The Second-Order Equation from the $(1/2,0)\oplus(0,1/2)$ Representation of the Poincaré Group*

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On the basis of the first principles we derive the Barut-Wilson-Fushchich second-order equation in the $(1/2,0)\oplus(0,1/2)$ representation. Then we discuss the possibility of the description of various mass and spin states in such a framework.

Few decades ago Prof. A. Barut and collaborators proposed to use the four-dimensional representation of the $O(4,2)$ group in order to solve the problem of the lepton mass splitting [1–3]. Similar research has been produced by Prof. V. Fushchich and collaborators [4].

The most general conserved current that is linear in the generators of the four-dimensional representation of the group $O(4,2)$ was given as [1,2]

$$j_\mu = \alpha_1\gamma_\mu + \alpha_2 P_\mu + \alpha_3\sigma_{\mu\nu}q^\nu \ , \tag{1}$$

where $P_\mu = p_{1\mu}+p_{2\mu}$ is the total momentum, $q_\mu = p_{1\mu}-p_{2\mu}$ is the momentum transfer and α_i are real constant coefficients which may depend on the internal degrees of freedom of leptons. The Lagrangian formalism and the secondary quantization scheme have been given in ref. [2]. A. Barut derived the mass spectrum of leptons [3] after taking into account the additional postulate that one has to fix the value of the anomalous magnetic moment of the particle by its *classical* value $g=2(2\alpha/3)$, α is the fine structure constant.

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Recently, we have come across the flexibility of the introduction of similar constructs in the $(j,0)\oplus(0,j)$ representations from very different standpoints [5,6]. Starting from the explicit form of the spinors which are eigenespinors of the helicity operator [7] one can reveal *two non-equivalent* relations between zero-momentum 2-spinors. The first form of the Ryder-Burgard relation[1] was given in ref. [13] (h is a quantum number answering for the helicity; in general, see the cited papers for the notation) :

$$[\phi_L^h(\overset{\circ}{p}^\mu)]^* = e^{-2i\vartheta_h}\Xi_{[1/2]}\phi_L^h(\overset{\circ}{p}^\mu) \;; \tag{2}$$

the second form, in ref. [14]:

$$[\phi_L^h(\overset{\circ}{p}^\mu)]^* = (-1)^{\frac{1}{2}-h}e^{-i(\vartheta_1+\vartheta_2)}\Theta_{[1/2]}\phi_L^{-h}(\overset{\circ}{p}^\mu) \;. \tag{3}$$

The matrices are defined in the $(1/2,0)\oplus(0,1/2)$ representation as follows:

$$\Theta_{[1/2]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;, \quad \Xi_{[1/2]} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \;, \tag{4}$$

here ϕ is the azimuthal angle related with $\mathbf{p}\to\mathbf{0}$.

In this paper we advocate the generalized Ryder-Burgard relation:

$$\phi_L^h(\overset{\circ}{p}^\mu) = a(-1)^{\frac{1}{2}-h}e^{i(\vartheta_1+\vartheta_2)}\Theta_{[1/2]}[\phi_L^{-h}(\overset{\circ}{p}^\mu)]^* + be^{2i\vartheta_h}\Xi^{-1}_{[1/2]}[\phi_L^h(\overset{\circ}{p}^\mu)]^* \quad , \tag{5}$$

with the *real* constant a and b being arbitrary at this stage. The relevant relations for ϕ_R are obtained after taking into account that

$$\phi_L^\uparrow(p^\mu) = -\Theta_{[1/2]}[\phi_R^\downarrow(p^\mu)]^* \quad , \quad \phi_L^\downarrow(p^\mu) = +\Theta_{[1/2]}[\phi_R^\uparrow(p^\mu)]^* \quad , \tag{6a}$$

$$\phi_R^\uparrow(p^\mu) = -\Theta_{[1/2]}[\phi_L^\downarrow(p^\mu)]^* \quad , \quad \phi_R^\downarrow(p^\mu) = +\Theta_{[1/2]}[\phi_L^\uparrow(p^\mu)]^* \;, \tag{6b}$$

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[1]This name was introduced by D. V. Ahluwalia when considering the $(1,0)\oplus(0,1)$ representation [8]. If one uses $\phi_R(\overset{\circ}{p}^\mu) = \pm\phi_L(\overset{\circ}{p}^\mu)$, cf. also [9,10], after application Wigner rules for the boosts of the 2-spinors to the momentum p^μ, one immediately arrives at the Bargmann-Wightman-Wigner type quantum field theory, ref. [11] (cf. also [12]), in this representation.

which are easily derived by means of the consideration of the explicit form of 4-spinors, e. g., ref. [10].[2] Next we apply the procedure outlined in ref. [13, footnote # 1]. Namely,

$$
\begin{aligned}
\phi_{_L}^h(p^\mu) &= \Lambda_{_L}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu))\phi_{_L}^h(\overset{\circ}{p}{}^\mu) = \Lambda_{_L}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\Big\{a(-1)^{\frac{1}{2}-h}e^{i(\vartheta_1+\vartheta_2)}\Theta_{[1/2]}[\phi_{_L}^{-h}(\overset{\circ}{p}{}^\mu)]^* + \\
&+ be^{2i\vartheta_h}\Xi^{-1}_{[1/2]}[\phi_{_L}^h(\overset{\circ}{p}{}^\mu)]^*\Big\} = -ae^{i(\vartheta_1+\vartheta_2)}\Lambda_{_L}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\Lambda_{_R}^{-1}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\phi_{_R}^h(p^\mu) + \\
&+ be^{2i\vartheta_h}(-1)^{\frac{1}{2}+h}\Theta_{[1/2]}\Xi_{[1/2]}\phi_{_R}^{-h}(p^\mu)\;. \qquad (8)
\end{aligned}
$$

As a consequence of Eqs. (6a,6b,7,8) after the choice of the phase factors (e. g., $\vartheta_1 = 0$, $\vartheta_2 = \pi$) one has

$$
\phi_{_L}^h(p^\mu) = a\frac{p_0 - \boldsymbol{\sigma}\cdot\mathbf{p}}{m}\phi_{_R}^h(p^\mu) + b(-1)^{\frac{1}{2}+h}\Theta_{[1/2]}\Xi_{[1/2]}\phi_{_R}^{-h}(p^\mu)\;, \qquad (9)
$$

$$
\phi_{_R}^h(p^\mu) = a\frac{p_0 + \boldsymbol{\sigma}\cdot\mathbf{p}}{m}\phi_{_L}^h(p^\mu) + b(-1)^{\frac{1}{2}+h}\Theta_{[1/2]}\Xi_{[1/2]}\phi_{_L}^{-h}(p^\mu)\;. \qquad (10)
$$

Thus, the momentum-space Dirac equation is generalized:

$$
(a\frac{\widehat{p}}{m} - 1\!\!1)u_h(p^\mu) + ib(-1)^{\frac{1}{2}-h}\gamma^5\mathcal{C}u^*_{-h}(p^\mu) = 0\;, \qquad (11)
$$

where

$$
\mathcal{C} = \begin{pmatrix} 0 & i\Theta_{[1/2]} \\ -i\Theta_{[1/2]} & 0 \end{pmatrix}\;, \qquad (12)
$$

the 4×4 matrix, which enters in the definition of the charge conjugation operation. The counterpart in the coordinate space is

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[2]In fact, we have certain room in the definitions of the right spinors due to the arbitrariness of the phase factors at this stage. But, if one chosen the spinorial basis as in [10] one can find the relevant spinors in any frame after the application of the Wigner rules

$$
\phi_{_{R,L}}(p^\mu) = \exp(\pm\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}/2)\phi_{_{R,L}}(\overset{\circ}{p}{}^\mu) \quad . \qquad (7)
$$

for right (left) spinors); $\cosh(\varphi) = E/m$, $\sinh(\varphi) = |\mathbf{p}|/m$, with $\widehat{\varphi} = \mathbf{p}/|\mathbf{p}|$ is the unit vector. In the subsequent papers we shall consider different choices of the phase factors between left- and right- spinors.

$$\left[a\,\frac{i\gamma^\mu\partial_\mu}{m}+b\,\mathcal{CK}-\mathbb{1}\right]\Psi(x^\mu)=0\ , \tag{13}$$

provided that in the operator formulation the creation (annihilation) operators are connected by

$$b_\downarrow(p^\mu)=-ia_\uparrow(p^\mu)\quad,\quad b_\uparrow(p^\mu)=+ia_\downarrow(p^\mu)\ , \tag{14}$$

We denote $\mathcal{K}$ to be the operation of the complex conjugation but one should imply that it acts as the Hermitian conjugation on the creation (annihilation) operators in the $q-$ numbers theories.

On the other hand we are aware about the possibility of dividing the Dirac function into the real and imaginary part [15]. The transformation from the Weyl representation of the γ^μ matrices is made by means of the unitary 4×4 matrix

$$U=\frac{1}{2}\begin{pmatrix}\mathbb{1}-i\Theta_{[1/2]} & \mathbb{1}+i\Theta_{[1/2]}\\ -\mathbb{1}-i\Theta_{[1/2]} & \mathbb{1}-i\Theta_{[1/2]}\end{pmatrix}\quad,\quad U^\dagger=\frac{1}{2}\begin{pmatrix}\mathbb{1}-i\Theta_{[1/2]} & -\mathbb{1}-i\Theta_{[1/2]}\\ \mathbb{1}+i\Theta_{[1/2]} & \mathbb{1}-i\Theta_{[1/2]}\end{pmatrix}\ . \tag{15}$$

The similar transformations can also be applied to the Weinberg (or Weinberg-Hammer-Tucker) spin-1 equation in order to divide it into the real and imaginary parts [16]. Presumably, this is the general property of all fundamental wave equations.

Let us try to write the generalized coordinate-space Dirac equation (13) in the Majorana representation:

$$\left[a\,\frac{i\gamma^\mu\partial_\mu}{m}-b\,\mathcal{K}-\mathbb{1}\right]\Psi(x^\mu)=0\ . \tag{16}$$

We used that $U\mathcal{CK}U^{-1}=-\mathcal{K}$. After Majorana we note that γ^μ matrices become to be *pure imaginary* and the Dirac function can be divided into $\Psi\equiv\Psi_1+i\Psi_2$. Hence, we have a set of two *real* equations:

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m}-b-\mathbb{1}\right]\Psi_1(x^\mu)=0\ , \tag{17a}$$

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m}+b-\mathbb{1}\right]\Psi_2(x^\mu)=0\ . \tag{17b}$$

Adding and subtracting the obtained equations we arrived at the set ($\phi=\Psi_1+\Psi_2$ and $\chi=\Psi_1-\Psi_2$)

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} - \mathbb{1}\right]\phi - b\,\chi = 0\ , \tag{18a}$$

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} - \mathbb{1}\right]\chi - b\,\phi = 0\ , \tag{18b}$$

which after multiplication by $b \neq 0$ yield the same second-order equations for ϕ and χ:

$$\left[2a\frac{i\gamma^\mu\partial_\mu}{m} + a^2\frac{\partial^\mu\partial_\mu}{m^2} + b^2 - \mathbb{1}\right]\left\{\begin{array}{c}\phi(x^\mu)\\ \\ \chi(x^\mu)\end{array}\right. = 0 \quad . \tag{19}$$

With the identification $\frac{a}{2m} \to \alpha_2$ and $\frac{1-b^2}{2a}m \to \kappa$ we obtain the equation (6) of ref. [3a]. This is the equation which Barut used to obtain the mass difference between a muon and an electron.

On using the similar technique and after the consideration of different *chirality* sub-spaces as independent ones one can come to the Fushchich equation [4c]. It has the interaction with the 4-vector potential which is the same as the $j = 0$ Sakata-Taketani particle. This problem is closely related with the recent discussion of the Dirac equation with *two* mass parameters [17]. It seems to me that the theoretical difference (comparing with the first-order Dirac equation), which we obtain after switched-on interactions, is caused by the induced asymmetry between evolutions toward and backward in the time (or, which might be equivalently, between different chirality/helicity sub-spaces).

In conclusion, we can now assert that the Barut guess was not some puzzled coincidence. In fact, the second-order equation can be derived on the basis of the minimal number of postulates (Lorentz invariance and relations between 2-spinors in the zero-momentum frame) and it is the natural consequence of the general structure of the $(j,0)\oplus(0,j)$ representation.

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