Power Law Subordinacy and Singular Spectra
I. Half Line Operators

Svetlana Jitomirskaya
Department of Mathematics
University of California
Irvine, California 92697

and

Yoram Last
Department of Mathematics
California Institute of Technology
Pasadena, California 91125

Abstract. We present an extension of the Gilbert-Pearson theory of subordinacy, which relates Hausdorff-dimensional spectral properties of one-dimensional Schrödinger operators to the behavior of solutions of the corresponding Schrödinger equation. We use this theory to analyze these properties for concrete examples. In particular, for every $\alpha \in (0,1)$ we construct potentials for which the spectral measure is of exact dimension α.

1. Introduction

In this paper we study one-dimensional Schrödinger operators on the "half line." We mainly discuss discrete operators on $\ell^2(\mathbb{Z}^+)$, defined by

$$(H_\theta\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n) \tag{1.1}$$

along with a phase boundary condition

$$\psi(0)\cos\theta + \psi(1)\sin\theta = 0, \tag{1.2}$$

where $-\pi/2 < \theta < \pi/2$. The potential $V = \{V(n)\}_{n=1}^{\infty}$ is a sequence of real numbers. While we discuss such discrete operators, our main results (namely, Theorems 1.1 and 1.2 below) are also valid for their continuous analogs of the form $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}^+)$, as long as the potential $V(x)$ is such that we are in the limit point case (so the operator is essentially self adjoint). The proofs for the discrete and continuous cases are essentially the same.

While we are mostly interested here in operators of the form (1.1), our core results are valid (and will be proven) for more general tridiagonal operators of the form

$$(H_\theta\psi)(n) = a(n)\psi(n+1) + a(n-1)\psi(n-1) + b(n)\psi(n), \tag{1.3}$$

where the $b(n)$'s are real numbers and the $a(n)$'s are real and $a(n) \neq 0 \;\; \forall n$. Moreover, we assume that $\sum_{n=1}^{\infty} |a(n)|^{-1} = \infty$, which is sufficient to insure that these operators are essentially self-adjoint [3]. The study of an operator of the form (1.3) along with the boundary condition (1.2) is equivalent to the study of this operator with a Dirichlet boundary condition,

$$\psi(0) = 0\,, \qquad \psi(1) = 1, \tag{1.4}$$

along with a rank-one perturbation at the origin

$$b(1) \to b(1) - a(0)\tan\theta. \tag{1.5}$$

Thus, without loss, we confine our discussion to operators of the form (1.3) acting on $\ell^2(\mathbb{Z}^+)$ (with $\mathbb{Z}^+ = \{1,2,3,\dots\}$ and thus $a(n)$ and $b(n)$ being defined for $n = 1,2,3,\dots$) along with the boundary condition (1.4) (so these are simply semi-infinite tridiagonal matrices), and interpert the boundary phase θ as the modification of $b(1)$ given by (1.5) (where we are free to choose $a(0)$, so we take $a(0)=1$). Our H_θ thus has the form

$$H_\theta \quad = \quad \begin{pmatrix} b(1)-\tan\theta & a(1) & 0 & 0 & 0 & \cdots \\ a(1) & b(2) & a(2) & 0 & 0 & \cdots \\ 0 & a(2) & b(3) & a(3) & 0 & \cdots \\ 0 & 0 & a(3) & b(4) & a(4) & \cdots \\ 0 & 0 & 0 & a(4) & b(5) & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

For such operators, the vector δ_1 ("the delta function at the origin") is cyclic, and so the spectral problem reduces to the study of the single spectral measure $\mu \equiv \mu_{\delta_1}$. The study of μ is related to the study of the Weyl-Titchmarsh m-function, which coincides with the Borel transform of μ, namely,

$$m(z) = \int \frac{d\mu(x)}{x-z}. \tag{1.6}$$

Our main goal here is to extend the Gilbert-Pearson theory of subordinacy [6,5] (also see [10] for the discrete case) which relates spectral properties of the operator (1.3) to properties of solutions of the corresponding Schrödinger equation

$$a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) = Eu(n). \tag{1.7}$$

Recall that a solution u of (1.7) is called *subordinate* if

$$\lim_{L\to\infty} \frac{\|u\|_L}{\|v\|_L} = 0 \tag{1.8}$$

for any other solution v of (1.7), where $\|\cdot\|_L$ denotes the norm of the solution over a lattice interval of length L. That is, we define

$$\|u\|_L \equiv \left[\sum_{n=1}^{[L]} |u(n)|^2 + (L-[L])|u([L]+1)|^2 \right]^{1/2}, \tag{1.9}$$

where $[L]$ denotes the integer part of L.

The Gilbert-Pearson theory relates the Lebesgue decomposition of the spectral measure μ to subordinacy of solutions as follows: The absolutely-continuous part of μ is supported on the set of energies E for which (1.7) has no subordinate solutions. (In fact, this set of energies is, up to a set of both Lebesgue and spectral measure zero, the set where μ has a finite non-vanishing derivative. That is, it can be identified as the essential support of the absolutely continuous part of μ.) The singular part of μ is supported on the set of energies for which the solutions which obey the appropriate boundary condition (namely, a Dirichlet boundary condition) are subordinate. (Moreover, this set coincides with the set of energies where μ has an infinite derivative, and can thus be identified as the essential support of the singular part of μ.)

Our primary purpose here is to provide tools for answering somewhat more delicate spectral questions. Explicitly, those arising when one tries to distinguish between different kinds of singular continuous spectra based on the classification of singular continuous measures with respect to Hausdorff measures and dimensions. Those questions are relevant to the study of quantum dynamics, as non-integral spectral Hausdorff dimensions are often connected with anomalous transport properties (see, e.g., [13]).

Given an H_θ of the form defined by (1.3), and $E \in \mathbb{R}$, we let u_1 be the solution of (1.7) which obeys the Dirichlet boundary condition

$$u_1(0) = 0\,, \qquad u_1(1) = 1, \tag{1.10}$$

and let u_2 be the solution of (1.7) which obeys the (orthogonal) boundary condition

$$u_2(0) = 1\,, \qquad u_2(1) = 0. \tag{1.11}$$

Now given any $\epsilon > 0$, we define a length $L(\epsilon) \in (0, \infty)$, by requiring the equality

$$\|u_1\|_{L(\epsilon)} \|u_2\|_{L(\epsilon)} = \frac{1}{2\epsilon}. \tag{1.12}$$

We note (see (1.15) below) that at most one of either u_1 or u_2 may be in ℓ^2, and so the l.h.s. of (1.12) is a monotonely increasing continuous function of L which vanishes for $L = 1$ and goes to infinity as L goes to infinity. Similarly, the r.h.s. of (1.12) is a monotonely decreasing continuous function of ϵ which goes to infinity as ϵ goes to 0. Thus the function $L(\epsilon)$ is well defined (by (1.12)), and it is a monotonely decreasing continuous function which goes to infinity as ϵ goes to 0. Our core result is the following inequality which relates the Weyl-Titchmarsh m-function (for z in the upper half plane) to the solutions u_1 and u_2.

Theorem 1.1. *Let H_θ be of the form* (1.3)*, and let $E \in \mathbb{R}$, $\epsilon > 0$ be given. Then the following inequality holds*

$$\frac{5-\sqrt{24}}{|m(E+i\epsilon)|} < \frac{\|u_1\|_{L(\epsilon)}}{\|u_2\|_{L(\epsilon)}} < \frac{5+\sqrt{24}}{|m(E+i\epsilon)|}.$$

¿From Theorem 1.1 (along with the theory of rank-one perturbations [18]), one can easily recover the original results of Gilbert-Pearson [6]. Moreover, our proof of this theorem is somewhat simpler than their analysis, so that we obtain a simplification (on top of a strengthening) of their theory. This is enabled in part due to our definition of $L(\epsilon)$ by (1.12).

A particular consequence of Theorem 1.1 is:

Theorem 1.2. *Let H_θ be of the form* (1.3)*, and let $E \in \mathbb{R}$ and $\alpha \in (0,1)$. Then*

$$\limsup_{\epsilon\to 0} \frac{\mu((x-\epsilon,\, x+\epsilon))}{(2\epsilon)^\alpha} = \infty$$

if and only if

$$\liminf_{L\to\infty} \frac{\|u_1\|_L}{\|u_2\|_L^\beta} = 0,$$

where $\beta = \alpha/(2-\alpha)$.

As we shall discuss in Section 2, properties such as singularity and continuity of measures with respect to dimensional Hausdorff measures are determined by local scaling properties of the kind that appear in Theorem 1.2. Thus, this theorem provides an effective tool for the analysis of such properties for concrete Schrödinger operators, as long as the nature of solutions of the corresponding Schrödinger equations is sufficiently well understood. It can also be combined with two further basic facts. The first is the existence of generalized eigenfunctions [3,14,17], from which it is known that for a.e. E with respect to the spectral measure μ, the solution u_1 must obey

$$\limsup_{L\to\infty} \frac{\|u_1\|_L}{L^{1/2}\ln L} < \infty \tag{1.13}$$

and

$$\liminf_{L\to\infty} \frac{\|u_1\|_L}{L^{1/2}} < \infty. \tag{1.14}$$

The second is the constancy of the Wronskian

$$a(n)(u_1(n+1)u_2(n) - u_2(n+1)u_1(n)) = 1 \ \ \forall n, \tag{1.15}$$

which implies that

$$\|u_1\|_L \|u_2\|_L \geq \frac{1}{2} \sum_{n=1}^{[L]-1} |a(n)|^{-1} + (L-[L])|a([L])|^{-1}. \tag{1.16}$$

As we shall see below, the combination of (1.13)–(1.16) with Theorem 1.2 leads to a number of fairly general results from which one can deduce spectral results using only partial information about the asymptotic behavior of solutions.

As a primary example of using Theorem 1.2, we shall apply it to study a family of Schrödinger operators of the form (1.1) with sparse barrier potentials. More explicitly, we consider potentials which vanish for all n's outside a sparse (fastly growing) sequence of points $\{L_n\}_{n=1}^{\infty}$ where $|V(L_n)| \to \infty$ as $n \to \infty$. Simon-Spencer [21] have shown that the Schrödinger operators corresponding to such potentials have no absolutely-continuous spectrum; and Gordon [7] has shown that if the $|V(L_n)|$'s grow sufficiently fast (compared to the growth of the L_n's), then for (Lebesgue) a.e. boundary phase θ, the corresponding operators have pure-point spectrum with exponentially decaying eigenfunctions (also see [11]). It is easy to see [22], however, that if the L_n's grow sufficiently fast (compared to the growth of the $|V(L_n)|$'s) then, for every boundary phase θ, the spectrum in $(-2,2)$ is purely singular-continuous; and Simon [19] has recently shown that if the growth is even faster, then the spectrum in $(-2,2)$ is purely one-dimensional, in the sense that the spectral measure does not give weight to sets of Hausdorff dimension less than 1. Here we will show:

Theorem 1.3. *Let $\alpha \in (0,1)$. Let $L_n = 2^{(n^n)}$ and define a potential $V(k)$ for $k > 0$ by $V(L_n) = L_n^{(1-\alpha)/(2\alpha)}$; $V(k) = 0$ if $k \notin \{L_n\}_{n=1}^{\infty}$. For each $\theta \in (-\pi/2, \pi/2)$, let H_θ be the Schrödinger operator on $\ell^2(\mathbb{Z}^+)$ defined by (1.1),(1.2). Then*

(i) *For every boundary phase θ, the spectrum of H_θ consists of the interval $[-2,2]$ (which is the essential spectrum) along with some discrete point spectrum outside this interval.*

(ii) *For every* θ, *the Hausdorff dimensionality of the spectrum in* $(-2,2)$ *is bounded between dimensions* α *and* $\beta \equiv 2\alpha/(1+\alpha)$, *in the sense that the restriction of the spectral measure to* $(-2,2)$ *is supported on a set of Hausdorff dimension* β *and does not give weight to sets of Hausdorff dimension less than* α.

(iii) *For Lebesgue a.e.* θ, *the spectrum in* $[-2,2]$ *is of exact dimension* α, *namely, the restriction of the spectral measure to* $[-2,2]$ *is supported on a set of Hausdorff dimension* α *and does not give weight to sets of Hausdorff dimension less than* α.

Remarks.

1. The result only requires the L_n's to be sufficiently sparse (namely, to grow sufficiently fast). The precise condition that our proof needs is that $\prod_{k=1}^{n-1} L_k < L_n^{\epsilon}$, where ϵ can be chosen arbitrarily small for n sufficiently large. $L_n = 2^{(n^n)}$ is just a particular choice for which this property holds.
2. As far as we know, Theorem 1.3 gives the first rigorous example of a Schrödinger operator with non-trivial exact spectral dimension.

The main results of this paper where previously announced in [8] (more explicitly, our Theorems 1.2 and 1.3 are essentially Theorems 1 and 2 of [8]). In a forthcoming paper [9] we will provide the technical details of extending the power-law subordinacy ideas introduced here to handle whole-line problems, and apply it to various quasiperiodic operators. The applications will include, in particular, the proof of zero-dimensionality of the spectral measure of the almost Mathieu operator for couplings above the critical (for *all* irrational rotations and *all* phases), and certain spectral continuity of the Fibonacci Hamiltonian, that is, proofs of Theorems 3 and 4 of [8].

The rest of this paper is organized as follows. In Section 2 we review some basic facts concerning Hausdorff measures and dimensional spectral properties. In Section 3 we prove Theorems 1.1 and 1.2. In Section 4 we discuss some general consequences of Theorem 1.2, and in Section 5 we prove Theorem 1.3.

2. Dimensional Spectral Properties

Recall that for any subset S of $\mathbb{R}$ and $\alpha \in [0,1]$, the α-dimensional Hausdorff measure, h^α, is given by

$$h^\alpha(S) \equiv \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha \,, \tag{2.1}$$

where a δ-cover is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ. (Technically, we consider h^α as being defined by (2.1) also for real α's outside [0,1], but the resulting h^α's are trivial in such case.) h^α, as defined by (2.1), is an outer measure on $\mathbb{R}$ and its restriction to Borel sets is a Borel measure. h^1 coincides with the Lebesgue measure and h^0 is the counting measure (assigning to each set the number of points in it), such that the family $\{h^\alpha \,|\, 0 \leq \alpha \leq 1\}$ can be viewed as a way of continuously interpolating between the counting measure and the Lebesgue measure. Given any $\emptyset \neq S \subseteq \mathbb{R}$, there exists a unique $\alpha(S) \in [0,\, 1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. This unique $\alpha(S)$ is called the Hausdorff dimension of S. A rich theory of decomposing measures with respect to Hausdorff measures had been developed by Rogers and Taylor [15,16]. Here we only discuss a small part of it. A much more detailed description had been given by Last [13].

Given α, a measure μ is called α-continuous (αc) if $\mu(S) = 0$ for every set S with $h^\alpha(S) = 0$. It is called α-singular (αs) if it is supported on some set S with $h^\alpha(S) = 0$. We say that μ is one-dimensional (od) if it is α-continuous for every $\alpha < 1$. We say that it is zero-dimensional (zd) if it is α-singular for every $\alpha > 0$. Measure μ is said to have exact dimension α if, for every $\epsilon > 0$, it is both $(\alpha - \epsilon)$-continuous and $(\alpha + \epsilon)$-singular.

Given a (positive, finite) measure μ and $\alpha \in [0, 1]$, we define

$$D_\mu^\alpha(x) \equiv \limsup_{\epsilon \to 0} \frac{\mu((x - \epsilon,\, x + \epsilon))}{(2\epsilon)^\alpha} \tag{2.2}$$

and $T_\infty \equiv \{x \,|\, D_\mu^\alpha(x) = \infty\}$. The restriction $\mu(T_\infty \cap \cdot\,) \equiv \mu_{\alpha s}$ is α-singular, and $\mu((\mathbb{R} \setminus T_\infty) \cap \cdot\,) \equiv \mu_{\alpha c}$ is α-continuous. Thus, each measure decomposes uniquely into an α-continuous part and an α-singular part: $\mu = \mu_{\alpha c} + \mu_{\alpha s}$. Moreover, an α-singular measure must have $D_\mu^\alpha(x) = \infty$ a.e. (with respect to it) and an α-continuous measure must have $D_\mu^\alpha(x) < \infty$ a.e.. It is important to note that $D_\mu^\alpha(x)$ is defined with a $\limsup$. The corresponding limit need not exist.

Consider now a separable Hilbert space $\mathcal{H}$ and a self-adjoint operator H. For each $\psi \in \mathcal{H}$, we denote by μ_ψ the spectral measure for ψ (and H). We let $\mathcal{H}_{\alpha c} \equiv \{\psi \,|\, \mu_\psi \text{ is } \alpha\text{-continuous}\}$ and $\mathcal{H}_{\alpha s} \equiv \{\psi \,|\, \mu_\psi \text{ is } \alpha\text{-singular}\}$. $\mathcal{H}_{\alpha c}$ and $\mathcal{H}_{\alpha s}$ are mutually orthogonal closed subspaces which are invariant under H, and $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{H}_{\alpha c} \oplus \mathcal{H}_{\alpha s}$. The α-continuous spectrum ($\sigma_{\alpha c}$) and α-singular spectrum ($\sigma_{\alpha s}$) are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\sigma = \sigma_{\alpha c} \cup \sigma_{\alpha s}$. Thus, the standard spectral theoretical scheme which uses the Lebesgue decomposition of a Borel measure into absolutely-continuous, singular-continuous, and pure-point parts can be extended to include further decompositions with respect to Hausdorff measures.

As described in [13], the full picture is somewhat richer than discussed above. For every dimension $\alpha \in (0,1)$, there is a natural unique decomposition (of a σ-finite Borel measure μ on $\mathbb{R}$) into five parts: one below the dimension α, one above it, and three within it—of which the middle one is absolutely continuous with respect to h^α. Furthermore, this picture can be extended to consider more general Hausdorff measures (namely, ones that do not come from a power law) and families of such measures—as originally discussed by Rogers-Taylor [15,16]. All of these measure decompositions lead to corresponding Hilbert space spectral decompositions. An important point for us in the current paper is that continuity and singularity with respect to Hausdorff measures are completely determined from the a.e. local scaling behavior of the measure. Knowing $D_\mu^\alpha(x)$ for every α in $[0,1]$ and a.e. x with respect to μ completely determines μ's decomposition with respect to dimensional Hausdorff measures. Knowing only the local dimension

$$\alpha_\mu(x) \equiv \liminf_{\epsilon\to 0} \frac{\log(\mu((x-\epsilon,\, x+\epsilon)))}{\log \epsilon} \tag{2.3}$$

(for a.e. x with respect to μ) determines its decomposition with respect to Hausdorff dimensions. In particular, μ is of exact dimension α if and only if $\alpha_\mu(x) = \alpha$ a.e. with respect to it.

3. Proof of Theorems 1.1 and 1.2

Let $z = E + i\epsilon$, and consider the equation

$$a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) = zu(n). \tag{3.1}$$

It is known that for $\epsilon > 0$, (3.1) has a unique (up to normalization) solution $\tilde{u}_z$ which is ℓ^2 at infinity, and moreover,

$$m(z) = -\frac{\tilde{u}_z(1)}{\tilde{u}_z(0)}. \tag{3.2}$$

We normalize $\tilde{u}_z$ by letting $\tilde{u}_z(0) = 1$, such that we have $\tilde{u}_z(1) = -m(z)$.

By considering an ℓ^2 solution u in (3.1), multiplying both sides of the equation by $u^*(n)$ (the complex conjugate of $u(n)$), taking imaginary parts, and summing both sides from 1 to ∞, we obtain the equality $\mathrm{Im}\,(a(0)u(0)u^*(1)) = \epsilon\sum_{n=1}^{\infty}|u(n)|^2$, which for $u = \tilde{u}_z$ implies

$$\frac{\mathrm{Im}\,m(z)}{\epsilon} = \sum_{n=1}^{\infty}|\tilde{u}_z(n)|^2. \tag{3.3}$$

We also have the following:

Lemma 3.1. *For any $n > 0$, $\tilde{u}_z(n)$ obeys the equality*

$$\tilde{u}_z(n) = u_2(n) - m(z)u_1(n) - i\epsilon u_2(n)\sum_{k=1}^{n}u_1(k)\tilde{u}_z(k) + i\epsilon u_1(n)\sum_{k=1}^{n}u_2(k)\tilde{u}_z(k), \tag{3.4}$$

where u_1 and u_2 are the solutions of equation (1.7) *obeying the boundary conditions* (1.10) *and* (1.11)*, respectively.*

Remark. Note that u_1 and u_2 in (3.4) are solutions of (1.7), namely, they solve (3.1), but with $\epsilon = 0$.

Proof. Let $\tilde{v}(n)$ be the r.h.s. of (3.4) for $n > 0$ and let $\tilde{v}(0) = 1$. By taking into account the Wronskian conservation (1.15), it is easy to verify that $\{\tilde{v}(n)\}_{n=0}^{\infty}$ obeys

$$a(n)\tilde{v}(n+1) = -a(n-1)\tilde{v}(n-1) + (E - b(n))\tilde{v}(n) + i\epsilon\tilde{u}_z(n) \tag{3.5}$$

for any $n > 0$. Since for $n > 0$,

$$a(n)\tilde{u}_z(n+1) = -a(n-1)\tilde{u}_z(n-1) + (E + i\epsilon - b(n))\tilde{u}_z(n), \tag{3.6}$$

and since (by (1.10), (1.11)) $\tilde{v}(0) = \tilde{u}_z(0)$, $\tilde{v}(1) = \tilde{u}_z(1)$, we see by induction that $\tilde{v}(n) = \tilde{u}_z(n)$ for any $n \geq 0$. $\square$

Proof of Theorem 1.1. ¿From (3.4) we see that for any $L > 1$,

$$\|\tilde{u}_z\|_L \geq \|u_2 - m(z)u_1\|_L - 2\epsilon\|u_1\|_L\|u_2\|_L\|\tilde{u}_z\|_L. \tag{3.7}$$

By considering $L = L(\epsilon)$, which implies $2\epsilon\|u_1\|_L\|u_2\|_L = 1$, we obtain

$$2\|\tilde{u}_z\|_{L(\epsilon)} \geq \|u_2 - m(z)u_1\|_{L(\epsilon)}. \tag{3.8}$$

Squaring the two sides of (3.8) and noting that by (3.3)

$$\|\tilde{u}_z\|^2_{L(\epsilon)} < \|\tilde{u}_z\|^2_\infty = \frac{\operatorname{Im} m(z)}{\epsilon}, \tag{3.9}$$

we obtain

$$\begin{aligned} \frac{4\operatorname{Im} m(z)}{\epsilon} &> \|u_2 - m(z)u_1\|^2_{L(\epsilon)} \\ &\geq \|u_2\|^2_{L(\epsilon)} + |m(z)|^2\|u_1\|^2_{L(\epsilon)} - 2|m(z)|\|u_2\|_{L(\epsilon)}\|u_1\|_{L(\epsilon)}. \end{aligned} \tag{3.10}$$

Now by (1.12), we have $\|u_2\|_{L(\epsilon)}\|u_1\|_{L(\epsilon)} = 1/2\epsilon$, and so multiplying the two sides of (3.10) by 2ϵ yields

$$8\operatorname{Im} m(z) > \frac{\|u_2\|_{L(\epsilon)}}{\|u_1\|_{L(\epsilon)}} + |m(z)|^2\frac{\|u_1\|_{L(\epsilon)}}{\|u_2\|_{L(\epsilon)}} - 2|m(z)|, \tag{3.11}$$

which implies

$$|m(z)|^2\frac{\|u_1\|_{L(\epsilon)}}{\|u_2\|_{L(\epsilon)}} - 10|m(z)| + \frac{\|u_2\|_{L(\epsilon)}}{\|u_1\|_{L(\epsilon)}} < 0. \tag{3.12}$$

Solving (3.12) as a quadratic inequality for the variable $|m(z)|$, one obtains

$$(5-\sqrt{24})\frac{\|u_2\|_{L(\epsilon)}}{\|u_1\|_{L(\epsilon)}} < |m(z)| < (5+\sqrt{24})\frac{\|u_2\|_{L(\epsilon)}}{\|u_1\|_{L(\epsilon)}}, \tag{3.13}$$

from which Theorem 1.1 follows. □

Proof of Theorem 1.2. For any two functions f and g of ϵ, we write $f \sim g$ if there are positive constants C_1 and C_2 such that $C_1 f < g < C_2 f$ for all $\epsilon > 0$. Using (1.12) and Theorem 1.1, we obtain

$$\epsilon^{1-\alpha}|m(E+i\epsilon)| \sim \|u_2\|^{\alpha-1}_{L(\epsilon)}\|u_1\|^{\alpha-1}_{L(\epsilon)}\frac{\|u_2\|_{L(\epsilon)}}{\|u_1\|_{L(\epsilon)}} = \left(\frac{\|u_2\|^\beta_{L(\epsilon)}}{\|u_1\|_{L(\epsilon)}}\right)^{2-\alpha}, \tag{3.14}$$

from which we see that

$$\limsup_{\epsilon\to 0} \epsilon^{1-\alpha}|m(E+i\epsilon)| = \infty \quad \Leftrightarrow \quad \liminf_{L\to\infty} \frac{\|u_1\|_{L(\epsilon)}}{\|u_2\|^\beta_{L(\epsilon)}} = 0. \tag{3.15}$$

At the same time, it is shown in [4] that

$$\limsup_{\epsilon\to 0} \epsilon^{1-\alpha}|m(E+i\epsilon)| = \infty \quad\Leftrightarrow\quad \limsup_{\epsilon\to 0} \frac{\mu((x-\epsilon,\, x+\epsilon))}{(2\epsilon)^\alpha} = \infty, \tag{3.16}$$

and so we obtain Theorem 1.2. $\square$

4. Some Consequences of Theorem 1.2

Our goal in this section is to discuss some consequences of Theorem 1.2 and the general principles (1.13)–(1.16) which permit one to deduce spectral results using only partial information about the asymptotic behavior of solutions. We start with the following:

Corollary 4.1. *Suppose that for some $\beta > 1$ and every E in some Borel set A, (1.7) has a solution v obeying*

$$\limsup_{L\to\infty} \frac{\|v\|_L^2}{L^\beta} > 0,$$

and let $\alpha = 2/(1+\beta)$. Then, for every $\epsilon > 0$, the restriction $\mu(A\cap\cdot\,)$ is $(\alpha+\epsilon)$-singular.

Proof. μ is supported on the set of energies E for which u_1 obeys (1.13), and so we only need to consider such E's. For every $E \in A$, v is some linear combination of u_1 and u_2, say $v = au_1 + bu_2$, such that for every L, $\|v\|_L \le |a|\|u_1\|_L + |b|\|u_2\|_L$. For every E where (1.13) holds, we see that we must have $b \neq 0$, and so $\|u_2\|_L \ge (\|v\|_L - |a|\|u_1\|_L)/|b|$ for every L. Thus, we must also have

$$\limsup_{L\to\infty} \frac{\|u_2\|_L^2}{L^\beta} > 0 \tag{4.1}$$

for such E's. Let $\epsilon > 0$ be given, and let

$$\gamma \equiv \frac{\alpha+\epsilon}{2-\alpha-\epsilon} = \frac{1+\epsilon(1+\beta)/2}{\beta-\epsilon(1+\beta)/2} > \frac{1}{\beta}. \tag{4.2}$$

Then we see that for a.e. E w.r.t. $\mu(A\cap\cdot\,)$, there is a constant C such that

$$\liminf_{L\to\infty} \frac{\|u_1\|_L}{\|u_2\|_L^\gamma} \le \liminf_{L\to\infty} \frac{CL^{1/2}\ln L}{L^{\gamma\beta/2}} = \liminf_{L\to\infty} CL^{(1-\gamma\beta)/2}\ln L = 0. \tag{4.3}$$

By Theorem 1.2, this implies that for a.e. E w.r.t. $\mu(A\cap\cdot)$, $D_\mu^{\alpha+\epsilon}(E)=\infty$, and so we obtain the corollary. $\square$

It is often convenient, in one-dimensional spectral theory, to formulate results in terms of the 2×2 transfer matrices $\Phi_n(E)\equiv T_n(E)T_{n-1}(E)\cdots T_1(E)$, where

$$T_n(E)\equiv\begin{pmatrix}\frac{E-V(n)}{a(n)} & -\frac{a(n-1)}{a(n)}\\ 1 & 0\end{pmatrix}. \tag{4.4}$$

The $\Phi_n(E)$'s are related to solutions of equation (1.7) by

$$\begin{pmatrix}u(n+1)\\ u(n)\end{pmatrix}=\Phi_n(E)\begin{pmatrix}u(1)\\ u(0)\end{pmatrix}, \tag{4.5}$$

and so

$$\Phi_n(E)=\begin{pmatrix}u_1(n+1) & u_2(n+1)\\ u_1(n) & u_2(n)\end{pmatrix}. \tag{4.6}$$

With these notations, Corollary 4.1 can now be reformulated as

Corollary 4.2. *Suppose that for some $\beta>1$ and every E in some Borel set A,*

$$\limsup_{L\to\infty}\frac{1}{L^\beta}\sum_{n=1}^{L}\|\Phi_n(E)\|^2>0,$$

and let $\alpha=2/(1+\beta)$. Then, for every $\epsilon>0$, the restriction $\mu(A\cap\cdot)$ is $(\alpha+\epsilon)$-singular.

Proof. ¿From (4.6) we see that $\|\Phi_n(E)\|^2\le|u_1(n+1)|^2+|u_1(n)|^2+|u_2(n+1)|^2+|u_2(n)|^2$, and thus

$$\sum_{n=1}^{L}\|\Phi_n(E)\|^2\le 2(\|u_1\|_{L+1}^2+\|u_2\|_{L+1}^2). \tag{4.7}$$

Since for a.e. E w.r.t. μ (1.13) holds, we deduce that for such E's, (4.1) must hold. Thus Corollary 4.1 implies Corollary 4.2. $\square$

Recall that for an equation of the form (1.7), the upper Lyapunov exponent $\overline{\gamma}(E)$ is defined by

$$\overline{\gamma}(E)\equiv\limsup_{n\to\infty}(1/n)\ln\|\Phi_n(E)\|.$$

A rather soft application of Corollary 4.2 gives

Corollary 4.3. *Suppose that $\overline{\gamma}(E) > 0$ for every E in some Borel set A, then the restriction $\mu(A \cap \cdot)$ is zero-dimensional.*

Proof. $\overline{\gamma}(E) > 0$ implies that $\limsup_{L\to\infty} \frac{1}{L^\beta} \sum_{n=1}^{L} \|\Phi_n(E)\|^2 > 0$ for any $\beta > 0$. Thus, by Corollary 4.2, $\mu(A \cap \cdot)$ is α-singular for any $\alpha > 0$. $\square$

Example. Consider an operator of the form (1.1) with $V(n) = \lambda \cos(2\pi\omega n + \varphi)$, where $\lambda > 2$, $\varphi \in [0, 2\pi]$, and ω is an irrational. We will show in [9] that in such a case, $\overline{\gamma}(E) > 0$ for any $E \in \mathbb{R}$. Thus, by Corollary 4.3, we obtain that for every boundary phase θ, the corresponding spectral measure μ is zero-dimensional.

Corollary 4.1 and its derivatives, Corollaries 4.2 and 4.3, say that the existence of fastly growing solutions imply corresponding singularity of the spectral measure. A weak inverse of this fact is given by the following

Corollary 4.4. *For any $L > 0$, let*

$$R_L \equiv \frac{1}{2} \sum_{n=1}^{[L]-1} |a(n)|^{-1} + (L - [L])|a([L])|^{-1}.$$

Suppose that for some $1 \le \beta < 2$ and every E in some Borel set A, every solution v of (1.7) *obeys*

$$\limsup_{L\to\infty} \frac{\|v\|_L^2}{R_L^\beta} < \infty.$$

Then the restriction $\mu(A \cap \cdot)$ is $(2-\beta)$-continuous.

Remark. Note that for $\beta = 1$ and $a(n) = 1$, $\forall n$, Corollary 4.4 implies in particular that energies for which (1.7) has only bounded solutions must be associated with the absolutely-continuous part of the spectral measure μ. This is a known fact, which is an immediate consequence of the Gilbert-Pearson theory, although it can also be shown by different means [20]. Corollary 4.4 is the natural generalization of this fact.

Proof. Let $E \in A$. By (1.16), we have $\|u_1\|_L \|u_2\|_L \ge R_L$, and since $\|u_2\|_L^2 < C R_L^\beta$ for some constant C, we see that $\|u_1\|_L > C^{-1/2} R_L^{1-\beta/2}$. Thus, if

$$\gamma \equiv \frac{2-\beta}{2-(2-\beta)} = \frac{2-\beta}{\beta}, \tag{4.8}$$

we have that

$$\frac{\|u_1\|_L}{\|u_2\|_L^\gamma} > C^{-(1+\gamma)/2} R_L^{1-\beta/2-\gamma\beta/2} = C^{-1/\beta} > 0. \tag{4.9}$$

By Theorem 1.2, it follows that $\mu(A \cap \cdot\,)$ is $(2-\beta)$-continuous. $\square$

Another general application of Theorem 1.2 relates averaged decay of generalized eigenfunctions (if it happens to occur) to dimensional Hausdorff properties:

Corollary 4.5. *Let R_L be as in Corollary 4.4. Suppose that*

$$\liminf_{L\to\infty} \frac{\|u_1\|_L^2}{R_L^\alpha} = 0$$

for every E in some Borel set A. Then the restriction $\mu(A \cap \cdot\,)$ is α-singular.

Remarks.

(i) Note, in particular, that if the generalized eigenfunctions are averagely decaying, then $\mu(A \cap \cdot\,)$ must be singular.

(ii) Kiselev-Last [12] have recently obtained a generalization of Corollary 4.5 to multi-dimensional Schrödinger operators.

Proof. Let $E \in A$ and $\beta \equiv \alpha/(2-\alpha)$. By (1.16), we have $\|u_1\|_L \|u_2\|_L \geq R_L$, and so $\|u_2\|_L^\beta \geq (R_L/\|u_1\|_L)^\beta$. Thus, we have

$$\liminf_{L\to\infty} \frac{\|u_1\|_L}{\|u_2\|_L^\beta} \leq \liminf_{L\to\infty} \frac{\|u_1\|_L^{1+\beta}}{R_L^\beta} = \liminf_{L\to\infty} \left(\frac{\|u_1\|_L^2}{R_L^\alpha} \right)^{1/(2-\alpha)} = 0. \tag{4.10}$$

By Theorem 1.2, it follows that $\mu(A \cap \cdot\,)$ is α-singular. $\square$

As a final remark to this section, we note that while Theorems 1.1 and 1.2 are also valid for continuous Schrödinger operators on $L^2(\mathbb{R}^+)$ (and so are suitable versions of Corollaries 4.1–4.3), Corollaries 4.4 and 4.5 are *not.* Their proofs use the fact that the Wronskian in the discrete case involves only solutions, as opposed to the continuous case where the Wronskian also involves derivatives. Continuous Schrödinger operators may have, for example, absolutely-continuous spectrum along with decaying eigenfunctions (e.g., the potential $V(x) = -x$, for which the eigenfunctions are decaying Airy functions).

5. Proof Theorem 1.3

Proof of (i). Clearly, $[-2,2]$ is in the essential spectrum, since the potential vanishes over arbitrarily long intervals, and so approximate eigenvectors for the free (discrete) Laplacian on $\ell^2(\mathbb{Z})$ are also approximate eigenvectors for H_θ. To see that the spectrum outside $[-2,2]$ consists only of some isolated eigenvalues, we consider the discrete Laplacian as a perturbation of the potential V. The spectrum of V consists of an infinite multiplicity eigenvalue at 0, along with simple isolated eigenvalues which correspond to the non-vanishing values of the potential. For any E outside $[-2,2]$, there are at most finitely many simple eigenvalues within a distance 2 of E. Thus, since the norm of the discrete Laplacian is 2, it follows from perturbation theory that there cannot be any essential spectrum outside $[-2,2]$. $\square$

Proof of (ii). Consider a closed interval $I=[a,b]\subset(-2,2)$. If we can show that $\mu(I\cap\cdot\,)$ has the desired properties for such an arbitrarily chosen I, it will follow that $\mu((-2,2)\cap\cdot\,)$ has them. For every $E\in I$, $m>k\geq 0$, let $\Phi_{k,m}(E)\equiv T_m(E)T_{m-1}(E)\;\cdots\;T_{k+1}(E)$, where the $T_n(E)$'s are defined by (4.4). $\Phi_{k,m}(E)$ is the transfer matrix from k to m, and we denote $\Phi_{0,m}(E)$ by $\Phi_m(E)$ as in Section 4. Since we always have $\det(\Phi_{k,m}(E))=1$, it follows that $\|\Phi_{k,m}^{-1}(E)\|=\|\Phi_{k,m}(E)\|$. For any $n\in\mathbb{Z}^+$, if $L_n\leq k<m<L_{n+1}$, then $\Phi_{k,m}(E)$ is the same as the corresponding transfer matrix for the free (discrete) Laplacian. In particular, there is a constant C_I, depending only on the interval I, such that $1\leq\|\Phi_{k,m}(E)\|<C_I$ for any such k,m and $E\in I$. Moreover, for any $n\in\mathbb{Z}^+$, we have

$$\Phi_{L_n-1,L_n}(E)=T_{L_n}(E)=\begin{pmatrix} E-V(L_n) & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.1}$$

and so

$$\max(1,V(L_n)-2)\leq\|T_{L_n}(E)\|\leq V(L_n)+3. \tag{5.2}$$

Consider now some $n\in\mathbb{Z}^+$ and $L_n\leq m<L_{n+1}$, then we have

$$\Phi_m(E)=\Phi_{L_n,m}(E)\,T_{L_n}(E)\,\Phi_{L_{n-1},L_n-1}(E)\,T_{L_{n-1}}(E)\;\cdots\;\Phi_{L_1,L_2-1}(E)T_{L_1}(E)\Phi_{L_1-1}(E), \tag{5.3}$$

and thus we see that

$$\|\Phi_m(E)\|\leq C_I^{n+1}\prod_{k=1}^{n}(V(L_k)+3)\leq C_1^n\left[\prod_{k=1}^{n}L_k\right]^{(1-\alpha)/2\alpha}, \tag{5.4}$$

where C_1 is some constant depending on C_I and α. Similarly, for large n, we also have

$$\|\Phi_m(E)\| \geq \left[C_I^{n+1}\prod_{k=1}^{n-1}(V(L_k)+3)\right]^{-1}(V(L_n)-2) \geq C_2^{-n}\left[\left[\prod_{k=1}^{n-1}L_k\right]^{-1}L_n\right]^{(1-\alpha)/2\alpha}, \tag{5.5}$$

where C_2 is some constant. Since $\prod_{k=1}^{n} L_k = 2^{\Sigma_{k=1}^{n}k^k}$, and since $\frac{1}{n^n}\sum_{k=1}^{n}k^k \to 1$ as $n\to\infty$, we see that for large n

$$L_n^{1-\epsilon} < \left[\left[\prod_{k=1}^{n-1}L_k\right]^{-1}L_n\right] < \left[\prod_{k=1}^{n}L_k\right] < L_n^{1+\epsilon}, \tag{5.6}$$

where ϵ can be made arbitrarily small by taking n sufficiently large. Similarly, for large n, we also have $C_1^n < L_n^{\epsilon}$ and $C_2^n < L_n^{\epsilon}$. Thus, we deduce that for any $L_n \leq m < L_{n+1}$,

$$L_n^{((1-\alpha)/2\alpha)-\epsilon} \leq \|\Phi_m(E)\| \leq L_n^{((1-\alpha)/2\alpha)+\epsilon}, \tag{5.7}$$

for any ϵ with sufficiently large n. By considering $m=2L_n$, it follows from (5.7) that

$$\sum_{k=1}^{m}\|\Phi_k(E)\|^2 \geq L_nL_n^{((1-\alpha)/\alpha)-2\epsilon} = L_n^{(1/\alpha)-2\epsilon} = (1/2)^{(1/\alpha)-2\epsilon}m^{(1/\alpha)-2\epsilon}. \tag{5.8}$$

Thus, by Corollary 4.2, we see that for any $\epsilon>0$, $\mu(I\cap\cdot)$ is $((2\alpha/(1+\alpha))+\epsilon)$-singular.

Next, we need to establish the desired continuity of $\mu(I\cap\cdot)$, namely, to show that for any $\epsilon>0$, $\mu(I\cap\cdot)$ is $(\alpha-\epsilon)$-continuous. Let $\beta=\alpha/(2-\alpha)$. By Theorem 1.2, it is sufficient to show that for every $E\in I$ and $\delta>0$,

$$\liminf_{m\to\infty}\frac{\|u_1\|_m^2}{\|u_2\|_m^{2(\beta-\delta)}} > 0. \tag{5.9}$$

Since for every m, $|u_1(m)|^2+|u_2(m)|^2 \geq \|\Phi_m(E)\|^{-2}$, we see, by (5.7), that for n large and $L_n\leq m<L_{n+1}$,

$$\|u_1\|_m^2 > \frac{1}{2}\left((L_n-L_{n-1})L_{n-1}^{-((1-\alpha)/\alpha)-\epsilon} + (m-L_n)L_n^{-((1-\alpha)/\alpha)-\epsilon}\right), \tag{5.10}$$

for any ϵ with large n. Letting $\ell \equiv m - L_n$, we see that (5.10) implies

$$\|u_1\|_m^2 > L_n^{1-\epsilon} + L_n^{-((1-\alpha)/\alpha)-\epsilon}\ell, \tag{5.11}$$

for n large. Similarly, we have

$$\|u_2\|_m^2 < L_n L_{n-1}^{((1-\alpha)/\alpha)+\epsilon} + (m - L_n) L_n^{((1-\alpha)/\alpha)+\epsilon}, \tag{5.12}$$

and thus for n large

$$\|u_2\|_m^2 < L_n^{1+\epsilon} + L_n^{((1-\alpha)/\alpha)+\epsilon}\ell. \tag{5.13}$$

By combining (5.11) and (5.13), and denoting $\gamma \equiv (1-\alpha)/\alpha$, we obtain

$$\frac{\|u_1\|_m^2}{\|u_2\|_m^{2(\beta-\delta)}} > \frac{L_n^{1-\epsilon} + L_n^{-\gamma-\epsilon}\ell}{(L_n^{1+\epsilon} + L_n^{\gamma+\epsilon}\ell)^{\beta-\delta}} \equiv f_{n,\epsilon,\delta}(\ell). \tag{5.14}$$

By computing the derivative of $f_{n,\epsilon,\delta}(\ell)$, one easily verifies that for a fixed $0 < \delta < \beta$, sufficiently large n, and sufficiently small ϵ, $f_{n,\epsilon,\delta}(\ell)$ has a single minimum on $[0, \infty)$, which occurs at the point where

$$L_n^{-\gamma-\epsilon} - (\beta - \delta) L_n^{\gamma+\epsilon} \frac{L_n^{1-\epsilon} + L_n^{-\gamma-\epsilon}\ell}{L_n^{1+\epsilon} + L_n^{\gamma+\epsilon}\ell} = 0. \tag{5.15}$$

(5.15) can be solved to yield

$$\ell = \frac{(\beta-\delta)L_n^{1+\gamma} - L_n^{1-\gamma}}{1 + \delta - \beta}, \tag{5.16}$$

and at the minimum point

$$L_n^{1-\epsilon} + L_n^{-\gamma-\epsilon}\ell = \frac{L_n^{-2\gamma-2\epsilon}}{\beta-\delta}(L_n^{1+\epsilon} + L_n^{\gamma+\epsilon}\ell), \tag{5.17}$$

and thus we have

$$\min_{\ell\in[0,\infty)} f_{n,\epsilon,\delta}(\ell) = \frac{L_n^{-2\gamma-2\epsilon}}{\beta-\delta}\left[L_n^{1+\epsilon} + L_n^{\gamma+\epsilon}\frac{(\beta-\delta)L_n^{1+\gamma} - L_n^{1-\gamma}}{1+\delta-\beta}\right]^{1+\delta-\beta}. \tag{5.18}$$

As $n \to \infty$ and $\epsilon \to 0$, the r.h.s. of (5.18) is of order $L_n^{(1+2\gamma+\epsilon)(1+\delta-\beta)-2\gamma-2\epsilon}$, and since $(1+2\gamma)(1-\beta)-2\gamma=0$, we see that it goes to infinity, and thus, in particular, it is bounded away from zero. By (5.14), this implies (5.9). $\square$

Proof of (iii). In order to prove (iii), we will establish that for Lebesgue a.e. E, (1.7) has solutions with appropriate decay properties. It will then follow from the theory of rank-one perturbations and from the uniqueness of subordinate solutions that for a.e. boundary phase θ and a.e. E w.r.t. μ, the solution u_1 must be one of these decaying solutions.

Fix some boundary phase θ_0, say $\theta_0 = 0$, and let H be H_θ for $\theta = \theta_0$. For each $m \in \mathbb{Z}^+$, let Γ_m be the operator on $\ell^2(\mathbb{Z}^+)$ with matrix elements

$$\langle \delta_i, \Gamma_m \delta_j \rangle = \delta_{i,m-1}\delta_{j,m} + \delta_{i,m}\delta_{j,m-1}, \tag{5.20}$$

where $\delta_{i,j}$ is the usual Kronecker symbol. For each $k \in \mathbb{Z}^+$, let $\tilde{H}_k = H - \Gamma_{L_k}$, $\hat{H}_k = H - \Gamma_{L_k} - \Gamma_{L_k+1}$, and for every $z \in \mathbb{C}$, let $G(z) = (H-z)^{-1}$, $\tilde{G}_k(z) = (\tilde{H}_k - z)^{-1}$ and $\hat{G}_k(z) = (\hat{H}_k - z)^{-1}$. Moreover, for every $i,j \in \mathbb{Z}^+$, let $G(i,j,z) = \langle \delta_i, G(z)\delta_j\rangle$, $\tilde{G}_k(i,j,z) = \langle \delta_i, \tilde{G}_k(z)\delta_j\rangle$, and $\hat{G}_k(i,j,z) = \langle \delta_i, \hat{G}_k(z)\delta_j\rangle$. Consider now some $n > L_k$. Using the resolvent identity $G(z) = \tilde{G}_k(z) - G(z)\,\Gamma_{L_k}\,\tilde{G}_k(z)$, we obtain that

$$G(1,n,z) = -G(1, L_k - 1, z)\tilde{G}_k(L_k, n, z). \tag{5.21}$$

Similarly, we have $\tilde{G}_k(z) = \hat{G}_k(z) - \hat{G}_k(z)\,\Gamma_{L_k+1}\,\tilde{G}_k(z)$, from which it follows that

$$\tilde{G}_k(L_k, n, z) = -\hat{G}_k(L_k, L_k, z)\tilde{G}_k(L_k+1, n, z) = \frac{-1}{V(L_k)-z}\,\tilde{G}_k(L_k+1, n, z). \tag{5.22}$$

By combining (5.21) and (5.22), we obtain

$$G(1,n,z) = G(1, L_k - 1, z)\tilde{G}_k(L_k+1, n, z)\,\frac{1}{V(L_k)-z}. \tag{5.23}$$

Consider now $E \in \mathbb{R}$ and $z = E + i\epsilon$ in the upper half plane. The various $G(i,j,z)$'s and $\tilde{G}_k(i,j,z)$'s are Borel transforms of signed measures, and so it is known that for Lebesgue a.e. E, they have finite non-tangential limits on the real axis: $G(i,j,E) \equiv G(i,j,E+i0)$ and $\tilde{G}_k(i,j,E) \equiv \tilde{G}_k(i,j,E+i0)$.

Recall Boole's equality [1,2,4], which says that if $f(E) = \int (x-E)^{-1}\, d\mu(x)$ for a positive singular measure μ with $\mu(\mathbb{R}) = 1$, then for any $\lambda > 0$, we have $|\{E \mid |f(E)| > \lambda\}| = 2/\lambda$, where $|\cdot|$ denotes Lebesgue measure. If μ is a signed measure with $|\mu|(\mathbb{R}) \le 1$, we can consider $f(E)$ as the difference of two integrals with positive measures, and obtain $|\{E \mid |f(E)| > \lambda\}| \le 4/\lambda$. Since we already know that the spectral measures of H (and thus also of $\tilde{H}$) for any vector δ_i are singular, we conclude that for any $i, j \in \mathbb{Z}^+$,

$$
\begin{aligned}
&|\{E \mid |G(i,j,E)| > \lambda\}| \le 4/\lambda, \\
&|\{E \mid |\tilde{G}_k(i,j,E)| > \lambda\}| \le 4/\lambda.
\end{aligned}
\tag{5.24}
$$

¿From (5.23) and (5.24) we deduce that for any $k > 1$, $n > L_k$, and $E \in (-2,2)$,

$$
\left| \left\{ E \;\middle|\; |G(1,n,E)| > \frac{\log^2(L_k)}{V(L_k) - 2} \right\} \right| \le \frac{8}{\log(L_k)}. \tag{5.25}
$$

Since $\sum_{k=2}^{\infty} (\log L_k)^{-1} < \infty$, it follows from the Borel-Cantelli lemma that for Lebesgue a.e. $E \in (-2,2)$, there exists a $K(E)$ such that for any $k > K(E)$ and $n = L_k + 1$ or $L_k + 2$,

$$
|G(1,n,E)| \le \frac{\log^2(L_k)}{V(L_k) - 2}. \tag{5.26}
$$

Now, for any E for which the sequence $\{G(1,n,E)\}_{n=1}^{\infty}$ exists, it solves (1.7) for $n > 2$. Thus $G(1,n,E)$, $L_k + 2 < n \le L_{k+1}$ can be obtained from $G(1, L_k+1, E), G(1, L_k + 2, E)$ by the action of the free transfer matrix:

$$
\begin{pmatrix} G(1,n+1,E) \\ G(1,n,E) \end{pmatrix} = \Phi_{L_k+2,n}(E) \begin{pmatrix} G(1,L_k+2,E) \\ G(1,L_k+1,E) \end{pmatrix},
$$

where, as in the proof of (ii), we have $\|\Phi_{L_k+2,n}(E)\| \le \hat{C}_E$ for $L_k + 2 < n < L_{k+1}$ and $E \in (-2,2)$. This implies that for the same full measure set of E as above and $k > K(E)$, (5.26) holds for all $L_k < n \le L_{k+1}$.

Thus, we deduce that for Lebesgue a.e. $E \in (-2,2)$, there exists a solution v of (1.7) (with $|v(0)|^2 + |v(1)|^2 = 1$) and a constant C_E, such that for sufficiently large k and $n > L_k$,

$$
|v(n)| < C_E \frac{\log^2(L_k)}{V(L_k)} = C_E \log^2(L_k)\, L_k^{-(1-\alpha)/2\alpha} < L_k^{-((1-\alpha)/2\alpha)+\epsilon}, \tag{5.27}
$$

where ϵ can be chosen arbitrarily small for k large. Moreover, since, by (1.16), there can be at most one solution of (1.7) which is decaying, v must also be the unique subordinate solution of (1.7).

We now go back to consider H_θ, where θ can vary. By the theory of rank-one perturbations (see, e.g., [18]), it is known that for any set $A \subset \mathbb{R}$ with $|A| = 0$, we have $\mu(A) = 0$ for (Lebesgue) a.e. boundary phase θ. Thus, it follows that for a.e. θ, μ is supported on the set of E's where the solution v of (5.27) exists. Moreover, since μ must also be supported on the set of E's for which u_1 is subordinate, it follows that for a.e. θ and a.e. E w.r.t. μ, u_1 must coincide with v of (5.27).

Let β and γ be as in the proof of (ii) above, and fix some $\delta > 0$. Consider an $E \in (-2, 2)$ where u_1 coincides with v of (5.27), a large $k \in \mathbb{Z}^+$, and let $n = L_k + L_k^{1+\gamma}$. By combining (1.13) with (5.27), we see that

$$\|u_1\|_n^2 < L_k^{1+\epsilon} + L_k^{1+\gamma} L_k^{-\gamma+\epsilon} = 2L_k^{1+\epsilon}, \tag{5.28}$$

and from (4.6), (5.7), and (5.27), we see that

$$\|u_2\|_n^2 > L_k^{1+\gamma} L_k^{\gamma-\epsilon} = L_k^{1+2\gamma-\epsilon}, \tag{5.29}$$

where, in both (5.28) and (5.29), ϵ can be made arbitrarily small by taking k sufficiently large. Thus, we have

$$\frac{\|u_1\|_n^2}{\|u_2\|_n^{2(\beta+\delta)}} < 2L_k^{1+\epsilon-(1+2\gamma-\epsilon)(\beta+\delta)} = 2L_k^{-\delta(1+2\gamma)+\epsilon(1+\beta+\delta)}, \tag{5.30}$$

and by considering $k \to \infty$ and $\epsilon \to 0$, we see that (5.30) implies

$$\liminf_{L\to\infty} \frac{\|u_1\|_L^2}{\|u_2\|_L^{2(\beta+\delta)}} = 0. \tag{5.31}$$

Since for any $\delta > 0$, (5.31) holds for a.e. θ, for a.e. E w.r.t. $\mu((-2,2) \cap \cdot)$, it follows from Theorem 1.2 that for a.e. θ, $\mu((-2,2) \cap \cdot)$ is $(\alpha + \epsilon)$-singular for any $\epsilon > 0$. Since we already know from (ii) that $\mu((-2,2) \cap \cdot)$ is $(\alpha - \epsilon)$-continuous (for any $\epsilon > 0$), it follows that it is of exact dimension α. $\square$

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