On the Stone-von Neumann Uniqueness Theorem and Its Ramifications

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July, 1998

Abstract

A brief history of the Stone-von Neumann uniqueness theorem and its ramifications is provided. The influence of this theorem on the development of quantum theory, which was its initial source of motivation, is emphasized. In addition, its impact upon mathematics itself is suggested by considering certain subsequent developments in originally unanticipated directions.

1. Introduction

In the mid to late 1920's, the emerging theory of quantum mechanics had two main competing (and, initially, mutually antagonistic) formalisms — the wave mechanics of E. Schrödinger [60] and the matrix mechanics of W. Heisenberg, M. Born and P. Jordan [27][2][3].[1] Though a connection between the two was quickly pointed out by Schrödinger himself — see paper III in [60] — among others, the folk-theoretic "equivalence" between wave and matrix mechanics continued to generate more detailed study, even into our times. One outgrowth of this was associated with the canonical commutation relations (CCR):

$$PQ - QP = \frac{h}{2\pi i}\mathbb{1} \, , \tag{1}$$

which had begun to play such an important role in quantum theory [9][27][2][3] and were particularly central in the matrix mechanics approach.

*To be published in *John von Neumann and the Foundations of Quantum Mechanics* (tentative title), edited by M. Redei and M. Stoelzner.

[1]Significant portions of [2] were obtained independently by P.A.M. Dirac [9].

Schrödinger found a representation of (1) in the context of his wave mechanics in paper III of [60]. Given in modern language, his Q is the multiplication operator

$$(Q\Psi)(x) = x\Psi(x) \ , \ x \in \mathbb{R} \ ,$$

on $L^2(\mathbb{R})$ and P is the differential operator

$$(P\Psi)(x) = -i\frac{h}{2\pi}\frac{d\Psi}{dx}(x) \ , \ x \in \mathbb{R} \ ,$$

on $L^2(\mathbb{R})$. Born and Jordan [2] had found another with P and Q formal matrices with infinitely many entries. Jordan [33] subsequently made a heuristic argument to the effect that these two representations of (1) are, in fact, equivalent in the sense described below. If that were indeed the case, it would be a very powerful confirmation that the physical content of matrix mechanics and wave mechanics coincided, since all physically relevant quantities can be expressed in terms of P and Q. And that, in turn, would enable physicists to employ with confidence whichever approach was most convenient.

However, much work remained to be done before this assertion could be mathematically well-formulated and then proven rigorously. First, quantum theory needed to be formulated in Hilbert space, a crucial step begun by D. Hilbert himself [30],[2] made explicit by von Neumann in [44], and reached culmination in von Neumann's book [46].

Then, because there is no realization of P and Q satisfying (1) as bounded operators on Hilbert space [70][73][3], one needed to address the fact that (1) could not be understood as an operator equation on all of Hilbert space. This difficulty was side-stepped by reformulating the problem [69]: formally, if P and Q satisfy (1), then, with $U(a)$ and $V(a)$ defined by

$$U(a) = e^{\frac{-i2\pi aP}{h}} \text{ and } V(a) = e^{\frac{-i2\pi aQ}{h}} \ ,$$

it follows that, for any $a, b \in \mathbb{R}$,

$$U(a)V(b) = e^{\frac{i2\pi ab}{h}} V(b)U(a) \ . \tag{2}$$

This is the Weyl form of the CCR for one degree of freedom. $U(a)$ and $V(a)$ are, formally, unitary operators and therefore bounded; hence, (2) *may* be understood as an operator equation on all of the Hilbert space of states. In the Schrödinger representation we have

$$(U(a)\Psi)(x) = \Psi(x-a) \text{ and } (V(b)\Psi)(x) = e^{\frac{-i2\pi bx}{h}}\Psi(x) \ , \tag{3}$$

[2] In the winter term of 1926-27, Hilbert gave lectures on the new quantum mechanics which were prepared in collaboration with his assistants, L. Nordheim and J. von Neumann. The notes of the lectures were written out by Nordheim, with von Neumann's assistance, and were published in [30].

[3] Of course, the founders of quantum theory did not have these later results, but they had realized that all of their examples were unbounded.

for any $\Psi \in L^2(\mathbb{R})$, and these are, indeed, unitary operators on $L^2(\mathbb{R})$.

In 1930, M.H. Stone [65] stated[4] and, in 1931, von Neumann [45] proved the following theorem. Note that a representation of the Weyl form of the CCR is said to be irreducible if the only subspaces of the Hilbert space $\mathcal{H}$ of states left invariant by the operators $\{U(a) \mid a \in \mathbb{R}\} \cup \{V(a) \mid a \in \mathbb{R}\}$ are $\{0\}$ and $\mathcal{H}$ itself.

Theorem 1 *If $\{\tilde{U}(a) \mid a \in \mathbb{R}\}$ and $\{\tilde{V}(a) \mid a \in \mathbb{R}\}$ are (weakly continuous[5]) families of unitary operators acting irreducibly on a (separable[6]) Hilbert space $\mathcal{H}$ such that*

$$\tilde{U}(a)\tilde{U}(b) = \tilde{U}(a+b) \quad , \quad \tilde{V}(a)\tilde{V}(b) = \tilde{V}(a+b) \; ,$$
$$\tilde{U}(a)\tilde{V}(b) \;=\; e^{\frac{i2\pi ab}{h}}\tilde{V}(b)\tilde{U}(a) \; ,$$

then there exists a Hilbert space isomorphism[7] $W : \mathcal{H} \to L^2(\mathbb{R})$ such that

$$W\tilde{U}(a)W^{-1} = U(a) \text{ and } W\tilde{V}(a)W^{-1} = V(a) \; ,$$

for all $a \in \mathbb{R}$, where $U(a)$ and $V(a)$ are the Weyl unitaries in the Schrödinger representation defined in (3).[8] If $\{\tilde{U}, \tilde{V}, \mathcal{H}\}$ is not irreducible but $\mathcal{H}$ is separable, then $\mathcal{H}$ decomposes into a direct sum of countably many closed subspaces, on each of which the restriction of $\{\tilde{U}, \tilde{V}\}$ is once again unitarily equivalent to the Schrödinger representation $\{U, V, L^2(\mathbb{R})\}$.

Hence, every irreducible Weyl representation of the CCR for one degree of freedom is unitarily equivalent to the Weyl form of the Schrödinger representation, and this is true, up to multiplicity, for reducible representations, as well. It therefore follows that the physical content of the irreducible representation $\{\tilde{U}, \tilde{V}, \mathcal{H}\}$ is identical to that of the Schrödinger representation $\{U, V, L^2(\mathbb{R})\}$. This theorem is usually referred to in the literature as the Stone-von Neumann uniqueness theorem.[9]

This discussion has been presented for one degree of freedom, but it may be reformulated for any finite number of degrees of freedom, and, there again, any irreducible representation of the Weyl form of the CCR for n degrees of freedom is unitarily equivalent to the corresponding Schrödinger representation (with analogous results in the reducible case) [45]. Hence, if one is considering a quantum system with only finitely many degrees of freedom, then it matters not which representation one chooses to work in, and the "equivalence" of matrix

[4]with an indication of the elements of a proof

[5]As shown by von Neumann [47], this continuity assumption may be replaced by mere weak Lebesgue-measurability as long as the Hilbert space is separable.

[6]Note that if the weakly continuous Weyl representation $\{\tilde{U}, \tilde{V}, \mathcal{H}\}$ is irreducible, then $\mathcal{H}$ must be separable.

[7]For Hilbert spaces, an isomorphism is a one-to-one linear norm-preserving transformation from one Hilbert space onto the other.

[8]If $\mathcal{H}$ is not separable and $\tilde{U}, \tilde{V}$ are not weakly continuous, then the conclusion of this portion of the theorem is false [13].

[9]Many authors refer to it simply as the von Neumann uniqueness theorem.

mechanics and wave mechanics is even more tightly knit. This seemed to satisfy the founders of quantum mechanics, though, much later, mathematical physicists found some clouds in this apparently brilliant sky when they refocussed their attention on the dynamical variables P and Q, as we shall see.

Returning to the unbounded operators P and Q, it should be noted that F. Rellich [52], followed by many authors (see [50][34] for references and recent results), provided sufficient conditions on the canonical conjugates P and Q in a representation of the CCR (1) which ensured that they are unitarily equivalent to the corresponding operators in the Schrödinger representation. The strategy ordinarily adopted was to find conditions on P and Q so that they may be exponentiated in such a way that (2) holds, and then to appeal to Theorem 1. As a useful and representative result of this type, we mention J. Dixmier's theorem, once again stated here only for one degree of freedom.

Theorem 2 ([11]) *Let P and Q be closed symmetric operators in a Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be a dense, linear subspace of $\mathcal{H}$ contained in the domains of both P and Q such that $P\mathcal{D} \subset \mathcal{D}$ and $Q\mathcal{D} \subset \mathcal{D}$. If (1) holds on $\mathcal{D}$ and the restriction of $P^2 + Q^2$ to $\mathcal{D}$ is essentially self-adjoint, then $\mathcal{H}$ decomposes into a direct sum of closed subspaces, on each of which the restrictions of P and Q are unitarily equivalent to the corresponding operators in the Schrödinger representation.*

On the other hand, there are many results (see [50] and [56] for references) to the effect that even if P and Q are essentially self-adjoint on a common invariant dense domain $\mathcal{D}$, on which they satisfy (1), they need *not* be unitarily equivalent to the Schrödinger representation. In fact, K. Schmüdgen [56] has produced an uncountable set of pairwise inequivalent representations of this type! Of course, by Theorem 1, when these operators are exponentiated, the resulting unitaries do *not* satisfy (2). Are all of these examples physically pathological? And even if so, could there be others which are not? The answer to this latter question is positive. H. Reeh [51] has provided such an example arising in the description of a charged particle in the exterior of an infinitely long cylinder with a magnetic flux running through it. This is therefore a physically meaningful representation of the CCR with finitely many degrees of freedom (two, after the idealization of letting the radius of the cylinder go to zero) which is not unitarily equivalent to the corresponding Schrödinger representation. Seventy years ago, this example would have been a bombshell; however, now that the developments described in the next section have accustomed us to the nonequivalence of physically relevant representations, Reeh's example[10] was hardly noticed. Nonetheless, even physicists should be a bit more careful when they proclaim the equivalence of the Heisenberg and Schrödinger representations in their quantum mechanics lectures.

We have been led to representations of the Weyl form of the CCR through the physically motivated interest in representations of conjugate P's and Q's. However, physically interesting applications have been found for representations

[10] There may well be other such physically motivated examples in the literature; we apologize in advance for not being aware of them.

$\{U, V, \mathcal{H}\}$ in nonseparable Hilbert spaces which have no connection with unbounded operators satisfying (1) at all — see the recent preprint [7] for references. In such representations the functions $a \mapsto U(a)$ and $a \mapsto V(a)$ are not weakly continuous; these representations are called nonregular. In [7] is given a generalization of Theorem 1 to the case of weakly measurable nonregular representations, which is sufficient to subsume the known physical models. We shall say no more about this interesting line of development here.

In this introduction, the mathematical level of the discussion has been deliberately held low. This will not be possible in the balance of the paper. We shall first consider the consequences of the fact that the analogue of Theorem 1 for infinitely many degrees of freedom is false; indeed, in that case, there is an enormously infinite number of unitarily inequivalent representations of the CCR in the Weyl form and, therefore, also of the original CCR. This fact was only slowly and painfully realized, because physicists chose to ignore the restriction in the hypothesis of the Stone-von Neumann uniqueness theorem. We shall indicate how this obduracy was overcome and what mathematical physicists have discovered in their exploration of this rich set of inequivalent representations in both its mathematical and physical aspects. We shall also discuss the correct generalization of Theorem 1 to infinitely many degrees of freedom. Finally, to trace another line of influence of the Stone-von Neumann uniqueness theorem, we shall briefly describe certain generalizations and their role in the harmonic analysis of locally compact groups, which has found particular application in such diverse fields as number theory, imaging science, communication theory and data/signal analysis. However, given the limitations of space imposed upon us, we have here no ambitions of completeness.

2. Infinitely Many Degrees of Freedom

Though the hypothesis of Theorem 1 clearly restricts its import to finitely many degrees of freedom and close examination of von Neumann's proof makes it evident that the argument loses its mathematical validity when extended to infinitely many degrees of freedom, physicists have always trusted their physical "intuition" more than mathematical proof. Indeed, that which a physicist calls a proof is often viewed by a mathematician as a plausibility argument, at best. Physicists are, however, often justified in not waiting for the mathematicians, whose concern for rigor they regard with impatience, to firmly bolster the physicists' ideas. If they were to do so, the natural sciences would not have advanced as rapidly as they have. Significantly, physicists have a source of conviction which mathematicians do not: mathematically unconstrained speculations can be checked, to a certain extent, in the laboratory. Nonetheless, important aspects of physicists' theories of nature — their attempts to formalize the physical intuitions gleaned from the complex feedback loop between theory and experiment — have often enough either remained vague or revealed themselves to be incorrect, if not nonsensical.

An example of this is the physicists' long-lived belief, based upon their experience with systems having finitely many degrees of freedom and the Stone-von Neumann uniqueness theorem, that the choice of representation of the CCR was merely a matter of convenience — one only needed to keep track of the number of degrees of freedom. It was realized quite early that quantum field theory necessitated infinitely many degrees of freedom in its canonical variables (see already [10]). When dealing with infinitely many degrees of freedom, they worked exclusively in the representation of the CCR associated with a Hilbert space containing a dense set of states describing only finitely many particles. This representation emerged heuristically in the first papers on quantum field theory by Heisenberg and W. Pauli [28] and was later formalized more completely by V. Fock [16] (see [8] for the first mathematically rigorous and Poincaré covariant presentation of this representation, now usually called the Fock[11] representation). Since the Fock representation, using annihilation and creation operators and a distinguished vacuum vector, is so well-known, and it is equally well-known that the Schrödinger representation can be re-written as a Fock representation with only finitely many annihilation and creation operators, we shall not interrupt the flow of our story with the details (but see [4] or [12], if necessary).

The Fock representation was therefore viewed as the natural generalization of the Schrödinger representation to infinitely many degrees of freedom and inherited its royal mantle of distinction. Hence, quantum field models were written in the Fock representation by theoretical physicists, insofar as a representation was actually specified, with the firm belief that it was the only representation they needed.

It is an interesting aside that von Neumann apparently did not appreciate systems of infinitely many degrees of freedom. He wrote in his treatment of radiation in [46]:

> Nun ist es formal unbequem und bedenklich, Systeme mit unendlich vielen Freiheitsgraden bzw. Wellenfunktionen mit unendlich vielen Argumenten zuzulassen.[12]

In what is effectively the Fock representation, he therefore considered N degrees of freedom, computed energy spectrum, and then let $N \to \infty$. In order to compute this spectrum, von Neumann performed a canonical transformation[13] to obtain a second representation of the CCR in which the transformed Hamiltonian has a simpler form. For finite N this transformation is, by Theorem 1, unitarily implementable. However, in the limit $N \to \infty$ the transformation is not unitarily implementable and the representations are unitarily inequivalent. In other words, without realizing it, von Neumann himself worked with unitarily inequivalent

[11] or Fock-Cook, among mathematicians

[12] Now it is inconvenient formally and of doubtful validity to admit systems with infinitely many degrees of freedom, or wave functions with infinitely many arguments. — See page 141, resp. page 265, of [46].

[13] in mathematical terms, a symplectic transformation

representations of the CCR. His argument about the energy spectrum is therefore suspect.[14]

From the very beginning of the subject, quantum field theory was plagued by divergences; when one source of infinity was heuristically taken care of, yet another was stumbled upon. This became such an apparently insurmountable problem, that some of the founders became quite pessimistic (particularly Bohr and Dirac) and decided that yet another conceptual revolution would be required to transcend quantum field theory and avoid its apparently inherent problems. However, some researchers had not yet given up on the possibility of getting sensible answers from quantum field theory and were trying to discern and then engage the various sources of these infinities from increasingly profound starting points.

Of direct relevance to our story, L. van Hove examined a simple model and argued that the origin of the divergences of perturbation theory (which is always carried out in Fock space) could be located in the fact that the state vectors of the interacting model were "orthogonal" to the state vectors in Fock space. In modern terms, what he argued was that the folium of states[15] of the interacting model was disjoint from the folium of states of the Fock representation. An immediate consequence of this observation would have been that the interacting representation for his model was unitarily inequivalent to the Fock representation. He did not quite get to this point.[16]

Also in the early 1950's, K.O. Friedrichs [17] undertook an influential attempt to reduce the hand-waving typical of quantum field theory up to that time. For our purposes here, the result of greatest interest was his construction of some representations of the CCR for infinitely many degrees of freedom which were *not* unitarily equivalent to the Fock representation. As he wrote:

> Accordingly, there are different — non-equivalent — realizations of the basic field operators, and consequently different — non-equivalent — kinds of fields, a fact which seems worth noticing.[17]

In point of fact, he constructed representations in which the number operator does not exist (cf. the discussion further below).[18]

Though it would appear that not many theoretical physicists did take notice of Friedrichs' results, at least a handful of mathematical physicists and mathematicians were paying attention. In particular, in the following year L. Gårding and A.S. Wightman [22], taking their cue from Friedrichs and trying to clas-

[14]We return to this point below.

[15]*i.e.* the set of states determined by the density matrices on the Hilbert space of the given representation

[16]It is of interest to note that van Hove perceived a connection between his model and von Neumann's infinite product spaces [48]. With the benefit of hindsight, we see that he was anticipating the theory of infinite product representations of the CCR [35].

[17]See p. 3 of [17].

[18]It would seem that Friedrichs was not aware of either van Hove's example nor von Neumann's paper [48] when he did this work; they were mentioned only in the Comments and Corrections at the very end of the book [17], added after the work had been completed.

sify representations of the CCR using properties of a number operator,[19] proved that there exists a large class of inequivalent representations of the CCR for infinitely many degrees of freedom.[20] Indeed, it slowly emerged that there exists an unimaginably infinite number of inequivalent representations — the space of unitary equivalence classes of such representations cannot even admit a separable Borel structure [40][19]. The task of classifying these representations would thus appear to be hopeless.

Another researcher who reacted to the examples of van Hove and Friedrichs was R. Haag. Aware of these preceding works, he presented an argument to the effect that the interaction representation, widely in use in quantum field theory on the basis of its prior success in quantum mechanical scattering theory, did not exist unless there was no interaction at all! This important assertion found a number of mathematically rigorous formulations and proofs, which can, perhaps, be summarized into two types, represented in [66] and [12]. We state Haag's theorem in a somewhat restricted form along the lines of [66].

Theorem 3 (Haag's Theorem) *Let $\phi(x)$ be a free hermitian scalar field*[21] *of mass $m > 0$, and let $\psi(x)$ be an irreducible local Poincaré-covariant field. If $\phi(x)$ and $\psi(x)$, resp. the canonical conjugates $\dot{\phi}(x)$ and $\dot{\psi}(x)$, are unitarily equivalent at some time t, then $\psi(x)$ is also a free field of mass m.*

Of course, the indicated hypothesis holds for the "free" field and the "interacting" field in the interacting representation. This was extremely inconvenient for the then-standard scattering theory for quantum fields. But it is clear that Haag's theorem also implies that the representations which are of physical interest, precisely *because* they involve interaction, are to be found among those *inequivalent* to the Fock representation. Therefore, by 1955, both the existence and the necessity of using representations inequivalent to the Fock representation had been firmly established — though not established in all theorists' minds: as late as 1961, a standard text on quantum field theory [61] could present the old scattering theory in the interaction representation with no mention of Haag's theorem[22].

Before we turn to a recounting of the progress made in constructing representations inequivalent to the Fock representation, we answer the natural question: which representations *are* equivalent to the Fock representation? It was evident to Friedrichs that a necessary condition for this equivalence is the existence of a number operator in the representation. A series of papers followed Friedrichs' lead and gave successively more general, rigorously proven content to the assertion "a representation of the CCR is unitarily equivalent to the Fock representation if and only if the number operator exists as a densely-defined self-adjoint positive

[19] They also made use of von Neumann's paper on infinite products.

[20] As straightforward an operation as multiplying all the P's by 2 and all the Q's by $\frac{1}{2}$ produces a representation of the CCR which is unitarily inequivalent to the initial representation.

[21] and therefore irreducible, local and Poincaré-covariant

[22] and yet still cite [24] for other purposes!

operator in the representation." However, as was emphasized by J.M. Chaiken [5], this result is very sensitive to the definition of "number operator".[23]

The work of Gårding and Wightman did not provide an explicit construction of inequivalent representations. Wightman and S.S. Schweber [72] later constructed some classes of inequivalent representations of the CCR, as did I.E. Segal (see a later account [62] and the references given there). Many further classes of inequivalent representations have been constructed and brought under mathematical control since then. We mention the infinite product representations [35], coherent representations [36], quasi-free representations [54], quadratic representations [49] and higher-order representations [15]. These various classes of representations have found physical application and will surely prove to be of further use in the future.

But the most ambitious and difficult constructions of representations of the CCR have been carried out under the rubric "constructive quantum field theory." This work was motivated by the desire to mathematically construct the sort of representations the quantum field theorists were tacitly referring to; in other words, to give some mathematical meaning to the quantum field models at the center of the theorists' discourse. This latter goal has been approached from two different directions — on the one hand, various axiom systems have been erected which hope to subsume basic principles common to large classes of quantum fields: then theorems are proven to establish physically interesting properties of all quantum fields satisfying the given axioms; and on the other hand, concrete models have been constructed to show that the axiom systems are not vacuous: of course, in this connection, valiant efforts have been made to construct the standard models of the quantum field theorists. The axiomatic approach will not be further discussed here.[24] Instead, we shall briefly indicate those results of constructive quantum field theory which are of direct relevance to the topic at hand.

We first discuss J. Glimm and A. Jaffe's construction of the $(\phi^4)_2$-model [20]. Let $\mathcal{H}_0$ be the Fock space for a scalar hermitian Bose field $\phi(x,t)$ of mass $m > 0$. Let $\pi(x,t) = \partial\phi(x,t)/\partial t$ and $\mathcal{D} \subset \mathcal{H}_0$ be the dense set of finite-particle vectors in $\mathcal{H}_0$. Then, for every f in a dense subspace $\mathcal{S}(\mathbb{R})$ of $L^2(\mathbb{R})$, the operator $\phi(f) \equiv \phi(f,0) = \int \phi(x,0)f(x)dx$ is essentially self-adjoint on $\mathcal{D}$ and $\phi(f)\mathcal{D} \subset \mathcal{D}$ (similarly for $\pi(f)$). Then one has on $\mathcal{D}$ the CCR[25]

$$\phi(f)\pi(g) - \pi(g)\phi(f) = i < f, g > \mathbb{I} \, , \tag{4}$$

$$\phi(f)\phi(g) - \phi(g)\phi(f) = 0 = \pi(f)\pi(g) - \pi(g)\pi(f) \, , \tag{5}$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. When exponentiated, these operators provide a Weyl representation of the CCR. For each bounded open subset $\mathbf{O} \subset \mathbb{R}$, denote by $\mathcal{A}(\mathbf{O})$

[23] See [4] for further references and [55] for a recent paper on those representations which have a "generalized number operator".

[24] We refer the interested reader to [66][25] and also, for a historical overview, [71].

[25] As is customary in quantum field theory, we adopt physical units in which $c = h/2\pi = 1$.

the von Neumann algebra[26] generated by the Weyl unitaries

$$\{e^{i\phi(f)}, e^{i\pi(f)} \mid f \in \mathcal{S}(\mathbb{R}) \ , \ \mathrm{supp}(f) \subset \mathbf{O}\} \ .$$

Note that, though there are many C^*-algebras associated with the CCR in the Fock representation,[27] they all have the same weak closure.

The total energy

$$H_0 = \frac{1}{2}\int \ : (\pi(x,0)^2 + \nabla\phi(x,0)^2 + m^2\phi(x,0)^2) : dx$$

of this field is a positive quadratic form on $\mathcal{D} \times \mathcal{D}$ and therefore determines uniquely a self-adjoint operator, which we also denote by H_0. With $g \in L^2(\mathbb{R})$ nonnegative of compact support, Glimm and Jaffe showed that, for each $\lambda > 0$, the cut-off interacting Hamilton operator

$$H(g) \equiv H_0 + \lambda \int : \phi(x,0)^4 : g(x)dx$$

is essentially self-adjoint on $\mathcal{D}$,[28] and its self-adjoint closure, also denoted by $H(g)$, is bounded from below. By adding a suitable multiple of the identity we may take 0 to be the minimum of its spectrum. Then, 0 is a simple eigenvalue of $H(g)$ with normalized eigenvector $\Omega(g) \in \mathcal{H}_0$.

For any $t \in \mathbb{R}$, let $\mathbf{O}_t$ denote the subset of $\mathbb{R}$ consisting of all points with distance less than $|t|$ to $\mathbf{O}$. By choosing the cutoff function g to be equal to 1 on $\mathbf{O}_t$, then for any $A \in \mathcal{A}(\mathbf{O})$ the operator

$$\sigma_t(A) \equiv e^{itH(g)}Ae^{-itH(g)}$$

is independent of g and is contained in $\mathcal{A}(\mathbf{O}_t)$. For any bounded open $\mathcal{O} \subset \mathbb{R}^2$ and $t \in \mathbb{R}$, let $\mathbf{O}(t) = \{x \in \mathbb{R} \mid (x,t) \in \mathcal{O}\}$ be the time t slice of $\mathcal{O}$. We define $\mathcal{A}(\mathcal{O})$ to be the von Neumann algebra generated by $\bigcup_s \sigma_s(\mathcal{A}(\mathbf{O}(s)))$.[29] Finally, we let $\mathcal{A}$ denote the closure in the operator norm of the union $\bigcup \mathcal{A}(\mathcal{O})$ over all open bounded $\mathcal{O} \subset \mathbb{R}^2$. Hence, σ_t is an automorphism on $\mathcal{A}$ and implements the time evolution associated with the interacting field. Similarly, "locally correct" generators for the Lorentz boosts and the spatial translations can be defined,

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[26] so-called because they were introduced in [43]

[27] As opposed to the case of the canonical anticommutation relations, the C^*-algebra obtained here depends upon the choice of the dense subspace $\mathcal{S}(\mathbb{R})$ of test functions — see [4] for a detailed discussion of this point. However, once the dense subspace of test functions has been fixed, the closure $\mathcal{A}_0$ in the operator norm of the algebra generated by the set $\{e^{i\phi(f)}, e^{i\pi(f)} \mid f \in \mathcal{S}(\mathbb{R})\}$ has the property that to each, not necessarily continuous representation of the CCR, (4)-(5), there corresponds a representation of $\mathcal{A}_0$ [64]. Moreover, $\mathcal{A}_0$ is simple [42][64], so it is representation-independent — see the discussion further below.

[28] Without the cutoff g, the interacting Hamilton operator is *not* densely defined in Fock space.

[29] One can then show that the algebra $\mathcal{A}(\mathcal{O})$ coincides with the von Neumann algebra generated by bounded functions of the self-adjoint field operators $\int \phi(x,t)f(x,t)dxdt$, with test functions $f(x,t)$ having support in $\mathcal{O}$.

resulting in an automorphic action on $\mathcal{A}$ of the entire Poincaré group in two spacetime dimensions.

For each $A \in \mathcal{A}$, we set $\omega_g(A) =< \Omega(g), A\Omega(g) >$ to define the locally correct vacuum state ω_g of the interacting field. Taking a limit as the cutoff function g approaches the constant function 1, Glimm and Jaffe showed that $\omega_g(A) \to \omega(A)$, for each $A \in \mathcal{A}$, defines a new (locally normal) state ω on $\mathcal{A}$ which is Poincaré invariant. By the GNS construction one then obtains a new Hilbert space $\mathcal{H}$, a representation ρ of $\mathcal{A}$ as a C^*-algebra acting on $\mathcal{H}$, and a vector $\Omega \in \mathcal{H}$ such that $\rho(\mathcal{A})\Omega$ is dense in $\mathcal{H}$ and

$$\omega(A) =< \Omega, \rho(A)\Omega > \; , \text{ for all } A \in \mathcal{A} \, .$$

In addition, one obtains a strongly continuous unitary representation of the Poincaré group in two spacetime dimensions under which the algebras $\rho(\mathcal{A}(\mathcal{O}))$ transform covariantly. The axioms of both the algebraic [25] and the field approach [66] have been verified for this model.

It is in this representation $(\rho, \mathcal{H})$ that the field equations for this model find a mathematically satisfactory interpretation [59]. And it is to the physically significant quantities in this representation that perturbation theory in λ is asymptotic — see the discussion in [21]. For this and other reasons, ω is interpreted as the exact vacuum state in the interacting theory, and its folium of states contains the physically admissible states of the interacting theory.

The generators of the strongly continuous Abelian unitary groups $\{\rho(e^{it\phi(f)}) \mid t \in \mathbb{R}\}$ and $\{\rho(e^{it\pi(f)}) \mid t \in \mathbb{R}\}$ satisfy the CCR (4). However, this representation of the CCR in $\mathcal{H}$ is not unitarily equivalent to the initial representation in Fock space. Indeed, by taking different values of the coupling constant λ in the above construction, one obtains an uncountably infinite family of mutually inequivalent representations of the CCR (4) (see [18])![30]

Similar constructions with similar results have been carried out for general polynomial interactions $P(\phi)$ and for the Yukawa model, both in two spacetime dimensions. For the sake of technical convenience, these constructions were redone in the Euclidean approach, and many additional models were constructed in that manner.[31] Although the program of constructing the standard models of quantum field theory in four spacetime dimensions is not completed, the lessons taught by the constructions of interacting field models in lower dimensions cannot be overlooked. In particular, quantum fields with different interactions are associated with mutually inequivalent representations of the CCR, which are in turn inequivalent to the Fock representation. The choice of representation would thus appear to be quite significant. Tersely summarized, one could say that the kinematics of the physical system fixes the CCR-algebra and the dynamics determines the representation.[32]

[30] Of relevance to quantum field theory, but particularly to quantum statistical mechanics, which also must face systems with infinitely many degrees of freedom [4], is the observation that also equilibrium states at different temperatures are associated with mutually inequivalent representations [67]. The Fock representation is associated with temperature zero.

[31] For an introduction to this work, as well as further references, see [21].

[32] In fact, in a certain sense, the representation also determines the dynamics — see [1].

Though the original problem was stated in terms of unitary equivalence, there are, in fact, weaker notions of equivalence which are also of physical relevance. A notion of physical equivalence introduced by Haag and D. Kastler [26] will be discussed next.

Since one can carry out only finitely many experiments which themselves have only a finite accuracy, the experimental situation strictly limits our ability to test the many idealizations which are implicit in any physical theory and which are particularly strongly present in quantum mechanics and quantum field theory. These limitations on measurement and the statistical interpretation of the basic objects in the theory induce a natural topology on the set of states on the algebra of observables $\mathcal{A}$. Let $\{A_i\}_{i=1}^n \subset \mathcal{A}$ be a set of observables of a system which has been prepared in the state ω. Let $\{a_i\}_{i=1}^n$ be their measured average values to within the (respective) errors $\{\epsilon_i\}_{i=1}^n$. Hence one has the n inequalities

$$|\omega(A_i) - a_i| < \epsilon_i \ , \ i = 1, \ldots, n \ .$$

On the other hand, recall that the $\sigma(\mathcal{A}^*, \mathcal{A})$-topology on the set $\mathcal{A}^*$ of all continuous, linear complex-valued functions on $\mathcal{A}$ is generated by the seminorms $N_A(\omega) \equiv |\omega(A)|$, for each $A \in \mathcal{A}$. In other words, the $\sigma(\mathcal{A}^*, \mathcal{A})$-topology is the locally convex topology with basis of neighborhoods at the origin given by

$$\{\mathcal{N}_{\{A_i\}_{i=1}^n, \{\epsilon_i\}_{i=1}^n} \mid n \in \mathbb{N}, \{A_i\}_{i=1}^n \subset \mathcal{A}, \{\epsilon_i\}_{i=1}^n \subset (0, \infty)\} \ ,$$

where

$$\mathcal{N}_{\{A_i\}_{i=1}^n, \{\epsilon_i\}_{i=1}^n} = \{\omega \in \mathcal{A}^* \mid |\omega(A_i)| < \epsilon_i, i = 1, \ldots, n\} \ .$$

Thus we see that any experiment (or set of experiments, necessarily finite) determines the state of the system only up to a neighborhood in the $\sigma(\mathcal{A}^*, \mathcal{A})$-topology.

For purely mathematical reasons, J.M.G. Fell introduced the following notion of equivalence of representations. If $(\rho, \mathcal{H})$ is a representation of $\mathcal{A}$, then its kernel is given by $\mathrm{Ker}(\rho) = \{A \in \mathcal{A} \mid \rho(A) = 0\}$.

Definition([14]) *Two representations* $(\rho_1, \mathcal{H}_1)$ *and* $(\rho_2, \mathcal{H}_2)$ of $\mathcal{A}$ *are said to be weakly equivalent if* $\mathrm{Ker}(\rho_1) = \mathrm{Ker}(\rho_2)$.

Unitary equivalence implies weak equivalence, but the converse is false. Fell showed that two representations $(\rho_1, \mathcal{H}_1)$ and $(\rho_2, \mathcal{H}_2)$ of $\mathcal{A}$ are weakly equivalent if and only if given every state ω_1 on $\mathcal{A}$ determined by a density matrix on $\mathcal{H}_1$ and given any $\sigma(\mathcal{A}^*, \mathcal{A})$-neighborhood $\mathcal{N}$ of ω_1, there exists a state $\omega_2 \in \mathcal{N}$ determined by a density matrix on $\mathcal{H}_2$. In other words, the $\sigma(\mathcal{A}^*, \mathcal{A})$-closure of the folium of states associated with $(\rho_1, \mathcal{H}_1)$ coincides with the $\sigma(\mathcal{A}^*, \mathcal{A})$-closure of the folium associated with $(\rho_2, \mathcal{H}_2)$.

Therefore, if the kernels of two representations of $\mathcal{A}$ coincide, then it is physically impossible to determine which representation one is in (and conversely)! But if $(\rho, \mathcal{H})$ is a representation of $\mathcal{A}$, then $\mathrm{Ker}(\rho)$ is a norm-closed two-sided ideal of $\mathcal{A}$. Thus it follows that whenever $\mathcal{A}$ is simple, every representation of $\mathcal{A}$ must be faithful, and hence all representations of a simple algebra $\mathcal{A}$ are physically equivalent. What is more, in quantum field theory the quasilocal algebras $\mathcal{A}$ are typically simple!

So, have we returned to the physicists' original point of view - the choice of representation is just a matter of convenience, even in systems with infinitely many degrees of freedom? Not exactly! Let us posit, once again, that we have chosen observables $\{A_i\}_{i=1}^n \subset \mathcal{A}$ and made measurements with results $\{a_i\}_{i=1}^n$ to within errors $\{\epsilon_i\}_{i=1}^n$, thereby determining a $\sigma(\mathcal{A}^*, \mathcal{A})$-neighborhood $\mathcal{N}$ of the actual state ω, normal in the true physical representation $(\rho, \mathcal{H})$. If $\mathcal{A}$ is simple, then Fell's theorem entails that we can find a state $\rho_{\mathcal{N}}$, normal in any other fixed representation of $\mathcal{A}$, which is contained in $\mathcal{N}$ and therefore yields predictions conforming with the results of this experiment. But the moment we improve the experiment, *i.e.* reduce the errors, or we change the experiment to include another set of observables (but still preparing the system in the same original state), then the neighborhood $\mathcal{N}$ changes (though ω does not change), and we must find *another* approximate state in the given "wrong" representation to reproduce the results of the new experiment. In other words, in order to have (correct) predictive power beyond the particular experiments to which one fitted the approximate state, one *needs* the correct state in the correct representation. This, surely, is not merely a matter of convenience![33]

J. Manuceau's [42] and J. Slawny's [64] observation that the minimal C^*-algebra $\mathcal{A}_0$ associated with the CCR is simple and hence all representations of $\mathcal{A}_0$ are isomorphic can be seen as the *correct* generalization of Theorem 1 to infinitely many degrees of freedom. The existence of this algebraic isomorphism implies unitary equivalence in the finite case but *not* in the infinite case. This independence of representation enables the rigorous study of the canonical transformations commonly employed in theoretical physics (by von Neumann himself — see further above) as automorphisms on the C^*-algebra $\mathcal{A}_0$. In a given representation of $\mathcal{A}_0$, the given automorphism may or may not be unitarily implementable. If it is, then the original Hamiltonian operator will have the same spectrum as the transformed (diagonalized) one; if not, then there need be no relation between the spectra of these operators.

To close this section, we remark that the significance of the Stone-von Neumann uniqueness theorem is further emphasized by the fact that for the other important types of algebraic relations — such as the canonical anticommutation relations and, more recently, supersymmetric commutation relations, p-adic commutation relations, and the deformed commutation relations of quantum groups — one of the first questions addressed is the validity of the counterpart of Theorem 1 in the given setting. For further reading, we mention the papers [57][37][23][32].

3. Generalizations to the Harmonic Analysis of Locally Compact Groups

[33] It is evident from this discussion that it is impossible to prove experimentally that a putative exact state is the correct one (and, thus, that the correct representation has been chosen). But at least it is logically possible to establish experimentally that it is *not* the correct one (if, in fact, it is not).

In 1949, G.W. Mackey [39] provided a generalization of the Stone-von Neumann uniqueness theorem to the setting of locally compact groups, which itself found many applications in mathematics and elsewhere and which may justifiably be seen as yet another impact of von Neumann's work. For simplicity, we shall restrict our attention to Abelian groups, though Mackey formulated and proved an analogous result for arbitrary locally compact groups. With G a locally compact Abelian group and μ a Haar measure on G, one can naturally define the Hilbert space $L^2(G, d\mu)$. If G^* is the topological character group of G, *i.e.* each $\tau \in G^*$ is a continuous homomorphism from G into the multiplicative group of complex numbers of modulus 1, and G^* is endowed with the natural induced topology (so that it, too, becomes a locally compact Abelian group), then the analogue of the Weyl form of the Schrödinger representation is described by the following unitary operators on $L^2(G, d\mu)$:

$$(U_S(g)\Psi)(x) = \Psi(g^{-1}x) \text{ and } (V_S(\tau)\Psi)(x) = \tau(x)\Psi(x) \ , \tag{6}$$

for any $\Psi \in L^2(G, d\mu)$. (Compare with (3).)

Theorem 4 ([39]) *Let G be an arbitrary separable*[34] *locally compact Abelian group, and let G^* be its topological character group. Let U be a weakly continuous representation of G in the (separable) Hilbert space $\mathcal{H}$. If V is a weakly continuous representation of G^* on $\mathcal{H}$ such that $U(g)V(\tau) = \tau(g)V(\tau)U(g)$, for all $g \in G$ and $\tau \in G^*$, then $\mathcal{H}$ decomposes into a direct sum of at most countably many closed subspaces $\mathcal{H}_n$, each invariant under $\{U(g) \mid g \in G\} \cup \{V(\tau) \mid \tau \in G^*\}$. Moreover, letting U_n, resp. V_n, denote the restriction of U, resp. V, to $\mathcal{H}_n$, there exists a Hilbert space isomorphism $W_n : \mathcal{H}_n \to L^2(G, d\mu)$ with*

$$W_n U_n(g) W_n^{-1} = U_S(g) \text{ and } W_n V_n(\tau) W_n^{-1} = V_S(\tau) \ ,$$

for all $g \in G$ and $\tau \in G^*$.

Subsequently, arguments which were more elementary than Mackey's original proof were found, as well as some additional reformulations - see, *e.g.* [63][53]. Theorem 4 is often called the Stone-von Neumann-Mackey theorem.[35] It has been placed by Mackey into the context of his theory of induced representations and there was seen to be a consequence of his imprimitivity theorem. The interested reader is referred to [41] for an introduction to this circle of ideas.

Theorem 1 is obtained as a special case of Theorem 4 by choosing G to be the additive group of reals (for more than one degree of freedom, G is chosen to be the additive group of vectors $\mathbb{R}^n$). Note that, in that case, G^* is isomorphic to G itself.

[34] Loomis [38] later showed that the assumption of separability of G could be dropped.

[35] It may be of interest to note that Theorem 4 was evoked in J. Slawny's proof [64] of the existence and properties of the minimal C^*-algebra associated with the CCR, which was mentioned in the previous section.

From Theorem 4 follows one of the most useful theorems in Abelian harmonic analysis, which is in turn a generalization of the crucial Plancherel theorem in Fourier analysis (a special instance of Abelian harmonic analysis). (See [63] for a proof.)

Theorem 5 *Let G be a locally compact Abelian group. Given any element $f \in L^1(G, d\mu) \cap L^2(G, d\mu)$, its Fourier transform $\hat{f}$, defined by*

$$\hat{f}(\tau) = \int \tau(g) f(g) d\mu(g) \ ,$$

is in $L^2(G^*, d\mu^*)$, and the mapping $f \mapsto \hat{f}$ extends uniquely to a Hilbert space isomorphism from $L^2(G, d\mu)$ onto $L^2(G^*, d\mu^*)$ (with suitable normalization of the Haar measure μ^*).

From this then follows the generalized Riemann-Lebesgue lemma: the Fourier transform of an integrable function on a locally compact Abelian group G vanishes at infinity on G^*. It is surely evident by now how central a result the Stone-von Neumann-Mackey theorem is in Abelian harmonic analysis.

To take yet another perspective on this topic, consider the n-dimensional Heisenberg group, which is the universal covering group of the non-Abelian group of unitary operators on $L^2(\mathbb{R}^n)$ generated by the translations

$$T_p f(x) = f(x+p) \ , \ p \in \mathbb{R}^n \ ,$$

and the multiplications

$$M_q f(x) = e^{iq \cdot x} f(x) \ , \ q \in \mathbb{R}^n \ .$$

It is evident from the discussion in the introduction that the Stone-von Neumann uniqueness theorem may be used to classify the irreducible representations of the Heisenberg group. This again permits the proof of a corresponding Plancherel theorem, *etc.*. We refer the reader to [68] for a development of this theory, as well as indications of the many sorts of applications which have arisen. Here we only mention one buzzword: wavelets.

Finally, just to hint at further realms, we mention that Theorem 1 has also found applications to number theory (see, for example, [6]), function theory (see [58]) and invariant subspace theory(see [29]).

Acknowledgements: We have found [12] and [71] particularly useful during the preparation of this paper and further recommend them to the reader's attention.

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