Invariant measures for Anosov maps with small holes

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Abstract

We study Anosov diffeomorphisms on surfaces with small holes. The points that are mapped into the holes disappear and never return. In our previous paper [6] we proved the existence of a conditionally invariant measure μ_+. Here we show that the iterations of any initially smooth measure, after renormalization, converge to μ_+. We construct the related invariant measure on the repeller and prove that it is ergodic and K-mixing. We prove the escape rate formula, relating the escape rate to the positive Lyapunov exponent and the entropy.

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1 Introduction

A pictorial model of a chaotic dynamical system with holes (also known as open dynamical systems) was proposed by Pianigiani and Yorke [14]. Imagine a Sinai billiard table (with dispersing boundary) in which the dynamics of the ball is strongly chaotic. Let one or more holes be cut in the table, so that the ball can fall through. One can also think of these holes as 'pockets' at the corners of the table. Let the initial position of the ball be chosen at random with some probability distribution. Denote by $P(t)$ the probability that the ball stays on the table for at least time t, and if it does, by $\rho(t)$ its (normalized) distribution on the table at time t. Some natural questions are: at what rate does $P(t)$ converge to zero as $t \to \infty$, what is the limit probability distribution $\lim_{t\to\infty} \rho(t)$, and does it depend on the initial distribution $\rho(0)$? These questions remain open.

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Open billiards and other open Hamiltonian systems have become very popular in physics under the name of chaotic scattering theory in the past ten years. They have been studied numerically and heuristically, see the survey [9] and the references therein. This has prompted mathematicians to study open systems as well. The first mathematical results have dealt with the case when the underlying system is uniformly hyperbolic and admits a finite Markov partition: expanding maps of the interval [14, 8], horseshoes [3], open billiard tables with no eclipse [12], and Anosov diffeomorphisms with Markov holes [4, 5]. In all these papers the holes are elements of a Markov partition.

This is a continuation of the paper [6] where we started the study of Anosov diffeomorphisms with small open holes. The relaxation of the Markov property of the holes is our main objective of this article as well as [6]. The main result of [6] was the existence and uniqueness of the conditionally invariant measure μ_+ with smooth distributions on unstable fibers. Here we prove that the iterations of any initially smooth measure, after renormalization, converge to μ_+. We also construct the related invariant measure, $\bar{\mu}_+$, on the repeller, which turned out to be ergodic and K-mixing. We then obtain an escape rate formula, relating the escape rate to the Lyapunov exponent and the entropy. Thus, the entire mathematical theory of open hyperbolic dynamical systems is here extended from examples with clear-cut Markov ('rectangular') repellers to Anosov diffeomorphisms with quite general small open holes. Our results have many promising physical applications to, e.g., open Lorentz gases, billiard tables with holes and pockets, and other models in the scattering theory.

This paper is closely connected to [6], even though the main ideas here are quite different. We often refer to [6] for notations and technical results, but we provide all necessary definitions here as well. We make an additional technical assumption on the holes, see Sect. 3, but our principal theme is unchanged – we work with small open holes of quite general nature (not even assuming the smoothness of their boundary).

2 Statements of main results

2.1 Let $\hat{T} : \hat{M} \to \hat{M}$ be a topologically transitive Anosov diffeomorphism of class $C^{1+\alpha}$ of a compact Riemannian surface $\hat{M}$. Let $H \subset \hat{M}$ be an open set with a finite number of connected components.

We denote $M = \hat{M} \setminus H$. For any $n \geq 0$ we put

$$M_n = \cap_{i=0}^n \hat{T}^i M \quad \text{and} \quad M_{-n} = \cap_{i=0}^n \hat{T}^{-i} M, \tag{2.1}$$

and also

$$M_+ = \cap_{n\geq 1} M_n, \quad M_- = \cap_{n\geq 1} M_{-n}, \quad \Omega = M_+ \cap M_- \tag{2.2}$$

Observe that all these sets are closed, $\hat{T}^{-1}M_+ \subset M_+$, $\hat{T}M_- \subset M_-$ and $\hat{T}\Omega = \hat{T}^{-1}\Omega = \Omega$. The set Ω is called a repeller.

We refer to the connected components of H as holes. We study the dynamics outside the holes, i.e. the trajectories that fall into H disappear and never return. We denote by T the restriction of $\hat{T}$ on M, which means that for any set $A \subset M$ and $n \geq 1$ we put $T^n A = \hat{T}^n(A \cap M_{-n})$ and $T^{-n} A = \hat{T}^{-n}(A \cap M_n)$.

Let W^u_x and W^s_x be local unstable and stable fibers through $x \in \hat{M}$. We denote by J^u_x and J^s_x the Jacobians of the map $\hat{T}$ restricted to W^u_x and W^s_x, respectively, at the point x. We put

$$\Lambda_{\min} = \min_{x \in \hat{M}} \{J^u_x, 1/J^s_x\} \quad \text{and} \quad \Lambda_{\max} = \max_{x \in \hat{M}} \{J^u_x, 1/J^s_x\}$$

For any two points $x, y \in W^s$ a holonomy map $h_{x,y} : W^u_x \to W^u_y$ is defined by sliding the points of W^u_x along local stable fibers (symmetrically, $h_{x,y} : W^s_x \to W^s_y$ is defined for $x, y \in W^u$).

A rectangle $R \subset \hat{M}$ is a small subset such that for any $x, y \in R$ we have $W^u_x \cap W^s_y \in R$. We consider only closed connected rectangles. Those are bounded by two stable and two unstable fibers (called stable and unstable sides of R). Segments of local unstable and stable fibers inside R that terminate, respectively, on the stable and unstable sides of R are called R-fibers. Any subrectangle $R' \subset R$ whose stable (unstable) sides are on the stable (unstable) sides of R is called a u-subrectangle (s-subrectangle).

Denote by $\mu_{\rm SBR}$ the unique Sinai-Bowen-Ruelle (SBR) measure of the diffeomorphism $\hat{T}$, cf. [17, 1, 15]. Its conditional distributions on local unstable fibers are smooth (with Hölder continuous densities). Motivated by this, we call the conditional distributions of $\mu_{\rm SBR}$ on unstable fibers u-SBR measures. Equivalently, for any local unstable fiber W^u its u-SBR measure is a probability measure, ν_{W^u}, on W^u whose density $\rho(x)$ with respect to the Riemannian length satisfies the equation

$$\frac{\rho(x)}{\rho(y)} = \lim_{n \to \infty} \frac{J^u_{T^{-1}y} \cdots J^u_{T^{-n}y}}{J^u_{T^{-1}x} \cdots J^u_{T^{-n}x}} \tag{2.3}$$

The u-SBR measures are $\hat{T}$-invariant, i.e. $\hat{T}_* \nu_{W^u} = \nu_{\hat{T}W^u}$.

For any $r > 0$ we denote by $D_1(r) \geq 1$ the supremum of all ratios $\rho(x)/\rho(y)$ in (2.3) for all $x, y \in W^u$ on all fibers $W^u \subset \hat{M}$ of length r (length always means the Riemannian length). Next, $D_2(r)$ denotes the supremum of all the Jacobians of holonomy maps $h_{x,y}$ for points $x, y \in W^{u,s}$ at distance $\leq r$ (measured along $W^{u,s}$). We put $D(r) = \max\{D_1(r), D_2(r)\}$. One can think of $D(r)$ as a general upper bound on distortions within the distance r in $\hat{M}$. Obviously, $D(r) \to 1$ as $r \to 0$.

2.2 We recall the assumptions on H made in [6]. First, there is a constant $B_0 > 0$ such that for any local unstable fiber W^u and any local stable fiber W^s that intersect only one hole H' (connected component of H) the sets $W^u \setminus H'$ and $W^s \setminus H'$ consist of not more than B_0 connected components.

Let N_H be the number of holes. We denote by $d_0(H)$ the minimum distance between the holes, if there is more than one hole. We also assume that $d_0(H)$ is smaller than a quarter of the length of the shortest closed geodesic on $\hat{M}$. In the case $N_H = 1$ this will be the definition of $d_0(H)$.

Let d_0 be any lower bound on $d_0(H)$, i.e. $d_0 \leq d_0(H)$.

We fix $D = D(2d_0)$, which will be the only bound on distortions that we use. We assumed in [6] that $\Lambda_{\min} > 64D^2$, which was not a restrictive assumption, because it can be always fulfilled by taking higher iterates of $\hat{T}$.

We denote by h the maximal size of holes defined as follows. For any hole $H' \subset H$ its size is

$$\sup_{x\in H'} \{\operatorname{diam} W_x^u \cap H',\ \operatorname{diam} W_x^s \cap H'\}$$

where the diameter is measured along the fibers $W_x^{u,s}$. We will need h to be small enough compared to d_0, i.e. $h < h_0 = h_0(\hat{T}, d_0, B_0)$. In Sections 2 and 3 of [6], we have assumed four specific upper bounds on h_0, but here we will just assume that h_0 is small enough whenever necessary.

We note that the assumption $\dim \hat{M} = 2$ is made mainly to simplify the proofs and can be possibly relaxed along the lines of [7]. On the contrary, the smallness of the holes is essential – for large holes the conditionally invariant measure may be not unique, and the dynamics on the repeller may be not ergodic.

2.3 For any finite Borel measure μ on M we define its norm by $||\mu|| = \mu(M)$. We denote by T_* the adjoint operator on the class of Borel measures on M defined by $(T_*\mu)(A) = \mu(T^{-1}(A \cap M_1))$ for any $A \subset M$. Due to the holes, the operator T_* does not preserve norm. We also denote by T_+ the (nonlinear) operator on the space of probability measures on M defined by $T_+\mu = T_*\mu/||T_*\mu||$, whenever $||T_*\mu|| \neq 0$.

Definition. A probability measure μ on M is said to be *conditionally invariant* under T if $T_+\mu = \mu$, i.e. if there is a $\lambda > 0$ such that $T_*\mu = \lambda\mu$. The factor λ is the *eigenvalue* of μ.

Obviously, any conditionally invariant measure μ is supported on M_+, and we have $\lambda = ||T_*\mu|| = \mu(M_{-1})$.

We are interested in measures whose conditional distributions on unstable fibers coincide with u-SBR measures. In addition, we assume a certain natural balance between long and short unstable fibers in terms of measures, see below.

First, for certain technical reasons it is convenient to limit the length of unstable fibers in M_m and M_+ by d_0. This can be done as in [6], Sect. 2, by subdividing longer unstable fibers into subfibers of length between $d_0/2$ and d_0. This can be accomplished by making a finite number of cuts in M along some local stable fibers, whose choice is not very important to us. Now, with these additional cuts, any maximal unstable fiber $W^u \subset M_m$, $m \geq 0$, and $W^u \subset M_+$, has length $\leq d_0$. We denote by $|W^u|$ the length of W^u.

Now, for $m \geq 0$, we denote by $\mathcal{W}_m^u$ the set of maximal unstable fibers $W^u \subset M_m$. For every unstable fiber W^u and $\varepsilon < |W^u|$ we denote by $W^u(\varepsilon) \subset W^u$ the union of two subsegments of W^u of length ε terminating at the endpoints of W^u, and put $W^u(\varepsilon) = W^u$ if $\varepsilon \geq |W^u|$. In other words, $W^u(\varepsilon)$ is the ε-neighborhood of the endpoints of W^u with respect to the length on that curve. Let $U_{m,\varepsilon} = \cup_{W^u \in \mathcal{W}_m^u} W^u(\varepsilon)$. Let $\mathcal{W}_{m,\varepsilon}^u = \{W^u \in \mathcal{W}_m^u : |W^u| < \varepsilon\}$. By replacing m with $+$ in the above formulas, we define $\mathcal{W}_+^u$, $U_{+,\varepsilon}$ and $\mathcal{W}_{+,\varepsilon}^u$.

Let $m \geq 0$. Denote by $\mathcal{M}_m$ the class of probability measures supported on M_m, such that for any $\mu \in \mathcal{M}_m$
(M1) its conditional measures on unstable fibers $W^u \in \mathcal{W}_m^u$ coincide with u-SBR measures on those fibers;
(M2) for any $\varepsilon > 0$ we have $\mu(U_{m,\varepsilon}) \leq C_1\varepsilon$.
Here $C_1 = 48D/d_0$ is the constant introduced in [6]. By replacing m with $+$, we define the class of measures $\mathcal{M}_+$. We call the above measures SBR-like measures.

Claim. For all $m \geq 0$ we have $T_+\mathcal{M}_m \subset \mathcal{M}_{m+1}$. Also, $T_+\mathcal{M}_+ \subset \mathcal{M}_+$.

Proof. The preservation of the property (M1) is obvious. That of (M2) follows from Theorem 2.1 in [6], along with the second remark after it for measures supported on a finite union of unstable fibers. Then taking weak limits of such measures automatically extends this claim to all measures in $\mathcal{M}_m$ and $\mathcal{M}_+$. □

The main result of [6] is the following.

Theorem 2.1 ([6]) *There is a unique SBR-like conditionally invariant measure $\mu_+ \in \mathcal{M}_+$, i.e. the operator $T_+ : \mathcal{M}_+ \to \mathcal{M}_+$ has a unique fixed point, μ_+.*

The eigenvalue λ_+ of the measure μ_+ satisfies the bound $\lambda_+ \geq 1 - C_2h$, where $C_2 = C_1D(\Lambda_{\max}/\Lambda_{\min} + 1)$. Technically, the uniqueness of μ_+ was proved in [6] under one extra assumption, but that one was proved there (in the statements 5.10 and 5.11) based on the remaining assumptions, so we drop it here.

Convention. We will use here the constants $C_1, \ldots, C_4$ and D_1, D_2 introduced in [6]. Generally, we will denote by C_i and D_i constants determined by the given Anosov diffeomorphism $\hat{T}$ and the parameters d_0, D, B_0, N_H, which we call *global parameters* (as opposed to the size h of the holes or any characteristics of the shape of the holes). The same goes for other constants: a_i, b, α_i, β_i, γ_i, l_0, r_0. All these constants will be independent of the size h of the holes or the particular shape of the holes. We call such constants *global constants.* (The only exception is g in Section 4, which is proportional to h.)

2.4 The SBR-like conditionally invariant measure μ_+ plays the same role in the theory of open systems (= systems with holes) as SBR measures play in conservative systems: they are the only physically observable measures. That means that taking a large $N \gg 1$ and a point $x \in M_{-N}$ at random (according to a smooth probability distribution), the orbit $T^n x$, $1 \leq n \leq N$, will be asymptotically distributed according to μ_+, as $N \to \infty$. Equivalently, if μ_0 is a smooth measure on M, then the sequence $\{T_+^n\mu_0\}$ weakly converges, as $n \to \infty$, to the conditionally invariant measure μ_+.

First of all, if the conditional measures of μ_0 on unstable fibers $W^u \in \mathcal{W}_0$ have smooth densities, then the densities of the measure $T_+^n\mu_0$ on fibers $W^u \in \mathcal{W}_n$ approach those of u-SBR measures exponentially fast in n, see [4]. Therefore, to find limit points of the sequence $\{T_+^n\mu_0\}$ we can restrict ourselves to the measures $\mu_0 \in \mathcal{M}_0$.

The weak convergence $T_+^n\mu_0 \to \mu_+$ was previously proved for various open systems with Markov 'rectangular' holes, cf. [4, 5]. Here we prove it in our context.

Theorem 2.2 *For any measure $\mu_0 \in \mathcal{M}_0$ the sequence of measures $T_+^n \mu_0$ weakly converges, as $n \to \infty$, to the conditionally invariant measure μ_+. Moreover, the sequence of measures $\lambda_+^{-n} \cdot T_*^n \mu_0$ weakly converges to $\rho(\mu_0) \cdot \mu_+$, where the function $\rho(\mu_0)$ is uniformly bounded on $\mathcal{M}_0$.*

Observe that $\mathcal{M}_0$ contains any measure μ_0 supported on a single unstable fiber $W^u \subset M$ of length $\geq d_0/2$, which coincides on it with the u-SBR measure ν_{W^u}. The above theorem also holds for measures supported on arbitrary short single fibers, provided they are 'eventually long', see Corollary 7.4.

We also estimate the speed of convergence in this theorem. Let f be any continuous function on M. Denote by $\delta_f(\varepsilon) = \sup\{|f(x) - f(y)| : \operatorname{dist}(x, y) < \varepsilon\}$ its modulus of continuity. Since M is compact, $f(x)$ is uniformly continuous on M, so $\delta_f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorem 2.3 *Using the notation of the previous theorem, let $\mu_n = T_+^n \mu_0$. For any continuous function $f \in C(M)$*

$$\left| \int_M f(x)\, d\mu_n - \int_M f(x)\, d\mu_+ \right| \leq \text{const} \cdot \left[\delta_f \left(e^{-a_1 n^{1/2}} \right) + ||f||_\infty e^{-a_2 n^{1/2}} \right] \tag{2.4}$$

for some global constants $a_1, a_2 > 0$ and const > 0.

In particular, if $f(x)$ is a Hölder continuous function, then $\delta_f(\varepsilon) = O(\varepsilon^a)$ with some $a > 0$, and we get the so-called stretched exponential convergence in the last theorem.

2.5 Next, since M_+ is invariant under T^{-1}, it makes sense to look for an invariant measure for the transformation $T : \Omega \to \Omega$ by taking a weak limit of $T_*^{-n} \mu_+$ as $n \to \infty$. Due to the conditional invariance of μ_+, $T_*^{-n} \mu_+$ simply coincides with the measure μ_+ conditioned on M_{-n} defined by

$$\mu_+(A/M_{-n}) = \mu_+(A \cap M_{-n})/\mu_+(M_{-n}) = \lambda_+^{-n} \cdot \mu_+(A \cap M_{-n})$$

Theorem 2.4 *The sequence $T_*^{-n} \mu_+$ weakly converges, as $n \to \infty$, to a T-invariant probability measure, called $\bar{\mu}_+$, supported on the repeller Ω. The measure $\bar{\mu}_+$ is ergodic and K-mixing.*

We also estimate the speed of convergence in this theorem.

Theorem 2.5 *In the notations of the previous theorem, let $\mu_{+,n} = T_*^{-n} \mu_+$. For any continuous function $f \in C(M)$*

$$\left| \int_M f(x)\, d\mu_{+,n} - \int_M f(x)\, d\bar{\mu}_+ \right| \leq \text{const} \cdot \left[\delta_f \left(e^{-a_3 n^{1/2}} \right) + ||f||_\infty e^{-a_4 n^{1/2}} \right] \tag{2.5}$$

for some global constants $a_3, a_4 > 0$ and const > 0.

We conjecture that our measure $\bar{\mu}_+$ is Bernoulli and enjoys strong statistical properties (fast decay of correlations and central limit theorem). It is quite clear, though, that $\bar{\mu}_+$ is not (necessarily) a Gibbs measure, because of the lack of a local product structure in its support.

The quantity $\gamma_+ = -\ln \lambda_+$ is known as the *escape rate*. It characterizes the rate of escape of the mass of any smooth measure μ_0 through holes under the iterations of T. Denote by χ_+ the positive Lyapunov exponent of the ergodic measure $\bar{\mu}_+$ and by $h(\bar{\mu}_+)$ its Kolmogorov-Sinai entropy. The following *escape rate formula* relates these three quantities: $\gamma_+ = \chi_+ - h(\bar{\mu}_+)$. It was previously proved for open systems with Markov rectangular holes, cf. [4, 5].

Theorem 2.6 $\gamma_+ = \chi_+ - h(\bar{\mu}_+)$.

Next we prove that the measures μ_+, $\bar{\mu}_+$ and the values of γ_+, $h(\bar{\mu}_+)$ depend continuously on the open hole H. For any open sets $H', H'' \subset \hat{M}$ consider the distance

$$d(H', H'') := \min\{\varepsilon > 0 : \ H' \setminus H'' \subset H''_\varepsilon \ \text{ and } \ H'' \setminus H' \subset H'_\varepsilon\} \tag{2.6}$$

where H_ε is the ε-neighborhood of ∂H in $\hat{M} \setminus H$.

Let $H_n \subset \hat{M}$ be a sequence of open sets. Assume that each H_n satisfies our assumptions on H with the same values of d_0, B_0, N_H, so for each H_n the measures $\mu_+[H_n]$ and $\bar{\mu}_+[H_n]$ and the quantities $\lambda_+[H_n]$ and $\gamma_+[H_n]$ are well defined.

Theorem 2.7 *If $d(H_n, H) \to 0$ as $n \to \infty$, then we have the weak convergence $\mu_+[H_n] \to \mu_+$ and $\bar{\mu}_+[H_n] \to \bar{\mu}_+$ and the convergence $\gamma_+[H_n] \to \gamma_+$ and $h(\bar{\mu}_+[H_n]) \to h(\bar{\mu}_+)$.*

Corollary 2.8 *Let $H_1 \supset H_2 \supset \cdots$ be a decreasing sequence of holes such that their intersection $\cap_n H_n$ consists of isolated curves or isolated points, or is just empty. Then $\gamma_+[H_n] \to 0$, as $n \to \infty$, and both measures $\mu_+[H_n]$ and $\bar{\mu}_+[H_n]$ weakly converge to the $\hat{T}$-invariant SBR measure $\mu_{\rm SBR}$.*

Back to a single open set H. By reversing the time, we can define the conditionally invariant measure μ_- on M_- for the map T^{-1}, whose conditional distributions on stable fibers $W^s \subset M_-$ are smooth. It has an eigenvalue, λ_-. We can then define the corresponding invariant measure $\bar{\mu}_-$ on the repeller Ω. Those also have all the properties described in the above theorems. The measure $\bar{\mu}_-$ and the value of λ_- are, generally, different from $\bar{\mu}_+$ and λ_+, respectively. However, there are important exceptions:

Theorem 2.9 *If for every periodic point $x \in \Omega$, $T^k x = x$, we have $|\det DT^k(x)| = 1$, then $\bar{\mu}_+ = \bar{\mu}_-$ and $\lambda_+ = \lambda_-$. In particular, this happens if the given Anosov diffeomorphism $\hat{T}$ preserves an absolutely continuous measure.*

The last theorem has potential applications to open Hamiltonian systems, including billiards, which preserve smooth (Liouville) measures.

3 Assumption on holes and preparatory lemmas

Definition. We say that an unstable fiber $W_1^u \subset \hat{M}$ is d_0-close to another fiber, $W_2^u \subset \hat{M}$ if for any $x \in W_1^u$ there is a local stable fiber W_x^s of length $\le d_0$ that meets W_2^u in exactly one point. In this case denote by $\tilde{h} : W_1^u \to W_2^u$ the holonomy map defined by $\tilde{h}(x) = W_x^s \cap W_2^u$.

We will say that a local unstable fiber W^u is d_0-close to a hole H' (a connected component of H) if for any $x \in H'$ there is a local stable fiber W_x^s of length $\le d_0$ that meets W^u in exactly one point. Then we define the holonomy projection of H' onto W^u by $\text{proj}_{W^u}(H') = \{W_x^s \cap W^u : x \in H'\}$. We put $|H'|_u = \sup_{W^u} |\text{proj}_{W^u}(H')|$, where the supremum is taken over all local unstable fibers d_0-close to H'. Now we put

$$|H|_u = \sum_{H'} |H'|_u \tag{3.1}$$

Remember that our holes $H' \subset H$ are assumed to be short (of size h) in stable and unstable directions. The values $|H'|_u$ and $|H|_u$ are not necessarily small, however. They can be large, for example, if the holes stretch 'diagonally', i.e., transversally to stable and unstable directions on $\hat{M}$. We additionally assume here that

Assumption H0. The diameter of every hole $H' \subset H$ in the Riemannian metric on $\hat{M}$ does not exceed $d_0/2$.

Under this assumption, for any hole H' there are local unstable fibers d_0-close to it. It also follows that $|H'|_u \le 3d_0$ and $|H|_u \le 3N_H d_0$.

Next, let W_1^u, W_2^u be two local unstable fibers d_0-close to a hole H'. We define

$$d_{H'}(W_1^u, W_2^u) = \max\{|(W_1^u \cap H') \setminus \tilde{h}^{-1}(W_2^u \cap H')|_1, |(W_2^u \cap H') \setminus \tilde{h}(W_1^u \cap H')|_2\}$$

where $|\cdot|_1$, $|\cdot|_2$ are the length measures on W_1^u and W_2^u, respectively. Clearly, $d_{H'}(\cdot,\cdot)$ is a pseudo-metric on the set of local unstable fibers d_0-close to H'. Now let

$$\text{Var}_u(H') = \sup_{k, W_1^u, \cdots, W_k^u} \left[d_{H'}(W_1^u, W_2^u) + d_{H'}(W_2^u, W_3^u) + \cdots + d_{H'}(W_{k-1}^u, W_k^u) \right]$$

where the supremum is taken over all finite collections of local unstable fibers d_0-close to H' naturally ordered in the stable direction.

Now, recall that any local stable and unstable fiber intersects any hole H' in at most $B_0 - 1$ open intervals on that fiber. This implies the following.

Lemma 3.1 *For any hole H' we have $Var_u(H') \le B_0 D |H'|_u$, so that $\sum_{H'} \text{Var}_u(H') \le B_0 D |H|_u$.*

Next, we prove two technical lemmas to be used later. Let $W^u \subset \hat{M}$ be an unstable fiber, and for any $n \ge 0$ let $W_{n,i}^u$ be the connected components of $T^n(W^u \cap M_{-n})$.

Remember that the lengths of W^u and all $W^u_{n,i}$ are bounded by d_0. Denote by ν the u-SBR measure on W^u and $\nu_n = T^n_* \nu$ for $n \geq 0$.

Let $B \geq 1$ be an integer. For every $n \geq 0$ let $\tilde{W}^u_{n,i,r} \subset W^u_{n,i}$ for $r = 1, \ldots, B$ be some disjoint subsegments of the fiber $W^u_{n,i}$. We assume that their total measure is less than 1/4:

$$\tilde{s}_n := \sum_i \sum_r \nu_n(\tilde{W}^u_{n,i,r}) < 1/4 \tag{3.2}$$

Lemma 3.2 *Let $0 \leq g < 1/3$. Let W^u be a fiber of length between $d_0/2$ and d_0. Then for every $n \geq 0$, every i, and any collection of subfibers $\tilde{W}^u_{n,i,r} \subset W^u_{n,i}$ we have*

$$\sum_i \sum_r \left[\nu_n(\tilde{W}^u_{n,i,r})\right]^{1-g} \leq C_g \Lambda^{ng}_{\max} \tilde{s}_n^{1-g} \left(\log_2 \tilde{s}_n^{-1}\right)^g \tag{3.3}$$

where $C_g = 200 D^{1+g} B^g$.

Proof. We start with the following obvious consequence of the Hölder inequality:

Sublemma 3.3 *Let $x_1, \ldots, x_m$ be positive real numbers. Then $x_1^{1-g} + \cdots + x_m^{1-g} \leq (x_1 + \cdots + x_m)^{1-g} m^g$.*

Let $\delta > 0$. Observe that

$$\#\{i : \delta < |W^u_{n,i}| \leq 2\delta\} \leq C_1 D d_0 \Lambda^n_{\max} \nu(W^u \cap M_{-n}) \leq 48 D^2 \Lambda^n_{\max} \tag{3.4}$$

Indeed, for any such i we have $\left|T^{-n} W^u_{n,i}\right| > \delta \Lambda^{-n}_{\max}$. On the other hand, $W^u_{n,i}$ coincides with the closure of $W^u_{n,i}(\delta)$ since $|W^u_{n,i}| \leq 2\delta$, and so the total length of the subfibers $T^{-n} W^u_{n,i}$, $i \geq 1$, of the fiber W^u does not exceed $D|W^u| C_1 \delta \nu(W^u \cap M_{-n})$ due to Theorem 2.1 in [6]. Finally, recall that $|W^u| \leq d_0$, and the first bound in (3.4) follows. In the second bound we replaced C_1 by its value, $48D/d_0$, and dropped $\nu(W^u \cap M_{-n})$, since that one was < 1.

We now prove Lemma 3.2. First of all, we bound the left hand side of (3.3) as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} \sum_{i: \frac{d_0}{2^k} < |W^u_{n,i}| \leq \frac{d_0}{2^{k-1}}} \sum_r \left[\nu_n(\tilde{W}^u_{n,i,r})\right]^{1-g} &\leq \sum_{k=1}^{\infty} \left(\sum_{i: \frac{d_0}{2^k} < |W^u_{n,i}| \leq \frac{d_0}{2^{k-1}}} \sum_r \nu_n(\tilde{W}^u_{n,i,r}) \right)^{1-g} \\
&\times \left(B \times \# \left\{ i : \frac{d_0}{2^k} < |W^u_{n,i}| \leq \frac{d_0}{2^{k-1}} \right\} \right)^g \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{i: \frac{d_0}{2^k} < |W^u_{n,i}| \leq \frac{d_0}{2^{k-1}}} \sum_r \nu_n(\tilde{W}^u_{n,i,r}) \right)^{1-g} \\
&\times \left[48\, B D^2 \Lambda^n_{\max} \right]^g
\end{aligned}$$

where we used Sublemma 3.3 and (3.4).

Now, let $k_0 \geq 1$. Then we have

$$\begin{aligned}\sum_{k=1}^{k_0}\left(\sum_{i:\frac{d_0}{2^k}<|W^u_{n,i}|\leq\frac{d_0}{2^{k-1}}}\sum_r \nu_n(\tilde{W}^u_{n,i,r})\right)^{1-g} &\leq \left(\sum_{i:\frac{d_0}{2^{k_0}}<|W^u_{n,i}|}\sum_r \nu_n(\tilde{W}^u_{n,i,r})\right)^{1-g}\times k_0^g \\ &\leq \tilde{s}_n^{1-g}k_0^g \qquad (3.5)\end{aligned}$$

where we again used Sublemma 3.3.

Theorem 2.1 in [6] implies that for any $k \geq k_0+1$

$$\begin{aligned}\sum_{i:\frac{d_0}{2^k}<|W^u_{n,i}|\leq\frac{d_0}{2^{k-1}}}\sum_r \nu_n(\tilde{W}^u_{n,i,r}) &\leq \sum_{i:\frac{d_0}{2^k}<|W^u_{n,i}|\leq\frac{d_0}{2^{k-1}}} \nu_n(W^u_{n,i}) \\ &\leq \sum_i \nu_n(W^u_{n,i}(d_0/2^k)) \\ &\leq 48\, D/2^k\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=k_0+1}^{\infty}\left(\sum_{i:\frac{d_0}{2^k}<|W^u_{n,i}|\leq\frac{d_0}{2^{k-1}}}\sum_r \nu_n(\tilde{W}^u_{n,i,r})\right)^{1-g} &\leq (48\,D)^{1-g}\cdot\left(\sum_{k=k_0+1}^{\infty}2^{-k(1-g)}\right) \\ &\leq (48\,D)^{1-g}\cdot 2^{-k_0(1-g)}\cdot(2^{1-g}-1)^{-1} \\ &\leq (48\,D)^{1-g}\cdot 2^{-k_0(1-g)}\cdot 2 \qquad (3.6)\end{aligned}$$

Here in the end we used the fact that $(2^{1-g}-1)^{-1} < 2$ since $g < 1/3$.

We now fix k_0 so that $2^{-k_0-1} \leq \tilde{s}_n \leq 2^{-k_0}$. Then $k_0 \leq \log_2 \tilde{s}_n^{-1}$, which we substitute in (3.5). Also,the right hand side of (3.6) does not exceed

$$(48\,D)^{1-g}\,2^{1-g}\,\tilde{s}_n^{1-g}\cdot 2 \leq (192\,D)^{1-g}\,\tilde{s}_n^{1-g}\cdot\left(\log_2\tilde{s}_n^{-1}\right)^g$$

where we bounded the factor of 2^g by $(\log_2 \tilde{s}_n^{-1})^g$, because $\tilde{s}_n < 1/4$. Now, combining all the above estimates gives Lemma 3.2. □

Lemma 3.4 *Let $R \subset \hat{M}$ be an arbitrary rectangle with at least one stable R-fiber of length between $d_0/2$ and d_0. Let $d^u_{\min}(R)$ and $d^u_{\max}(R)$ be the minimum and maximum length of unstable R-fibers, and suppose $d^u_{\max}(R) < d_0$. Then for any $\mu \in \mathcal{M}_0$ we have*

$$C_5^{-1}d^u_{\max}(R) \leq \left(T_*^{k_1}\mu\right)(R) \leq C_5 d^u_{\min}(R)$$

with some global constant $C_5 > 0$. Here k_1 is the constant appearing in Theorem 3.6 of [6].

Proof. The upper bound follows from the requirement (M2) on the class of measures $\mathcal{M}_0$, with any $C_5 \geq C_1 D^2$. The lower bound follows from Lemma 5.9 in [6], see also 5.10 and 5.11 there. □

Remark. Examining the proof of Theorem 3.6 in [6] shows that the constant k_1 depends only on $\hat{T}$, d_0, and N_H, so k_1 is a global constant. Also, in this paper, for technical reasons, we restrict the lengths of unstable fibers by d_0 rather than by $2d_0$, as we did in [6]. So, we need to replace d_0 in Theorem 3.6 of [6] by $d_0/2$, which is clearly possible.

4 Measures on long unstable fibers

Denote by $\mathcal{W}^u(d_0)$ the collection of all unstable fibers in $\hat{M}$ of length between $d_0\Lambda_{\max}^{-1}$ and d_0. This restriction is convenient since for any unstable fiber $W^u \subset \hat{M}$ there is an $n \in \mathbb{Z}$ such that $\hat{T}^n W^u \in \mathcal{W}^u(d_0)$.

Theorem 4.1 *Let $W_1^u, W_2^u \in \mathcal{W}^u(d_0)$, and let ν_1, ν_2 be their u-SBR measures, respectively. Then for any $n \geq 0$*

$$0 < C_6^{-1} \leq \frac{\nu_1(W_1^u \cap M_{-n})}{\nu_2(W_2^u \cap M_{-n})} \leq C_6 < \infty \tag{4.1}$$

where $C_6 > 1$ is a global constant.

Notations. Put $\lambda_h = 1 - C_2 h$, as in Theorem 2.2 of [6]. Since h is small, we can write

$$\lambda_h = e^{-C_2' h} \tag{4.2}$$

with some $C_2' > C_2$ which is certainly close to C_2 for small h. Put also

$$g = C_2' h / \ln \Lambda_{\min} \tag{4.3}$$

Observe that C_2' and g are *not* global constants.

Remark. According to the above theorem, the sequence $\delta_n(W^u) := \nu_{W^u}(W^u \cap M_{-n})$ has the same asymptotics for all $W^u \in \mathcal{W}^u(d_0)$. Below we will show that $\delta_n(W^u) \sim \lambda_+^n$, where λ_+ is the eigenvalue of the conditionally invariant measure μ_+. For now, we fix a $W_0^u \in \mathcal{W}^u(d_0)$ and set $\delta_n = \nu_{W_0^u}(W_0^u \cap M_{-n})$. Theorem 2.2 in [6] implies that for all $n \geq l \geq 0$

$$\delta_n \geq \lambda_h^n \qquad \text{and} \qquad \delta_{n-l} \leq \lambda_h^{-l} \delta_n \tag{4.4}$$

The rest of this section is devoted to the proof of the above theorem. Readers interested in getting quickly to the proofs of the main results of this article can skip it. Only the last remark in this section will be actually used, just once, in the subsequent sections.

Proof. Due to an obvious symmetry, it is enough to prove the upper bound in (4.1).

There is an $m_0 \geq 0$ and depending only on d_0 such that the fiber $\tilde{W}_2^u = \hat{T}^{m_0} W_2^u$ is long enough so that any fiber $W^u \in \mathcal{W}^u(d_0)$ is d_0-close to $\tilde{W}_2^u$, as defined in the previous section. Denote by $\tilde{h}_{W^u} : W^u \to \tilde{W}_2^u$ be the holonomy map, i.e. the sliding along local stable fibers the distance $\leq d_0$. Since d_0 is assumed to be less than a quarter of the shortest closed geodesic in $\hat{M}$ [6], the rectangle with the unstable sides W^u and $\tilde{h}_{W^u}(W^u) \subset W_2^u$ is well defined (without overlaps).

Applying Theorem 2.2 in [6] shows that it is enough to prove the upper bound in (4.1) for $\tilde{W}_2^u$ instead of W_2^u, so we will simply assume that original fiber W_2^u is long enough, so that any fiber $W^u \subset \mathcal{W}^u(d_0)$ is d_0-close to W_2^u.

Note that $\mathrm{dist}(x, \tilde{h}_{W^u} x) \leq d_0$ for any $x \in W^u$. Therefore, the jacobian of the map $\tilde{h}_{W^u}$ with respect to the length on W^u and W_2^u is bounded by D. For $k \geq 1$ denote by $\tilde{h}_{W^u,k} = \hat{T}^k \circ \tilde{h} \circ \hat{T}^{-k}$ the induced holonomy map $\hat{T}^k W^u \to \hat{T}^k W_2^u$. Since T^k makes stable fibers shorter, the jacobian of $\tilde{h}_{W^u,k}$ is also bounded by D.

Now, let $W_1^u \in \mathcal{W}^u(d_0)$ and $n \geq 0$. We will define the generation $\mathrm{gen}(x, n, W_1^u)$ for every point $x \in W_1^u \cap M_{-n}$ as follows. If $\tilde{h}_{W_1^u}(x) \in W_2^u \cap M_{-n}$, then $\mathrm{gen}(x, n, W_1^u) = 0$. If not, we set $k = k(x) = \min\{k' \geq 1 : \hat{T}^{k'}(\tilde{h}_{W_1^u}(x)) \in H\}$ (obviously, $k(x) \leq n$). Let H' be the particular hole (connected component of H) that contains the point $\hat{T}^k(\tilde{h}_{W_1^u}(x))$, and let $W_{2,k,x}^u$ be the component of $(\hat{T}^k W_2^u) \cap H'$ containing that point. We denote the connected component of the set $\tilde{h}^{-1}_{W_1^u,k}(W_{2,k,x}^u) \setminus H'$ containing the point $T^k x$ by $\tilde{W}_{1,k,x}^u$. This entire segment will be our trouble, since its counterpart, $\tilde{h}_{W_1^u,k}(\tilde{W}_{1,k,x}^u)$ falls through the hole H'.

Now, let $l = l(x) = \min\{l' > k = k(x) : \hat{T}^{l'-k} \tilde{W}_{1,k,x}^u \in \mathcal{W}^u(d_0)\}$, and let $\hat{W}_{1,l,x}^u = \hat{T}^{l-k} \tilde{W}_{1,k,x}^u$. Observe that

$$l - k \leq \frac{\ln d_0 - \ln |\tilde{W}_{1,k,x}^u|}{\ln \Lambda_{\min}} \tag{4.5}$$

(since $|\hat{W}_{1,l,x}^u| \leq d_0$) and

$$\hat{T}^{l-k}(\tilde{W}_{1,k,x}^u \cap M_{-n+k}) \subset \hat{W}_{1,l,x}^u \cap M_{-n+l} \tag{4.6}$$

Now, if $l(x) \geq n$, we set $\mathrm{gen}(x, n, W_1^u) = 1$. Otherwise the equation

$$\mathrm{gen}(x, n, W_1^u) = 1 + \mathrm{gen}(T^l x, n - l, \hat{W}_{1,l,x}^u) \tag{4.7}$$

defines $\mathrm{gen}(x, n, W_1^u)$ inductively on n (for all fibers in $\mathcal{W}^u(d_0)$). In particular, $\mathrm{gen}(x, n, W_1^u) = 1$ if $\tilde{h}_{\hat{W}_{1,l,x}^u}(\hat{T}^l x) \in W_2^u \cap M_{-n+l}$.

We now estimate $\nu_1(W_1^u \cap M_{-n})$. Let ν_{2,W^u} be the measure on W_2^u obtained by taking ν_{W^u} from W^u under the holonomy map $\tilde{h}_{W^u}$ to W_2^u. Clearly, $d\nu_{2,W^u}/d\nu_2 \leq D_3$ for some constant $D_3 = D_3(d_0) > 0$. Therefore,

$$\nu_1\{x : \mathrm{gen}(x, n, W_u^1) = 0\} \leq \nu_{2,W_1^u}(W_2^u \cap M_{-n}) \leq D_3 \nu_2(W_2^u \cap M_{-n}) \tag{4.8}$$

We now consider the points $x \in W_1^u \cap M_{-n}$ with $\mathrm{gen}(x, n, W_u^1) = 1$. For every such point we have defined the segment $\tilde{W}^u_{1,k,x}$ on the curve $\hat{T}^k W_1^u$ with $k = k(x) \le n$. Obviously, for each $k = 1, \ldots, n$ there is only a finite number of distinct segments $\tilde{W}^u_{1,k,x}$ for points x with $\mathrm{gen}(x, n, W_1^u) = 1$. We denote these segments by $\tilde{W}^u_{1,k,j}$, $j \ge 1$. Denote by $\nu_{1,k}$ the u-SBR measure on $\hat{T}^k W_1^u$ (the image of ν_1 under $\hat{T}^k$). Recall also that for every segment $\tilde{W}^u_{1,k,j}$ we previously defined an $l = l(k, j)$ by $l = \min\{l' > k : \hat{T}^{l'-k}\tilde{W}^u_{1,k,j} \in \mathcal{W}^u(d_0)\}$. Put also $\hat{W}^u_{1,l,j} = \hat{T}^{l-k}\tilde{W}^u_{1,k,j}$, and we keep in mind that $l = l(k, j)$.

Claim. We have

$$\nu_1\{x : \mathrm{gen}(x, n, W_1^u) = 1\} \le \sum_{k=1}^{n} \sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j}) \cdot D_3\, \lambda_h^{-l}\, \nu_2(W_2^u \cap M_{-n}) \tag{4.9}$$

Proof. First note that if $l = l(k, j) \ge n$, then $\lambda_h^{-l}\nu_2(W_2^u \cap M_{-n}) \ge 1$ in view of Theorem 2.2 in [6]. For those k, j that $l < n$ we have

$$\begin{aligned}
\nu_1\{x : \mathrm{gen}(x, n, W_1^u) = 1\} &\le \sum_{k=1}^{n} \sum_j \nu_{1,k}\left\{y \in \tilde{W}^u_{1,k,j} : \tilde{h}_{\hat{W}^u_{1,l,j}}(\hat{T}^{l-k}y) \in W_2^u \cap M_{-n+l}\right\} \\
&\le \sum_{k=1}^{n} \sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j}) \cdot \nu_{\hat{W}^u_{1,l,j}}\left\{z \in \hat{W}^u_{1,l,j} : \tilde{h}_{\hat{W}^u_{1,l,j}}(z) \in W_2^u \cap M_{-n+l}\right\} \\
&\le \sum_{k=1}^{n} \sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j}) \cdot D_3\ \nu_2(W_2^u \cap M_{-n+l}) \\
&\le \sum_{k=1}^{n} \sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j}) \cdot D_3\, \lambda_h^{-l}\, \nu_2(W_2^u \cap M_{-n})
\end{aligned}$$

In the last step we used the inequality

$$\nu_2(W_2^u \cap M_{-n+l}) \le \lambda_h^{-l}\, \nu_2(W_2^u \cap M_{-n}) \tag{4.10}$$

which follows from Theorem 2.2 in [6]. This proves the claim.

Due to (4.5) we have

$$\lambda_h^{-l} \le \lambda_h^{-k} \cdot \lambda_h^{\frac{\ln|\tilde{W}^u_{1,k,j}| - \ln d_0}{\ln \Lambda_{\min}}} \tag{4.11}$$

In order to estimate $|\tilde{W}^u_{1,k,j}|$ here, we observe that

$$\begin{aligned}
\nu_{1,k}(\tilde{W}^u_{1,k,j}) = \nu_1(\hat{T}^{-k}\tilde{W}^u_{1,k,j}) &\le D\, |\hat{T}^{-k}\tilde{W}^u_{1,k,j}|/|W_1^u| \\
&\le D\, |\tilde{W}^u_{1,k,j}|\Lambda_{\min}^{-k}/(\Lambda_{\max}^{-1} d_0)
\end{aligned} \tag{4.12}$$

Therefore, $\ln|\tilde{W}^u_{1,k,j}| \ge \ln \nu_{1,k}(\tilde{W}^u_{1,k,j}) - \ln(Dd_0^{-1}\Lambda_{\max}) + k \ln \Lambda_{\min}$, and so (4.11) implies

$$\lambda_h^{-l} \le \lambda_h^{-\frac{\ln(D\Lambda_{\max})}{\ln \Lambda_{\min}}} \cdot \lambda_h^{\frac{\ln\left(\nu_{1,k}(\tilde{W}^u_{1,k,j})\right)}{\ln \Lambda_{\min}}}$$

$$
\begin{aligned}
&= \exp\left(C_2' h \, \frac{\ln(D\Lambda_{\max})}{\ln \Lambda_{\min}}\right) \cdot \exp\left(-C_2' h \, \frac{\ln(\nu_{1,k}(\tilde{W}^u_{1,k,j})}{\ln \Lambda_{\min}}\right) \\
&= (D\Lambda_{\max})^g \cdot \left[\nu_{1,k}(\tilde{W}^u_{1,k,j})\right]^{-g} \qquad (4.13)
\end{aligned}
$$

where we used (4.3). The main bound (4.9) then yields

$$
\nu_1\{x : \operatorname{gen}(x) = 1\} \le D_3\,(D\Lambda_{\max})^g \nu_2(W^u_2 \cap M_{-n}) \cdot \sum_{k=1}^{n} \sum_j \left[\nu_{1,k}(\tilde{W}^u_{1,k,j})\right]^{1-g} \qquad (4.14)
$$

In the spirit of notation in Lemma 3.2, we denote by $W^u_{1,k,i}$ the components of $T^k W^u_1$. Each curve $\tilde{W}^u_{1,k,j}$ lies on some component $W^u_{1,k,i}$, and we denote this fact by a shorthand $j \in i$. Recall that the components $W^u_{1,k,i}$ have length $\le 2d_0$, so that every curve $\tilde{h}_{W^u_1,k}(W^u_{1,k,i}) \subset \hat{T}^k W^u_2$ has length $\le 2Dd_0$ and thus can cross at most $2D+1$ holes. The intersection of that curve with any hole consists of $\le B_0 - 1$ segments. Therefore, every curve $W^u_{1,k,i}$ contains no more than $(2D+1)(B_0-1) \le 3DB_0$ subcurves $\tilde{W}^u_{1,k,j}$, i.e. for any i we have

$$
\#\{j : j \in i\} \le 3DB_0 \qquad (4.15)
$$

In addition, the intersection of the curve $\tilde{h}_{W^u_1,k}(W^u_{1,k,i})$ with any hole has total length $\le h$. Therefore, $|\tilde{h}_{W^u_1,k}(\tilde{W}^u_{1,k,i}) \cap H| \le (2D+1)h \le 3Dh$. Mapping this intersection back on $W^u_{1,k,i}$ gives $\sum_{j\in i} |\tilde{W}^u_{1,k,j}| \le 3D^2 h$, so that

$$
\sum_{j\in i} \nu_{1,k}(\tilde{W}^u_{1,k,j}) \le D\nu_{1,k}(W_{1,k,i}(2D^2 h))
$$

where $W_{1,k,i}(\varepsilon)$ stands for the ε-neighborhood of the endpoints of $W_{1,k,i}$ within this curve. Applying Theorem 2.1 in [6] gives

$$
\sum_i \sum_{j\in i} \nu_{1,k}(\tilde{W}^u_{1,k,j}) \le 2C_1 D^3 h \le \bar{D}h/d_0 \qquad (4.16)
$$

where $\bar{D} = 2C_1 D^3 d_0 = 96D^4$. On the other hand, between any component $W^u_{1,k,i} \subset T^k W^u_1$ and its counterpart $\tilde{h}_{W^u_1,k}(W^u_{1,k,i}) \subset \hat{T}^k W^u_2$ there is no other parts of $\hat{T}^k W^u_1$ or $\hat{T}^k W^u_2$. Therefore, Lemma 3.1 applies and gives

$$
\sum_j |\tilde{W}^u_{1,k,j}| \le B_0 D |H|_u
$$

so that using (4.12)

$$
\sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j}) \le B_0 D^2 |H|_u d_0^{-1} \Lambda_{\max} \Lambda_{\min}^{-k} \qquad (4.17)
$$

Now we fix k_* so that $\Lambda_{\min}^{-k_*-1} \le h/|H|_u \le \Lambda_{\min}^{-k_*}$, i.e.

$$
k_* \approx \ln(|H|_u/h)/\ln \Lambda_{\min}
$$

We will use (4.16) for $k < k_*$ and (4.17) for $k \ge k_*$.

We can now apply Lemma 3.2 with $B = 3DB_0$, cf. (4.15). But first, just to simplify our calculations, we will assume that h (and hence, g) is small enough, so that, e.g., $(\bar{D}h/d_0)^{-g} = 1 + o(1) \le 2$ and $\left[\log_2(\bar{D}h/d_0)^{-1}\right]^g = 1 + o(1) \le 2$, $(BD)^g = 1 + o(1) \le 2$, etc. Then Lemma 3.2 combined with (4.15) and (4.16) yields

$$\begin{aligned}\sum_{k=1}^{k_*-1}\sum_j \left[\nu_{1,k}(\tilde{W}^u_{1,k,j})\right]^{1-g} &\le 800\, D \sum_{k=1}^{k_*-1} \Lambda_{\max}^{kg}\, (\bar{D}h/d_0)^{1-g} \left[\log_2(\bar{D}h/d_0)^{-1}\right]^g \\ &\le 3200\, D\bar{D} \cdot \frac{h}{d_0}\, \frac{\Lambda_{\max}^{k_* g} - 1}{\Lambda_{\max}^{g} - 1}\end{aligned}$$

Note that both g and $k_* g$ approach zero as $h \to 0$. Thus, if h is sufficiently small, we have

$$\frac{\Lambda_{\max}^{k_* g} - 1}{\Lambda_{\max}^{g} - 1} = k_* + o(k_*) \le 2k_*$$

and hence

$$\sum_{k=1}^{k_*-1}\sum_j \left[\nu_{1,k}(\tilde{W}^u_{1,k,j})\right]^{1-g} \le D'(h/d_0)\, \ln(|H|_u/h) \tag{4.18}$$

where $D' = 10^6 D^5/\ln \Lambda_{\min}$.

Next, using Lemma 3.2 and (4.17) gives

$$\sum_{k=k_*}^{\infty}\sum_j \left[\nu_{1,k}(\tilde{W}^u_{1,k,j})\right]^{1-g} \le C_g \sum_{k=k_*}^{\infty} \Lambda_{\max}^{kg} \left(\frac{B_0 D^2 \Lambda_{\max}|H|_u}{\Lambda_{\min}^k d_0}\right)^{1-g} \left[\log_2\left(\frac{\Lambda_{\min}^k d_0}{B_0 D^2 \Lambda_{\max}|H|_u}\right)\right]^g$$

We again assume that h is small enough, then it is simple estimation that

$$\sum_{k=k_*}^{\infty}\sum_j \left[\nu_{1,k}(\tilde{W}^u_{1,k,j})\right]^{1-g} \le D''(h/d_0)\, \ln(|H|_u/h) \tag{4.19}$$

where $D'' = 2000\, D^3 B_0 \Lambda_{\max}$. (A crucial point here is to observe that $t = \Lambda_{\max}^g/\Lambda_{\min}^{1-g} < 1$, and $\sum_{k_*}^{\infty} k^g t^k < \int_{k_*}^{\infty} x t^x\, dx < 2k_* t^{k_*} \ln t^{-1}$ with $t^{k_*} \le 2h/|H|_u$.)

Combining (4.3), (4.14), (4.18) and (4.19) (and assuming h is small enough) gives

$$\nu_1\{x :\, \mathrm{gen}(x, n, W_1^u) = 1\} \le D_3 D_4\, \nu_2(W_2^u \cap M_{-n}) \cdot (h/d_0)\, \ln(|H|_u/h) \tag{4.20}$$

with $D_4 = 2(D' + D'')$. We assume h is so small that

$$h_1 \stackrel{\text{def}}{=} (h/d_0)\, \ln(|H|_u/h) < D_4^{-1}/2 \tag{4.21}$$

Next, observe that we actually proved that the right hand side of (4.9) was smaller than that of (4.20), i.e.,

$$\sum_{k=1}^{n}\sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j})\lambda_h^{-l} \le D_4 h_1 < 1/2 \tag{4.22}$$

Observe also that this bound is uniform in n.

Now, due to (4.7), we can estimate the measure of points of the second generation as follows:

$$\begin{aligned}
\nu_1\{x:\ \text{gen}(x,n,W_1^u)=2\} &\leq \sum_{k=1}^{n}\sum_j \nu_{1,l}\{y\in\hat{W}^u_{1,l,j}:\ \text{gen}(y,n-l,\hat{W}^u_{1,l,j})=1\}\\
&\leq \sum_{k=1}^{n}\sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j})\cdot\nu_{\hat{W}^u_{1,l,j}}\{y\in\hat{W}^u_{1,l,j}:\ \text{gen}(y,n-l,\hat{W}^u_{1,l,j})=1\}\\
&\leq \sum_{k=1}^{n}\sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j})\cdot D_3D_4h_1\,\nu_2(W_2^u\cap M_{-n+l})\\
&\leq \sum_{k=1}^{n}\sum_j \nu_{1,k}(\tilde{W}^u_{1,k,j})\cdot D_3D_4h_1\lambda_h^{-l}\,\nu_2(W_2^u\cap M_{-n})\\
&\leq D_3D_4^2h_1^2\,\nu_2(W_2^u\cap M_{-n}) \qquad (4.23)
\end{aligned}$$

Here we subsequently used (4.20), (4.10) and (4.22).

For higher generations, we obtain inductively that

$$\nu_1\{x:\ \text{gen}(x,n,W_1^u)=r\}\leq D_3D_4^rh_1^r\,\nu_2(W_2^u\cap M_{-n}) \qquad (4.24)$$

In particular, for $r=0$ we recover (4.8).

Summing up over $r\geq 0$ and using (4.21) completes the proof of Theorem 4.1 and gives $C_6=2D_3$. $\square$

Remark. Summing (4.24) over all $r\geq 1$ gives

$$\nu_1\{x:\ \text{gen}(x,n,W_1^u)\geq 1\}\leq \frac{D_3D_4h_1}{1-D_4h_1}\,\nu_2(W_2^u\cap M_{-n}) \qquad (4.25)$$

Assuming that h, and hence h_1, are small enough, we will have $D_3D_4h_1/(1-D_4h_1)\leq (2C_6)^{-1}$. Then, combining (4.25) and (4.1) gives

$$\nu_1\{x:\ \text{gen}(x,n,W_1^u)=0\}\geq \frac{1}{2}\,\nu_1(W_1^u\cap M_{-n})$$

In other words, assuming that the fiber W_1^u is d_0-close to W_2^u, we have that $\forall n\geq 0$, at least 50% of the points in $W_1^u\cap M_{-n}$ can be connected by stable fibers of length $\leq d_0$ with points of $W_2^u\cap M_{-n}$.

5 Measures on short unstable fibers

Theorem 5.1 *Let $W^u\subset\hat{M}$ be an unstable fiber of length $\leq d_0$, and ν its u-SBR measure. For any subfiber $W_1^u\subset W^u$ and any $n\geq 0$ we have*

$$\nu(W_1^u\cap M_{-n})\leq C_7\delta_n|W_1^u|^{1-g}/|W^u| \qquad (5.1)$$

where g is given by (4.3) and $C_7=4C_6D^2$.

Proof. First, we assume that $W^u \in \mathcal{W}^u(d_0)$, i.e. $|W^u| \geq d_0/\Lambda_{\max}$. Let $l_1 = \min\{l' \geq 0 : \hat{T}^{l'} W_1^u \in \mathcal{W}^u(d_0)\}$. Since $|\hat{T}^{l_1} W_1^u| \leq d_0$,

$$l_1 \leq \frac{\ln d_0 - \ln |W_1^u|}{\ln \Lambda_{\min}}$$

Consider the u-SBR measure $\nu_{W_1^u}$ on W_1^u. Observe that $d\nu_{W_1^u}/d\nu \geq D^{-1}|W^u|/|W_1^u|$ on W_1^u. Thus

$$\begin{aligned}
\nu(W_1^u \cap M_{-n}) &\leq D\,|W_1^u|\,|W^u|^{-1}\nu_{W_1^u}(W_1^u \cap M_{-n}) \\
&\leq D\,|W_1^u|\,|W^u|^{-1}\nu_{\hat{T}^{l_1}W_1^u}(\hat{T}^{l_1}W_1^u \cap M_{-n+l_1}) \\
&\leq D\,|W_1^u|\,|W^u|^{-1}C_6\nu_{W_0^u}(W_0^u \cap M_{-n+l_1}) \\
&\leq D\delta_n|W_1^u|\,|W^u|^{-1}C_6\lambda_h^{-l_1} \qquad (5.2)
\end{aligned}$$

Here we subsequently used Theorem 4.1 and (4.4).

Using (4.2) and (4.3) gives

$$\lambda_h^{-l_1} \leq e^{C_2' h \frac{\ln d_0 - \ln |W_1^u|}{\ln \Lambda_{\min}}} = d_0^g\,|W_1^u|^{-g}$$

For h small enough, $d_0^g = 1 + o(1) \leq 2$. Combining the last bound with (5.2) yields

$$\nu(W_1^u \cap M_{-n}) \leq 2C_6 D\delta_n|W_1^u|^{1-g}/|W^u| \qquad (5.3)$$

Next, let $|W^u| < d_0/\Lambda_{\max}$. Let $l = \min\{l' \geq 0 : \hat{T}^{l'}W^u \in \mathcal{W}^u(d_0)\}$. Again, as above,

$$l \leq \frac{\ln d_0 - \ln |W^u|}{\ln \Lambda_{\min}}$$

and so

$$\lambda_h^{-l} \leq e^{C_2' h \frac{\ln d_0 - \ln |W^u|}{\ln \Lambda_{\min}}} = d_0^g\,|W^u|^{-g} \qquad (5.4)$$

We now have

$$\begin{aligned}
\nu(W_1^u \cap M_{-n}) &\leq \nu_{\hat{T}^l W^u}(\hat{T}^l W_1^u \cap M_{-n+l}) \\
&\leq 2C_6 D\delta_{n-l}|\hat{T}^l W_1^u|^{1-g}/|\hat{T}^l W^u| \\
&\leq 2C_6 D\delta_n\lambda_h^{-l}\left(|\hat{T}^l W_1^u|/|\hat{T}^l W^u|\right)^{1-g}|\hat{T}^l W^u|^{-g} \\
&\leq 2C_6 D\delta_n d_0^g|W^u|^{-g}\left(D|W_1^u|/|W^u|\right)^{1-g}(d_0/\Lambda_{\max})^{-g} \\
&\leq 4C_6 D^2\delta_n|W_1^u|^{1-g}/|W^u|
\end{aligned}$$

where we subsequently used (5.3), (4.4), (5.4) and the fact that $\Lambda_{\max}^g = 1 + o(1) \leq 2$ for small h. Theorem 5.1 is now proved. □

In particular, setting $W_1^u = W^u$ gives the following corollary:

Corollary 5.2 *Let $W^u \subset \hat{M}$ be an unstable fiber of length $\leq d_0$, and ν its u-SBR measure. For any $n \geq 0$ we have*

$$\nu(W^u \cap M_{-n}) \leq C_7 \delta_n |W^u|^{-g} \tag{5.5}$$

Convention. In the rest of this section, we use the notations $\mathcal{M}_m$, $\mathcal{W}^u_m$, $\mathcal{W}^u_{m,\varepsilon}$, $U_{m,\varepsilon}$ with either $m \geq 0$ and $m = +$, see Section 2.3.

Let $B \geq 1$, $\varepsilon > 0$, and $G \subset M$ be an arbitrary subset such that for every maximal unstable fiber $W^u \in \mathcal{W}^u_m$ the intersection $W^u \cap G$ is a union of no more than $2B$ subfibers of W^u, and each of those subfibers has length $\leq \varepsilon$. One can think of $G = U_{m,\varepsilon}$, as an example, in which $B = 1$. Loosely speaking, the set G is $(2B\varepsilon)$-thick in the unstable direction.

Theorem 5.3 *For any $\mu \in \mathcal{M}_m$, $B \geq 1$, $\varepsilon > 0$, and any set G described above, we have for all $n \geq 0$*

$$\mu(G \cap M_{-n}) \leq C_8 B \delta_n \varepsilon^{1-g} \tag{5.6}$$

where $C_8 = 2C_1C_7D$.

Proof. Consider the factor measure μ^f induced by μ on the set of maximal unstable fibers $\mathcal{W}^u_m$. Then

$$\begin{aligned} \mu(U_{m,B\varepsilon}) &= \int_{\mathcal{W}^u_m} \nu_{W^u}(W^u \cap U_{m,B\varepsilon})\, d\mu^f(W^u) \\ &\geq \mu\left(\cup W^u : W^u \in \mathcal{W}^u_{m,2B\varepsilon}\right) \\ &+ \int_{\mathcal{W}^u \setminus \mathcal{W}^u_{m,2B\varepsilon}} D^{-1}(2B\varepsilon/|W^u|) d\mu^f(W^u) \end{aligned} \tag{5.7}$$

Now, let

$$F_\mu(y) = \mu\left(\cup W^u : W^u \in \mathcal{W}^u_{m,y}\right) \tag{5.8}$$

The function $F_\mu(y)$ is a kind of distribution function of the length of maximal fibers $W^u \in \mathcal{W}^u_m$ with respect to the given measure $\mu \in \mathcal{M}_m$. Then (5.7) and the properties (M1)-(M2) of μ imply that for any $\varepsilon > 0$ we have

$$F_\mu(2B\varepsilon) + \int_{2B\varepsilon}^{d_0} D^{-1}(2B\varepsilon/y)\, dF_\mu(y) \leq C_1 B\varepsilon \tag{5.9}$$

In particular,

$$F_\mu(y) \leq \frac{1}{2} C_1 y \tag{5.10}$$

In the following estimate, we apply Theorem 5.1 to fibers of length $> 2B\varepsilon$ and a subfiber of length $2B\varepsilon$, and Corollary 5.2 to fibers smaller than $2B\varepsilon$:

$$\begin{aligned}
\mu(G\cap M_{-n}) &\leq \int_{2B\varepsilon}^{d_0} 2BC_7\delta_n\varepsilon^{1-g}/y\, dF_\mu(y) + \int_0^{2B\varepsilon} C_7\delta_n y^{-g}\, dF_\mu(y) \\
&= C_7D\delta_n\varepsilon^{-g}\int_{2B\varepsilon}^{d_0} D^{-1}(2B\varepsilon/y)\, dF_\mu(y) + C_7\delta_n(2B\varepsilon)^{-g}F_\mu(2B\varepsilon) \\
&+ C_7 g\delta_n\int_0^{2B\varepsilon} y^{-1-g}F_\mu(y)\, dy
\end{aligned} \tag{5.11}$$

Here we applied the integration by parts to the integral from 0 to $2B\varepsilon$. Due to (5.10), the last integral in (5.11) does not exceed $C_1B\varepsilon^{1-g}/(1-g)$. To the first two terms on the right hand side of (5.11) we apply (5.9) and get

$$\mu(G\cap M_{-n}) \leq C_7D\delta_n\varepsilon^{-g}C_1B\varepsilon + C_1C_7Bg\delta_n\varepsilon^{1-g}/(1-g)$$

For h small enough, we have $g/(1-g) < D$, which completes the proof of Theorem 5.3. □

In particular, if $B=1$ and $\varepsilon = d_0/2$, we can take $G = M_m$. Assuming g be sufficiently small, we have

$$\mu(M_{-n}) \leq C_8 d_0\, \delta_n \tag{5.12}$$

On the other hand,

$$\begin{aligned}
\mu(M_{-n}) &\geq \mu\left(\cup(W^u\cap M_{-n}) : |W^u| > d_0\Lambda_{\max}^{-1}\right) \\
&\geq C_6^{-1}\delta_n\,\mu\left(\cup W^u : |W^u| > d_0\Lambda_{\max}^{-1}\right)
\end{aligned} \tag{5.13}$$

Here we used Theorem 4.1 (recall that $|W^u| \leq d_0$, $\forall W^u \in \mathcal{W}^u$). Using the property (M2) with $\varepsilon = (2C_1)^{-1}$ gives

$$\mu\left(\cup W^u : |W^u| > C_1^{-1}\right) \geq 1/2$$

Since $C_1 = 48D/d_0$ and $\Lambda_{\max} \geq \Lambda_{\min} \geq 64D^2$, cf. Section 2.2, then

$$\mu\left(\cup W^u : |W^u| > d_0\Lambda_{\max}^{-1}\right) \geq 1/2$$

so that (5.13) implies

$$\mu(M_{-n}) \geq \delta_n/(2C_6) \tag{5.14}$$

Lastly, recall that $\mu_+(M_{-n}) = ||T_*^n\mu_+|| = \lambda_+^n$. Combining (5.12) and (5.14) gives the following:

Corollary 5.4 *For any $\mu \in \mathcal{M}_m$ and $n \geq 0$ we have*

$$C_9^{-2}\lambda_+^n \leq \mu(M_{-n}) \leq C_9^2\lambda_+^n \tag{5.15}$$

and

$$C_9^{-1}\lambda_+^n \leq \delta_n \leq C_9\lambda_+^n \tag{5.16}$$

with $C_9 = \max\{2C_6, d_0C_8\}$. The last bound holds true for any $\delta_n(W^u)$, $W^u \in \mathcal{W}^u(d_0)$.

This also gives a corollary to Lemma 3.4:

Corollary 5.5 *Using notation of Lemma 3.4, we have for any* $n \geq k_1$

$$C_{10}^{-1}\lambda_+^{n-k_1} d^u_{\max}(R) \leq (T_*^n \mu)(R) \leq C_{10}\lambda_+^{n-k_1} d^u_{\min}(R)$$

with $C_{10} = C_5 C_9^2$.

Recall that for any $\varepsilon > 0$ we denote by $H_\varepsilon \subset M$ the ε-neighborhood of ∂H in M.

Theorem 5.6 *Let* $\varepsilon > 0$ *and put*

$$q_\varepsilon = \ln(D^{-1}d_0/\varepsilon)/\ln \Lambda_{\max}$$

Then for any $\mu \in \mathcal{M}_m$ *and* $q \geq q_\varepsilon$, $n \geq 0$, *we have*

$$(T_+^q \mu)(H_\varepsilon \cap M_{-n}) \leq C_{11}\varepsilon^{b(1-4g)} (T_+^q \mu)(M_{-n}) \tag{5.17}$$

where $b = \ln \Lambda_{\min}/(2\ln \Lambda_{\max})$ *and* $C_{11} > 0$ *is a global constant.*

Proof. Assume for a moment that ∂H consists of a finite number of smooth curves transversal to the unstable direction. Then the set H_ε is at most $(B_0\varepsilon)$-thick in the unstable direction, and a combination of Theorem 5.3 and (5.14) applied to the measure $T_+^q\mu$ imply (5.17) with $b = 1/2$ and all $q \geq 0$. Similarly, if ∂H consists of curves which either are transversal to the unstable direction or have curvature larger than that of unstable fibers, the set H_ε is (const$\cdot\varepsilon^{1/2}$)-thick in the unstable direction and we again get (5.17) with $b = 1/2$ and all $q \geq 0$.

There are, however, reasons why the theorem cannot always hold for all $q \geq 0$. For example, let a hole $H' \subset H$ be a rectangle of size h in the unstable direction, and μ be supported on a single unstable fiber of length d_0 that is $(\varepsilon/2)$-close to an unstable side of H', then $\mu(H_\varepsilon) \geq h/d_0$ independently of ε.

Now, in the general case, we consider maximal connected unstable fibers $W^u \subset H_\varepsilon$ and call them long if $|W^u| \geq \varepsilon^b$. To any long fiber $W^u \subset H_\varepsilon$ we associate the rectangle $R = R(W^u)$ such that (i) the fiber W^u is an R-fiber [6] (i.e., it terminates on the stable sides of R) and (ii) the minimal distance from W^u to each unstable side of $R(W^u)$, measured along stable fibers, is 2ε (the maximal distance is then $< 2D\varepsilon$). We now pick any long fiber $W_1^u \subset H_\varepsilon$, then any long fiber $W_2^u \subset H_\varepsilon \setminus R(W_1^u)$, etc. In that way we will find a finite collection of rectangle $R(W_r^u)$, $r = 1, \ldots, \bar{r}$ such that the residual set $H_\varepsilon \setminus (\cup_r R(W_r^u))$ consists of short fibers. That residual set is then at most $(2B_0\varepsilon^b)$-thick in the unstable direction, and the above argument applies giving the right estimate for it. It remains to consider the set $\cup_r R(W_r^u)$.

The regularity of our holes $H' \subset H$ (the fact that any local stable/unstable fiber intersects any hole in no more than B_0 intervals) implies that

$$\sum_{r=1}^{\bar{r}} |W_r^u| \leq B_1 |H|_u$$

with some $B_1 = \mathrm{const}(B_0) > 0$, cf. (3.1). Therefore,

$$\bar{r} \le B_1 |H|_u \varepsilon^{-b} \tag{5.18}$$

For any $r = 1, \dots, \bar{r}$ the rectangle $R'_r = \hat{T}^{-q_\varepsilon} R(W^u_r)$ has length $\le 4d_0$ in the stable direction and $\le |W^u_r| \cdot \Lambda_{\min}^{-q_\varepsilon} \le |H|_u (D\varepsilon/d_0)^{2b}$ in the unstable direction. Let $\mu' = T_+^{q-q_\varepsilon}\mu$ (note that $\mu' \in \mathcal{M}_{m+q-q_\varepsilon}$). Observe that

$$(T^q_+\mu)\,(R(W^u_r)\cap M_{-n}) = (T^{q_\varepsilon}_+\mu')\,(R(W^u_r)\cap M_{-n}) = \frac{(T^{q_\varepsilon}_*\mu')\,(R(W^u_r)\cap M_{-n})}{||T^{q_\varepsilon}_*\mu'||} \le \frac{\mu'(R'_r\cap M_{-n-q_\varepsilon})}{||T^{q_\varepsilon}_*\mu'||}$$

Applying Theorem 5.3 with $B = 1$ to R'_r gives

$$\mu'(R'_r \cap M_{-n-q_\varepsilon}) \le C_8 \delta_{n+q_\varepsilon} |H|_u^{1-g} (D\varepsilon/d_0)^{2b(1-g)} \le 2C_8\delta_n |H|_u (D\varepsilon/d_0)^{2b(1-g)}$$

According to (5.14), (4.4), (4.3), and (4.2), we have

$$||T^{q_\varepsilon}_*\mu'|| = \mu'(M_{-q_\varepsilon}) \ge \delta_{q_\varepsilon}/(2C_6) \ge (2C_6)^{-1} e^{-C'_2 h q_\varepsilon} \ge (2C_6)^{-1} (D\varepsilon/d_0)^{2bg}$$

Combining the above estimates gives

$$(T^q_+\mu)\,(R(W^u_r)\cap M_{-n}) \le 4C_6C_8\delta_n|H|_u(D\varepsilon/d_0)^{2b(1-2g)}$$

Together with (5.18) this yields

$$(T^q_+\mu)\,([\cup_r R(W^u_r)]\cap M_{-n}) \le 4C_6C_8DB_1\delta_n|H|_u^2\varepsilon^{b(1-4g)}/d_0^{2b(1-2g)}$$

Theorem 5.6 is now proved. □

Let $W^u \subset \hat{M}$ be an unstable fiber. For every $n \ge 0$ and $x \in W^u \cap M_{-n}$ denote by $W^u_n(x)$ the smooth component of $T^n(W^u \cap M_{-n})$ containing $T^n x$ and by $|W^u_n(x)|$ its length. For every $x \in W^u \cap M_{-n}$ let $l_n(x) = \min\{l \in [0,n] : |W^u_l(x)| \ge d_0/2\}$, and if $|W^u_l(x)| < d_0/2$ for all $l = 0, \dots, n$, we set $l_n(x) = n+1$.

Lemma 5.7 *There are global constants $\beta_1 \in (0,1)$ and $C_{13} > 0$ such that for every $n \ge m \ge 0$*

$$\nu_{W^u}\{x \in W^u \cap M_{-n} : l_n(x) \ge m\} \le C_{13}\,\beta_1^m \lambda_+^n/|W^u|$$

Proof. For each $i = 0, \dots, m$ let $W^u_i = \{x \in W^u \cap M_{-i} : l_i(x) = i\}$. The points $x \in W^u_i$ have their first $i-1$ images under T in short components (of length $< d_0/2$) in the images of the curve W^u. The proof of Lemma 3.3 in [6] can be easily modified to produce the bound $\nu_{W^u}(W^u_i) \le \mathrm{const}\cdot(B_0/\Lambda_{\min})^i/|W^u|$, where the constant is global. At the i-th iteration, however, all the points $x \in W^u_i$ get into long (of length $\ge d_0/2$) components of $T^i(W^u \cap M_{-i})$. Due to (5.16) we have $\nu_{W^u}(W^u_i \cap M_{-n}) \le \mathrm{const}\cdot(B_0/\Lambda_{\min})^i\lambda_+^{n-i}/|W^u|$. We can find a global constant $\beta_1 \in (0,1)$ such that

$$\beta_1 \ge B_0\Lambda_{\min}^{-1}/\lambda_+ \tag{5.19}$$

cf. Assumption H2 in [6]. Hence, $\nu_{W^u}(W^u_i \cap M_{-n}) \le \mathrm{const}\cdot\beta_1^i\lambda_+^n/|W^u|$. Summing up over $i = m, \dots, n$ gives the lemma. □

6 Iterations of SBR-like measures

In [6] we defined 'rectangular' holes $H^{(k)}$ approximating the given holes H as follows. We fixed an arbitrary Markov partition $\mathcal{R}$, refined it in a standard way, $\mathcal{R}^{(k)} = \vee_{i=-k}^{k} T^i \mathcal{R}$, and defined $H^{(k)}$ to be the union of the interiors of all rectangles $R \in \mathcal{R}^{(k)}$ that intersected H. In other words, $H^{(k)}$ consisted of 'rectangular holes' approximating H 'from outside'. We proved that for all sufficiently large k, $k \geq k_0$, the map $T^{(k)}$ (the restriction of $\hat{T}$ to the set $M^{(k)} = \hat{M} \setminus H^{(k)}$) had a unique conditionally invariant SBR-like measure, $\mu_+^{(k)}$, with an eigenvalue $\lambda_+^{(k)}$ and a unique invariant measure $\bar{\mu}_+^{(k)}$ on the repeller $\Omega^{(k)}$. We also showed that $\lambda_+^{(k)} \to \lambda_+$ and $\mu_+^{(k)}$ weakly converged to μ_+ as $k \to \infty$.

The rectangular holes $H^{(k)}$, for sufficiently large k, are very close to H. So, they satisfy all the assumptions we have made in [6] and in the present paper, hence we can assume that the values of the parameters d_0, B_0, N_H are the same for all the holes $H^{(k)}$, $k \geq k_0$.

Observe that

$$H^{(k)} \setminus H \subset H_{\varepsilon_k/2} \tag{6.1}$$

where

$$\varepsilon_k = 2D_5 \Lambda_{\min}^{-k} \tag{6.2}$$

and D_5 stands for the maximal diameter of the atoms $R \in \mathcal{R}$ of the Markov partition $\mathcal{R}$.

In all that follows, unless specified otherwise, we will use rectangular holes $H^{(k)}$ defined in a more flexible way.

Definition. Let $k \geq k_0$ and $H^{(k)}$ be the union of interiors of some $R \in \mathcal{R}^{(k)}$. We say that $H^{(k)}$ properly approximates the given hole H if
(i) $d(H^{(k)}, H) \leq \varepsilon_k$ in the sense of (2.6),
(ii) the holes $H^{(k)}$ satisfy all our assumptions on H with the same values of d_0, B_0, N_H.

The results of [6] and this paper apply to the map $T^{(k)}$ on $M^{(k)} = \hat{M} \setminus H^{(k)}$ if $H^{(k)}$ properly approximates H. But remember, the convergence $\lambda_+^{(k)} \to \lambda_+$ and $\mu_+^{(k)} \to \mu_+$ is proved in [6] only for the approximation 'from outside' as described in the first paragraph of this section.

Let $\mu_0 \in \mathcal{M}_0$. Let $k \geq k_0$ and $q \geq 1$ to be specified later. Assume that $H^{(k)}$ properly approximate H. This will be a standing assumption for the rest of the paper. For any $n \geq 1$ denote $\mu_n = T_+^n \mu_0$. For $n \geq q$ we can write this as

$$\mu_n = T_+^{n-q} \mu_q = \frac{T_*^{n-q} \mu_q}{||T_*^{n-q} \mu_q||} = \frac{T_*^{n-q} \mu_q}{\mu_q(M_{-n+q})} \tag{6.3}$$

Consider also the measure

$$\mu_n^{(k)} = [T_+^{(k)}]^{n-q} \mu_q = \frac{[T_*^{(k)}]^{n-q} \mu_q}{||\,[T_*^{(k)}]^{n-q} \mu_q||} = \frac{[T_*^{(k)}]^{n-q} \mu_q}{\mu_q(M_{-n+q}^{(k)})} \tag{6.4}$$

The superscript (k) is always the index of the Markov approximation $T^{(k)}$ to the map T.

For $k \geq k_0$ we set q to be $q = q(k) = k + l_0$, where $l_0 = [\ln(d_0/DD_5)/\ln \Lambda_{\max}] + 1$. Observe that

$$q = q(k) \geq \ln(D^{-1}d_0/\varepsilon_k)/\ln \Lambda_{\max} \tag{6.5}$$

so that we can apply Theorem 5.6. Suppose $n > q$. Since

$$M_{-n+q} \setminus M_{-n+q}^{(k)} = \cup_{i=0}^{n-q} \hat{T}^{-i}(H^{(k)} \setminus H) \subset \cup_{i=0}^{n-q} \hat{T}^{-i} H_{\varepsilon_k}$$

we have the following estimate:

$$\begin{aligned}
\mu_q \left(M_{-n+q} \setminus M_{-n+q}^{(k)} \right) &\leq \sum_{i=0}^{n-q} \mu_q(M_{-n+q} \cap \hat{T}^{-i} H_{\varepsilon_k}) \\
&\leq \sum_{i=0}^{n-q} C_{11} \varepsilon_k^{b(1-4g)} \mu_q(M_{-n+q}) \\
&= C_{11}(n-q+1)\varepsilon_k^{b(1-4g)} \mu_q(M_{-n+q})
\end{aligned} \tag{6.6}$$

Here we used the following estimate, based on (6.5) and Theorem 5.6:

$$\begin{aligned}
\mu_q \left(M_{-n+q} \cap \hat{T}^{-i} H_{\varepsilon_k} \right) &= \mu_q \left[\hat{T}^{-i} \left(M_{-n+q+i} \cap H_{\varepsilon_k} \right) \cap M_{-i} \right] \\
&= \left(T_*^i \mu_q \right) \left(M_{-n+q+i} \cap H_{\varepsilon_k} \right) \\
&= \left(T_+^i \mu_q \right) \left(M_{-n+q+i} \cap H_{\varepsilon_k} \right) \cdot ||T_*^i \mu_q|| \\
&\leq C_{11} \varepsilon_k^{b(1-4g)} \left(T_+^i \mu_q \right) \left(M_{-n+q+i} \right) \cdot ||T_*^i \mu_q|| \\
&= C_{11} \varepsilon_k^{b(1-4g)} \cdot \left(T_*^i \mu_q \right) \left(M_{-n+q+i} \right) \\
&= C_{11} \varepsilon_k^{b(1-g)} \mu_q(M_{-n+q})
\end{aligned}$$

for any $i = 0, 1, \ldots, n-q$. The following bound is symmetric to (6.6), its proof is similar:

$$\mu_q^{(k)} \left(M_{-n+q}^{(k)} \setminus M_{-n+q} \right) \leq C_{11}(n-q+1)\varepsilon_k^{b(1-4g)} \mu_q^{(k)}(M_{-n+q}^{(k)}) \tag{6.7}$$

Observe that the measure μ_n is supported on M_{n-q} and the measure $\mu_n^{(k)}$ is supported on $M_{n-q}^{(k)}$. On the common part of their supports, $M_{n-q} \cap M_{n-q}^{(k)}$, these two measures are proportional, so their conditional measures on that common part coincide, we call that conditional measure $\hat{\mu}_+^{(k)}$. Therefore,

$$\mu_n = (1 - \sigma_n)\hat{\mu}_n^{(k)} + \sigma_n \mu' \tag{6.8}$$

and

$$\mu_n^{(k)} = (1 - \sigma_n^{(k)})\hat{\mu}_n^{(k)} + \sigma_n^{(k)} \mu'' \tag{6.9}$$

for some $\sigma_n, \sigma_n^{(k)} > 0$ and some probability measures μ', μ'' supported on $M_{n-q} \setminus M_{n-q}^{(k)}$ and $M_{n-q}^{(k)} \setminus M_{n-q}$, respectively. Moreover, it follows from (6.6) and (6.7) that

$$\sigma_n + \sigma_n^{(k)} \leq 2C_{11} n \varepsilon_k^{b(1-4g)} \leq C_{12} n \alpha_1^k \tag{6.10}$$

with $C_{12} = 2C_{11} D_5^{b/2}$ and $\alpha_1 = \Lambda_{\min}^{-b/2} < 1$ (recall that g, as well as h, are very small, so we can assume that $g < 1/8$). Therefore, for any bounded function $f(x)$ on M we have

$$\left| \int_M f(x)\, d\mu_n - \int_M f(x)\, d\mu_n^{(k)} \right| \leq 2||f||_\infty (\sigma_n + \sigma_n^{(k)}) \leq 2||f||_\infty C_{12} n \alpha_1^k \tag{6.11}$$

Definition. We say that two sequences of probability measures, $\{\mu_n'\}$ and $\{\mu_n''\}$ on M are (asymptotically) equivalent (denoted by $\mu_n' \sim \mu_n''$) if for any continuous function $f(x)$ on M

$$\lim_{n\to\infty} \left| \int_M f(x)\, d\mu_n' - \int_M f(x)\, d\mu_n'' \right| = 0$$

Note that if $\mu_n' \sim \mu_n''$ and μ_n' converges weakly to a limit measure, μ_∞, then μ_n'' also converges weakly to μ_∞.

Proposition 6.1 *Let $k = k_n$ be a sequence that grows faster than any logarithmic function but slower than any linear function: $k_n = o(n)$ and $1/k_n = o(1/\ln n)$ as $n \to \infty$. Then $\mu_n \sim \mu_n^{(k_n)}$.*

Proof. For large n we have $n > q = k_n + l_0$, so that (6.8)-(6.11) hold, and $n\alpha_1^{k_n} \to 0$ as $n \to \infty$. □

Our next goal is to show that there is a sequence $k_n \to \infty$ such that $k_n = o(n)$ and $1/k_n = o(1/\ln n)$, for which $\mu_n^{(k_n)} \sim \mu_+^{(k_n)}$.

We say that a sequence of numbers $\{L_k\}$ majorizes another sequence $\{M_k\}$ if $L_k \geq M_k$, $\forall k \geq 1$. The above goal will be achieved if we show the following:

There is a sequence $\{M_k\}$ such that for any majorizing sequence $\{L_k\}$ the sequence of measures $\mu_{L_k}^{(k)} = [T_+^{(k)}]^{L_k - k - l_0} \mu_{k+l_0}$ is equivalent to $\mu_+^{(k)}$, i.e.

$$\lim_{k\to\infty} \left| \int_M f(x)\, d\mu_{L_k}^{(k)} - \int_M f(x)\, d\mu_+^{(k)} \right| = 0 \tag{6.12}$$

for every $f \in C(M)$. The sequence $\{M_k\}$ has to grow faster than any linear function Ak, $A > 0$, and slower than any exponential function e^{ak}, $a > 0$.

The weak convergence of $\mu_L^{(k)}$ to $\mu_+^{(k)}$, as $L \to \infty$, for all $k \geq k_0$ was proved in [6]. It implies that for any continuous function $f \in C(M)$ there is a sequence $M_k = M_k(f)$ such that (6.12) holds for any majorizing sequence $\{L_k\}$. All we need to prove is that one can choose M_k uniformly in f and so that $M_k = o(e^{ak})$, $\forall a > 0$.

Recall that the measures $\mu_L^{(k)}$ and $\mu_+^{(k)}$ are supported on $M_{L-q}^{(k)} \subset M^{(k)}$ and $M_+^{(k)} \subset M^{(k)}$, respectively. Recall that $\mathrm{diam} R \leq \varepsilon_k = D_5 \Lambda_{\min}^{-k}$, $\forall R \in \mathcal{R}^{(k)}$. The oscillation of $f(x)$ on any rectangle $R \in \mathcal{R}^{(k)}$ does not exceed $\delta_f(\varepsilon_k)$, and $\delta_f(\varepsilon_k) \to 0$ as $k \to \infty$.

For any two probability measures, μ' and μ'', on $M^{(k)}$ let

$$|\mu' - \mu''|_k := \sum_{R \in \mathcal{R}^{(k)}} |\mu'(R) - \mu''(R)|$$

It is then a simple calculation that

$$\left| \int_M f(x)\, d\mu_L^{(k)} - \int_M f(x)\, d\mu_+^{(k)} \right| \le 2\delta_f(\varepsilon_k) + ||f||_\infty \cdot \left| \mu_L^{(k)} - \mu_+^{(k)} \right|_k \tag{6.13}$$

Therefore, to establish (6.12) it is enough to prove the following:

Proposition 6.2 *There is a sequence $M_k \to \infty$ such that for any majorizing sequence L_k*

$$\lim_{k \to \infty} \left| \mu_{L_k}^{(k)} - \mu_+^{(k)} \right|_k = 0$$

The sequence $\{M_k\}$ has to grow faster than any linear function Ak, $A > 0$, and slower than any exponential function e^{ak}, $a > 0$.

We will also sharpen this proposition as follows:

Proposition 6.3 *There are global constants $r \ge 1$, $C_{14} > 0$ and $\alpha_2 \in (0,1)$ such that for all $k \ge k_0$ and $L \ge rk^2 + k + l_0$*

$$|\mu_L^{(k)} - \mu_+^{(k)}| \le C_{14}\alpha_2^k$$

These two propositions are only concerned with properties of conditionally invariant measures $\mu_+^{(k)}$ for the Anosov maps $T^{(k)}$ with rectangular holes, $H^{(k)}$. So, we can apply the finite-dimensional approximations and matrix techniques developed in [3, 4, 5]. A complete proof of Proposition 6.3 is provided in Section 8, while 6.2 immediately follows from 6.3.

7 Proofs of the main theorems

Proof of Theorem 2.2. To prove the weak convergence of $T_+^n \mu_0$ to μ_+, it is enough to combine Proposition 6.1, (6.12) and the weak convergence of $\mu_+^{(k)}$ to μ_+ proved in [6] for the specific rectangular holes $H^{(k)}$ described in the first paragraph of the previous section. The second statement of the theorem will follow from Corollary 7.2 below. □

Proof of Theorem 2.3. For any large n, i.e. $n \ge n_0 := rk_0^2 + k_0 + l_0$, we take $k = k_n = \max\{k : n \ge rk^2 + k + l_0\}$. Then we combine (6.11) and (6.13) with Proposition 6.3, in which we set $L = n$. As a result,

$$\left| \int_M f(x)\, d\mu_n - \int_M f(x)\, d\mu_+^{(k_n)} \right| \le \text{const} \cdot \left[\delta_f \left(e^{-a_1 n^{1/2}} \right) + ||f||_\infty e^{-a_2 n^{1/2}} \right] \tag{7.1}$$

with some global constants $a_1, a_2 > 0$ and const> 0. Since this bound holds for any $\mu_0 \in \mathcal{M}_0$, it holds, in particular, for the conditionally invariant measure μ_+, for which we have $(\mu_+)_n = T_+^n\mu_+ = \mu_+$. Now, we combine the above bound for $\mu_n = T_+^n\mu_0$ with that same bound for $\mu_+ = T_+^n\mu_+$ and use the triangle inequality. That concludes the proof of Theorem 2.3 for $n \geq n_0$. To enforce that theorem for all $n \geq 0$, it is enough to adjust the constant coefficient in (2.4). □

Remark. For each $k \geq k_0$ put $n = rk^2 + k + l_0$ in (7.1), and then combining it with the just proved Theorem 2.3 yields

$$\left|\int_M f(x)\, d\mu_+^{(k)} - \int_M f(x)\, d\mu_+\right| \leq \text{const} \cdot \left[\delta_f\left(e^{-a_1 k}\right) + ||f||_\infty e^{-a_2 k}\right] \tag{7.2}$$

Note that the measures $\mu_+^{(k)}$ here correspond to any rectangular holes $H^{(k)}$ that properly approximate H.

Let $\mu_0 \in \mathcal{M}_0$ and $\mu_n = T_+^n\mu_0$. Recall that $||T_*\mu_n|| = \mu_n(M_{-1})$.

Lemma 7.1 *There are global constants $C_{15}, a_5 > 0$ such that $\forall n \geq 0$*

$$\exp(-C_{15}e^{-a_5 n^{1/2}}) \leq \mu_n(M_{-1})/\lambda_+ \leq \exp(C_{15}e^{-a_5 n^{1/2}})$$

Proof. Let again $n \geq n_0$ and $k = k_n = \max\{k : n \geq rk^2 + k + l_0\}$. Then (6.8-6.10), and Proposition 6.3 allow us to compare the measures μ_n and $\mu_+ = (\mu_+)_n$ with the measure $\mu_+^{(k)}$ and imply that $|\mu_n - \mu_+|_k \leq Ce^{-ak}$ with some global constants $C, a > 0$. Therefore,

$$\left|\sum_{R\subset M_{-1}} \mu_n(R) - \sum_{R\subset M_{-1}} \mu_+(R)\right| \leq Ce^{-ak}$$

where the sums are taken over $R \in \mathcal{R}^{(k)}$.

Furthermore,

$$\sum_{R\cap\partial M_{-1}\neq\emptyset} \mu_n(R) \leq [T_*\mu_n](H_{\varepsilon'_k}) \leq C_{11}\left(\varepsilon'_k\right)^{b(1-4g)}$$

with $\varepsilon'_k = \varepsilon_k\Lambda_{\max}$ (observe that if $R \cap \partial M_{-1} \neq \emptyset$, then R lies in an ε_k-neighborhood of $\hat{T}^{-1}(\partial H)$, and so $\hat{T}R \subset H_{\varepsilon'_k}$). The second bound follows from (5.17), by setting $n = 0$ there. The above estimate also holds for μ_+. Hence,

$$|\mu_n(M_{-1}) - \mu_+(M_{-1})| \leq Ce^{-ak} + 2C_{11}\left(\varepsilon'_k\right)^{b(1-4g)}$$

for all $n \geq n_0$. Lemma 7.1 is then proven for $n \geq n_0$. To enforce it for all $n \geq 0$, it is enough to adjust the value of C_{15}. □

Now recall that $||T_*^n\mu_0|| = \mu_0(M_{-n}) = \prod_{i=0}^{n-1}\mu_i(M_{-1})$.

Corollary 7.2 *For any measure $\mu_0 \in \mathcal{M}_0$ there is a finite positive limit*

$$\lim_{n\to\infty} \mu_0(M_{-n})/\lambda_+^n := \rho(\mu_0) \tag{7.3}$$

Furthermore,

$$0 < C_{16}^{-1} \leq \rho(\mu_0) \leq C_{16} < \infty, \qquad \forall \mu_0 \in \mathcal{M}_0 \tag{7.4}$$

with some global constant $C_{16} > 0$. In addition, $\forall n \geq 0$

$$|\mu_0(M_{-n})/\lambda_+^n - \rho(\mu_0)| \leq \gamma_n' := C_{15}C_{16} \sum_{j=n}^{\infty} e^{-a_5 j^{1/2}} \tag{7.5}$$

Now, let $W^u \subset \hat{M}$ be an unstable fiber and ν its u-SBR measure. We will use the functions $l_n(x)$, $n \geq 0$, on W^u introduced before Lemma 5.7. In terms of these functions, the fiber W^u is said to be eventually long, cf. Section 3 in [6], if $l_n(x) \leq n$ for some $n \geq 0$ and some $x \in W^u \cap M_{-n}$, otherwise the fiber W^u is said to be forever short.

Theorem 7.3 *For every local unstable fiber $W^u \subset \hat{M}$ there is a finite limit*

$$\lim_{n\to\infty} \nu(W^u \cap M_{-n})/\lambda_+^n := \rho(W^u) \tag{7.6}$$

which is positive if and only if the fiber W^u is eventually long. In addition,

$$|\nu(W^u \cap M_{-n})/\lambda_+^n - \rho(W^u)| \leq \gamma_n''/|W^u| \tag{7.7}$$

where

$$\gamma_n'' := C_{17} \sum_{j=0}^{n} \gamma_{n-j}' \beta_1^j \, + \, C_{18} \sum_{j=n}^{\infty} \beta_1^j \leq C_{19} e^{-a_6 n^{1/2}} \tag{7.8}$$

with some global constants $a_6, C_{17}, C_{18}, C_{19} > 0$ and γ_n' defined in (7.5).

Remark. Due to Corollaries 5.2 and 5.4, we have $\nu(W^u \cap M_{-n})/\lambda_+^n \leq C_{20}|W^u|^{-g}$ for every $n \geq 0$, and so $\rho(W^u) \leq C_{20}|W^u|^{-g}$. Here $C_{20} = C_7 C_9$

Remark. If $W^u \subset M_+$, then $\rho(T^{-n}W^u) = \lambda_+^{-n}\rho(W^u)$ for all $n \geq 0$.

Proof. For fibers W^u of length between $d_0/2$ and d_0, we have $\nu \in \mathcal{M}_0$, thus the theorem follows from Corollary 7.2, and we actually get a slightly better estimate: $\gamma_n'' = d_0\gamma_n'$. Assume now that $|W^u| < d_0/2$. For every $j \geq 0$ let $\tilde{W}_j^u = \{x \in W^u : l_j(x) = j\}$. Observe that the sets $\tilde{W}_j^u$, $j \geq 0$, are pairwise disjoint, and for any $j \geq 0$ the set $T^j\tilde{W}_j^u$ is a finite union of unstable fibers of length $\geq d_0/2$, which we denote by $\tilde{W}_{j,s}^u$, $s = 1, 2, \ldots$. Denote by $\nu_j = T_*^j\nu$ the induced u-SBR measure on $T^j(W^u \cap M_{-j})$. Put $\kappa_n = \nu\{x \in W^u \cap M_{-n} : l_n(x) = n+1\}$. Now, for any $n \geq 0$

$$\begin{aligned} \nu(W^u \cap M_{-n}) &= \sum_{j=0}^{n} \nu_j \left(T^j\tilde{W}_j^u \cap M_{-n+j}\right) + \kappa_n \\ &= \sum_{j=0}^{n} \sum_{s} \nu_j \left(\tilde{W}_{j,s}^u \cap M_{-n+j}\right) + \kappa_n \end{aligned}$$

We will show that

$$\rho(W^u) = \sum_{j=0}^{\infty} \sum_s \nu_j(\tilde{W}^u_{j,s}) \rho(\tilde{W}^u_{j,s}) \lambda_+^{-j}$$

Taking this as the definition of $\rho(W^u)$ yields

$$\begin{aligned} |\nu(W^u \cap M_{-n})/\lambda_+^n - \rho(W^u)| \ \leq \ & \sum_{j=n+1}^{\infty} \sum_s \nu_j(\tilde{W}^u_{j,s}) \rho(\tilde{W}^u_{j,s}) \lambda_+^{-j} \ + \ \kappa_n \lambda_+^{-n} \\ & + \ \sum_{j=0}^{n} \sum_s \nu_j(\tilde{W}^u_{j,s}) \left| \nu_{\tilde{W}^u_{j,s}}(\tilde{W}^u_{j,s} \cap M_{-n+j})/\lambda_+^{n-j} \ - \ \rho(\tilde{W}^u_{j,s}) \right| \lambda_+^{-j} \end{aligned}$$

To each $\tilde{W}^u_{j,s}$, $0 \leq j \leq n$, we apply the already proven bound (7.7) with $\gamma''_n = d_0\gamma'_n$, as noted above. To each $\tilde{W}^u_{j,s}$, $j \geq n+1$, we use the bound $\rho(\tilde{W}^u_{j,s}) \leq C_{16}$, cf. (7.4). Then, for every $j \geq 0$ we have $\sum_s \nu_j(\tilde{W}^u_{j,s}) \leq C_{13}\beta_1^j\lambda_+^j/|W^u|$ according to Lemma 5.7 (setting $m = n = j$ there). We also have $\kappa_n \leq C_{13}\beta_1^n\lambda_+^n/|W^u|$, due to the same lemma (setting $m = n$ there). The rest is direct substitution. Theorem 7.3 is proven. □

Corollary 7.4 *Let W^u be an eventually long unstable fiber, and let μ_0 be the measure concentrated on W^u and coinciding with the u-SBR measure on that fiber. Then the sequence $T_+^n\mu_0$ weakly converges to μ_+.*

Proof of Theorems 2.4 and 2.5. For every unstable fiber W^u and $n \geq 0$ denote by $\nu_{W^u,n}$ its u-SBR measure ν_{W^u} conditioned on $W^u \cap M_{-n}$. Recall that the measure $\mu_{+,n} := T_*^{-n}\mu_+$ coincides with the measure μ_+ conditioned on M_{-n}, cf. Sect. 2.5. Hence, for every $n \geq 0$ the measure $\mu_{+,n}$ has conditional measure on $W^u \in \mathcal{W}^u_+$ which coincides with $\nu_{W^u,n}$. Denote by μ_+^f the factor measure on $\mathcal{W}^u_+$ induced by μ_+, and by $\mu_{+,n}^f$ the factor measure on $\mathcal{W}^u_+$ induced by $\mu_{+,n}$. It is an easy exercise that the measure $\mu_{+,n}^f$ is absolutely continuous with respect to μ_+^f, and

$$\frac{d\mu_{+,n}^f}{d\mu_+^f}(W^u) = \frac{\nu_{W^u}(W^u \cap M_{-n})}{\lambda_+^n} := \rho_n(W^u) \qquad \forall W^u \in \mathcal{W}^u_+ \tag{7.9}$$

Next, we will define the measure $\bar{\mu}_+$ by first introducing its conditional measures on $W^u \in \mathcal{W}^u_+$.

For every eventually long unstable fiber $W^u \in \mathcal{W}^u_+$ define a probability measure $\bar{\nu}_{W^u}$ on it by

$$\bar{\nu}_{W^u}(W_1^u) := \nu_{W^u}(W_1^u)\rho(W_1^u)/\rho(W^u) \tag{7.10}$$

for every subfiber $W_1^u \subset W^u$. Theorem 7.3 implies that $\bar{\nu}_{W^u}$ is a weak limit of $\nu_{W^u,n}$ as $n \to \infty$. In particular, $\bar{\nu}_{W^u}$ is a probability measure supported on $W^u \cap \Omega$. For forever short unstable fibers $W^u \in \mathcal{W}^u_+$, let $\bar{\nu}_{W^u}$ be an arbitrary probability measure on W^u.

Next, define a measure, $\bar{\mu}_+^f$, on $\mathcal{W}^u_+$ by

$$\frac{d\bar{\mu}_+^f}{d\mu_+^f}(W^u) = \rho(W^u) \tag{7.11}$$

Lastly, let $\bar{\mu}_+$ be a measure on M_+ whose conditional measures on $W^u \subset \mathcal{W}^u_+$ coincide with $\bar{\nu}_{W^u}$ and whose factor measure on $\mathcal{W}^u_+$ is $\bar{\mu}^f_+$. Clearly, $\bar{\mu}_+$ is supported on Ω.

First, we show that the measure $\bar{\mu}^f_+$, and hence, $\bar{\mu}_+$, are finite. We use the first remark after Theorem 7.3 and the function $F_{\mu_+}(y)$ defined in (5.8), and get a bound

$$\int_{\mathcal{W}^u_+} \rho(W^u)\, d\mu^f_+ \le C_{20} \int_0^{d_0} y^{-g}\, dF_{\mu_+}(y)$$

Integrating by parts and using (5.10) gives

$$\int_0^{d_0} y^{-g}\, dF_{\mu_+}(y) \le d_0^{-g} + gC_1 d_0^{1-g} < \infty \tag{7.12}$$

In the same way, $\forall \varepsilon > 0$

$$\int_{\mathcal{W}^u_{+,\varepsilon}} \rho(W^u)\, d\mu^f_+ \le C_{20} \int_0^{\varepsilon} y^{-g}\, dF_{\mu_+}(y) \le C_{20} C_1 \varepsilon^{1-g} \tag{7.13}$$

Observe that the above bound also holds if we substitute $\rho_n(W^u)$ for $\rho(W^u)$.

We now prove (2.5), which will imply the weak convergence of $\mu_{+,n}$ to $\bar{\mu}_+$. For every $n \ge 1$ we take $\varepsilon = \varepsilon_n = e^{-a_7 n^{1/2}}$ with $a_7 = a_6/2$. Then according to (7.8), $\gamma''_n/\varepsilon \le C_{19} e^{-a_7 n^{1/2}}$. We put, for brevity, $\mathcal{W}^{u,c}_{+,\varepsilon} = \mathcal{W}^u_+ \setminus \mathcal{W}^u_{+,\varepsilon}$. We have

$$\begin{aligned}
\left|\int_M f\, d\mu_{+,n} - \int_M f\, d\bar{\mu}_+\right| &= \left|\int_{\mathcal{W}^u_+} d\mu^f_{+,n} \int_{W^u} f\, d\nu_{W^u,n} - \int_{\mathcal{W}^u_+} d\bar{\mu}^f_+ \int_{W^u} f\, d\bar{\nu}_{W^u}\right| \\
&\le \left|\int_{\mathcal{W}^u_{+,\varepsilon}} d\mu^f_{+,n} \int_{W^u} f\, d\nu_{W^u,n}\right| + \left|\int_{\mathcal{W}^u_{+,\varepsilon}} d\bar{\mu}^f_+ \int_{W^u} f\, d\bar{\nu}_{W^u}\right| \\
&+ \left|\int_{\mathcal{W}^{u,c}_{+,\varepsilon}} d\mu^f_{+,n} \int_{W^u} f\, d\nu_{W^u,n} - \int_{\mathcal{W}^{u,c}_{+,\varepsilon}} d\bar{\mu}^f_+ \int_{W^u} f\, d\bar{\nu}_{W^u}\right|
\end{aligned} \tag{7.14}$$

The first two integrals on the right hand side of (7.14) do not exceed

$$2||f||_\infty C_{20} \int_0^{\varepsilon} y^{-g}\, dF_{\mu_+}(y) \le 2||f||_\infty C_{20} C_1 \varepsilon^{1/2}$$

according to (7.13).

Using (7.9) and (7.11) we see that the last term on the right hand side of (7.14) does not exceed $I_1 + I_2$, where

$$I_1 = \int_{\mathcal{W}^{u,c}_{+,\varepsilon}} \left|\int_{W^u} f(x)\, d\nu_{W^u,n} - \int_{W^u} f(x)\, d\bar{\nu}_{W^u}\right| \rho(W^u)\, d\mu^f_+$$

and

$$I_2 = \int_{\mathcal{W}^{u,c}_{+,\varepsilon}} \left|\int_{W^u} f(x)\, d\nu_{W^u,n}\right| \cdot |\rho_n(W^u) - \rho(W^u)|\, d\mu^f_+$$

The bound (7.7) immediately implies that $I_2 \le ||f||_\infty \gamma_n''/\varepsilon$.

We now bound I_1. Observe that $\rho(W^u) = 0$ for forever short fibers, so they can be ignored. Let $W^u \in \mathcal{W}^{u,c}_{+,\varepsilon}$ be an eventually long fiber, $|W^u| > \varepsilon$. Partition it into some subfibers W_i^u, $i \ge 1$, of length between $\varepsilon/2$ and ε. On each W_i^u, pick a point $x_i \in W_i^u$. Then for any probability measure ν on W^u we have

$$\left| \int_{W^u} f(x)\, d\nu \; - \; \sum_i f(x_i) \nu(W_i) \right| \le \delta_f(\varepsilon)$$

Therefore,

$$\begin{aligned} \left| \int_{W^u} f\, d\nu_{W^u,n} - \int_{W^u} f\, d\bar{\nu}_{W^u} \right| &\le \left| \sum_i f(x_i) \left(\nu_{W^u,n}(W_i^u) - \bar{\nu}_{W^u}(W_i^u) \right) \right| + 2\delta_f(\varepsilon) \\ &\le ||f||_\infty \sum_i |\nu_{W^u,n}(W_i^u) - \bar{\nu}_{W^u}(W_i^u)| + 2\delta_f(\varepsilon) \quad (7.15) \end{aligned}$$

For each subfiber W_i^u, we have

$$\nu_{W^u,n}(W_i^u) = \frac{\nu_{W^u}(W_i^u)\, \nu_{W_i^u}(W_i^u \cap M_{-n})}{\nu_{W^u}(W^u \cap M_{-n})}$$

Using (7.9) and (7.10) yield

$$\begin{aligned} |\nu_{W^u,n}(W_i^u) - \bar{\nu}_{W^u}(W_i^u)| &= \left| \nu_{W^u}(W_i^u) \frac{\rho_n(W_i^u)}{\rho_n(W^u)} \; - \; \nu_{W^u}(W_i^u) \frac{\rho(W_i^u)}{\rho(W^u)} \right| \\ &= \nu_{W^u}(W_i^u) \left| \frac{\rho_n(W_i^u)\rho(W^u) - \rho_n(W^u)\rho(W_i^u)}{\rho_n(W^u)\rho(W^u)} \right| \end{aligned}$$

Now, using (7.7)

$$\begin{aligned} |\rho_n(W_i^u)\rho(W^u) - \rho_n(W^u)\rho(W_i^u)| &= |\rho_n(W_i^u)(\rho(W^u) - \rho_n(W^u)) + \rho_n(W^u)(\rho_n(W_i^u) - \rho(W_i^u))| \\ &\le \rho_n(W_i^u)\gamma_n''/|W^u| + 2\rho_n(W^u)\gamma_n''/\varepsilon \end{aligned}$$

Hence,

$$|\nu_{W^u,n}(W_i^u) - \bar{\nu}_{W^u}(W_i^u)| \le \nu_{W^u}(W_i^u) \cdot \frac{\rho_n(W_i^u)\gamma_n''}{\rho_n(W^u)\rho(W^u)\varepsilon} \; + \; \nu_{W^u}(W_i^u) \cdot \frac{2\gamma_n''}{\varepsilon \rho(W^u)}$$

Note that

$$\sum_i \nu_{W^u}(W_i^u) = \nu_{W^u}(W^u) = 1$$

and

$$\sum_i \nu_{W^u}(W_i^u)\rho_n(W_i^u)/\rho_n(W^u) = \nu_{W^u,n}(W^u) = 1$$

Therefore,

$$\sum_i |\nu_{W^u,n}(W_i^u) - \bar{\nu}_{W^u}(W_i^u)| \le \frac{3\gamma_n''}{\varepsilon\rho(W^u)}$$

Combining this with (7.15) and substituting in the integral formula for I_1 give the bound

$$\begin{aligned} I_1 &\le 3||f||_\infty \gamma_n''/\varepsilon + 2\delta_f(\varepsilon)\int_{\mathcal{W}^{u,c}_{+,\varepsilon}} \rho(W^u)\, d\mu_+^f \\ &\le 3||f||_\infty \gamma_n''/\varepsilon + 2\delta_f(\varepsilon)\left(d_0^{-g} + gC_1 d_0^{1-g}\right) \end{aligned}$$

where we used (7.12).

The proof of Theorem 2.5 is completed. This implies the weak convergence of the measures $\mu_{+,n}$ to $\bar{\mu}_+$. In particular, $\bar{\mu}_+$, and hence, $\bar{\mu}_+^f$, are probability measures.

We now continue the proof of Theorem 2.4. The invariance of the measure $\bar{\mu}_+$ follows from the above weak convergence.

To prove the ergodicity of $\bar{\mu}_+$ we employ the classical Hopf technique of connecting any two generic points of Ω by a finite chain of stable and unstable fibers, see, e.g., [2] for a self contained introduction to the Hopf technique in Hopf's original setting. The crucial step in this technique is to ensure some analogue of the absolute continuity of the foliations of Ω by stable and unstable fibers. Precisely, we need to show that for any two nearby parallel unstable fibers $W_1^u, W_2^u \subset M_+$ the holonomy map (sliding along stable fibers) is not completely singular with respect to the conditional measures $\bar{\nu}_{W_1^u}, \bar{\nu}_{W_2^u}$.

Let $W_1^u, W_2^u \subset M_+$ be two unstable fibers of lengths $\ge d_0/4$, and let ν_1, ν_2 be their u-SBR measures, respectively. Assume that W_1^u is d_0-close to W_2^u, see Sect. 3. By shortening W_2^u, if necessary, we can make it d_0-close to W_1^u, too. In particular, the holonomy map $\tilde{h}$ (sliding along stable fibers of length $\le d_0$) will be then a bijection of W_1^u to W_2^u. For any $n \ge 0$ let $\tilde{W}_{1,n}^u = \{x \in W_1^u \cap M_{-n} : \tilde{h}(x) \in W_2^u \cap M_{-n}\}$, and $\tilde{W}_{2,n}^u := \tilde{h}(\tilde{W}_{1,n}^u) \subset W_2^u \cap M_{-n}$. The sets $\tilde{W}_{1,n}^u$ and $\tilde{W}_{2,n}^u$ are finite unions of closed subfibers in W_1^u, W_2^u, respectively, and the jacobian of $\tilde{h} : \tilde{W}_{1,n}^u \to \tilde{W}_{2,n}^u$ is uniformly bounded away from zero and infinity (by D^{-1} and D). The remark in the end of Section 4 shows that $\nu_i(\tilde{W}_{i,n}^u) \ge \nu_i(W_i^u \cap M_{-n})/2$, i.e.

$$\nu_{i,n}(\tilde{W}_{i,n}^u) = \nu_i(\tilde{W}_{i,n}^u / M_{-n}) \ge 1/2 \tag{7.16}$$

for $i = 1, 2$ and all $n \ge 0$.

Claim. The holonomy map $\tilde{h} : W_1^u \to W_2^u$ is not singular with respect to the measures $\bar{\nu}_{W_i^u}$, $i = 1, 2$.

Proof. By way of contradiction, assume that $\exists A \subset W_1^u$ such that $\bar{\nu}_{W_1^u}(A) = 0$ and $\bar{\nu}_{W_2^u}(\tilde{h}(A)) = 1$. Then $\forall \delta > 0$ there is a countable union of disjoint open subintervals $I_i \subset W_1^u$ such that $A \subset \cup I_i$ and $\bar{\nu}_{W_1^u}(\cup I_i) < \delta$. Since $\bar{\nu}_{W_2^u}(\tilde{h}(\cup I_i)) = 1$, we can find a finite subunion $G_j = \cup_{i=1}^j I_i$ for some $j < \infty$ such that $\bar{\nu}_{W_2^u}(\tilde{h}(G_j)) \ge 0.99$. Recall that $\nu_{i,n}$ weakly converge, as $n \to \infty$, to $\bar{\nu}_{W_i^u}$ for $i = 1, 2$. Since $\tilde{h}(G_j)$ is an open subset of

W_2^u, there is an $n_0 \geq 1$ such that $\nu_{2,n}(\tilde{h}(G_j)) \geq 0.98$ for all $n \geq n_0$. Due to (7.16), we have

$$\nu_{2,n}(\tilde{h}(G_j \cap \tilde{W}_{1,n}^u)) = \nu_{2,n}(\tilde{h}(G_j) \cap \tilde{W}_{2,n}^u) \geq 0.48$$

for all $n \geq n_0$. Hence, $\nu_{1,n}(G_j \cap \tilde{W}_{1,n}^u) \geq 0.48\,(C_6 D)^{-1} > 0$, where we used Theorem 4.1. Hence, $\nu_{1,n}(G_j) \geq 0.48\,(C_6 D)^{-1}$ for all $n \geq n_0$. Consider the closure $\bar{G}_j$ of G_j. The weak convergence $\nu_{1,n} \to \bar{\nu}_{W_1^u}$ implies that $\bar{\nu}_{W_1^u}(\bar{G}_j) \geq 0.48\,(C_6 D)^{-1}$. Note that $\bar{\nu}_{W_1^u}$ is a nonatomic measure, cf. (7.10) and the first remark after Theorem 7.3. Hence, $\bar{\nu}_{W_1^u}(G_j) \geq 0.48\,(C_6 D)^{-1}$, a contradiction that proves the claim.

This allows us to link W_1^u and W_2^u by stable fibers and make one Hopf chain that contains both W_1^u and W_2^u.

Next, the unstable fibers $W^u \subset M_+$ of length $\geq d_0/4$ are rather dense in $\hat{M}$. As it was shown in [6] (Lemma 3.4), any stable fiber $W^s \subset \hat{M}$ of length d_2 (with $d_2 \sim h$) crosses at least one unstable fiber $W^u \subset M_+$ whose endpoints are the distance $\geq d_0/3$ away from the point of intersection, $W^s \cap W^u$. Therefore, any two unstable fibers $W_1^u, W_2^u \subset M_+$ of length $\geq d_0/4$ can be connected by one finite Hopf chain of stable and unstable fibers, so they all belong (mod 0) in one ergodic component of the measure $\bar{\mu}_+$.

Consider now short unstable fibers $W^u \subset M_+$. If W^u is forever short then $\rho(W^u) = 0$, and hence the union of such fibers has zero $\bar{\mu}_+$ measure. If W^u is eventually long, then, in the same way, only its parts that become long under the iterations of T can carry positive mass of $\bar{\mu}_+$. Those parts, however, belong (mod 0) to the same ergodic component as above, since that component is T-invariant. The ergodicity of $\bar{\mu}_+$ is proved.

It is standard that the K-property on each ergodic component follows from the existence of stable and unstable fibers at a.e. point, see, e.g., [10] and [11] (Theorem B). Since our measure $\bar{\mu}_+$ is ergodic, it is K-mixing. □

Remark. Due to (5.17),

$$\mu_{+,n}(H_\varepsilon) \leq C_{11}\varepsilon^{b/2} \qquad \forall \varepsilon > 0 \text{ and } n \geq 0 \tag{7.17}$$

Hence, $\bar{\mu}_+(H_\varepsilon) \leq C_{11}\varepsilon^{b/2}$. It is then a standard application of Borel-Cantelli lemma to show that at $\bar{\mu}_+$-a.e. point $x \in \Omega$ there are fibers $W_x^u \subset M_+$ and $W_x^s \subset M_-$ of nonzero length. Furthermore, for any $\varepsilon > 0$ the set of points whose fibers $W_x^u \subset M_+$ and $W_x^s \subset M_-$ are shorter than ε has $\bar{\mu}_+$-measure $\leq \text{const} \cdot \varepsilon^{b/2}$.

Next, the estimates on convergence in Theorem 2.5 are independent of h and the shape of the holes. Therefore, the weak convergence of the measures $\mu_{+,n}^{(k)} := [T^{(k)}]_*^{-n}\mu_+^{(k)}$ to $\bar{\mu}_+^{(k)}$, proved in [4],[5] is actually *uniform in* k for all $k \geq k_0$. Hence, for any sequence $n_k \to \infty$ as $k \to \infty$ the sequences of measures $\{\mu_{+,n_k}^{(k)}\}$ is equivalent to $\bar{\mu}_+^{(k)}$, i.e. $\mu_{+,n_k}^{(k)} \sim \bar{\mu}_+^{(k)}$ as defined in the previous section.

Recall that the sequence $\mu_+^{(k)}$ converges, as $k \to \infty$, to μ_+, cf. [6]. Since T^{-1} acts smoothly on M_1, the measures $\mu_{+,n}^{(k)} = T_*^{-n}\mu_+^{(k)}$ remain close to $\mu_{+,n} = T_*^{-n}\mu_+$ in the weak topology for sufficiently small n (compared to k). It is, however, clear, that n may be taken larger as k grows, i.e. there is a sequence $n_k \to \infty$ such that $\mu^{(k)}_{+,n_k} \sim \mu_{+,n_k}$ as $k \to \infty$.

Combining the above observations with Theorem 2.4 gives

Corollary 7.5 *The sequence of measures $\bar{\mu}^{(k)}_+$ supported on the repellers $\Omega^{(k)} \subset \Omega$ weakly converges, as $k \to \infty$, to the measure $\bar{\mu}_+$ supported on Ω. Note that all these measures are $\hat{T}$-invariant, ergodic and K-mixing.*

Proof of Theorem 2.6. We start with a formula for the Kolmogorov-Sinai entropy of smooth hyperbolic maps on surfaces:

$$h(\mu) = \chi_+(\mu) \cdot \lim_{\varepsilon \to 0} \frac{\log \mu^u(B^u(x,\varepsilon))}{\log \varepsilon} \tag{7.18}$$

where μ is an ergodic invariant measure, $\chi_+(\mu)$ is its positive Lyapunov exponent, x is a μ-generic point, μ^u is the conditional measure induced by μ on the local unstable fiber W^u_x, and $B^u(x,\varepsilon)$ is the ε-ball in W^u_x centered at x (i.e., the interval on W^u_x of length 2ε centered at x). This formula was proved by Ledrappier and Young, even in the nonuniformly hyperbolic case, see pp. 545, 559 in Part II of [11].

For any $\bar{\mu}_+$-generic point $x \in \Omega$ there is a sequence $n_i \to \infty$ such that the points $x_i := T^{n_i}x$ have long unstable fibers in M_+, i.e. the distance from x_i to the endpoints of $W^u_{x_i} \subset M_+$ is at least $d_0/8$. For every sufficiently large i we find an $\varepsilon_i > 0$ such that $W^u_i := T^{n_i}B^u(x,\varepsilon_i)$ is a subsegment of $W^u_{x_i}$ of length $\geq d/8$. Due to the standard distortion estimates,

$$(8D)^{-1} \leq \frac{d_0/\varepsilon_i}{J^u_x J^u_{Tx} \cdots J^u_{T^{n_i-1}x}} \leq 8D$$

Next, let $W^u \subset M_+$ be the maximal unstable fiber containing x. Then

$$\begin{aligned}\bar{\nu}_{W^u}(B^u(x,\varepsilon_i)) &= \nu_{W^u}(B^u(x,\varepsilon_i))\, \rho(B^u(x,\varepsilon_i))/\rho(W^u)\\ &= \nu_{W^u}(B^u(x,\varepsilon_i))\, \lambda_+^{-n_i}\rho(T^{n_i}B^u(x,\varepsilon_i))/\rho(W^u)\end{aligned}$$

where we used (7.10) and the second remark after Theorem 7.3. Observe that $\nu_{W^u}(B^u(x,\varepsilon_i)) \sim \varepsilon_i/|W^u|$, and $\rho(T^{n_i}B^u(x,\varepsilon_i)) \sim 1$. Therefore,

$$\begin{aligned}h(\bar{\mu}_+) &= \chi_+ \cdot \lim_{i\to\infty} \frac{\log \varepsilon_i - \log \lambda_+^{n_i}}{\log \varepsilon_i}\\ &= \chi_+ \cdot \left(1 - \lim_{i\to\infty} \frac{\log \lambda_+}{n_i^{-1}\log \varepsilon_i}\right)\\ &= \chi_+ \cdot \left(1 - \lim_{i\to\infty} \frac{\gamma_+}{n_i^{-1}\log J^u_x \cdots J^u_{T^{n_i-1}x}}\right)\\ &= \chi_+ \cdot \left(1 - \frac{\gamma_+}{\chi_+}\right)\end{aligned}$$

Theorem 2.6 is proved. □

Corollary 7.6 (see [13]) *Let $G_{\bar{\mu}_+} \subset \Omega$ be the set of $\bar{\mu}_+$-generic points $x \in \Omega$. Then the Hausdorff dimension, denoted by δ^u, of $G_{\bar{\mu}_+} \cap W^u(x)$ is independent of $x \in \Omega$, and*

$$\delta^u = h(\bar{\mu}_+)/\chi_+ = \lim_{\varepsilon \to 0} \frac{\log \mu^u(B^u(x,\varepsilon))}{\log \varepsilon}$$

Proof of Theorem 2.7. Let now, for a moment, $H^{(k)}$ be rectangular holes approximating H 'from outside' as described in the first paragraph of Section 6. They satisfy $d(H^{(k)}, H) \le \varepsilon_k/2$, see (6.1). Hence, $d(H^{(k)}, H_n) \le d(H_n, H) + \varepsilon_k/2$. Let $k = k_n = \max\{k \ge k_0 : \varepsilon_k \ge 2d(H_n, H)\}$. Obviously, $k_n \to \infty$ as $n \to \infty$. Then $d(H^{(k_n)}, H_n) \le \varepsilon_{k_n}$, so the rectangular holes $H^{(k_n)}$ properly approximate H_n. Hence, we can apply (7.2) with μ_+ replaced by $\mu_+[H_n]$ and $k = k_n$. This proves the weak convergence $\mu_+[H_n] \to \mu_+$. The convergence $\bar{\mu}_+[H_n] \to \bar{\mu}_+$ is proved exactly as Corollary 7.5.

Now, $\lambda_+[H_n] = 1 - \mu_+[H_n](\hat{T}^{-1}H_n)$. The weak convergence $\mu_+[H_n] \to \mu_+$, the assumption $d(H_n, H) \to 0$, and the bound (7.17), in which one sets $n = 0$, imply that $\lambda_+[H_n] \to \lambda_+$, and hence $\gamma_+[H_n] \to \gamma_+$. The escape rate formula holds for all measures $\bar{\mu}_+[H_n]$ and implies the convergence of the entropies $h(\bar{\mu}_+[H_n]) \to h(\bar{\mu}_+)$. Observe that, generally, the entropy $h(\mu)$ in not (!) a continuous function of the invariant measure μ for T. $\square$

Proof of Theorem 2.9. That theorem was proved in [4] for Anosov diffeomorphisms with rectangular holes. We can apply it to $T^{(k)}$, because $\Omega^{(k)} \subset \Omega$, so the necessary assumptions are fulfilled on $\Omega^{(k)}$. Thus, we get $\bar{\mu}_+^{(k)} = \bar{\mu}_-^{(k)}$ and $\lambda_+^{(k)} = \lambda_-^{(k)}$ for all $k \ge k_0$. Taking the limit as $k \to \infty$ proves Theorem 2.9. $\square$

8 Proof of Proposition 6.3

The matrix techniques for proving the weak convergence of measures is based on the following. Let $\xi^{(1)} < \xi^{(2)} < \cdots$ be an increasing sequence of finite partitions of the underlying space, $\xi^{(k)} = \{A_1^{(k)}, \ldots, A_{m_k}^{(k)}\}$, that converges to a partition into single points. Then one can represent any measure μ by a sequence of (row) vectors $p^{(k)}$ with components $p_i^{(k)} = \mu(A_i^{(k)})$, $1 \le i \le m_k$. The norm of the measure is given by $|\mu| = |p^{(k)}| = \sum_i p_i^{(k)}$. Then, under certain regularity conditions, the weak convergence of a sequence of measures, $\mu_n \to \mu_\infty$, is equivalent to the componentwise convergence of the sequence of vectors $p^{(k)}(\mu_n) \to p^{(k)}(\mu_\infty)$, as $n \to \infty$, for all $k \ge 1$. Similarly, the transformation of measures $\mu' = T_*\mu$ is equivalent to the right multiplication by matrices, $p^{(k)}(\mu') = p^{(k)}(\mu)\Pi^{(k)}(\mu)$, with components

$$\Pi_{ij}^{(k)}(\mu) = \mu\left(A_i^{(k)} \cap T^{-1}A_j^{(k)}\right)/\mu(A_i^{(k)})$$

which is a nonnegative substochastic matrix for every k. More details of our techniques can be found in [4, 5, 16].

Fix a $k \geq k_0$. We represent any measure μ on $M^{(k)}$ by the row vector $p(\mu)$ with components $\{\mu(R) : R \in \mathcal{R}^{(k)}\}$. We always assume, for simplicity, that $\mu(\cup_R \partial R) = 0$, so that $|\mu| = |p(\mu)|$.

For $n \geq q$ denote by $p_n = \{p_n(R)\}$ the vector representing the measure $[T_*^{(k)}]^{n-q}\mu_q$, $q = k + l_0$ (this is not a probability measure for $n > q$, of course). The normalized vector, $p_n/|p_n|$, will then represent the measure $\mu_n^{(k)}$.

Recall that the conditional distributions of μ_n on unstable R-fibers, $R \in \mathcal{R}^{(k)}$, coincide with u-SRB measures on those fibers. Then the sequence of vectors p_n can be well approximated by the product of the vector p_{n_0} (representing μ_{n_0}) for some $n_0 \geq q$ and certain substochastic matrices defined as follows. In every rectangle $R \in \mathcal{R}^{(k)}$ pick an arbitrary unstable R-fiber $U(R)$. For any $m \geq 1$ define the substochastic matrix Π_m with components

$$\Pi_m(R', R'') = \nu_{U(R')}\left(U(R') \cap [T^{(k)}]^{-m}(R'' \cap M_m^{(k)})\right) \tag{8.1}$$

where $\nu_{U(R')}$ is the u-SBR measure on $U(R')$.

Since the rectangles in $\mathcal{R}^{(k)}$ are exponentially small (in k), then for any $n \geq m + q$ and R''

$$e^{-C\alpha^k} \leq \frac{\sum_{R'} p_{n-m}(R')\,\Pi_m(R', R'')}{p_n(R'')} \leq e^{C\alpha^k}$$

and for any R', R'' and $m_1, m_2 \geq 1$

$$e^{-C\alpha^k} \leq \frac{\sum_{R'''} \Pi_{m_1}(R', R''')\,\Pi_{m_2}(R''', R'')}{\Pi_{m_1+m_2}(R', R'')} \leq e^{C\alpha^k}$$

for some global constants $C > 0$ and $\alpha \in (0, 1)$. Let $\bar{m} = m_1 + \cdots + m_t \leq n - q$ and put

$$\hat{p}_n = p_{n-\bar{m}}\Pi_{m_1} \cdots \Pi_{m_t}$$

Then, for any $R \in \mathcal{R}^{(k)}$

$$e^{-Ct\alpha^k} \leq \frac{\hat{p}_n(R)}{p_n(R)} \leq e^{Ct\alpha^k}$$

As a result, $e^{-Ct\alpha^k} \leq |\hat{p}_n|/|p_n| \leq e^{Ct\alpha^k}$, so that the normalized vectors $\hat{p}_n/|\hat{p}_n|$ and $p_n/|p_n|$ are close as well:

$$e^{-2Ct\alpha^k} \leq \frac{\hat{p}_n(R)/|\hat{p}_n|}{p_n(R)/|p_n|} \leq e^{2Ct\alpha^k} \tag{8.2}$$

According to (8.2), the vectors $\hat{p}_n/|\hat{p}_n|$ approximately represent the measure $\mu_n^{(k)}$ as long as $t \ll \alpha^{-k}$. In our further considerations, t will be actually equal to k, so $\hat{p}_n$ will always approximate p_n well enough. In fact, we will find an integer $r \geq 1$ such that such that the matrix Π_{rk} has good mixing properties uniformly in k, see below. Then for any $n \geq rk^2 + q$ we approximate $\mu_n^{(k)}$ by

$$\hat{p}_n := p_{n-rk^2}\Pi_{rk}^k \tag{8.3}$$

Due to (8.2), for any $n \geq rk^2 + q$

$$\sum_{R \in \mathcal{R}^{(k)}} \left| \frac{p_n(R)}{|p_n|} - \frac{\hat{p}_n(R)}{|\hat{p}_n|} \right| \leq e^{2Ck\alpha^k} - 1 = O(k\alpha^k) \tag{8.4}$$

Remark. The vector $p_n/|p_n|$ represents the measure $\mu_n^{(k)}$. According to (8.4), the vector $\hat{p}_n/|\hat{p}_n|$ approximately represents the same measure. The vector $\hat{p}_n/|\hat{p}_n|$ will not change if we normalize the vector p_{n-rk^2} in (8.3), i.e. if we assume that p_{n-rk^2} represents the measure $\mu_{n-rk^2}^{(k)}$ rather than $[T_*^{(k)}]^{n-rk^2-q}\mu_q$.

We will need certain matrix estimates similar to those in [16]. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ nonnegative $N \times N$ matrices and X, Y nonnegative row vectors of length N with the norm $|X| = \sum x_i$, the distance $|X - Y| = \sum |x_i - y_i|$, and the normed distance

$$||X - Y|| = \sum_{i=1}^{N} \left| \frac{x_i}{|X|} - \frac{y_i}{|Y|} \right|$$

Let $\tilde{X} = X + X'$ and $\tilde{Y} = Y + Y'$, where $\tilde{X}, X', \tilde{Y}, Y'$ are some nonnegative vectors. Denote $\varepsilon_X = |X'|/|\tilde{X}|$ and $\varepsilon_Y = |Y'|/|\tilde{Y}|$. The following estimate is a result of direct calculations:

$$||\tilde{X} - \tilde{Y}|| \leq ||X - Y|| + 2(\varepsilon_X + \varepsilon_Y) \tag{8.5}$$

Let $1 \leq J \leq N$. Assume that $x_j > 0$ iff $j \leq J$ and $y_j > 0$ iff $j \leq J$. Let

$$d(X, Y) = \ln \left(\frac{\max_{1 \leq i \leq J}(x_i/y_i)}{\min_{1 \leq i \leq J}(x_i/y_i)} \right) = \max_{1 \leq i,j \leq J} \ln \left(\frac{x_i y_j}{x_j y_i} \right)$$

be the projective distance between vectors, based on their first J components. Obviously, $x_i/x_j \leq e^{d(X,Y)} y_i/y_j$ for all $1 \leq i, j \leq J$. Summing over $i = 1, \ldots, J$ gives $y_j/|Y| \leq e^{d(X,Y)} x_j/|X|$, hence

$$||X - Y|| \leq e^{d(X,Y)} - 1 \tag{8.6}$$

Note that $d(X, Y) = 0$ (hence, $||X - Y|| = 0$) iff $X = \lambda Y$ for some $\lambda > 0$.

Now, let $1 \leq I \leq J$. Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are nonnegative matrices. For the matrix $\mathcal{B}$, assume that

(B) $b_{ij} > 0$ iff $i \in \{1, \ldots, I\} \cup I'$ and $j \in \{1, \ldots, J\}$,

where $I' \subset \{J + 1, \ldots, N\}$ is an arbitrary subset of indices. Denote by $\mathcal{B}_n^*$ the $J \times J$ principal minor of the matrix $\mathcal{B}^n$, $n \geq 1$. Observe that $\mathcal{B}_n^* = (\mathcal{B}_1^*)^n$.

For any nonnegative vector $\tilde{X} = \{\tilde{x}_i\}$ we put $\tilde{X}_n = \tilde{X}\mathcal{A}^n$, $X_n = \tilde{X}\mathcal{B}^n$ and $X_n' = \tilde{X}_n - X_n$. We say that $\tilde{X}$ is admissible if $\tilde{x}_i > 0$ for at least one $i \in \{1, \ldots, I\} \cup I'$. In that case $x_j^{(1)} > 0$, $\forall j \leq J$, where $x_j^{(1)}$ are the components of $X_1 = \tilde{X}\mathcal{B}$. Therefore, for any two admissible vectors $\tilde{X}$, $\tilde{Y}$ we have $d(X_n, Y_n) < \infty$, $\forall n \geq 1$.

It is known that $d(X_n, Y_n) \leq d(X_{n-1}, Y_{n-1})$, cf. [16]. Let

$$\tau(\mathcal{B}) = \sup_{X \neq \lambda Y} \frac{d(X\mathcal{B}, Y\mathcal{B})}{d(X, Y)}$$

where the supremum is taken over vectors X, Y whose components x_j, y_j are positive if and only if $j \le J$. $\tau(\mathcal{B})$ is called Birkhoff contraction coefficient. It is known that $\tau(\mathcal{B}^n) \le [\tau(\mathcal{B})]^n$, cf. [16]. There is an explicit formula for $\tau(\mathcal{B})$, see [16], pp. 101–106:

$$\tau(\mathcal{B}) = \frac{1-\varphi^{1/2}}{1+\varphi^{1/2}}, \qquad \varphi = \varphi(\mathcal{B}) = \min_{i,j\le I,\ k,l\le J} \frac{b_{ik}b_{jl}}{b_{jk}b_{il}}$$

In particular, $\tau(\mathcal{B}) = 0$ iff the $J \times J$ principal minor of $\mathcal{B}$ coincides with that of the matrix $U^T V$, where U and V are some nonnegative row vectors such that $u_i > 0$ iff $i \le I$ and $v_j > 0$ iff $j \le J$, and U^T is the transpose of U.

Assume that $\exists \beta_0 > 0$ such that

$$0 < \beta_0 \le \frac{b_{ik}b_{jl}}{b_{jk}b_{il}} \le \beta_0^{-1} \tag{8.7}$$

for all $i, j \le I$ and $k, l \le J$. Then $\tau(\mathcal{B}) \le \beta := (1-\sqrt{\beta_0})/(1+\sqrt{\beta_0}) < 1$, and hence

$$d(X_n, Y_n) \le d(X_2, Y_2) \cdot \beta^{n-2}$$

It is also a direct calculation that

$$d(X_2, Y_2) \le d(X_1\mathcal{B}, Y_1\mathcal{B}) \le 2\ln\beta_0^{-1}$$

for any two admissible vectors $\tilde{X}$, $\tilde{Y}$. Summarizing (8.5), (8.6) and (8.7) gives

$$\begin{aligned} ||\tilde{X}_n - \tilde{Y}_n|| &\le \left(\beta_0^{-2\beta^{n-2}} - 1\right) + 2(\varepsilon_{X_n} + \varepsilon_{Y_n}) \\ &\le \mathrm{const}(\beta_0)\cdot\beta^n + 2(\varepsilon_{X_n} + \varepsilon_{Y_n}) \end{aligned} \tag{8.8}$$

To prove (8.7), it is enough to show that there exist two nonnegative vectors U and V such that $u_i > 0$ iff $i \le I$ and $v_j > 0$ iff $j \le J$ and the components b_{ij} of the matrix $\mathcal{B}$ admit the decoupling $b_{ij} \sim u_i v_j$, i.e. for some $\gamma_0 > 0$

$$0 < \gamma_0 < \frac{b_{ij}}{u_i v_j} \le \gamma_0^{-1} \tag{8.9}$$

for all $i \le I$, $j \le J$. Then (8.7) follows with $\beta_0 = \gamma_0^4$.

Next, we will define the matrices $\mathcal{A}$ and $\mathcal{B}$ related to the transition matrix Π_m in (8.1) and show (8.9). For brevity, we will suppress the superscript (k) in $\mathcal{R}^{(k)}$, $T^{(k)}$, and $M^{(k)}$.

Let $R \in \mathcal{R}$ and U be an unstable R-fiber. Consider the functions $l_n(x)$, $n \ge 0$, on the fiber U as defined before Lemma 5.7. Let S be a stable R-fiber. In a similar way, define functions $l_n^-(x)$, $n \ge 0$, on S. For every $n \ge 0$ and $x \in S \cap M_n$ denote by $S_n(x)$ the smooth component of $T^{-n}(S \cap M_n)$ containing $T^{-n}x$. For every $x \in S \cap M_{-n}$ let $l_n^-(x) = \min\{l \in [0, n] : |S_l(x)| \ge d_0/2\}$ (if $|S_l(x)| < d_0/2$ for all $l = 0, \dots, n$, we set $l_n^-(x) = n + 1$).

Let $r > r_0 > 1$ to be specified below. We set $\mathcal{A} = \Pi_{rk}$, see (8.1), and will now define $\mathcal{B}$ so that $\mathcal{C} = \mathcal{A} - \mathcal{B}$ will be a nonnegative matrix. For any two rectangles $R', R'' \in \mathcal{R}$ we define $\mathcal{B}(R', R'')$ as follows. Let $U(R')$ be the unstable R'-fiber fixed in (8.1). Let $\tilde{U}(R') = \{x \in U(R') \cap M_{-rk} : l_{rk}(x) \le r_0 k\}$. Now, let $S(R'')$ be an arbitrary stable R''-fiber. Let $\tilde{S}(R'') = \{x \in S(R'') \cap M_{rk} : l^-_{rk}(x) \le r_0 k\}$. Now, let $\tilde{R}'' \subset R''$ be the union of unstable R''-fibers that cross $\tilde{S}(R'')$. Obviously, $\tilde{R}''$ is a finite union of u-subrectangles in R''. Observe that $T^{-rk}\tilde{R}'' \cap R'$ consists of a finite number of s-subrectangles in R'. We now define

$$\mathcal{B}(R', R'') = \nu_{U(R')}[\tilde{U}(R') \cap T^{-rk}(\tilde{R}'')] \tag{8.10}$$

Comparing this to (8.1) shows that $\mathcal{B}(R', R'') \le \Pi_{rk}(R', R'') = \mathcal{A}(R', R'')$, so that the matrix $\mathcal{C} = \mathcal{A} - \mathcal{B}$ is, indeed, nonnegative.

We now prove (8.9). Let again $R', R'' \in \mathcal{R}$. Denote by $\tilde{U}_i$, $i \ge 1$, all distinct curves $U_{l_{rk}(x)}(x)$ for points $x \in \tilde{U}(R')$. If $\tilde{U}_i = U_{l_{rk}(x)}(x)$ for some x, we put $l_i = l_{rk}(x)$, the iteration associated with $\tilde{U}_i$. There are finitely many of those curves, and their lengths are $\ge d_0/2$ and $\le d_0$. Observe that $T^{-l_i}\tilde{U}_i$, $i \ge 1$, are disjoint subsegments of $U(R')$, and their union covers $\tilde{U}(R')$.

Similarly, we define stable fibers $\tilde{S}_j$, $j \ge 1$, and $l^-_j \le r_0 k$ such that $T^{l^-_j}\tilde{S}_j$ are disjoint subsegments of $S(R'')$, their union covers $\tilde{S}(R'')$, and the length of every $\tilde{S}_j$ is between $d_0/2$ and d_0. Denote by $\tilde{R}''_j$ the u-subrectangle in $T^{-l^-_j}R''$ that consists of unstable fibers crossing the curve $\tilde{S}_j$. Let d^u_j be the maximum length of unstable $\tilde{R}''_j$-fibers.

Assume that $r > 2r_0$, so that $rk > 2r_0 k + k_1$ for all sufficiently large k. Denote by ν_i the u-SBR measure on the fiber $\tilde{U}_i$ and put $w_i = \nu_{U(R')}(T^{-l_i}\tilde{U}_i)$. In the following, we use Corollary 5.5:

$$\begin{aligned}
\mathcal{B}(R', R'') &= \sum_{i,j} w_i \cdot \left[T_*^{rk-l_i-l^-_j}\nu_i\right](\tilde{R}''_j) \\
&\sim \sum_{i,j} w_i \cdot [\lambda_+^{(k)}]^{rk-l_i-l^-_j-k_1} \cdot d^u_j \\
&= \left(\sum_i w_i \cdot [\lambda_+^{(k)}]^{rk-l_i}\right)\left(\sum_j [\lambda_+^{(k)}]^{-l^-_j-k_1} \cdot d^u_j\right)
\end{aligned} \tag{8.11}$$

where $a \sim b$ means that the ratio a/b is bounded above and below by two positive global constants, in this case the constants are C_{10} and C_{10}^{-1}. This proves (8.9) with $\gamma_0 = C_{10}^{-1}$.

Remark. Observe that if $\tilde{U}(R') = \emptyset$ or $\tilde{S}(R'') = \emptyset$, then $\mathcal{B}(R', R'') = 0$. Otherwise $\mathcal{B}(R', R'') > 0$, as the above calculation shows. We can number the rectangles $R_i \in \mathcal{R}$, $1 \le i \le N$, so that $\tilde{S}(R_i) \ne \emptyset$ iff $i \le J$ for some $J \le N$, and furthermore, for $i \le J$ we require $\tilde{U}(R_i) \ne \emptyset$ iff $i \le I$ for some $I \le J$. Then the matrix $\mathcal{B}$ will satisfy our early assumption (B), if only $I \ge 1$, i.e. if there is at least one rectangle R with $\tilde{U}(R) \ne \emptyset$ and $\tilde{S}(R) \ne \emptyset$ simultaneously. This will be ensured by our choice of r_0 and r below. Here we just observe that if $I = 0$, nonetheless, then $\mathcal{B}^2 = 0$, so $X_n = \tilde{X}\mathcal{B}^n$ will be zero vectors for all $n \geq 2$.

We now turn to the proof og Proposition 6.3. Let $L \geq rk^2 + q$. Let $\tilde{X}$ represent the measure $\mu^{(k)}_{L-rk^2}$ (cf. the remark after (8.4)), and let $\tilde{X}_k = \tilde{X}\Pi^k_{rk}$. Then the vector $\tilde{X}_k/|\tilde{X}_k|$ approximately represents the measure $\mu^{(k)}_L$, according to (8.4). Also, let $\tilde{Y}$ represent the measure $\mu^{(k)}_+$. Then $\tilde{Y}_k/|\tilde{Y}_k|$ approximates the same measure $\mu^{(k)}_+$. Therefore, due to (8.4)

$$\left|\mu^{(k)}_L - \mu^{(k)}_+\right|_k = ||\tilde{X}_k - \tilde{Y}_k|| + O(k\alpha^k)$$

Here and further on, $a_k = O(b_k)$ means that $a_k \leq Cb_k$, with some global constant $C > 0$.

Next, we use (8.8) with $n = k$. If $I \geq 1$, then the just proven bound (8.9) ensures that the first term on the right hand side of (8.8) is exponentially small in k, i.e.

$$\left|\mu^{(k)}_L - \mu^{(k)}_+\right|_k \leq 2(\varepsilon_{X_k} + \varepsilon_{Y_k}) + O(\beta^k) + O(k\alpha^k)$$

with $\beta = (1-\gamma_0^2)/(1+\gamma_0^2) < 1$. If $I = 0$, however, then $X_k = Y_k = 0$, and the first term in (8.8) can be simply omitted, but then $\varepsilon_{X_k} = \varepsilon_{Y_k} = 1$.

Therefore, it remains to estimate the quantities ε_{X_k}, ε_{Y_k}. We will show that they are exponentially small in k by choosing r_0 and r properly. In particular, that will imply that $I \geq 1$, i.e. the matrix $\mathcal{B}$ will be nontrivial ($\mathcal{B}^2 \neq 0$).

Again, for brevity we will suppress the superscript (k) in $\mathcal{R}^{(k)}$, $T^{(k)}$ and $M^{(k)}$. Let $R \in \mathcal{R}$ and U be an unstable R-fiber. Denote by ν_U the u-SBR measure on U.

Lemma 8.1 *There is a global constant $C_{21} > 0$ such that for every $n \geq m \geq 0$*

$$\nu_U\{x \in U \cap M_{-n} : \, l_n(x) \geq m\} \leq C_{13}\,\beta_1^m[\lambda^{(k)}_+]^n/|U| \leq C_{21}\,\beta_1^m[\lambda^{(k)}_+]^n\Lambda^k_{\max}$$

Proof. The first bound is claimed in Lemma 5.7 for the original holes H, so, it applies here provided

$$\beta_1 > B_0\Lambda^{-1}_{\min}/\lambda^{(k)}_+$$

for all sufficiently large k, which is true, see (5.19). The second inequality in the lemma follows from the obvious bound $|U| \geq \text{const} \cdot \Lambda^{-k}_{\max}$. $\square$

Corollary 8.2 *By setting $m = 0$ we get $\nu_U(U \cap M_{-n}) \leq C_{13}\,[\lambda^{(k)}_+]^n/|U| \leq C_{21}\,[\lambda^{(k)}_+]^n\Lambda^k_{\max}$.*

Now, let $R \in \mathcal{R}$ and S be a stable R-fiber. Let $m \geq 1$ and $S_m = \{x \in S \cap M_m : l^-_m(x) = m+1\}$. These are points in S whose first m backward images (under T^{-1}) are in short components (of length $< d_0/2$) in the backward images of the curve S. Denote by R_m the union of unstable R-fibers that cross S_m. Let $G_m = \cup_{R\in\mathcal{R}}R_m$.

Lemma 8.3 *Let $\mu \in \mathcal{M}_0$ and $n, m \ge 0$. Then for some global constant $C_{22} > 0$*

$$(T_*^m \mu)\,(G_m \cap M_{-n}) \le C_{22}\,\beta_1^m [\lambda_+^{(k)}]^{n+m} \Lambda_{\max}^{2k}$$

Proof. Let $R \in \mathcal{R}$ and $d^u(R)$, $d^s(R)$ be the maximum lengths of unstable and stable R-fibers, respectively. The set $T^{-m}R_m$ consists of at most B_0^m connected rectangles, denote them by $R_{m,i}$, $1 \le i \le B_0^m$. Each $R_{m,i}$ has length in the stable direction $< d_0/2$ (measured along the stable fiber $R_{m,i} \cap T^{-m}S_m$). The length of unstable fibers in $R_{m,i}$ does not exceed $\Lambda_{\min}^{-m}\, d^u(R)$. Therefore, $\mu(R_{m,i}) \le C_1 D \Lambda_{\min}^{-m}\, d^u(R)$, and so $(T_*^m\mu)(R_m) \le C_1 D B_0^m \Lambda_{\min}^{-m}\, d^u(R)$. Observe that $\sum_{R\in\mathcal{R}} d^u(R) d^s(R) \le \text{const} \cdot \text{Area}(\hat{M})$. Also, $d^s(R) \ge \text{const} \cdot \Lambda_{\max}^{-k}$. Therefore,

$$\begin{aligned}
(T_*^m \mu)\,(G_m) &\le \text{const} \cdot \sum_{R\in\mathcal{R}} B_0^m \Lambda_{\min}^{-m}\, d^u(R)\, d^s(R)/d^s(R) \\
&\le \text{const} \cdot B_0^m \Lambda_{\min}^{-m} \Lambda_{\max}^{k} \\
&\le \text{const} \cdot \beta_1^m [\lambda_+^{(k)}]^m \Lambda_{\max}^{k}
\end{aligned}$$

Now, we apply Corollary 8.2 to each unstable R-fiber in R_m for all $R \in \mathcal{R}$ and complete the proof of Lemma 8.3. □

We will apply Lemmas 8.1 and 8.3 with $m = r_0 k$ and $n = rkl$ with $l = 1, \dots, k$. We now fix r_0 large enough, so that $\beta_2 := \beta_1^{r_0} \Lambda_{\max}^2 < 1$. Then, the right hand side of the bound in the Lemma 8.1 will be const$\cdot\beta_2^k [\lambda_+^{(k)}]^n$, and that of Lemma 8.3 will be const$\cdot\beta_2^k [\lambda_+^{(k)}]^{m+n}$, where the constants are global. The value of r is simply chosen to be $> 2r_0$, so that $rk > 2r_0 k + k_1$ for all sufficiently large k.

We now estimate the value $\varepsilon_{X_k} = |\tilde{X}_k - X_k|/|\tilde{X}_k|$, i.e. the relative difference between the vectors $\tilde{X}_k$ and X_k. Recall that $\tilde{X}$ represents the probability measure $\mu_{L-rk^2}^{(k)}$ and that $\tilde{X}_k = \tilde{X}\Pi_{rk}^k$, cf. Remark after (8.4). Observe that

$$|\tilde{X}_k| = ||T_*^{rk^2} \mu_{L-rk^2}^{(k)}|| \cdot \Big(1 + O(e^{Ck\alpha^k})\Big) \sim [\lambda_+^{(k)}]^{rk^2}$$

We now make the last crucial observation. The difference between $\tilde{X}_k$ and X_k results from the differences between the matrices $\mathcal{A} = \Pi_{rk}$ and $\mathcal{B}$, compare (8.1) setting $m = rk$ there and (8.10). We have defined the components $\mathcal{B}(R', R'')$ by removing certain parts from R' and R'' that entered the definition of $\Pi_{rk} = \mathcal{A}$. Those parts are precisely described by Lemmas 8.1 and 8.3, where one sets $m = r_0 k$ and $n = rk, 2rk, \dots, rk^2$. Our choice of r_0 ensures that the relative losses incurred by the removal of those parts from all $R', R'' \in \mathcal{R}$ are exponentially small in k (the losses are bounded by const$\cdot\beta_2^k$). The vector $X_k = \tilde{X}\mathcal{B}^k$ suffers these losses every time we replace $\mathcal{A}$ by $\mathcal{B}$ in the product $\tilde{X}_k = \tilde{X}\mathcal{A}^k$. This happens k times during rk^2 iterations of T, so the total relative difference between

$\tilde{X}_k$ and X_k will be bounded by const$\cdot k\beta_2^k$. The argument for the vector $\tilde{Y}_k$ is the same. Therefore,

$$\varepsilon_{X_k} + \varepsilon_{Y_k} \le \text{const} \cdot k\beta_2^k$$

which is exponentially small in k. This completes the proof of Proposition 6.3. $\square$

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