'} = \omega\nu_{\ell_{m'}(\ell)}$ are the quantities with respect to which the propagator $g_{n_\ell}(\nu_\ell(\mathbf{t}))$ is derived.

Given a derived line ℓ, let V be the minimal resonance containing it. If the line ℓ is derived once, then let W be the resonance for which $\ell_m(\ell)$ is an entering line; if instead ℓ is derived twice, let $W, W' \subseteq W$ be the resonances for which the lines $\ell_m(\ell)$, $\ell_{m'}(\ell)$ respectively are entering lines.

In the first case, let W_i, $i = 0, \ldots, p$ the resonances contained by W and containing V, ordered naturally by inclusion:

$$V = W_0 \subset W_1 \subset \cdots \subset W_p = W. \quad (4.10)$$

We shall call the set $\mathbf{W}(\ell) = \{W_0,\ldots,W_p\} >$ the *simple cloud* of ℓ.

In the second case, let W_i, $i = 0, \ldots, p$, the resonances contained by W and containing V, ordered naturally by inclusion:

$$V = W_0 \subset W_1 \subset \cdots \subset W_{p'} = W' \subset \cdots \subset W_p = W, \quad (4.11)$$

with $p' \le p$. We shall say that $\mathbf{W}_-(\ell) = \{W_0,\ldots,W_{p'}\}$ is the *minor cloud* of ℓ while $\mathbf{W}_+(\ell) = \{W_0,\ldots,W_p\}$ is the *major cloud* of V.

When the renormalization of a resonance $V \in \mathbf{V}_{j+1}$ is performed, a tree $\vartheta_0^V \in \mathcal{F}_{V_j}(\vartheta)$, with $\vartheta \in \mathcal{T}_{\nu,k}$, is replaced by the action of the group $\mathcal{P}_V$ with a new tree ϑ^V. As this replacement is performed iteratively, one has the constraint that if V_1 and V_2 are two resonance such that V_1 is the minimal resonance containing V_2, then $\vartheta^{V_1} = \vartheta_0^{V_2}$. At the end, the original tree $\vartheta_0 \in \mathcal{T}_{\nu,k}$ is replaced with a tree $\vartheta \in \mathcal{T}^*_{\nu,k}$. On each resonance $V \in \mathbf{V}$ of ϑ the renormalization operator $\mathcal{R}$ acts: a tree whose resonance factors have been all renormalized will be called a *renormalized* (or *resummed*) *tree*.

As the replacement corresponding to each resonance settles a conjugation between lines of ϑ_0^V and those of ϑ^V, in the end for each line of ϑ there will be a conjugate line of ϑ_0.

Note that, as the transformations of the groups $\mathcal{P}_V$, $V \in \mathbf{V}$, do not modify the scales of ϑ_0 (see remark 6), the scales of the lines of ϑ are the same as those of the conjugate lines of the tree ϑ_0, so that, in order to apply lemma 5, we have only to verify that (2.11) is verified for the lines in ϑ: this will be done below (after remark 12).

Now, we shall show that:

- the localized resonance factors can be neglected (in a sense that will appear clear shortly, see lemma 8 below),
- for any (renormalized) resonance we obtain a factor:
$$\left(768 q_{n_V+1}\right)^2 ||\omega\nu_{\ell_V^0}||^2, \tag{4.12}$$
and
- the number of terms generated by the renormalization procedure is bounded by a costant to the power k,

so that the bound (2.20) can be replaced by a bound which leads to (2.21), as anticipated in sect. 2.

Note firstly that the localized part of the resonance factors can be dealt with as in sect. 3, when only first generation resonances were considered. More formally, we have the following result, which is proved in sect. 6.

Lemma 8. *Given a tree ϑ and a resonance $V \in \vartheta$, the localized resonance factor $\mathcal{L}\mathcal{V}_V(\vartheta)$ gives zero when the values of the trees belonging to the same resonance family $\mathcal{F}_V(\vartheta)$ are summed together.*

Define the map Λ:
$$\Lambda\colon \mathbf{V} \mapsto \Lambda\mathbf{V} = \left\{ z_V, \ell_V^1, \ell_V^2, \{\ell_m^V, \ell_{m'}^V\}^* \right\}_{V\in\mathbf{V}}, \tag{4.13}$$
which associates to each resonance $V \in \mathbf{V}$ the derived lines ℓ_V^1, ℓ_V^2 and the lines in the set $\{\ell_m^V, \ell_{m'}^V\}^*$ defined as:
$$\{\ell_m^V, \ell_{m'}^V\}^* = \begin{cases} \{\ell_m^V, \ell_{m'}^V\}, & \text{if } z_V = 2, \\ \ell_m^V, & \text{if } z_V = 1, \\ \emptyset, & \text{if } z_V = 0, \end{cases} \tag{4.14}$$
where $m, m' = 1, \ldots, m_V$ and $\ell_1^V, \ldots, \ell_{m_V}^V$ are the lines entering V.

Note that that the map Λ gives a natural decomposition of the set L of all lines of ϑ into $L = L_0 \cup L_1 \cup L_2$, where L_j is the set of lines derived j times.

Then, by using also lemma 6, one has

$$\begin{aligned}\mathrm{Val}(\vartheta) = \sum_{\Lambda \mathbf{V}} \left(\prod_{V \in \mathbf{V}} \int_0^1 \pi_{z_V}(t_V)\, \mathrm{d}t_V \right) \left[\prod_{u \in \vartheta} \frac{\nu_u^{m_u+1}}{m_u!} \right] \cdot \\ \cdot \left(\prod_{\ell \in L_0} g_{n_\ell}(\nu_\ell(\mathbf{t})) \right) \left(\prod_{\ell \in L_1} \omega \nu_{\ell_m(\ell)} \frac{\partial}{\partial \mu_m} g_{n_\ell}(\nu_\ell(\mathbf{t})) \right) \cdot \\ \cdot \left(\prod_{\ell \in L_2} \omega \nu_{\ell_m(\ell)} \omega \nu_{\ell_{m'}(\ell)} \frac{\partial^2}{\partial \mu_m \partial \mu_{m'}} g_{n_\ell}(\nu_\ell(\mathbf{t})) \right).\end{aligned} \tag{4.15}$$

Remark 12. Note that no propagator is derived more than twice: this fact is essential for our proof since we have no control on the growth rate of the derivatives of the compact support functions (2.6).

After the renormalization procedure has been applied for all resonances, one check that the momenta of the lines in ϑ have changed, with respect to the original tree ϑ_0 with nonvanishing value, in such a way that the bound (2.11) still hold.

Lemma 9. *Consider a renormalized tree $\vartheta \in \mathcal{T}^*_{\nu,k}$, obtained from $\vartheta \in \mathcal{T}_{\nu,k}$ by the iterative replacements, described above, that take place each time a resonance appears. Then the lines of ϑ inherit the scales of the conjugate lines of ϑ_0 and lemma 5 applies to ϑ.*

Proof. The first assertion follows by construction. The second one can be seen by induction on the generation of the resonances, by taking into account that for the first generation resonances the result has been already proved in sect. 3. So let us suppose that (2.14) holds for resonances of any generation j', with $j' < j$. Consider a line ℓ contained inside a resonance $V \in \mathbf{V}_j$ and outside all resonances in $\mathbf{V}_{j+1}$ contained inside V: then there will be j resonances $V \equiv W_1 \subset \ldots \subset W_j$ containing ℓ. Each renormalization produces a change on the momentum flowing through the line ℓ, such that, if $\tilde{\nu}_\ell$ is the momentum flowing through the line ℓ in ϑ_0 and ν_ℓ is the momentum flowing through the conjugate line ℓ in ϑ, then

$$\frac{1}{96 q_{n_\ell+1}} - \sum_{i=1}^{j} \frac{1}{4 q_{n^R_{W_i}}} \le ||\omega \tilde{\nu}_\ell|| \le \frac{1}{48 q_{n_\ell}} + \sum_{i=1}^{j} \frac{1}{4 q_{n^R_{W_i}}}. \tag{4.16}$$

Call $\vartheta_0^V \in \mathcal{F}_{\mathbf{V}_j}(\vartheta_0)$ the tree containing V (which is not, in general, the originary tree ϑ_0) and ϑ^V the tree in $\mathcal{F}_V(\vartheta_0^V)$ obtained by the action of the group $\mathcal{P}_V$. As (2.11) is supposed to hold before renormalizing V, for all lines ℓ_m, $m = 1, \ldots, m_V$, entering V one has $||\omega \nu_{\ell_m}|| < 1/8 q_{n_{\ell_m}}$, so that, by reasoning as in sect. 3 to prove lemma 7, we can conclude that

$$\big| ||\omega \nu_\ell|| - ||\omega \tilde{\nu}_\ell|| \big| \le \frac{1}{4 q_{n^R_V}}, \qquad ||\omega \nu_\ell|| \ge \frac{1}{4 q_{n^R_V}}, \qquad ||\omega \tilde{\nu}_\ell|| \ge \frac{1}{4 q_{n^R_V}}, \tag{4.17}$$

where ν_ℓ is the momentum flowing through the line ℓ in ϑ^V.

In order that ℓ be contained inside $V = W_1$, one must have $1/48 q_{n_\ell} \ge 1/4 q_{n^R_V}$; moreover if $j_1 = \lfloor (j-1)/2 \rfloor$ and $j_2 = \lfloor j/2 \rfloor$ (here $\lfloor \cdot \rfloor$ denotes the integer part), one has

$$q_{n_{W_1}} \le \frac{q_{n_{W_3}}}{2} \le \ldots \le \frac{q_{n_{W_{j_1}}}}{2^{j_1}}, \qquad q_{n_{W_2}} \le \frac{q_{n_{W_4}}}{2} \le \ldots \le \frac{q_{n_{W_{j_2}}}}{2^{j_2}}, \tag{4.18}$$

(simply use that $q_{n+1} \geq q_n$ and $q_{n+2} \geq 2q_n$ for any $n \geq 0$). Then one can write

$$||\omega\nu_\ell|| \leq \frac{1}{48q_{n_\ell}} + \frac{1}{4q_{n_V^R}}\Big(\sum_{i=1}^{j_1}\frac{1}{2^i} + \sum_{i=1}^{j_2}\frac{1}{2^i}+\Big) \leq \frac{1}{48q_{n_\ell}} + \frac{1}{q_{n_V^R}}; \tag{4.19}$$

this is bounded from above by $5/48q_{n_\ell}$. Likewise one finds

$$||\omega\nu_\ell|| \geq \frac{1}{96q_{n_\ell+1}} - \frac{1}{4q_{n_V^R}}\Big(\sum_{i=1}^{j_1}\frac{1}{2^i} + \sum_{i=1}^{j_2}\frac{1}{2^i}+\Big) \geq \frac{1}{96q_{n_\ell+1}} - \frac{1}{q_{n_V^R}}; \tag{4.20}$$

this is bounded from below by $1/192q_{n_\ell+1}$ if $1/96q_{n_\ell+1} > 2/q_{n_V^R}$ and by $1/768q_{n_\ell+1}$ if $1/96q_{n_\ell+1} \leq 2/q_{n_V^R}$.

Then (2.14) holds also for any line ℓ contained inside V_0, if V is a resonance in $\mathbf{V}_j$. As any next renormalization is on resonances $V \in \mathbf{V}_{j'}$, with $j' > j$, so that it does not shift the line ℓ, the momentum ν_ℓ changes no more, so that the inductive proof is complete. □

Then in (4.15) we can bound, for $\ell \in L_1$:

$$\begin{aligned} &\left|\omega\nu_{\ell_m(\ell)}\frac{\partial}{\partial\mu_m}g_{n_\ell}(\nu_\ell(\mathbf{t}))\right| \leq \\ &\qquad \leq D_9||\omega\nu_{\ell_m(\ell)}||\big(768q_{n_\ell+1}\big)^3 \\ &\qquad \leq D_9||\omega\nu_{\ell_m(\ell)}||\big(768q_{n_\ell+1}\big)^3\prod_{i=0}^{p-1}\frac{||\omega\nu_{\ell^0_{W_i}}||}{||\omega\nu_{\ell^0_{W_i}}||} \\ &\qquad \leq D_9\big(768q_{n_\ell+1}\big)^2\left[\prod_{i=0}^{p}||\omega\nu_{\ell^0_{W_i}}||\right]\left[\prod_{i=0}^{p}\big(768q_{n_{W_i}+1}\big)\right], \end{aligned} \tag{4.21}$$

where $\mathbf{W}(\ell) = \{W_0, \dots, W_p\}$ is the simple cloud of ℓ, and, for $\ell \in L_2$:

$$\begin{aligned} &\left|\omega\nu_{\ell_m(\ell)}\omega\nu_{\ell_{m'}(\ell)}\frac{\partial^2}{\partial\mu_m\partial\mu_{m'}}g_{n_\ell}(\nu_\ell(\mathbf{t}))\right| \leq \\ &\qquad \leq D_9||\omega\nu_{\ell_m(\ell)}||\,||\omega\nu_{\ell_{m'}(\ell)}||\big(768q_{n_\ell+1}\big)^4 \\ &\qquad \leq D_9||\omega\nu_{\ell_m(\ell)}||\,||\omega\nu_{\ell_{m'}(\ell)}||\big(768q_{n_\ell+1}\big)^4\prod_{i=0}^{p-1}\frac{||\omega\nu_{\ell^0_{W_i}}||}{||\omega\nu_{\ell^0_{W_i}}||}\prod_{i'=0}^{p'-1}\frac{||\omega\nu_{\ell^0_{W_{i'}}}||}{||\omega\nu_{\ell^0_{W_{i'}}}||} \\ &\qquad \leq D_9\big(768q_{n_\ell+1}\big)^2\left[\prod_{i=0}^{p}||\omega\nu_{\ell^0_{W_i}}||\right]\left[\prod_{i=0}^{p}\big(768q_{n_{W_i}+1}\big)\right] \\ &\qquad\quad \left[\prod_{i'=0}^{p'}||\omega\nu_{\ell^0_{W_{i'}}}||\right]\left[\prod_{i'=0}^{p'}\big(768q_{n_{W_{i'}}+1}\big)\right], \end{aligned} \tag{4.22}$$

where $\mathbf{W}_-(\ell) = \{W_0, \dots, W_{p'}\}$ is the minor cloud and $\mathbf{W}_+(\ell) = \{W_0, \dots, W_p\}$ is the major cloud of ℓ.

Note that (4.21) and (4.22) give a factor

$$||\omega\nu_{\ell^0_{W_i}}||\big(768q_{n_{W_i}+1}\big) \tag{4.23}$$

for each resonance W_i belonging to the (simple or minor or major) cloud of ℓ. As each resonance belongs to the cloud of some line internal to it and each resonance contains two derived lines or one line derived twice (by definition of the renormalization procedure), then one concludes that a factor equal to the square of (4.23) is obtained for each resonance.

If we note that each underived propagator can be bounded again using (3.28) with $p = 0$, then we can summarize the bounds (4.21) $\div$ (4.22) stating that, for each resummed tree ϑ, we have:

- for each resonance V, a factor $||\omega\nu_{\ell_V^0}||^2$ times a factor $(768 q_{n_V+1})^2$;
- for each line ℓ, a factor $D_9(768 q_{n_\ell+1})^2$ (as the factors $(768 q_{n_\ell+1})^p$, $p = 1, 2$, appearing when the corresponding propagator is derived, are taken into account by the factors associated to the resonances, see the item above);

Then the statement concerning (4.12) is proved.

Once the single summand in (4.15) has been bounded, one is left with the problem of bounding the number of terms on which the sum is performed.

For each first generation resonance V at most m_V^2 times k_V^2 summands are generated by the renormalization procedure (see (3.13)). In general, for each (renormalized) resonance, we have to sum over the entering lines $\{\ell_m^V, \ell_{m'}^V\}^*$ (corresponding to the quantities μ_m, $m = 1, \ldots, m_V$, in terms of which the renormalized resonance factor is considered a function) and over the internal lines $\{\ell_V^1, \ell_V^2\}$ (corresponding to the factors on which the derivatives act). An estimates on the number of summands generated by the renormalization procedure can be obtained by using the counting lemma 6.

If $V \in \mathbf{V}_j$, $j \geq 1$, let $\mathcal{N}_V$ be the number of $(j+1)$-th generation resonances contained inside V. Recall that V_0 is the set of lines internal to V which are outside any resonance contained in V, and denote by k_{V_0} the number of elements in V_0.

The renormalization procedure, for each renormalized resonance, generates a single or double sum over the entering lines whose momenta appear in the quantities $\omega\nu_{\ell_1}(\mathbf{t}), \ldots, \omega\nu_{\ell_{m_V}}(\mathbf{t})$, in terms of which the resonance factor is expanded: the sum is single if the localization is to first order and double if the localization is to second order (see (4.7) and (4.8)).

Then we find, using lemma 6, that in the renormalization procedure each sum over the entering lines of a first generation resonance V is on m_V terms, each sum over the entering lines of all second order resonances $V' \subset V$ is on $k_{V_0} + \mathcal{N}_V$ terms, each sum over the entering lines of all third generation resonances $V'' \subset V' \subset V$ is on $k_{V_0'} + \mathcal{N}_{V'}$, and so on; in general, each sum over the entering lines of all the resonances $V' \in \mathbf{V}_{j+1}$ contained inside a resonance $V \in \mathbf{V}_j$ is bounded by $k_{V_0} + \mathcal{N}_V$.

Once all generations of resonances have been considered, the overall number of summands generated by the renormalization procedure – by taking also into account the sum over the derived lines and using remark 12 – is bounded by:

$$\left[\prod_{V \in \mathbf{V}_1} k_V^2\right] \left[\Big(\prod_{V \in \mathbf{V}_1} m_V^2\Big)\Big(\prod_{V \in \mathbf{V}} (k_{V_0} + \mathcal{N}_V)^2\Big)\right] \leq e^{6k}, \tag{4.24}$$

where k is the order of the tree ϑ. In fact, just use $x \le e^x$ and the obvious inequalities:

$$\begin{aligned} &\sum_{V\in\mathbf{V}_1} k_V \le k, \\ &\sum_{V\in\mathbf{V}_1} m_V + \sum_{V\in\mathbf{V}} k_{V_0} \le k, \\ &\sum_{V\in\mathbf{V}} \mathcal{N}_V \le k. \end{aligned} \tag{4.25}$$

Then the statement after (4.12) is proved and the constant D_3 is e^6.

Finally one has to count the number of trees. The bound given in sect. 2 is no more valid, as a line $\ell \in \vartheta$ can have more than two scale labels. However lemma 4 proves that to each line at most $D_{10} = 17$ scale labels can be associated, so that the number of trees in $\mathcal{T}^*_{\nu,k}$ is bounded by $2^{3k} D_{10}^k$. Then the bound (2.21) follows, with $D_4 = 2^3 D_3 D_9 D_{10}$: this concludes the proof of the theorem.

5. Proof of lemma 5

We shall prove inductively on the order k the following bounds:

$$M_n(\vartheta) = 0, \qquad \text{if } k < q_n, \tag{5.1a}$$

$$M_n(\vartheta) \le \frac{2k}{q_n} - 1 + N_n^R(\vartheta), \qquad \text{if } k \ge q_n, \tag{5.1b}$$

for any $n \ge 0$, and:

$$M_n(\vartheta) = 0, \qquad \text{if } k < q_n, \tag{5.2a}$$

$$M_n(\vartheta) \le \frac{k}{q_n} + N_n^R(\vartheta), \qquad \text{if } q_n \le k < \frac{q_{n+1}}{4}, \tag{5.2b}$$

$$M_n(\vartheta) \le \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + N_n^R(\vartheta), \qquad \text{if } k \ge \frac{q_{n+1}}{4}, \tag{5.2c}$$

for $q_{n+1} > 4q_n$, where k is the order of the tree ϑ.

Note that (5.1a) and (5.2a) are simply a consequence of lemma 2 of sect. 2, so we have to prove only (5.1b), (5.2b) and (5.2c).

Remark 13. If we were only interested in proving the analyticity of the invariant curves for rotation numbers satisfying the Bryuno condition, then equations (5.1) would be sufficient – as it would be easy to check by proceeding along the lines of sect. 3 and 4. However, in order to find the optimal dependence of the radius of convergence $\rho(\omega)$ on ω, which is the main focus of this paper, the more refined bounds (5.2) are necessary.

Remark 14. The proof of (5.1) is easier, as it is obvious since it is a weaker result. After dealing with (5.2), the proof of (5.1) could be left as an exercise: we shall prove it explicitely for completeness, and as it could be read as an introduction to the more involved proof of (5.2).

We shall prove first (5.2) (case $q_{n+1} > 4q_n$) in cases [1] $\div$ [3] below, then (5.1) in items [4] $\div$ [6] below. We proceed by induction, and assuming that (5.1), (5.2) hold for any $k' < k$ we shall show that they hold for k also; their validity for $k = 1$ being trivial, lemma 5 is proved. Recall also remark 5 in sect. 2 about the way of counting the resonances on scale n and the resonances with resonance-scale n.

- So consider first $q_{n+1} > 4q_n$.

[1] If the root line ℓ of ϑ has scale $\neq n$ and it is not the exiting line of a resonance on scale n, let us denote with $\ell_1, \ldots, \ell_m$ the lines entering the last node u_0 of ϑ and $\vartheta_1, \ldots, \vartheta_m$ the subtrees of ϑ whose root lines are those lines. By construction $M_n(\vartheta) = M_n(\vartheta_1) + \cdots + M_n(\vartheta_m)$ and $N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m)$: the bounds (5.2) follow inductively by noting that for $k \geq q_{n+1}/4$ one has $8k/q_{n+1} - 1 \geq 1$.

[2] If the root line ℓ of ϑ has scale n, then we can reason as follows. Let us denote with $\ell_1, \ldots, \ell_m$ the lines on scale $\geq n$ which are the nearest to the root line of ϑ,[2] and let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees with root lines $\ell_1, \ldots, \ell_m$. If $m = 0$ then (5.2) follow immediately from lemma 2 of sect. 2; so let us suppose that $m \geq 1$. Then the lines $\ell_1, \ldots, \ell_m$ are the entering lines of a cluster T (which can degenerate to a single point) having the root line of ϑ as the exiting line. As ℓ cannot be the exiting line of a resonance on scale n, one has:

$$M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m). \tag{5.3}$$

In general $\tilde{m}$ subtrees among the m considered have orders $\geq q_{n+1}/4$, with $0 \leq \tilde{m} \leq m$, while the remaining $m_0 = m - \tilde{m}$ have orders $< q_{n+1}/4$. Let us numerate the subtrees so that the first $\tilde{m}$ have orders $\geq q_{n+1}/4$.

Let us distinguish the cases $k < q_{n+1}/4$ and $k \geq q_{n+1}/4$.

[2.1] If $k < q_{n+1}/4$, then $\tilde{m} = 0$ and each line entering T, by lemma 1 of sect. 2, has a momentum which is a multiple of q_n and, by lemma 2, has a scale label n. Therefore the momentum flowing through the root line is $\nu = \nu_T + s_0 q_n$, for some $s_0 \in \mathbb{Z}$, with:

$$\nu_T \equiv \sum_{u \in T} \nu_u. \tag{5.4}$$

Moreover also the root line of ϑ has scale n, by assumption, and momentum $\nu = sq_n$ for some $s \in \mathbb{Z}$, by lemma 1, so that $\nu_T = (s - s_0)q_n = s'q_n$, for some integer s'.

[2.1.1] If $s' \neq 0$, then $k_T \geq |\nu_T| \geq q_n$, giving:

$$\begin{aligned} M_n(\vartheta) \leq 1 + \frac{k_1 + \cdots + k_m}{q_n} + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \leq \\ 1 + \frac{k - k_T}{q_n} + N_n^R(\vartheta) \leq \frac{k}{q_n} + N_n^R(\vartheta), \end{aligned} \tag{5.5}$$

as $N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m)$, and (5.2b) follows.

[2.1.2] If $s' = 0$ and $k_T \geq q_n$, one can reason as in case [2.1.1].

[2] That is, such that no other line along the paths connecting the lines $\ell_1, \ldots, \ell_m$ to the root line is on scale $\geq n$.

[2.1.3] If $s' = 0$ and $k_T < q_n$, then T is a resonance with resonance-scale n, and:

$$M_n(\vartheta) \le 1 + \frac{k_1 + \cdots + k_m}{q_n} + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \le$$
$$\le 1 + \frac{k}{q_n} + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \le \frac{k}{q_n} + N_n^R(\vartheta), \quad (5.6)$$

as $N_n^R(\vartheta) = 1 + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m)$, and again (5.2b) follows.

[2.2] If $k \ge q_{n+1}/4$, assume again inductively the bounds (5.2). From (5.3) we have:

$$M_n(\vartheta) \le 1 + \sum_{j=1}^{\tilde{m}} \Big(\frac{k_j}{q_n} + \frac{8k_j}{q_{n+1}} - 1 \Big) + \sum_{j=\tilde{m}+1}^{m} \frac{k_j}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j), \quad (5.7)$$

where k_j is the order of the subtree ϑ_j, $j = 1, \ldots, m$.

[2.2.1] If $\tilde{m} \ge 2$, then (5.2c) follows immediately.

[2.2.2] If $\tilde{m} = 0$, then (5.7) gives:

$$M_n(\vartheta) \le 1 + \frac{k_1 + \cdots + k_m}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) \le 1 + \frac{k}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) \le$$
$$\le \frac{8k}{q_{n+1}} - 1 + \frac{k}{q_n} + N_n^R(\vartheta), \quad (5.8)$$

as we are considering k such that $1 \le 8k/q_{n+1} - 1$ and $N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$.

[2.2.3] If $\tilde{m} = 1$, then (5.7) gives:

$$M_n(\vartheta) \le 1 + \Big(\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} - 1 \Big) + \sum_{j=2}^{m} \frac{k_j}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) =$$
$$= \frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j), \quad (5.9)$$

where $k_0 = k_2 + \cdots + k_m$.

[2.2.3.1] If in such case $k_0 \ge q_{n+1}/8$, then we can bound in (5.9):

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \le \frac{k_1 + k_0}{q_n} + \frac{8(k_1 + k_0)}{q_{n+1}} - \frac{8k_0}{q_{n+1}} \le \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1, \quad (5.10)$$

and $N_n^R(\vartheta_1 + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$, so that (5.2c) follows.

[2.2.3.2] If $k_0 < q_{n+1}/8$, then, denoting with ν and ν_1 the momenta flowing through the root line ℓ of ϑ and the root line ℓ_1 of ϑ_1 respectively, one has:

$$||\omega(\nu - \nu_1)|| \le ||\omega\nu|| + ||\omega\nu_1|| \le \frac{1}{4q_n}, \quad (5.11)$$

as both ℓ and ℓ_1 are on scale $\ge n$ (see remark 2 in sect. 2 and use (2.14)). Then either $|\nu - \nu_1| \ge q_{n+1}/4$ or $\nu - \nu_1 = \tilde{s} q_n$, $\tilde{s} \in \mathbb{Z}$, by lemma 1 of sect. 2.

[2.2.3.2.1] If $|\nu - \nu_1| \ge q_{n+1}/4$, noting that $\nu = \nu_1 + \nu_T + \nu_0$, where $\nu_0 = s_0 q_n$ (with $s_0 \in \mathbb{Z}$ and $|\nu_0| \le k_0 < q_{n+1}/8$) is the sum of the momenta flowing through the root lines of the m_0 subtrees entering T with orders $< q_{n+1}/4$ and ν_T is defined by (5.4), one has:

$$k_T \geq |\nu_T| \geq |\nu - \nu_1| - |\nu_0| \geq \frac{q_{n+1}}{8}, \tag{5.12}$$

so that in (5.9) one can bound:

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \leq \frac{k - k_T}{q_n} + \frac{8(k - k_0 - k_T)}{q_{n+1}} \leq \frac{k}{q_n} + \frac{8(k - k_T)}{q_{n+1}} \leq$$
$$\leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1, \tag{5.13}$$

and $N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$, so that (5.2c) follows again.

[2.2.3.2.2] If $\nu - \nu_1 = \tilde{s} q_n$, $\tilde{s} \in \mathbb{Z}$, then:

$$\nu_T = \nu - \nu_1 - \nu_0 = (\tilde{s} - s_0) \equiv s q_n, \tag{5.14}$$

where $s \in \mathbb{Z}$.

[2.2.3.2.2.1] If $s \neq 0$, then $k_T \geq q_n$, so that in (5.3) one has:

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \leq \frac{k - k_T}{q_n} - \frac{8k}{q_{n+1}} \leq \frac{k}{q_n} - 1 + \frac{8k}{q_{n+1}}, \tag{5.15}$$

and $N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$, so implying (5.2c).

[2.2.3.2.2.2] If $s = 0$ (*i.e.* $\nu_T = 0$) and $k_T \geq q_n$, one can proceed as in case [2.2.3.2.2.1].

[2.2.3.2.2.3] If $s = 0$ and $k_T < q_n$, then T is a resonance with resonance-scale n,[3] so that $N_n^R(\vartheta) = 1 + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m)$, hence (5.9) gives:

$$M_n(\vartheta) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + 1 + \sum_{j=1}^{m} N_n^R(\vartheta_j) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + N_n^R(\vartheta), \tag{5.16}$$

and (5.2c) follows.

[3] If the root line ℓ of ϑ is on scale $> n$ and it is the exiting line of a resonance V_n on scale n, let us denote with $\ell_1, \ldots, \ell_m$ the lines on scale $\geq n$ which are the nearest to the root line of ϑ, and let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees with root lines $\ell_1, \ldots, \ell_m$; some of these lines – at least one – are lines on scale n inside V_n.[4] Let T be the cluster which the lines $\ell_1, \ldots, \ell_m$ enter; of course $T \subset V_n$ and T can degenerate into a single point. As in case [2], let $\tilde{m}$ be the number of subtrees among the m considered which have orders $\geq q_{n+1}/4$, and again let us numerate the subtrees in such a way that the ones with orders $\geq q_{n+1}/4$ are the first $\tilde{m}$.

Note that $k \geq q_{n+1}$ (otherwise ℓ could not be on scale $> n$) and

$$M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m), \tag{5.17}$$

as the root line ℓ contributes one unit to $P_n(\vartheta)$ and does not contribute to $N_n(\vartheta)$. Note also that if T is a resonance then its resonance scale is n.

[3] If $m_0 = 0$, then $n \equiv \nu_\ell = \nu_{\ell_1}$ so that $n_\ell \leq n_{\ell_1} \leq n_\ell + 1$, by construction and by remark 2.

[4] Otherwise V_n would not contain any line on scale n, so that it would not be a resonance on scale n as we are supposing.

[3.1] If T is not a resonance, then:

$$N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m). \tag{5.18}$$

By induction (5.2) and (5.17) imply:

$$M_n(\vartheta) \le 1 + \sum_{j=1}^{\tilde m} \left(\frac{k_j}{q_n} + \frac{8k_j}{q_{n+1}} - 1 \right) + \sum_{j=1}^{m} \frac{k_j}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j), \tag{5.19}$$

where k_j are the orders of the subtrees ϑ_j, $j = 1, \ldots, m$.

[3.1.1] If $\tilde m = 2$, then (5.2c) follows immediately.

[3.1.2] The case $\tilde m = 0$ is impossible because T is contained inside a resonance V_n on scale n, so that at least one of the subtrees entering T must have order $\ge q_{n+1}/4$ – otherwise no line on scale $> n$ could enter V_n, see lemma 2.

[3.1.3] If $\tilde m = 1$ let $k_0 = k_2 + \cdots + k_m$; then the case $k_0 \ge q_{n+1}/8$ can be dealt with as in case [2.2.3.1]; if $k_0 < q_{n+1}/8$, we deduce from lemma 1 that either $|\nu - \nu_1| \ge q_{n+1}/4$ or $\nu - \nu_1 = \tilde s q_n$, using the same notations of case [2.2.3.2].

The first case can be discussed as in case [2.2.3.2.1], while in the second case we find, as in case [2.2.3.2.2], that $\nu_T = \nu - \nu_1 - \nu_0 = s q_n$, with either $s \ne 0$ or $s = 0$ and $k_T \ge q_n$ (otherwise T would be a resonance), so that the conclusions in cases [2.2.3.2.2.1] and [2.2.3.2.2.2] can be inherited in the present case and (5.2c) follows again.

[3.2] If T is a resonance, then its resonance-scale is n (and all its entering lines are on scale n; see item 1 in the definition of resonance), so that:

$$N_n^R(\vartheta) = 1 + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m). \tag{5.20}$$

The discussion goes on as in case [3.1] above, with the only difference that now, when $\tilde m = 1$ (and $k_T < q_n$, $k_0 < q_{n+1}/8$), the case $\nu_T = 0$ (*i.e.* $\nu_T = s q_n$, with $s = 0$) is the only possible since T is a resonance. In such a case:

$$M_n(\vartheta) \le 1 + \frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} - 1 + \frac{k_0}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) \le \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + N_n^R(\vartheta), \tag{5.21}$$

and (5.2c) follows once more.

• Now we prove (5.1).

[4] If the root line ℓ of ϑ as scale $\ne n$ and it is not the entering line of a resonance on scale n, let us denote with $\ell_1, \ldots, \ell_m$ the lines entering the last node u_0 of ϑ. By construction $M_n(\vartheta) = M_n(\vartheta_1) + \cdots + M_n(\vartheta_m)$ and $N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m)$ so that the bound (5.1) follows immediately by induction.

[5] If the root line ℓ of ϑ has scale n, using the same notations as in case [2], denote with $\ell_1, \ldots, \ell_m$ the lines on scale $\ge n$ which are nearest to the root line of ϑ, and let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees with these lines as root lines. Then such lines are the entering lines of a cluster T (which can degenerate into a single point) having the root line of ϑ as the exiting line. We have:

$$M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m). \tag{5.22}$$

Assuming again inductively the bounds (5.1), from (5.22) we have:

$$M_n(\vartheta) \le 1 + \sum_{j=1}^{m} \left(\frac{2k_j}{q_n} - 1 \right) + \sum_{j=1}^{m} N_n^R(\vartheta_j), \tag{5.23}$$

where k_j is the order of the subtree ϑ_j, $j = 1, \ldots, m$.

[5.1] If $m \ge 2$, then (5.1b) follows immediately.

[5.2] If $m = 0$, then $M_n(\vartheta) = 1$. As ℓ is on scale n, the order k of ϑ has to be $k \ge q_n$, so that:

$$M_n(\vartheta) = 1 \le \frac{2k}{q_n} - 1, \quad N_n^R(\vartheta) = 0, \tag{5.24}$$

and (5.1b) follows again.

[5.3] If $m = 1$, then (5.23) gives:

$$M_n(\vartheta) \le 1 + \left(\frac{2k_1}{q_n} - 1 \right) + N_n^R(\vartheta_1) = \frac{2k_1}{q_n} + N_n^R(\vartheta_1). \tag{5.25}$$

Denoting with ν and ν_1 the momenta flowing, respectively, through the root line ℓ of ϑ and through the root line ℓ_1 of ϑ_1, we have:

$$||\omega(\nu - \nu_1)|| \le ||\omega\nu|| + ||\omega\nu_1|| \le \frac{1}{4q_n}, \tag{5.26}$$

as both ℓ and ℓ_1 are on scale $\ge n$ (see remark 2 in page 4 and use (2.14)). Then, as $\nu_T = \nu - \nu_1$, either $|\nu_T| \ge q_n$ or $\nu_T = 0$.

[5.3.1] If $|\nu_T| \ge q_n$, then $k_T \ge |\nu_T| \ge q_n$ and $N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$ (since T is not a resonance), so that (5.25) gives:

$$M_n(\vartheta) \le \frac{2k}{q_n} - \frac{2k_T}{q_n} + N_n^R(\vartheta_1) \le \frac{2k}{q_n} - 1 + N_n^R(\vartheta_1), \tag{5.27}$$

and (5.1b) follows.

[5.3.2] If $\nu_T = 0$ and $k_T \ge q_n$, one can reason as in case [5.3.1].

[5.3.3] If $\nu_T = 0$ and $k_T < q_n$, then $\nu_1 = \nu$ and either $n_{\ell_1} = n$ or $n_{\ell_1} = n + 1$ (see remark 1 in page 4): then T is a resonance with resonance scale n, so that $1 + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$, hence (5.25) gives:

$$M_n(\vartheta) \le \left(\frac{2k}{q_n} - 1 \right) + 1 + N_n^R(\vartheta_1) \le \frac{2k}{q_n} - 1 + N_n^R(\vartheta_1), \tag{5.28}$$

and (5.1) follows again.

[6] If the root line ℓ of ϑ is on scale $> n$ and it is the exiting line of a resonance V_n, as in case [3] above, denote with $\ell_1, \ldots, \ell_m$ the lines on scale $\ge n$ wich are nearest to the root line of ϑ, and let $\vartheta_1, \ldots, \vartheta_m$ be the subtree of ϑ of which these lines are root lines. Some of these lines – at least one – are lines on scale n inside V_n. Let T be the cluster which the lines $\ell_1, \ldots, \ell_m$ enter; of course $T \subset V_n$, and T can degenerate into a single point.

Note that as in case [3]:

$$M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m), \tag{5.29}$$

as the root line ℓ contributes one unit to $P_n(\vartheta)$ and does not contribute to $N_n(\vartheta)$, and that if T is a resonance then its resonance scale is n.

[6.1] If T is not a resonance, then:

$$N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m). \tag{5.30}$$

By induction, (5.1) and (5.29) imply:

$$M_n(\vartheta) \le 1 + \sum_{j=1}^{m} \left(\frac{2k_j}{q_n} - 1 \right) + \sum_{j=1}^{m} N_n^R(\vartheta_j), \tag{5.31}$$

where k_j are the orders of the subtrees ϑ_j, $j = 1, \ldots, m$.

[6.1.1] If $m = 2$, then (5.1b) follows immediately.

[6.1.2] The case $m = 0$ is impossible (see case [3.1.2]).

[6.1.3] If $m = 1$ in (5.31), we have $\nu_T = \nu - \nu_1$, so that $|\nu_T| \ge q_n$ (as $\nu_T \neq 0$, otherwise T would be a resonance). Then we can go on along the lines of case [5.3.1] in order to obtain (5.1b).

[6.2] If T is a resonance, then its resonance scale is n, so that:

$$N_n^R(\vartheta) = 1 + N_n^R(\vartheta_1), \tag{5.32}$$

and the discussion goes on as in case [6.1], with the only difference that now, for $m = 1$, the case $\nu_T = 0$ is the only possible as T is supposed to be a resonance. In such a case:

$$M_n(\vartheta) \le 1 + \left(\frac{2k}{q_n} - 1 \right) + N_n^R(\vartheta_1) \le \frac{2k}{q_n} - 1 + N_n^R(\vartheta), \tag{5.33}$$

implying again (5.1b).

• Finally, to deduce (2.19) from (5.1) and (5.2), simply note that, for $q_{n+1} \le 4q_n$, we have $2k/q_n \le 8k/q_{n+1}$; them lemma 5 follows.

Remark 15. Note that the correspondence between momenta and scale labels has been used only through the inequality (2.11). As we have seen in sect. 4 the renormalization procedure can shift the "original" momenta flowing through the lines of a bounded quantity which does not alter such an inequality. This allow us to apply lemma 4 also to the renormalized trees, as it was repeatedly claimed in the previous sections.

6. Proof of lemma 8

As far as only the localized resonance factor is involved, the momenta flowing through the lines entering any resonance are set to zero, so that it does not matter if such momenta are interpolated or not (*i.e.* if they are of the form $\boldsymbol{\nu}$ or $\boldsymbol{\nu}(\mathbf{t})$). In particular, the case of first generation resonances (discussed in sect. 3) is included in lemma 8.

A basic property of the trees belonging to the resonance family $\mathcal{F}_V(\vartheta)$ is that the difference between their values is only in the resonance factor: for any tree $\vartheta' \in \mathcal{F}_V(\vartheta)$, we can write:

$$\mathrm{Val}(\vartheta') = \mathcal{A}(\vartheta)\mathcal{V}_V(\vartheta'), \tag{6.1}$$

for some factor $\mathcal{A}(\vartheta)$ which is the same for all $\vartheta' \in \mathcal{F}_V(\vartheta)$. This simply follows from the fact that the transformations in $\mathcal{P}_V$ do not touch the part of the tree ϑ which is outside the resonance V. Therefore a cancellation between localized resonance factors yields a cancellation between tree values (in which the resonance factor has been localized of course).

By item 1 in the definition of resonance and by definition of V_0, one has

$$\sum_{u\in V_0} \nu_u = 0; \tag{6.2}$$

moreover, given an entering line ℓ_m of V, if $\ell_m \in L_V^R$ and $\tilde{V}_0 = V_0(\ell_m)$, then

$$\sum_{u\in \tilde{V}_0} \nu_u \equiv \sum_{u\in V_0(\ell_m)} \nu_u = 0. \tag{6.3}$$

In general we can write, for any tree $\vartheta' \in \mathcal{F}_V(\vartheta)$,

$$\mathcal{L}\mathcal{V}_V(\vartheta') = \mathcal{B}(\vartheta')\mathcal{L}\mathcal{V}_{V_0}(\vartheta') \prod_{\ell\in L_V^R} \mathcal{L}\mathcal{V}_{V(\ell)}(\vartheta'), \tag{6.4}$$

where $\mathcal{V}_{V_0}(\vartheta')$ and $\mathcal{V}_{V(\ell)}(\vartheta')$ are defined as the resonance factor $\mathcal{V}_V(\vartheta')$, but with the product ranging only over nodes and lines internal to V_0 and $V(\ell)$, respectively, while $\mathcal{L}\mathcal{V}_{V_0}(\vartheta')$ and $\mathcal{L}\mathcal{V}_{V(\ell)}(\vartheta')$ are obtained from $\mathcal{V}_{V_0}(\vartheta')$ and $\mathcal{V}_{V(\ell)}(\vartheta')$, respectively, by replacing ν_ℓ with ν_ℓ^0 in V, for all lines $\ell \in V$. In (6.4) $\mathcal{B}(\vartheta')$ takes into account all other factors (if there are any), alwyas evaluated with ν_ℓ replaced with ν_ℓ^0, $\ell \in V$. Note that, as $\mathcal{A}(\vartheta)$ in (6.1), also $\mathcal{B}(\vartheta')$ is the same for all $\vartheta' \in \mathcal{F}_V(\vartheta)$, so that one can set $\mathcal{B}(\vartheta') = \mathcal{B}(\vartheta)$ and write:

$$\mathrm{Val}(\vartheta') = \mathcal{A}(\vartheta)\mathcal{V}_V(\vartheta'), \qquad \mathcal{L}\mathcal{V}_V(\vartheta') = \mathcal{B}(\vartheta)\mathcal{L}\mathcal{V}_{V_0}(\vartheta') \prod_{\ell\in L_V^R} \mathcal{L}\mathcal{V}_{V(\ell)}(\vartheta'). \tag{6.5}$$

[1] If $z_V = 1$ the localized resonance factor is given by the resonance factor computed for $\mu_1 = \cdots = \mu_m = 0$.

Summing the localized resonance factors corresponding to the trees belonging to $\mathcal{F}_V(\vartheta)$, we can group them into subfamilies of inequivalent trees whose contributions are different as for each node $u \in V$ there is a factor;

$$\frac{1}{m_u!}\binom{m_u}{s_u} = \frac{1}{s_u!}\frac{1}{r_u!}, \tag{6.6}$$

as all terms which are obtained by permutations are summed together (this gives the binomial coefficient in the left hand side of the above equation), times a factor:

$$\nu_u^{m_u+1} = \nu_u^{(s_u+1)+r_u}, \tag{6.7}$$

times a propagator $g_{n_{\ell_u}}(\nu_{\ell_u}^0)$ (the last factor is missing if corresponding to the line exiting V; see definitions (4.3)÷(4.6)).

Then for $\mu_1 = \cdots = \mu_m = 0$ we can write:

$$
\begin{aligned}
\sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathcal{L}\mathcal{V}_V(\vartheta') = & \sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \left[\prod_{u \in V} \frac{\nu_u^{s_u+1}}{s_u!} \right] \left[\prod_{\ell \in V} g_{n_\ell}(\nu_\ell^0) \right] \cdot \\
& \cdot \left(\prod_{u \in V_0} \frac{\nu_u^{r_u}}{r_u!} \right) \left(\prod_{\ell \in L_V^R} \prod_{u \in V_0(\ell)} \frac{\nu_u^{r_u}}{r_u!} \right) = \\
= & \left[\prod_{u \in V} \frac{\nu_u^{s_u+1}}{s_u!} \right] \left[\prod_{\ell \in V} g_{n_\ell}(\nu_\ell^0) \right] \cdot \\
& \cdot \sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \left(\prod_{u \in V_0} \frac{\nu_u^{r_u}}{r_u!} \right) \left(\prod_{\ell \in L_V^R} \prod_{u \in V_0(\ell)} \frac{\nu_u^{r_u}}{r_u!} \right),
\end{aligned}
\tag{6.8}
$$

where we have used the fact that for $\mu_1 = \cdots = \mu_m = 0$ the factors in square brackets have the same value for all $\vartheta' \in \mathcal{F}_V(\vartheta)$ (see (3.11) and take into account what observed at the beginning of this section). The last sum in (6.8) can be rewritten as:

$$
\begin{aligned}
& \sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \left(\prod_{u \in V_0} \frac{\nu_u^{r_u}}{r_u!} \right) \left(\prod_{\ell \in L_V^R} \prod_{u \in V_0(\ell)} \frac{\nu_u^{r_u}}{r_u!} \right) = \\
& = \left(\sum_{\substack{\{r_u \ge 0\} \\ \sum_{u \in V_0} r_u = m_{V_0}}} \prod_{u \in V} \frac{\nu_u^{r_u}}{r_u!} \right) \left(\prod_{\tilde{V} \in \tilde{\mathbf{V}}(V)} \sum_{\substack{\{r_u \ge 0\} \\ \sum_{u \in \tilde{V}_0} r_u = 1}} \prod_{u \in \tilde{V}_0} \frac{\nu_u^{r_u}}{r_u!} \right) = \\
& = \frac{1}{m_{V_0}!} \left(\sum_{u \in V_0} \nu_u \right)^{m_{V_0}} \prod_{\tilde{V} \in \tilde{\mathbf{V}}(V)} \left(\sum_{u \in \tilde{V}_0} \nu_u \right),
\end{aligned}
\tag{6.9}
$$

which is zero by definition of resonance (see (6.2) and (6.3) above).

[2] If $z_V = 2$ the localized resonance factor, with respect to the previous case, contains also the first order terms (again computed in $\mu_1 = \cdots = \mu_m = 0$).

The zero-th order contribution can be discussed as for the case $z_V = 1$, and the same result holds. Also the second order contribution vanishes, after summing over the trees $\vartheta' \in \mathcal{F}_V(\vartheta)$. To prove this we shall consider separately the cases $m_V = 2$ and $m_V = 1$.

In the first case, when the derivative $(\partial/\partial\mu_m)\mathcal{V}_V(\vartheta; 0, \ldots, 0)$ is considered, let us compare all the trees ϑ' in the subfamily of $\mathcal{F}_V(\vartheta)$ in which the line ℓ_m is kept fixed (call $\bar{u}$ the node which such a line enters), while all other lines are shifted (*i.e.* detached and reattached to all nodes inside the resonance). The difference with respect to the previous case, discussed above, is that the line with momentum ν_{ℓ_m} can be choosen in $r_{\bar{u}}$ ways among the $r_{\bar{u}}$ lines entering the node $\bar{u} \in V$ and outside V. This means that we can write:

$$
\frac{\nu_u^{m_u+1}}{m_u!} \binom{m_u}{s_u} = \frac{\nu_u^{(s_u+1)+r_u}}{s_u! r_u!}
\tag{6.10}
$$

for all nodes $u \neq \bar{u}$, and:

$$
\frac{\nu_{\bar{u}}^{m_{\bar{u}}}}{m_{\bar{u}}!} \binom{m_u}{s_u} r_{\bar{u}} = \frac{\nu_{\bar{u}}^{(s_{\bar{u}}+1)+(r_{\bar{u}}-1)}}{s_{\bar{u}}!(r_{\bar{u}}-1)!}
\tag{6.11}
$$

for $\bar{u}$. Then we have an expression analogous to (6.8), with the only difference that the labels $\{r_u\}$ have to be replaced with labels $\{r'_u\}$, defined as:

$$r'_u = r_u - \delta_{u\bar{u}}, \qquad \forall u \text{ either in } V_0 \text{ or in } \bigcup_{\tilde{V}\in\tilde{\mathbf{V}}(V)} \tilde{V}_0, \tag{6.12}$$

such that

$$\sum_{u\in V_0} r'_u + \sum_{\tilde{V}\in\tilde{\mathbf{V}}(V)} \sum_{u\in\tilde{V}_0} r'_u = m_V - 1; \tag{6.13}$$

so the last sum in the second line of (6.8) has to be replaced by:

$$\begin{aligned}
&\sum_{\vartheta'\in\mathcal{F}_V(\vartheta)} \left(\prod_{u\in V_0} \frac{\nu_u^{r_u}}{r_u!}\right)\left(\prod_{\ell\in L_V^R}\prod_{u\in V_0(\ell)} \frac{\nu_u^{r_u}}{r_u!}\right)\nu_{\bar{u}} = \\
&= \left(\sum_{\substack{\{r_u\ge 0\}\\ \sum_{u\in V_0} r_u = m^*_{V_0}}} \prod_{u\in V} \frac{\nu_u^{r_u}}{r_u!}\right)\left(\prod_{\tilde{V}\in\tilde{\mathbf{V}}(V)} \sum_{\substack{\{r_u\ge 0\}\\ \sum_{u\in\tilde{V}_0} r_u = \zeta^*(\ell)}} \prod_{u\in\tilde{V}_0} \frac{\nu_u^{r_u}}{r_u!}\right) = \\
&= \frac{1}{m^*_{V_0}!}\left(\sum_{u\in V_0}\nu_u\right)^{m^*_{V_0}} \prod_{\tilde{V}\in\tilde{\mathbf{V}}(V)} \left(\sum_{u\in\tilde{V}}\nu_u\right)^{\zeta^*(\tilde{V})},
\end{aligned} \tag{6.14}$$

where

$$m^*_{V_0} = \begin{cases} m_{V_0}, & \text{if} \quad \bar{u}\notin V_0, \\ m_{V_0}-1, & \text{if} \quad \bar{u}\in V_0, \end{cases} \qquad \zeta^*(\tilde{V}) = \begin{cases} 1, & \text{if} \quad \bar{u}\notin \tilde{V}_0, \\ 0, & \text{if} \quad \bar{u}\in \tilde{V}_0, \end{cases} \tag{6.15}$$

so that we have again vanishing contributions (as $m_V \ge 2$).

On the contrary, if $m_V = 1$, the above reasoning does not apply, as there is only one entering line. Anyway the function $(\partial/\partial\mu_1)\mathcal{V}_V(\vartheta;0)$ is an odd function, as all the propagators are even in their arguments, so that the derived one[5] becomes odd, and the numerator contains an even number of ν_u's. Then by reversing the signs of the labels ν_u, $u \in V$, the numerator will not change, while the overall sign of the denominator will change, so that the sum over the first order contributions of the localized resonance factors of the two tree values being considered vanishes.[6]

[3] Finally if $z_V = 0$ the localization operator $\mathcal{L}$ gives zero when acting on the resonance factors, so that nothing has to be proved.

[5]If $z_V = 2$, then there is only one derived propagator, arising from the renormalization of the resonance V itself.

[6]Note that the renormalization transformations of type 3 are explicitly used in order to implement the cancellation mechanism *only* in the case of a resonance V with $z_V = 2$ and $m_V = 1$. In general not all the transformations are used for all resonances: in particular, when $z_V = 0$, we consider separately all terms generated by the action of the group $\mathcal{P}_V$, as there is no need of additional renormalizations.

7. Conclusions

Our theorem can be related to the result and the methods of [1]. There we proved that, for $\omega \in \mathbb{C}$, if ω tends to a rational number p/q through a path in the complex plane non-tangential to the real axis, then the radius of convergence satisfies:

$$\left|\log \rho(\omega) + \frac{2}{q} \log\left|\omega - \frac{p}{q}\right|\right| < C_4 \tag{7.1}$$

for some constant C_4.

If instead we consider a sequence of *real*, irrational numbers tending to a rational value p/q, the situation is quite more complex. In fact, the limit and its very existence may depend on the arithmetic properties of the numbers of the sequence we consider, *and on their uniformity in* k; namely:

1. The sequence $\{\omega_k\}$ can tend to p/q but, though all the ω_k are irrational, some of them are not Bryuno numbers so that for those $B(\omega_k) = +\infty$ and $\rho(\omega_k) = 0$.
2. The sequence $\{\omega_k\}$ can tend to p/q through Bryuno numbers, or even Diophantine numbers, but they are not uniformly such in k so that $B(\omega_k)$ diverges *faster* than $\log\left(|\omega_k - p/q|^{1/q}\right)$ (and so $\rho(\omega_k)$ tends to zero *faster* than $|\omega_k - p/q|^{2/q}$). An example can be the sequence of Diophantine (actually even "noble") numbers:
$$\omega_k = \cfrac{1}{k + \cfrac{1}{2^{k^2} + \gamma}}, \tag{7.2}$$
where γ denotes the "golden mean":
$$\gamma = \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = \frac{\sqrt{5} - 1}{2}; \tag{7.3}$$
a simple calculation using the recursion relation (1.7) shows that indeed $B(\omega_k) = O(k)$ while $\omega_k = O(1/k)$, so that, by taking into account also logarithmic corrections in $B(\omega_k)$, $\rho(\omega_k) = O(\omega_k^2 e^{-2/\omega_k})$, that is *much faster* than ω_k^2.
3. Finally, the sequence $\{\omega_k\}$ can tend to p/q through a sequence of Bryuno numbers satisfying uniform estimates in k, so that an estimate like (7.1) holds (note that decays slower than $|\omega_k - p/q|^{2/q}$ are not possible); an example can be given by the sequence:
$$\omega_k = \frac{1}{k + \gamma}, \tag{7.4}$$
where again γ is the golden mean (7.3).

Notice that in the numerical calculations of [14] only real sequences of type 3 were considered, and that sequences of type 2 are practically inaccessible from the numerical point of view.

Finally, one may ask how much these results can be extended to more complicated, and realistic, symplectic maps and continuous time Hamiltonian systems. We believe that while some additional complications may arise, the really hard problem (*i.e.* how to handle resonances) is already present in the standard map and it was solved by carefully using the trees formalism and the multiscale decomposition of the propagators. More general maps and Hamiltonian systems, though, as already pointed out in [1], may have different, more complicated interpolation properties for the radius of convergence of their Lindstedt series: this is an area where still much work has to be done.

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