STABILITY OF INFINITE CLUSTERS IN SUPERCRITICAL PERCOLATION

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Abstract. A recent theorem by Häggström and Peres concerning independent percolation is extended to all the quasi-transitive graphs. This theorem states that if $0 < p_1 < p_2 \leq 1$ and percolation occurs at level p_1, then every infinite cluster at level p_2 contains some infinite cluster at level p_1. Consequences are the continuity of the percolation probability above the percolation threshold and the monotonicity of the uniqueness of the infinite cluster, i.e., if at level p_1 there is a unique infinite cluster then the same holds at level p_2. These results are further generalized to graphs with a "uniform percolation" property. The threshold for uniqueness of the infinite cluster is characterized in terms of connectivities between large balls.

1. Introduction and main results. In recent years increasing attention has been dedicated to the study of interacting particle systems, statistical mechanics models and percolation on fairly arbitrary graphs. A sample of these efforts can be found in the papers [BLPS1], [BLPS2], [BS], [GN], [HP], [JS], [Lal], [NW], [SS1], [SS2], [ST], and references therein. Motivation for this trend is at least twofold. On one hand, it is becoming clear, e.g., from results in the papers quoted above, that on graphs different from the most commonly studied square lattices and homogeneous trees there is additional mathematical structure to be found. On the other hand the techniques and results obtained from the study of the processes on more general graphs in some instances have led to results which were not yet available even for the most studied particular graphs.

In this paper we study independent site percolation on graphs. (Independent bond percolation is a particular case thereof, since the edges of a graph can be considered as the sites of another graph.) We start by introducing standard definitions and terminology.

Let $G = (V, E)$ be an infinite connected graph of locally bounded degree. Here V is the set of vertices (also called sites) of G, and E is the set of edges (also called bonds) of G. Sites will be said to be neighbors if they belong to a common edge. A chain is a sequence $x_0, x_1, ..., x_n$, of distinct sites in which for each i, x_i is neighbor to x_{i+1}; the length of this chain is n, and x_0 and x_n are its end-points. The distance

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$d(y,z)$ between sites y and z is the minimal length of the chains which have y and z as end-points. The distance between a site y and a set of sites S is the minimal distance between y and each site in S. A site of G will be singled out and called its root, denoted by r. The ball of center $x \in V$ and radius N is the set

$$B(x,N) = \{y \in V : d(x,y) \le N\}.$$

We will use the abbreviation $B(r,N) = B(N)$. An automorphism of G is a one-to-one mapping from V onto V which preserves the graph structure, i.e., such that the set of edges of G is identical to the set of pairs of images of vertices of G which belong to a common edge. A graph is said to be transitive (or homogeneous) if for each pair x and y of its vertices there is an automorphism of the graph which maps x into y. A graph is quasi-transitive if there is a finite set of vertices, V_0, with the property that each vertex of the graph can be mapped into one of the vertices of V_0 by an automorphism.

Declare each site of a graph G to be independently occupied with probability p and to be vacant otherwise. We will refer to this scenario as density p percolation, or level p percolation. The corresponding probability will be denoted by $\mathbb{P}_p$. The connected components of the graph which has as vertices the occupied sites and as edges the edges in E which connect any two of these vertices are called clusters. Each cluster is a subgraph of G, but in a common abuse of terminology and notation, we will also identify each cluster with its set of vertices. The boundary of a cluster is the set of vacant sites at distance 1 from this cluster. In case the site x is occupied, the cluster to which it belongs will be denoted by $C(x)$, and its boundary will be denoted by $\partial C(x)$. If x is vacant, then we set $C(x) = \emptyset$ and $\partial C(x) = \emptyset$. Given $S \subset V$, set

$$\theta_G(p,S) = \mathbb{P}_p(S \text{ intersects an infinite cluster}),$$

and abbreviate

$$\theta_G(p,\{x\}) = \theta_G(p,x), \qquad \theta_G(p,r) = \theta_G(p).$$

The threshold for percolation is defined by

$$p_c(G) = \inf\{p : \theta_G(p) > 0\} = \inf\{p : \mathbb{P}_p - \text{a.s. there is an infinite cluster}\},$$

where the second equality is a consequence of Kolmogorov's 0-1 law.

In order to present the basic result in this paper it is convenient to first recall the standard way in which percolation at all levels $p \in [0,1]$ can be constructed on the same probability space in a "monotone" fashion. To each $x \in V$ associate an independent random variable, U_x, uniformly distributed between 0 and 1. Say then that x is occupied at level p if $U_x < p$. We will refer to this construction as the standard monotone coupling.

Suppose now that we have two fixed levels $0 < p_1 < p_2 \le 1$ and that we are using the standard monotone coupling. An infinite cluster at level p_2 will be said to be p_1-stable if some of the sites of this cluster also belong to infinite clusters at level p_1. Note that while we are using the standard monotone coupling in this definition, the same concept can be defined directly, by just looking at percolation exclusively at level p_2 and saying that an infinite cluster C at this level is p_1-stable in case $\theta_C(p_1/p_2) > 0$. The first definition has the advantage that it is more transparent, while the second one has the advantage of being intrinsic to percolation at a single level.

We are now ready to state the following result.

Theorem 1.1. *Suppose that G is a quasi-transitive graph, that $0 < p_1 < p_2 \leq 1$, and that $\theta_G(p_1) > 0$. Then a.s. all the infinite clusters at level p_2 are p_1-stable.*

This result has recently been proved in [HP] under the extra assumption that the graph satisfies a certain algebraic condition, namely that it is unimodular. One of the main points of the current paper is to provide a self-contained proof without requiring this extra hypothesis. A further extension of the theorem will be presented in Section 3.

Two immediate consequences of Theorem 1.1 are the following.

Corollary 1.1. *Suppose that G is a quasi-transitive graph. Then for each site x, the function $\theta_G(p,x)$ is continuous on $(p_c(G),1]$.*

The right continuity of $\theta_G(p,x)$ for all G and at all p is a well known and easy to prove fact (see, e.g., [Gri], Lemma 6.36). Corollary 1.1 follows then from Theorem 1 and the theorem in [BK] which states that $\theta_G(p,x)$ is discontinuous from the left precisely at the values of p at which there is positive probability that x belongs to an infinite cluster which is p'-unstable for every $p' < p$.

Corollary 1.2. *Suppose that G is a quasi-transitive graph, that $0 < p_1 < p_2 \leq 1$, and that at level p_1 there is a.s. a unique infinite cluster. Then also at level p_2 there is a.s. a unique infinite cluster.*

Note that Corollary 1.2 is immediate from Theorem 1.1.

It is well known that the argument in [NS] can be adapted to an arbitrary quasi-transitive graph, to show that for each value of p one of the following three alternatives must hold: either a.s. there is no infinite cluster, or a.s. there is a unique infinite cluster, or a.s there are infinitely many infinite clusters. If, as in [BS], we define

$$p_u = \inf\{p \colon \mathbb{P}_p\text{-a.s. there is a unique infinite cluster}\},$$

then Corollary 1.2 tells us that for all $p > p_u$, $\mathbb{P}_p$-a.s. there is a unique infinite cluster. This answers positively to the first part of Question 5 in [BS]. The "general picture" for percolation on quasi-transitive graphs is therefore that there are two critical points, p_c and p_u (which in some particular cases coincide). Below p_c there is no infinite cluster a.s., between p_c and p_u there are infinitely many infinite clusters a.s., and above p_u there is a unique infinite cluster a.s..

When $p_c < p_u < 1$, it is natural to ask whether at p_u there is a.s. uniqueness or not (this is the second part of Question 5 in [BS]). According to [BLPS2], Benjamini and Schramm showed that this is the case if the graph G is planar (quoted there as work in preparation). In a forthcoming publication we will show that for the example introduced in [GN], in which G is a direct product of $\mathbb{Z}$ and a homogeneous tree (other than $\mathbb{Z}$) the answer is negative, so that at p_u there are a.s. infinitely many infinite clusters.

A characterization of p_u in terms of connectivities between large balls will be provided in Section 3.

The rest of this paper is organized as follows. Section 2 contains the proof of Theorem 1.1, and Section 3 contains further results related to the issues discussed above.

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2. Proof of Theorem 1.1. Before we can start the proof of Theorem 1.1, we need to introduce some tools.

We will need to construct the percolation process at levels p_1 and p_2 on the same probability space, in a way different from the one given by the standard monotone coupling, but providing the same joint distribution for these two processes. To each site $x \in V$ associate independently two independent random variables, $\eta(x)$ and $\alpha(x)$, each one taking only the values 0 and 1. The probabilities that $\eta(x)$ and $\alpha(x)$ take the value 1 are respectively p_1 and $(p_2 - p_1)/(1 - p_1)$. Set also $\zeta(x) = \max(\eta(x), \alpha(x))$, so that the probability that $\zeta(x)$ takes the value 1 is p_2. The random field $\eta = (\eta(x))_{x \in V}$ is a Bernoulli random field with density p_1, while the random field $\zeta = (\zeta(x))_{x \in V}$ is a Bernoulli random field with density p_2. These two random fields also have the desired joint distribution, since $\eta(x) \leq \zeta(x)$ for all $x \in V$. We will use the notation $C^{p_j}(x)$, $j = 1, 2$ to denote the cluster of the site x at level p_j in this construction. We will also refer to these clusters as the η-cluster of x and the ζ-cluster of x.

We will need to use a technique known as "growing the cluster of the site x", and we review now what this means. Say that we want to do this at level p_1. A random sequence of two finite subsets of V, $C_i^{p_1}(x)$ and $\partial C_i^{p_1}(x)$, $i = 0, 1, \ldots$, will be defined, with the first one growing towards the η-cluster of x and the second one growing towards the boundary of this cluster. For this purpose order the sites of V in an arbitrary way. Now, in case $\eta(x) = 0$ set

$$C_0^{p_1}(x) = \emptyset \qquad \text{and} \qquad \partial C_0^{p_1}(x) = \emptyset,$$

while in case $\eta(x) = 1$ set

$$C_0^{p_1}(x) = \{x\} \qquad \text{and} \qquad \partial C_0^{p_1}(x) = \emptyset.$$

Then recursively proceed as follows. In the i-th step ask whether there is any site which is at distance 1 from $C_{i-1}^{p_1}(x)$ but is not in $\partial C_{i-1}^{p_1}(x)$. In case no such site exists, the construction has been completed, our sequences are finite ones, $C^{p_1}(x) = C_{i-1}^{p_1}(x)$ and $\partial C^{p_1}(x) = \partial C_{i-1}^{p_1}(x)$. Otherwise let X_i be the first such site and proceed as follows. In case $\eta(X_i) = 0$, set

$$C_i^{p_1}(x) = C_{i-1}^{p_1}(x) \qquad \text{and} \qquad \partial C_i^{p_1}(x) = \partial C_{i-1}^{p_1}(x) \cup \{X_i\}.$$

In case $\eta(X_i) = 1$, set

$$C_i^{p_1}(x) = C_{i-1}^{p_1}(x) \cup \{X_i\} \qquad \text{and} \qquad \partial C_i^{p_1}(x) = \partial C_{i-1}^{p_1}(x).$$

Note that, as desired, the sites which will eventually be contained in $C_i^{p_1}(x)$ (resp. $\partial C_i^{p_1}(x)$) are precisely the sites of $C^{p_1}(x)$ (resp. $\partial C^{p_1}(x)$).

Proof of Theorem 1.1. Let E_x be the event that the cluster of x at level p_2 is infinite but is p_1-unstable. Let E denote the event that we want to show to have probability 0, i.e.,

$$E = \bigcup_{x \in V} E_x.$$

To each set $C \subset V(G)$ we associate the random variable $W(C)$ which gives the number of sites which are at distance 1 from C and are also at distance 1 from some infinite cluster at level p_1.

Given $x \in V$ define the events

$$E_x^0 = E_x \cap \{W(C^{p_2}(x)) = 0\}, \qquad E^0 = \bigcup_{x \in V} E_x^0.$$

So E^0 is the event that there is an infinite cluster at level p_2 which is p_1-unstable with no site in its boundary being at distance 1 from any infinite cluster at level p_1. Similarly define the events

$$E_x^{\mathrm{f}} = E_x \cap \{W(C^{p_2}(x)) < \infty\}, \qquad E^{\mathrm{f}} = \bigcup_{x \in V} E_x^{\mathrm{f}}.$$

So E^{f} is the event that there is an infinite cluster at level p_2 which is p_1-unstable with only finitely many sites in its boundary being at distance 1 from any infinite cluster at level p_1. Finally define

$$E_x^{\infty} = E_x \cap \{W(C^{p_2}(x)) = \infty\}, \qquad E^{\infty} = \bigcup_{x \in V} E_x^{\infty}.$$

So E^{∞} is the event that there is an infinite cluster at level p_2 which is p_1-unstable with infinitely many sites in its boundary being at distance 1 from infinite clusters at level p_1.

Our proof will consist of three steps. In the first one we will prove that

$$\mathbb{P}(E^0) = 0. \tag{2.1}$$

In the second step this result will be used to show the following stronger result:

$$\mathbb{P}(E^{\mathrm{f}}) = 0. \tag{2.2}$$

In the third step a separate argument will be used to show that

$$\mathbb{P}(E^{\infty}) = 0. \tag{2.3}$$

Clearly (2.2) and (2.3) prove the theorem, since $E = E^{\mathrm{f}} \cup E^{\infty}$.

Step 1: Given a site $x \in V$ and a number N, we will construct the random fields η and ζ in a particular way described next. This construction will rely on a "duplication trick", i.e., we suppose given two independent copies of the random field (η, ζ). A prime will be used to denote objects related to the first copy and a double prime will be used for objects related to the second copy.

We start by growing the ζ'-cluster of x, $C'^{p_2}(x)$, as explained before we started this proof, but using this time the random field ζ'. We stop nevertheless at step i if the set $C_i'^{p_2}(x)$ contains a ball of radius N. If this happens we say that event F has happened, we denote by Y the random location of the center of this ball (if there is more than one choice for Y, take the smallest one according to the ordering of the sites), and by I the random time i. Since a.s. any infinite cluster contains balls of arbitrarily large radii (this is intuitively clear, and formally proved in Lemma 1, at the end of this section) we know that a.s. on F^c the ζ'-cluster of x is finite. In this case let I be defined as the total number of steps needed to grow this cluster, i.e., $C'^{p_2}(x) = C_I'^{p_2}(x)$ and $\partial C'^{p_2}(x) = \partial C_I'^{p_2}(x)$.

Once we stop the construction above we set $\eta_z = \eta_z'$, $\zeta_z = \zeta_z'$ for $z \in C_I'^{p_2}(x) \cup \partial C_I'^{p_2}(x)$; and $\eta_z = \eta_z''$, $\zeta_z = \zeta_z''$ for $z \in V \backslash (C_I'^{p_2}(x) \cup \partial C_I'^{p_2}(x))$. This defines the random fields η and ζ (which have the correct joint distribution), and all the clusters to which we refer below correspond to these random fields, unless stated otherwise. In particular $C^{p_2}(x)$ will denote the ζ-cluster of x. Note that $C_I'^{p_2}(x) \subset C^{p_2}(x)$, and $\partial C_I'^{p_2}(x) \subset \partial C^{p_2}(x)$. Moreover, on F^c a.s. $C^{p_2}(x) = C'^{p_2}(x) = C_I'^{p_2}(x)$ is a finite cluster.

Let $\widetilde{F}$ be the event that F happens and $B(Y, N)$ intersects an infinite η''-cluster. We will argue next that

$$\mathbb{P}(E_x^0) \le \mathbb{P}(F \cap \widetilde{F}^c). \tag{2.4}$$

Since on E_x^0 the cluster $C^{p_2}(x)$ is infinite, it follows that up to a set of measure 0 this event is contained in F. Now, if we are on $\widetilde{F}$, then clearly either $C^{p_2}(x)$ contains an infinite η-cluster, and hence we are not in E_x^0 because $C^{p_2}(x)$ is p_1-stable, or else there is a site in the boundary of $C^{p_2}(x)$ which is at distance 1 from an infinite η-cluster, and hence we are not in E_x^0 because $W(C^{p_2}(x)) > 0$. This proves (2.4).

Since η'' is independent of ζ', by conditioning on the location of Y we obtain from (2.4)

$$\mathbb{P}(E_x^0) \le \mathbb{P}(\widetilde{F}^c | F) \le 1 - \inf_{y \in V} \theta(p_1, B(y, N)). \tag{2.5}$$

Since $\theta(p_1) > 0$, there are infinite clusters at level p_1 a.s., and since G is quasi-transitive,

$$\mathbb{P}(E_x^0) \le \lim_{N \to \infty} (1 - \inf_{y \in V} \theta(p_1, B(y, N))) = \lim_{N \to \infty} (1 - \min_{y \in V_0} \theta(p_1, B(y, N))) = 0. \tag{2.6}$$

Summing over $x \in V$, we obtain (2.1).

Step 2: Our goal now is to prove (2.2). We will show that if $\mathbb{P}(E^{\mathrm{f}}) > 0$ then also $\mathbb{P}(E^0) > 0$. Therefore (2.2) will follow from (2.1), proved in Step 1. The argument is standard and intuitive. Any configuration of occupied and vacant sites at levels p_1 and p_2 which is in E^{f} can be mapped by a modification which affects only finitely many sites into a configuration in E^0. Since on a finite set of sites any configuration has positive probability, the positivity of $\mathbb{P}(E^{\mathrm{f}})$ should imply the positivity of $\mathbb{P}(E^0)$.

To be more precise, we introduce the event $E^{\mathrm{f},M}$ that there is an infinite cluster at level p_2 which is p_1-unstable and such that any site which is in its boundary and is also at distance 1 from an infinite cluster at level p_1 is contained in the ball $B(M)$. If $\mathbb{P}(E^{\mathrm{f}}) > 0$, then for some finite $M > 0$ we have $\mathbb{P}(E^{\mathrm{f},M}) \geq \mathbb{P}(E^{\mathrm{f}})/2 > 0$. We use now again a duplication trick, so that two copies of the random fields η and ζ are being used; a prime will be used again to denote the first one and a double prime for the second one. This time we will take our final random fields η and ζ as follows: on $B(M)$ they are identical to η' and ζ', and outside $B(M)$ they are identical to η'' and ζ''. It is clear that if the event $\{\eta'(x) = \zeta'(x) = 0 \text{ for all } x \in B(2M)\}$ happens and the event $E^{\mathrm{f},M}$ for the random fields η'' and ζ'' also happens, then E^0 will happen for the random fields η and ζ. Therefore

$$\mathbb{P}(E^0) \geq \mathbb{P}(\eta'(x) = \zeta'(x) = 0 \text{ for all } x \in B(2M))\, \mathbb{P}(E^{\mathrm{f},M}) > 0.$$

This provides the contradiction which proves (2.2).

Step 3: Our goal now is to prove (2.3). For this we will grow the cluster of x at level p_2 in a somewhat different way than done before. The idea is to start with the full percolation process at level p_1 given, and then grow the cluster of x at level p_2 on top of this process using the random field α. We will use the notation $\mathcal{S}(y)$ for the set of sites which contains the site y and all its neighbors.

First, in case $\alpha(x) = 0$ set

$$C_0^{p_2}(x) = C^{p_1}(x) \qquad \text{and} \qquad \partial C_0^{p_2}(x) = \emptyset,$$

while in case $\alpha(x) = 1$ set

$$C_0^{p_2}(x) = \bigcup_{y \in \mathcal{S}(x)} C^{p_1}(y) \qquad \text{and} \qquad \partial C_0^{p_2}(x) = \emptyset.$$

Then recursively proceed as follows. In the i-th step ask whether there is any site which is at distance 1 from $C_{i-1}^{p_2}(x)$ but is not in $\partial C_{i-1}^{p_2}(x)$. In case no such site exists, the construction has been completed, $C^{p_2}(x) = C_{i-1}^{p_2}(x)$ and $\partial C^{p_2}(x) = \partial C_{i-1}^{p_2}(x)$. Otherwise let X_i be the first such site and proceed as follows. In case $\alpha(X_i) = 0$, set

$$C_i^{p_2}(x) = C_{i-1}^{p_2}(x) \qquad \text{and} \qquad \partial C_i^{p_2}(x) = \partial C_{i-1}^{p_2}(x) \cup \{X_i\}.$$

In case $\alpha(X_i) = 1$, set

$$C_i^{p_2}(x) = C_{i-1}^{p_2}(x) \cup \Big(\bigcup_{y \in \mathcal{S}(X_i)} C^{p_1}(y) \Big) \qquad \text{and} \qquad \partial C_i^{p_2}(x) = \partial C_{i-1}^{p_2}(x).$$

Note that, as desired, the sites which will eventually be contained in $C_i^{p_2}(x)$ (resp. $\partial C_i^{p_2}(x)$) are precisely the sites of $C^{p_2}(x)$ (resp. $\partial C^{p_2}(x)$).

Let $I_1, I_2, \ldots$ be the sequence, in increasing order, of steps at which X_i is at distance 1 from an infinite η-cluster. The random variables $\alpha(X_{I_1}), \alpha(X_{I_2}), \ldots$ are then i.i.d., and each takes the value 1 with probability $(p_2 - p_1)/(1 - p_1)$. This sequence of random variables can be finite, but note that on the event E_x^∞ this sequence is infinite and moreover it is a.s. a sequence of 0s. Since this event has probability 0, we conclude that also $\mathbb{P}(E_x^\infty) = 0$. Summing over $x \in V$, we obtain (2.3).

■

Next we state and prove a lemma needed in Step 1 of the proof above. Note that it is not hard to see that the hypothesis on the graph G in this lemma cannot be relaxed to just requiring that G be of locally bounded degree (for a counterexample, attach leaves to each vertex of $\mathbb{Z}^2$, with the number of added leaves growing rapidly with the distance from the root).

Lemma 1.1. *For every infinite connected graph of uniformly bounded degree G, every $0 < p \le 1$ and every $N \in \{1, 2, 3, \ldots\}$, $\mathbb{P}_p$-a.s. every infinite cluster contains a ball of radius N.*

Proof. The assumption on G implies that for some finite $L(N)$ we have $|B(x, N)| \le L(N)$ for all $x \in V$.

Let $\eta = (\eta(x))_{x \in V}$ be a product random field with $\mathbb{P}(\eta(x) = 1) = p = 1 - \mathbb{P}(\eta(x) = 0)$. We consider percolation with respect to η.

The restriction of η to a set $S \subset V$ will be denoted by $\eta \mid_S$. The symbol η_S will denote the configuration which is identical to η on S and identically 0 on S^c.

For $x \in V$ let $E_{N,x}$ be the event that the cluster of x is infinite and does not contain any ball of radius N. We want to show that for all x

$$\mathbb{P}(E_{N,x}) = 0.$$

We will look successively inside the balls of center x and radii which are multiples of $4N$. Let $E_{N,x}^K$ be the event that in the configuration $\eta_{B(x,4NK)}$ the cluster of x contains some site at distance $4NK$ from x and contains no ball of radius N. Clearly

$$\mathbb{P}(E_{N,x}) = \lim_{K \to \infty} \mathbb{P}(E_{N,x}^K).$$

Therefore it is enough to show that for some $\epsilon_N > 0$, for all K,

$$\mathbb{P}(E_{N,x}^{K+1} | E_{N,x}^K) \le 1 - \epsilon_N. \tag{2.7}$$

The event $E_{N,x}^K$ can be written as $\{\eta \mid_{B(x,4NK)} \in \mathcal{R}_{N,x}^K\}$, for a certain set $\mathcal{R}_{N,x}^K \subset \{0,1\}^{B(x,4NK)}$. For each $a \in \mathcal{R}_{N,x}^K$,

$$\mathbb{P}(E_{N,x}^{K+1} \mid \eta \mid_{B(x,4NK)} = a) = \mathbb{P}(\eta \mid_{B(x,4N(K+1)) \setminus B(x,4NK)} \in \mathcal{S}_{N,x}^{K,a}),$$

for a certain set $\mathcal{S}_{N,x}^{K,a} \subset \{0,1\}^{B(x,4N(K+1)) \setminus B(x,4NK)}$. Therefore (2.7) will follow once we argue that for all $a \in \mathcal{R}_{N,x}^K$,

$$\mathbb{P}(\eta \mid_{B(x,4N(K+1)) \setminus B(x,4NK)} \in \mathcal{S}_{N,x}^{K,a}) \le 1 - \epsilon_N. \tag{2.8}$$

To see why (2.8) is true, we introduce the notation $a \star b$ to denote the configuration in $\{0,1\}^{B(x,4N(K+1))}$ which is identical to a on $B(x,4NK)$ and identical to b on $B(x,4N(K+1))\backslash B(x,4NK)$. Note that if $a \in \mathcal{R}^K_{N,x}$ and $b \in \mathcal{S}^{K,a}_{N,x}$, then in the configuration $a \star b$ there is a site at distance $4NK+2N$ from x which is connected to x by a chain of occupied sites. If there is more than 1 such site call the first one z (using an arbitrary order on V), and let b' be the configuration in $\{0,1\}^{B(x,4N(K+1))\backslash B(x,4NK)}$ which is obtained from b by making all the sites in $B(z,N)$ occupied. It is clear that $b' \notin \mathcal{S}^{K,a}_{N,x}$. Since the transformation $b \longrightarrow b'$ affects the configuration at at most $L(N)$ sites, and at most $2^{L(N)}$ configurations b will be mapped into the same b', one can easily conclude that (2.8) holds with $1-\epsilon_N = (1+(p/2)^{L(N)})^{-1}$. ■

3. Further results. The hypothesis of quasi-transitivity of the graph G is a very natural hypothesis in Theorem 1.1, Corollary 1.1 and Corollary 1.2. In the absence of this hypothesis it is not hard to construct examples of graphs which violate the conclusions of these statements. For instance, if a graph is obtained by connecting the root of a copy of $\mathbb{Z}^2$ to the root of a disjoint copy of $\mathbb{Z}^3$, then a.s. uniqueness of the infinite cluster occurs only for $p \in (p_c(\mathbb{Z}^3), p_c(\mathbb{Z}^2)]$. Still, there is a useful direction in which Theorem 1.1 can be extended, without making the proof more complicated. For this purpose say that on the graph $G=(V,E)$ there is uniform percolation at level p if

$$\lim_{N\to\infty} \inf_{x\in V} \theta_G(p, B(x,N)) = 1.$$

Obviously quasi-transitive graphs have this property at any level p at which $\theta_G(p) > 0$. An inspection of the proof of Theorem 1.1 shows that the hypothesis that G is quasi-transitive was only used to extract (2.6) from (2.4). It is clear that for this purpose uniform percolation at level p_1 is what is really needed. So we can state the following generalizations of Theorem 1.1, Corollary 1.1 and Corollary 1.2. The proofs of the corollaries from the theorem are identical to the ones explained in Section 1.

Theorem 3.1. *Suppose that G is an infinite connected graph of uniformly bounded degree, that $0 < p_1 < p_2 \leq 1$, and that there is uniform percolation at level p_1. Then a.s. all the infinite clusters at level p_2 are p_1-stable.*

Corollary 3.1. *Suppose that G is an infinite connected graph of uniformly bounded degree and that for all $p > p_c(G)$ there is uniform percolation at level p on G. Then for each site x, the function $\theta_G(p,x)$ is continuous on $(p_c(G),1]$.*

Corollary 3.2. *Suppose that G is an infinite connected graph of uniformly bounded degree, that $0 < p_1 < p_2 \leq 1$, that there is uniform percolation at level p_1 and that at level p_1 there is a.s. a unique infinite cluster. Then also at level p_2 there is a.s. a unique infinite cluster. In particular if for all $p > p_c(G)$ there is uniform percolation at level p on G, then for all $p > p_u(G)$ there is a.s. a unique infinite cluster.*

An interesting class of graphs which are not quasi-transitive, but for which there is uniform percolation at each level $p > p_c$ can be obtained as follows. For a pair of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, let $G_1 \times G_2$ be the graph which has as vertices the elements of the Cartesian product $V_1 \times V_2$ and an edge connects (x, u) to (y, v) iff either $x = y$ and $\{u, v\} \in E_2$ or else $\{x, y\} \in E_1$ and $u = v$. Suppose now that G_1 is a quasi-transitive graph and that G_2 is the semi-infinite linear chain, i.e., $V_2 = \{0, 1, 2, ...\}$ and edges connect vertices in this set which differ by 1. It is clear that in this case $G_1 \times G_2$ has the desired property.

Next we characterize p_u in terms of connectivities between large balls. For $S_1, S_2 \subset V$ let $\{S_1 \longleftrightarrow S_2\}$ be the event that there is a chain of occupied sites with one endpoint in the set S_1 and the other in the set S_2. Set now

$$p_{BB}(G) = \inf \left\{ p \colon \lim_{N \to \infty} \inf_{x,y \in V} \mathbb{P}_p(B(x, N) \longleftrightarrow B(y, N)) = 1 \right\}.$$

Note that clearly $p_{BB}(G) \geq p_c(G)$.

Theorem 3.2. *For every quasi-transitive graph G,*

$$p_u(G) = p_{BB}(G).$$

The proof of the harder part of this theorem is very similar to that of Theorem 1.1, so we simply sketch it next. This harder part amounts to proving that $p_u \leq p_{BB}$. Suppose that $p_{BB} < p_1 < p_2$. We want to show that at level p_2 there is then a.s. a unique infinite cluster. Theorem 1.1 tells us that each infinite cluster at level p_2 contains some infinite cluster at level p_1 and therefore, for our purpose it is enough to show that for every $x, y \in V$,

$$\mathbb{P}\left(|C^{p_1}(x)| = \infty, |C^{p_1}(y)| = \infty, \left(x \stackrel{p_2}{\longleftrightarrow} y \right)^c \right) = 0, \tag{2.1}$$

where $\{x \stackrel{p_2}{\longleftrightarrow} y\}$ is the event that at level p_2 there is a chain of occupied sites with one endpoint at x and the other endpoint at y. The proof of (2.1) can be obtained in three steps, which are similar to the three steps in the proof of Theorem 1.1.

An argument similar to the one in Step 1 shows that a.s. on the event that the clusters at level p_1 of two sites u and v are both infinite but distinct from each other, there will be some site u' in the boundary of $C^{p_1}(u)$ and some site v' in the boundary of $C^{p_1}(v)$ such that either u' and v' coincide, or they are neighbors to each other, or else a neighbor of u' and a neighbor of v' are connected by a chain of occupied sites at level p_1. For this use the random field η' to grow the cluster of u at level p_1, until a ball of radius N is contained in this cluster, then use the same random field to grow the cluster of v at level p_1, until a ball of radius N is also contained in this cluster, check then if in the random field η'' these two balls of radius N are connected by an occupied chain, finally obtain the random field η from the random fields η' and η'' in essentially the same way as in the proof of Theorem 1.1. Call the triple formed by the sites u', v' and the level p_1 occupied chain between neighbors of u' and v' a boundary connection at level p_1 between the two infinite clusters (the third component in this triple is the empty set in case u' and v' coincide, or if they are neighbors to each other).

Now an argument similar to that used in Step 2 shows that a.s. on the event that there exist distinct infinite clusters at level p_1, there is an unbounded sequence of such boundary connections at level p_1 between any two of these clusters. More precisely, for any two of these infinite clusters at level p_1, given any M, we can find a boundary connection at level p_1 between them outside of $B(M)$.

Step 3 is now slightly different from that in the proof of Theorem 1.1. Suppose that the clusters at level p_1 of the sites x and y are both infinite but distinct from each other. By the arguments sketched above, there is also a random sequence $M_1, M_2, \ldots$ such that in each set of the form $B(M_{i+1})\backslash B(M_i)$ there is a boundary connection at level p_1 between these infinite clusters at level p_1. By updating in increasing order of i, say, the probability of occupancy of all the sites in each set $B(M_{i+1})\backslash B(M_i)$, from p_1 to p_2, one can easily conclude that (2.1) holds.

For the proof that $p_u \geq p_{BB}$, let $p > p_u$. Then we know from Corollary 1.2 that at level p there is a.s. a unique infinite cluster and therefore, using the Harris inequality and the fact that G is quasi-transitive,

$$\begin{aligned}\mathbb{P}_p(B(x,N) \longleftrightarrow B(y,N)) &\geq \mathbb{P}_p(B(x,N) \text{ and } B(y,N) \text{ intersect the infinite cluster})\\ &\geq \theta(p, B(x,N))\,\theta(p, B(y,N)) \geq (\min_{z\in V_0} \theta(p, B(z,N)))^2.\end{aligned}$$

Hence

$$\lim_{N\to\infty} \inf_{x,y\in V} \mathbb{P}_p(B(x,N) \longleftrightarrow B(y,N)) = 1,$$

and therefore $p \geq p_{BB}$, completing the argumentation for Theorem 3.2.

It is very interesting to point out the way in which results in this paper are related to results and techniques in [SS1] and [SS2]. Theorem 1.1 and Theorem 3.1 had their proofs inspired by the proof of Theorem 3.4.1 in [SS2]. The definition of p_{BB} was motivated by relation (1.5) in [SS1], which was showed there to be equivalent to a contact process property denoted by *r&pc* and which in the case of quasi-transitive graphs was showed there to be equivalent to the well known property of complete convergence with survival, denoted *s&cc* in that paper. These facts were used in [SS1] to show that *s&cc* is a monotone increasing property of the infection parameter λ (the analogue of p in percolation) in the case of quasi-transitive graphs. Indeed, the analogies between complete convergence for the contact process and uniqueness of the infinite cluster in percolation are certainly worth further investigation.

References

[BLPS1] I. Benjamini, R. Lyons, Y. Peres and O. Schramm, *Group-invariant percolation on graphs*, preprint (1997).

[BLPS2] I. Benjamini, R. Lyons, Y. Peres and O. Schramm, *Critical percolation on any nonamenable group has no infinite clusters*, The Annals of Probability (to appear).

[BS] I. Benjamini and O. Schramm, *Percolation beyond* $\mathbb{Z}^d$*, many questions and a few answers*, Electronic Communications in Probability **1** (1996), 71–82.

[BK] J. van den Berg and M. Keane, *On the continuity of the percolation probability function*, in *Conference in Modern Analysis and Probability*, Contemporary Mathematics 26, R. Beals et al., editors, American Mathematical Society, Providence, RI (1984), 61–65.

[Gri] G. R. Grimmett, *Percolation*, Springer-Verlag, New York-Berlin, 1989.

[GN] G. R. Grimmett and C. M. Newman, *Percolation in* $\infty + 1$ *dimensions*, in *Disorder in physical systems*, G. R. Grimmett and D. J. A. Welsh, editors, Claremond Press, Oxford (1990), 219–240.

[HP] O. Häggström and Y. Peres, *Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously*, Probability Theory and Related Fields (to appear).

[JS] J. Jonasson and J. E. Steif, *Amenability and phase transition in the Ising model*, Journal of Theoretical Probability (to appear).

[Lal] S. P. Lalley, *Percolation on Fuchsian groups*, Annales de L'Institut Henri Poincaré (Probability and Statistics) **34** (1998), 151–177.

[NS] C. M. Newman and L. S. Schulman, *Infinite clusters in percolation models*, Journal of Statistical Physics **26** (1981), 613–628.

[NW] C. M. Newman and C. C. Wu, *Markov fields on branching planes*, Probability Theory and Related Fields **85** (1990), 539–552.

[SS1] M. Salzano and R. H. Schonmann, *The second lowest extremal invariant measure of the contact process*, The Annals of Probability **25** (1997), 1846–1871.

[SS2] M. Salzano and R. H. Schonmann, *The second lowest extremal invariant measure of the contact process II*, preprint (1997).

[ST] R. H. Schonmann and N. I. Tanaka, *Lack of monotonicity in ferromagnetic Ising model phase diagrams*, The Annals of Applied Probability **8** (1998), 234–245.

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