RANDOM SCHRÖDINGER OPERATORS WITH POTENTIALS HAVING STATIONARY INCREMENTS

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Abstract. We study one dimensional Schrödinger operators $H_\omega = -\frac{d^2}{dx^2} + V_\omega$ where V_ω is a stochastic process with stationary, independent increments.

1. Introduction

In this paper we study spectral properties of one dimensional Schrödinger operators

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega \tag{1.1}$$

on $L^2([0,\infty])$ (resp. $L^2(\mathbb{R})$) and its analog on $l^2(\mathbb{Z})$. The potentials V_ω we consider are stochastic processes with stationary and independent increments. Thus Brownian motion is a typical example for the continuous case (i.e. L^2) Our potentials differ from the models commonly considered so far in that they are not stationary (ergodic). Consequently the general theory of ergodic potentials like non randomness of the spectra, existence of the density of states, positivity of Lyapunov exponents etc. is not applicable.

We are led to consider such random potentials by various physical motivations. Our first one is the Burgers equation with a random stationary force term $f(x,\omega)$, i.e.

$$\frac{\partial u}{\partial t} + uu_x = \eta u_{xx} + f(x,\omega) \tag{1.2}$$

The standard Cole-Hopf substitution $u = -2\eta(\ln\varphi)'$ reduces Burgers's equation (1.2) to the (linear) heat equation

$$\frac{\partial \varphi}{\partial t} = \eta \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{\eta} V_\omega(x)\varphi \tag{1.3}$$

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where

$$V_\omega(x) = \int_0^x f(y,\omega)\,dy \tag{1.4}$$

If $f(x,\omega)$ is *stationary* the random potential V_ω (1.4) in equation (1.3) will have *stationary increments.* The solution $\varphi(t) = e^{-tH_\omega}\varphi_0$ of (1.3) is consequently intimately connected with the spectral properties of the operator (1.1). For recent papers on Burgers equation we refer to [12], [1], [6], [3] and references therein.

Another motivation to study potentials with stationary increments comes from quantum mechanics itself. If we consider an electric field $E_\omega(x)$ which is stationary then - as above - the corresponding random potential $V_\omega(x)$ will have stationary increments.

In the following we will describe a few basic examples of potentials with stationary increments

- **Model I:** We start with the discrete case, i.e. the Hilbert space $l^2(\mathbb{Z}_+)$. The free Hamiltonian is defined as usual by

$$h_0\ u(n) = -u(n+1) - u(n-1) \tag{1.5}$$

  (for $n > 0$ and $h_0\ u(0) = -u(1)$).

  Let $\xi_n,\ n \in \mathbb{Z}^+$ be a sequence of independent, identically distributed random variables. We set

$$V_\omega(n) = \sum_{j=0}^{n} \xi_j \tag{1.6}$$

  This model can be considered on $l^2(\mathbb{Z})$ instead if we take $\xi_n, n < 0$ independent from each other and from the $\xi_n, n \geq 0$ and with the same distribution as ξ_0. For $n < 0$ we set $V_\omega(n) = \sum_{j=-1}^{n} \xi_j$.
- **Model II:** Next we describe an analogous model on $L^2(\mathbb{R}_+)$ by setting

$$V_\omega(x) = \sum_{j=0}^{[x]} \xi_j \tag{1.7}$$

  where again the ξ_n are independent, identically distributed random variables and $[x]$ is the largest integer not exceeding x. This process has independent, stationary increments in the sense that $V_\omega(x+n+1) - V_\omega(x+n)$ are independent identically distributed random variables for $n \in \mathbb{Z}_+$.

  In what follows we will always assume $E(\xi_0) = 0$ Moreover we suppose that $E(\xi_0^2) < \infty$, without loss we set $E(\xi_0^2) = 1$.

- **Model III:** Perhaps the prototype of a process with stationary increments is the Brownian motion β_s, $s \geq 0$, so we consider

$$V_\omega(x) = \beta_x(\omega) \tag{1.8}$$

with e.g. $\beta_0 = 0$. This model (like Model II) can be extended to all of $\mathbb{R}$ by adding an independent copy of the process V_ω to the left of the origin, i.e. by setting

$$V_\omega(x) = \begin{cases} \beta_x(\omega) & \text{for } x \geq 0 \\ \widetilde{\beta}_{-x}(\omega) & \text{for } x \leq 0 \end{cases}$$

where $\widetilde{\beta}$ is another Brownian motion with $\widetilde{\beta}_0 = 0$ independent of β. An analogous method works for model II.

In the following, we consider the above models on the halfline $\mathbb{R}_+$ (resp. $\mathbb{Z}_+$) with Dirichlet boundary conditions at 0 unless otherwise stated. Nevertheless, analogous results can be proven for the line in very much the same way.

We remark that for model II

$$|V_\omega(x)| \leq \sum_{i=0}^{[x]} \xi_i \leq \sqrt{x \sum_{i=0}^{[x]} \xi_i^2} \leq cx \sqrt{\frac{1}{[x]+1} \sum_{i=0}^{[x]} \xi_i^2}$$

Since $\frac{1}{[x]+1} \sum_{i=0}^{[x]} \xi_i^2 \to E(\xi_0^2)$ a.s. by the ergodic theorem, we have therefore $|V_\omega(x)| \leq C_0 + C_1 |x|$ almost surely. It follows that the operator $H_\omega = H_0 + V_\omega$ is essentially self adjoint (see e.g.[10]). The law of iterated logarithm (see e.g. [4] or [11]) allows a similar conclusion for model III (see also the discussion in section 3).

All the above models have the property that $E\left(V_\omega(x)\right) = 0$ for all x. This is a serious restriction, since $E\left(V_\omega(x)\right) \neq 0$ is equivalent to adding a homogeneous electric field to the potential. It is known that this additional term can change pure point spectrum to absolutely continous spectrum in the stationary case [2]. We will consider the more complicated case $E\left(V_\omega(x)\right) \neq 0$ in a subsequent publication [7].

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2. The integrated density of states

The integrated density of states is an important characteristic of quantum mechanical disordered systems (for a review from the mathematical point of view see [5]). For *stationary* potentials (in the one dimensional case) it is defined by the following procedure:

Let H_L denote the random Hamiltonian H_ω restricted to the Hilbert Space $L^2([0,L])$ (resp. $L^2(\{0,\ldots,L-1\})$) with appropriate (say Dirichlet) boundary conditions. By $E_n(H_L)$ we denote the n-th eigenvalue of H_L (arranged in increasing order counting multiplicity). We set

$$N_L(E) = \#\left\{n \,|E_n(H_L) \le E\right\} \tag{2.1}$$

Then (for a huge class of stationary potentials)

$$\frac{1}{L} N_L(E) \to N(E). \tag{2.2}$$

The limit $N(E)$ is a nonrandom function which we call the integrated density of state. This convergence is a.s. convergence at each continuity point E of N, i.e. vague convergence of the corresponding Borel measures.

Let us try to do a similar procedure for potentials with independent increments. We consider the discrete case (model I) first. We may assume that $E(\xi_j)=0$ and $E(\xi_j^2)=1$. Since h_0 is a bounded operator, in fact $\|h_0\|=2$, we have $E_n(H_L)\le E_n(V_\omega)+2$ and $E_n(H_L)\ge E_n(V_\omega)-2$, where $E_n(V_\omega)$ are the eigenvalues of the multiplication operator V_ω. Because $V_\omega(k)=\sum_{j=0}^{k}\xi_j =: S_k$ the eigenvalues $E_n(V_\omega)$ are just the values of S_k, properly ordered. Thus

$$\begin{aligned}\frac{1}{L+1}N_L(E) &\le \frac{1}{L+1}\#\left\{E_n(V_\omega)\,|E_n(V_\omega)\le E+2\right\}\\ &\le \frac{1}{L+1}\#\left\{S_k\,|S_k\le E+2\,, k=0,\ldots,L\right\}\\ &= \frac{1}{L+1}\#\left\{S_k\,\middle|\,\frac{1}{\sqrt{L}}S_k \le \frac{E+2}{\sqrt{L}}\,, k=0,\ldots,L\right\}\end{aligned} \tag{2.3}$$

In the same way we prove:

$$\frac{1}{L+1}N_L(E) \ge \frac{1}{L+1}\#\left\{S_k\,\middle|\,\frac{1}{\sqrt{L}}S_k \le \frac{E-2}{\sqrt{L}}\,, k=0,\ldots,L\right\} \tag{2.4}$$

Since $\frac{1}{\sqrt{L}}S_k$ converges in law to a standard Brownian motion β_S by Donsker's theorem (see e.g.[4], [11]) we have that

$$\frac{1}{L+1}\#\left\{S_k\,\middle|\,\frac{1}{\sqrt{L}}S_k \le a\,, k=0,\ldots,L\right\} \overset{\mathcal{L}}{\Rightarrow} \int_0^1 \chi_{(-\infty,a]}(\beta_s)\,ds \tag{2.5}$$

Consequently

$$\frac{1}{L+1} N_L(E) \Rightarrow \int_0^1 \chi_{(-\infty,0]}(\beta_s)\, ds \tag{2.6}$$

Equation (2.6) has a number of "strange" aspects. First of all the limit of $\frac{1}{L+1} N_L(E)$ is *not* independent of the random parameters. This is to be expected because the random field V_ω is not ergodic. But what is surprising, indeed, is the fact that the limit is *independent* of the energy, which means that this quantity is physically meaningless. In mathematical terms it means that the scaling and/or the normalization chosen is incorrect! In fact, an inspection of formulas (2.3 and 2.4) above leads us to define

$$M_L(E) = \#\left\{E_n(H_L) \leq \sqrt{L}E\right\} = N_L(\sqrt{L}E) \tag{2.7}$$

since the expected value of the potential $|V_\omega(L)|$ is of the order $\sqrt{L}$.The reasoning above leads us to the following result:

Theorem 2.1. *If $V_\omega(n) = \sum_{j=0}^n \xi_j$, ξ_i being independent, identically distributed with $E(\xi_i) = 0, E(\xi_i^2) = 1$ and if H_L is the restriction of $H_\omega = H_0 + V_\omega$ on $l^2(\mathbb{Z})$ to the interval$\{0, 1, \ldots, L\}$ then*

$$\frac{1}{L+1} M_L(E) \stackrel{\mathcal{L}}{\Rightarrow} \int_0^1 \chi_{(-\infty,E)}(\beta_s) ds$$

where β_s is a standard Brownian motion.

Let us now look at the continuous case, more precisely at model II. Introducing Dirichlet boundary conditions at the integers n changes the operator H_ω on $[0, L]$ to the direct sum of the operators H_ω on $[n, n+1]$ (with Dirichlet boundary conditions at each endpoint). Let us denote the latter operator by $H_\omega^{(n)}$. Since the potential is constant between n and $n+1$ the eigenvalues of $H_\omega^{(n)}$ are given by $m^2 + S_n$, with $S_n = \sum_{i=0}^n \xi_i$. Since introducing boundary conditions changes the resolvent by a finite rank operator the quantity $N_L(E)$ (or $M_L(E)$) are changed by at most $O(L)$. Let us estimate the expectation $N_L(E)$ for

$E = 0$. Our somewhat formal manipulations will be justified below:

$$\mathbf{E}\left(N_L(0)\right) = \sum_{n=0}^{L-1} \mathbf{E}\left(\#\left\{m \mid m^2 + S_n < 0\right\}\right) + O(L) \tag{2.8}$$

$$= \sum_{n=0}^{L-1} \mathbf{E}\left(\#\left\{m \mid m < \sqrt{(S_n)_+}\right\}\right) + O(L) \tag{2.9}$$

$$= \sum_{n=0}^{L-1} \mathbf{E}\left(\sqrt{(S_n)_+}\right) + O(L) \tag{2.10}$$

$$= \sum_{n=0}^{L-1} n^{\frac{1}{4}}\ \mathbf{E}\left(\sqrt{\left(\frac{1}{\sqrt{n}}S_n\right)_+}\right) + O(L) \tag{2.11}$$

$$\approx O\left(L^{\frac{5}{4}}\right) \tag{2.12}$$

From this considerations we learn that $\frac{1}{L}N_L$ or $\frac{1}{L}M_L$ have no chance to converge. It is rather $\frac{1}{L^{\frac{5}{4}}}M_L$ that is suggested by our computation. In fact, we have

Theorem 2.2. *If $V_\omega(x) = \sum_{j=0}^{n} \xi_j$, ξ_i being independent, identically distributed with $E(\xi_i) = 0, E(\xi_i^2) = 1$ and if H_L is the restriction of $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R})$ to the interval $[0, L]$ then*

$$\frac{1}{L^{\frac{5}{4}}} M_L(E) \overset{\mathcal{L}}{\Rightarrow} \int_0^1 \sqrt{(\beta_s - E)_+}\, ds \tag{2.13}$$

where β_s is a standard Brownian motion.

Proof. As in the above consideration we obtain

$$\frac{1}{L^{\frac{5}{4}}} M_L(E) \quad = \quad \frac{1}{L^{\frac{5}{4}}} \sum_{n=0}^{L-1} \surd$$

$\left(S_n - \sqrt{L}E\right)_+ + o(1)$

$=1 \overline{L \sum_{n=0}^{L-1} \sqrt{\left(\frac{1}{\sqrt{L}}\right.}}$

$\mathrm{S}_n - E_+ + o(1)$(2.14)By Donsker's theorem (see e.g. [4] or [11]) the stochastic process X_s on $[0,1]$ defined by $X_{\frac{n}{L}} = \frac{1}{\sqrt{L}}S_n$ and by linear interpolation in between converges weakly to standard Brownian motion. The sum in 2.14 is actually a Riemann sum for $\int_0^1$

$\sqrt{(X_s - E)_+}\, ds$. Consequently the whole term converges weakly to $\int_0^1 \sqrt{(\beta_s - E)_+} ds$. □

We have an analog for the above result for the Brownian motion potential (Model III):

Theorem 2.3. *If $V_\omega(x) = \beta_x$ is the Brownian motion potential then*

$$\frac{1}{L^{\frac{5}{4}}} M_L(E) \overset{\mathcal{L}}{\Rightarrow} \int_0^1 \sqrt{(\beta_s - E)_+}\, ds \tag{2.15}$$

Proof. The proof is similar to the above case. Defining

$$Y_n = \sup_{0 \le s \le 1} |\beta_s - \beta_n|$$

we get as above:

$$\frac{1}{L^{\frac{5}{4}}} M_L(E) = \sum_{n=0}^{L-1} \frac{1}{L} \sqrt{\left(\frac{1}{\sqrt{L}}\beta_n - E \pm \frac{Y_n}{\sqrt{L}}\right)_+} + o(1)$$

Since $\frac{Y_n}{\sqrt{L}} \to 0$ almost surely, the above sum converges to

$$\int_0^1 \sqrt{(\beta_s - E)_+}\, ds.$$

□

3. Localization

In this section we prove Anderson localization for the model I-III. Since the proofs for model I and II are similar to the one for model III but more standard we give the details of the proof only in the latter case. We also restrict ourselves to the case of the half line, the case of $L^2(\mathbf{R})$ being similar.

Theorem 3.1. *a) Model I: If the distribution of the random variable ξ_0 in (1.6) is absolutely continuous (w.r.t. Lebesgue measure) with a bounded density, then the operator*

$$h_\omega = h_0 + \sum_{j=0}^{n} \xi_j \qquad on\ l^2(\mathbf{N})$$

has pure point spectrum. The same holds for h_ω on $l^2(\mathbf{Z})$.
b) Model II: If the distribution of the random variable ξ_0 in (1.7) is absolutely continuous with a bounded density, then the operator

$$H_\omega = H_0 + \sum_{j=0}^{[x]} \xi_j \qquad on\ L^2(\mathbf{R}_+)$$

has pure point spectrum. The same holds for H_ω on $L^2(\mathbf{R})$.
c) Model III: The operator $H_\omega = H_0 + \beta_x$ with a Brownian motion β on

$L^2(\mathbf{R}_+)$ has pure point spectrum. The same is true for H_ω on $L^2(\mathbf{R})$ if β_x for $x \leq 0$ is a Brownian motion independent of the Brownian motion β_x on $\mathbf{R}_+$.

As mentioned above we give the proof for model III on $\mathbf{R}_+$. The proof consists of two steps. In the first step we prove that the resolvent is square integrable in the sense

$$\sup_{\varepsilon>0} \int |R_{E+i\varepsilon}(a,x)| \; dx < \infty \tag{3.1}$$

for any $a \in \mathbf{R}_+$ where $R_{E+i\varepsilon}(x,y)$ is the kernel of the resolvent of H_ω. To prove this we employ the paper[8]. We will give details below. To deduce localization from (3.1) we need a procedure of Simon-Wolff-type [**?**]. For this we refer to the paper of Kotani and Simon[9]. They prove localization from (3.1) if the Prüfer angle ϕ at a point $a > 0$ has a conditional distribution with a bounded density. To prove this we use the technique Kotani-Simon use in example 2 of section 3 of their work[9]. We refer to this paper for details. Let us now prove (3.1). In[8] the following theorem is proved:

Theorem 3.2. *Suppose the potential V satisfies,*

$$V(x) \geq -c(1+x)^{-\alpha} \tag{3.2}$$

for some $\alpha < 2$ and denote by R_z the resolvent kernel of the operator $H = H_0 + V$ on $L^2(\mathbf{R}_+)$ (with some boundary condition at 0) for the energy $z \in \mathbf{C}$. Assume there are sequences $V_n, x_n \to \infty$ and $h_n > 0$ with $\sqrt{V_n}h_n \to \infty$ such that:

$$V(x) \geq V_n \qquad \text{for } x \in [x_n - h_n, x_n + h_n]$$

If

$$\sum x_{n+1}^{\frac{3+\alpha}{2}} e^{-\gamma\sqrt{V_n}h_n} < \infty \qquad \text{for a } \gamma < 1$$

then

$$\sup_{\varepsilon>0} \int_a^\infty |R_{e+i\varepsilon}(a,x)|^2 \, dx < \infty$$

Proof. The theorem is essentially proved in[8]. The only difference being that[8] assume $V(x) \geq 0$, i.e. $\alpha = 0$. In our more general case we have to change the estimate of

$$N_L := \#\{\text{eigenvalues of } H_L \leq E\}$$

in the lemma of section 5 in [8]. We may estimate

$$N_L \leq C \int_{x_n}^{x_{n+1}} \left(1 + V_-(x)\right) \; dx$$

V_- is the negative part of V. By 3.2 we obtain

$$N_L \leq C_2 x_{n+1}^{\alpha} \left(x_{n+1} - x_n\right) \leq C_2 x_{n+1}^{1+\alpha}$$

Inserting this bound in [8] we obtain the desired result. $\square$

Now we prove that the Brownian motion potential $V(x) = \beta_x$ satisfies the assumptions of 3.2.
First, it is an immediate consequence of the law of iterated logarithm (see e.g [4]or[11]) that with probability one:

$$|V(x)| = |\beta_x| \leq |x|^{\frac{1}{2}+\varepsilon} \tag{3.3}$$

for any $\varepsilon > 0$ and all x large enough. Hence (3.2) is satisfied for $\alpha = \frac{1}{2} + \varepsilon$. Moreover we prove:

Proposition 3.3. *With probability one there is a sequence $x_n \uparrow \infty$ such that*

$$\beta_y \geq C\sqrt{x_k} \qquad \textit{for } x_k \leq y \leq x_k + 1$$

Remark 3.1. *Assertion and proof of the about proposition are very much in the spirit of the law of iterated logarithm.*

Proof. The set $\Omega_0 = \{\beta |\ \beta_x \leq |x|^{\frac{2}{3}}$ for x large$\}$ has probability one (see 3.3). Look at the points $t_n = 2^n$ and define

2 and $\sup\limits_{0 \leq s \leq 1} \left|\beta_{2^{m+1}+s} - \beta_{2^{m+1}}\right| \leq 2^{\frac{m-1}{2}}$

These events are independent and satisfy

$$P(A_m) \ = \ P\left(\beta_{2^{m+1}} - \beta_{2^m+1} \geq 2\right.$$

m+1 $\overline{2P\left(\sup\limits_{0\leq s\leq 1} |\beta_s - \beta_0| \leq 2^{\frac{m-1}{2}}\right) \geq \frac{1}{2} P\left(\frac{\beta_{2^{m+1}} - \beta_{2^m+1}}{\surd}\right.}$

$2^{m+1} - 2^m - 1 \geq \frac{2^{\frac{1}{2}}}{\sqrt{1-2^{-m}}}$

$\geq \frac{1}{2} P(\beta_1 - \beta_0 \geq 2) \geq C$if m is large enough. Hence $\sum P(A_m) = \infty$, which implies that P-almost surely there exists a sequence $n_k \uparrow \infty$ such that $\beta \in A_{n_k}$. Without loss we may assume that
$\mathrm{n}_{k+1} > 2\, n_k$. Consequently with probability one:

$$\begin{aligned} \beta_{2^{n_{k+1}}} &\geq \left(\beta_{2^{n_{k+1}}} - \beta_{2^{n_k}+1}\right) + \beta_{2^{n_k}+1} \\ &\geq 2^{\frac{n_{k+1}}{2}} - \left(2^{n_k} + 1\right)^{\frac{2}{3}} \\ &\geq 2^{\frac{n_{k+1}}{2}} - 2 \cdot 2^{\frac{2}{3} n_k} \\ &= \left(1 - 2 \cdot 2^{\frac{2}{3} n_k - \frac{n_{k+1}}{2}}\right) 2^{n_{k+1}} \end{aligned}$$

$\frac{}{2 \geq \frac{1}{2} 2^{\frac{n_{k+1}}{2}}}$ for large k. Moreover, for $0 \leq s \leq 1$

$$\begin{aligned} \beta_{2^{n_{k+1}}+s} &\geq (\beta_{2^{n_{k+1}}+s} - \beta_{2^{n_{k+1}}}) + (\beta_{2^{n_{k+1}}} - \beta_{2^{n_k}+1}) + \beta_{2^{n_k}+1} \\ &\geq 2^{\frac{n_{k+1}}{2}} - (2^{n_k} + 1)^{\frac{2}{3}} - 2 \end{aligned}$$

$\mathrm{n}_{k+1} - 2 \frac{}{2 = 2^{\frac{n_{k+1}}{2}} \left(1 - 2 \cdot 2^{\frac{2}{3} n_k - \frac{n_{k+1}}{2} - \frac{1}{2}}\right) \geq \frac{1}{3} \cdot 2^{\frac{n_{k+1}}{2}}}$ (k large enough). This proves the theorem. □

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