Finite size scaling in three-dimensional bootstrap percolation

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Abstract

We consider the problem of bootstrap percolation on a three dimensional lattice and we study its finite size scaling behavior. Bootstrap percolation is an example of Cellular Automata defined on the d-dimensional lattice $\{1, 2, ..., L\}^d$ in which each site can be empty or occupied by a single particle; in the starting configuration each site is occupied with probability p, occupied sites remain occupied for ever, while empty sites are occupied by a particle if at least ℓ among their $2d$ nearest neighbor sites are occupied. When d is fixed, the most interesting case is the one $\ell = d$: this is a sort of threshold, in the sense that the critical probability p_c for the dynamics on the infinite lattice $\mathbb{Z}^d$ switches from zero to one when this limit is crossed. Finite size effects in the three-dimensional case are already known in the cases $\ell \leq 2$: in this paper we discuss the case $\ell = 3$ and we show that the finite size scaling function for this problem is of the form $f(L) = \text{const}/\ln\ln L$. We prove a conjecture proposed by A.C.D. van Enter.

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Cellular Automata are dynamical systems defined on the d-dimensional lattice $\mathbb{Z}^d$ in which each site of the lattice is occupied by one of finitely many *types* at each time t. An updating rule is defined, which is *homogeneous*, all the sites follow the same rule, and *local*, transitions are determined by the configuration of types on a finite set of neighboring sites [18, 13].

These models can be thought as interacting particle systems and their connections with statistical mechanics models have been widely studied in past years (see, for instance, [9, 17, 19, 20]). A particular example of cellular automata, known as *bootstrap percolation*, has been introduced in [5] to model some magnetic systems. More informations on the physical relevance of this model are given in [10, 4].

In bootstrap percolation only two different types are associated to each site: each site can be occupied or not by a particle. In the starting configuration each site is independently occupied with probability p, occupied sites remain occupied for ever, while empty sites are occupied by a particle if at least ℓ among their $2d$ nearest neighbor sites are occupied. The object of primary interest is the probability $p_{\text{full}}(\ell)$ that at the end of the dynamics, that is in the infinite time configuration, all the sites are occupied. The basic question that has been addressed in physics literature is whether by changing the value of the parameter p the system exhibits a sort of phase transition, that is whether there exists a critical value $p_c(\ell) \in [0, 1]$ such that if $p \geq p_c(\ell)$ then $p_{\text{full}}(\ell) = 1$, otherwise $p_{\text{full}}(\ell) < 1$.

Fixed the dimension d, the smaller ℓ the easier empty sites are occupied, hence one expects that $p_c(\ell)$ is an increasing function of ℓ. The first rigorous result on this topic is due to van Enter, who used the idea of the Straley's argument [10] to prove that in the case $d = 2$ and $\ell = 2$ the critical probability is equal to zero. In [14, 16] Schonmann has proved that $p_c(\ell) \in \{0, 1\}$, more precisely $p_c(\ell) = 0$ if $\ell \leq d$, otherwise $p_c(\ell) = 1$; these results suggest that the most peculiar case is $\ell = d$.

Before these rigorous results the phase transition scenario in bootstrap percolation models was not clear. The technique that had been used to measure the critical probability $p_c(\ell)$ was the finite size scaling: a finite volume estimate of the critical probability was found by means of Monte Carlo simulations on a finite lattice $\Lambda_L = \{1, 2, ...L\}^d$, for instance the probability $p_L^{0.5}$ that one half of the samples were completely filled at the end of the dynamics, and the critical value $p_c(\ell)$ was extrapolated by means of a suitable scaling function $f(L)$. That is the expression

$$p_L^{0.5} - p_c(\ell) \overset{L\to\infty}{\sim} f(L) \tag{1}$$

was supposed to be valid and Monte Carlo data were properly fitted by means of the function $f(L)$ (see [1] and references therein).

It is rather clear that the estimate of p_c strongly depends on the choice of the scaling function $f(L)$: the typical choice, when critical effects in second order phase transition are studied, is $f(L) = \text{const} \times L^{-1/\nu}$ where ν is a suitable exponent. Actually this choice with $1/\nu = d$ is correct in the case $\ell = 1$, while estimations of $p_c(\ell)$ in the cases $\ell = 2$ and $d \geq 2$ obtained by means of Monte Carlo data analyzed through this function $f(L)$ did not fit in the rigorous scenario depicted by Schonmann's results [1]: the problem is that the power law $L^{-1/\nu}$ approaches zero too quickly and must be replaced by a slower function $f(L) = \text{const} \times (\ln L)^{-(d-1)}$ as suggested by the finite volume Aizenman and Lebowitz's results [3, 11]. Indeed the analysis of old and new data performed via the correct scaling function yields the correct estimate of the critical probability [2, 7, 8].

In [3] bootstrap percolation on finite lattices Λ_L is considered in the case $\ell = 2$ and $d \geq 2$ and it is observed that if p is kept fixed, then in the limit $L \to \infty$ the probability $p_{\text{full}}^{L,p}$ to fill $\{1, ..., L\}^d$ tends to one whatever the value of p is. But if $p \to 0$ together with $L \to \infty$, then it is possible to find a particular regime in which the probability to fill everything tends to zero. Indeed they prove that there exist two constants $c_+ > c_- > 0$ such that if $p \geq c_+/(\ln L)^{d-1}$ then $p_{\text{full}}^{L,p} \to 1$ when $p \to 0$ and $L \to \infty$, while if $p \leq c_-/(\ln L)^{d-1}$ then in the same limit $p_{\text{full}}^{L,p} \to 0$.

Let us focus on the case $d = 3$: the choice $\ell = 2$ is not the most delicate one, indeed according to [16] even in the case $\ell = 3$, that is even in a situation in which it is more difficult to fill empty sites, the critical probability is still zero. Hence one can guess that in the three-dimensional case if $\ell = 3$ then the finite scaling function is no more the Aizenman-Lebowitz one, but a function approaching zero more slowly. Our aim is to study this case and to show that results similar to those in [3], and conjectured by A.C.D. van Enter, can be proved if the scaling function is replaced by $f(L) = \text{const}/\ln\ln L$. This problem has been proposed in [16] as Problem 3.1; we notice also that related problems have been discussed in [12, 15].

An interesting follow up would be the generalization of our results to the d-dimensional case by considering arbitrary dimension d and $\ell = d$. In this case one expects $f(L) = \text{const}/(\ln\ln\ \dots\ \ln L)$ with the logarithm applied $d-1$ times.

We next define the particular model of bootstrap percolation that we are going to study and we introduce some notations. Let us consider the lattice $\mathbb{Z}^3$ and the discrete time variable $t = 0, 1, 2, \dots$. To each site $x \in \mathbb{Z}^3$ we associate at each instant of time t a random variable $X_t(x)$ which takes values in $\{0,1\}$; depending on $X_t(x)$ equals 0 or 1 we say that the site x is empty or occupied. We denote by $\Omega := \{0,1\}^{\mathbb{Z}^3}$ the space of configurations and by $X_t \in \Omega$ the configuration of the system at time t. The initial configuration X_0 is chosen by occupying independently each site of the lattice with probability p (initial density). Then the system evolves according to the following deterministic rules:

- if $X_t(x) = 1$, then $X_{t+1}(x) = 1$ (1's are stable);
- if $X_t(x) = 0$ and x has at least three occupied sites among its six nearest neighbors, then $X_{t+1}(x) = 1$;
- $X_{t+1}(x) = 0$ otherwise.

We omit in our notations the dependence of the process X_t on the initial density p; when the initial density will be different from p it will be clearly stated.

The primary object of interest for this problem is the final configuration

$$X := \lim_{t\to\infty} X_t \quad , \tag{2}$$

that is the configuration attained by the system when the dynamics stops.

Let us consider a subset $\Lambda \subset \mathbb{Z}^3$, we denote by $X_{\Lambda,t}$ the process restricted to Λ with free boundary conditions, that is without taking into account sites outside Λ. In this case $X_{\Lambda,t}$, $X_{\Lambda,t}(x)$ and X_Λ will respectively denote the configuration at time t, the value of the random variable at time t and site x, and the final configuration.

Given two arbitrary sets $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^3$ we will consider the process $X^{\Lambda_2}_{\Lambda_1,t}$ restricted to Λ_1 and with occupied sites in Λ_2 (the sites in $\mathbb{Z}^3 \setminus (\Lambda_1 \cup \Lambda_2)$ are not taken into account). We will obviously omit Λ_1 in the notation if $\Lambda_1 = \mathbb{Z}^3$, while we will omit Λ_2 if $\Lambda_2 = \emptyset$.

Definition 1 *Following [3] we say that a set $\Lambda \subset \mathbb{Z}^3$ is internally spanned if it is entirely covered in the final configuration of the dynamics restricted to Λ, that is if*

$$\forall x \in \Lambda \qquad X_\Lambda(x) = 1 \quad . \tag{3}$$

We have now introduced the bootstrap percolation model in dimension $d = 3$ and with parameter $\ell = 3$; as we have already stated, this is the most delicate three-dimensional bootstrap percolation model. Indeed $\ell = 3$ is the highest value of the parameter ℓ such that the critical probability in infinite volume is equal to zero, that is for each positive initial density p the probability that the whole lattice will be completely occupied by particles at the end of the dynamics is equal to one.

In this particular case we examine the question of finite size scaling, that is, following [3], we consider the process $X_{\Lambda_L,t}$ on the finite cube $\Lambda_L = \{1, ..., L\}^3$ of size L and we perform the limit $L \to \infty$ and $p \to 0$. We prove that there exists a particular regime in which the probability that the cube Λ_L is internally spanned is zero in the limit $L \to \infty$ and $p \to 0$.

Theorem 2 *Let us consider the cube Λ_L of side length L and the process X_t with initial density p; let us denote by $R(L,p)$ the probability that the cube Λ_L is internally spanned. There exist two constants $c_+ > c_- > 0$ such that*

- *$R(L,p) \longrightarrow 1$ if $(L,p) \to (\infty,0)$ in the regime $p > c_+/\ln\ln L$*

- *$R(L,p) \longrightarrow 0$ if $(L,p) \to (\infty,0)$ in the regime $p < c_-/\ln\ln L$*

We start to prove the first part of Theorem 2: this is the easy part of the theorem and its proof has already been sketched in [7]. The idea of the proof relies on the notion of *critical length*: suppose p is small, if an occupied cube has size large enough, then the probability to find on its faces two-dimensional occupied square droplets so large to cover the whole faces of the cube (two-dimensional critical droplets) is close to one. Obviously if p goes to 0, then the size of the cube must diverge and this must happen fastly enough; in this way one will find that the critical length is of order $\exp(\mathrm{const}/p)$. Then, one estimates that in order to have a high probability of finding a critical droplet, the size L of the cube must be of order $\exp\exp(\mathrm{const}/p)$.

Now, we come to the proof: from results in [16] one easily obtains that there exists a constant $c_1 > 0$ such that, given a cube Λ_l, if $p \geq c_1/\ln l$ then there exists a constant $a_1 > 0$ such that

$$P(\Lambda_l \text{ covers } \mathbb{Z}^3 \mid \Lambda_l \text{ occupied at } t=0) \geq \prod_{k=l}^{\infty}(1-\mathrm{e}^{-a_1 k}) \quad . \tag{4}$$

We consider a large cube Λ_L and we estimate $R(L,p)$:

$$R(L,p) \geq P(\text{in } \Lambda_L\ \exists\Lambda_l \text{ occupied at } t=0,\ \Lambda_l \text{ covers } \mathbb{Z}^3) =$$

$$P(\ \Lambda_l \text{ covers } \mathbb{Z}^3 \mid \text{in } \Lambda_L\ \exists\Lambda_l \text{ occupied at } t=0)\ P(\text{in } \Lambda_L\ \exists\Lambda_l \text{ occupied at } t=0) \tag{5}$$

Now, from equation (4) it follows that if $l \geq \exp(c_1/p)$ then the first factor in (5) tends to 1 when p goes to 0. Then, we just have to prove that with such an l it is possible to choose L such that the second factor of (5) tends to 1, as well. Indeed by partitioning the cube Λ_L in disjoint cubes of size l one has

$$P(\text{in } \Lambda_L\ \exists\Lambda_l \text{ occupied at } t=0) \geq 1-(1-p^{l^3})^{L/l}; \tag{6}$$

by choosing $c_+ > 3c_1 > 0$ and $L > \exp\exp(c_+/p)$, the right hand term in the equation above tends to 1 when p goes to 0. This completes the proof of the first part of Theorem 2.

Next we prove the second part of Theorem 2 by finding a suitable upper bound to the probability that a cube Λ_L is internally spanned. First of all we give some more definitions: given a site $x \in \mathbb{Z}^3$ we denote by (x_1, x_2, x_3) its three coordinates and given a set $\Lambda \subset \mathbb{Z}^3$ we define its diameter

$$d(\Lambda) := \sup\{|x_i - y_i| : \; x \in \Lambda, \; y \in \Lambda, \; i \in \{1,2,3\}\} \quad , \tag{7}$$

that is $d(\Lambda)$ is the side length of the minimal cube surrounding the set Λ. We say that $\Lambda \subset \mathbb{Z}^3$ is a region of $\mathbb{Z}^3$ if and only if it is nearest neighbors connected. We note that if Λ_1 and Λ_2 are two regions of $\mathbb{Z}^3$ and $\Lambda_1 \cup \Lambda_2$ is a region as well, then $d(\Lambda_1 \cup \Lambda_2) \leq d(\Lambda_1) + d(\Lambda_2)$.

Now we adapt to our situation a key Lemma of [3] that describes what happens at a smaller scale inside an internally spanned region.

Lemma 3 *Let Λ_1 be a region of $\mathbb{Z}^3$. If Λ_1 is internally spanned, then for all κ such that $1 \leq \kappa$ and $2\kappa + 1 \leq d(\Lambda_1)$ there exists at least a region Λ_2 included in Λ_1 which is internally spanned and such that $\kappa \leq d(\Lambda_2) < 2\kappa + 1$.*

Proof. We build the configuration X_{Λ_1} by means of the following algorithmic procedure. Let $\mathcal{C}_0$ be the collection of the sites occupied at time zero. Suppose we have built a collection of regions internally spanned $\mathcal{C}_n$, we define a rule to build the collection $\mathcal{C}_{n+1}$:

- If there exist two regions A, B of $\mathcal{C}_n$ such that $A \cup B$ is still a region, then we set

$$\mathcal{C}_{n+1} := \mathcal{C}_n \cup \{A \cup B\} \setminus \{A, B\} \quad , \tag{8}$$

  that is $\mathcal{C}_{n+1}$ is obtained by replacing in $\mathcal{C}_n$ the two elements A and B by $A \cup B$.

- If no such regions exist then we choose a site x not belonging to any set in $\mathcal{C}_n$ and having three neighbors in the set $\bigcup_{A \in \mathcal{C}_n} A$. We denote by A_i with $1 \leq i \leq r$ the r regions of $\mathcal{C}_n$ containing a neighbor of x, and we set

$$\mathcal{C}_{n+1} := \mathcal{C}_n \cup \left\{ \bigcup_{i=1}^{r} A_i \cup \{x\} \right\} \setminus \{A_1, ..., A_r\} \quad . \tag{9}$$

- If no such site x exists, the algorithm stops.

Notice that for each n, the regions of $\mathcal{C}_n$ are internally spanned. Since Λ_1 is internally spanned the procedure ends for some m such that $\mathcal{C}_m = \{\Lambda_1\}$. Moreover we have that $\max\{d(A) : A \in \mathcal{C}_0\} = 1$, $\max\{d(A) : A \in \mathcal{C}_m\} = d(\Lambda_1)$ and for any $n \leq m-1$

$$\max\{d(A) : A \in \mathcal{C}_{n+1}\} \leq 2 \max\{d(A) : A \in \mathcal{C}_n\} + 1 \quad . \tag{10}$$

Hence there exists n such that:

$$\kappa \leq \max\{d(A) : A \in \mathcal{C}_n\} < 2\kappa + 1 \quad , \tag{11}$$

which means that in $\mathcal{C}_n$ there is an internally spanned region A of diameter $d(A)$ such that $\kappa \leq d(A) < 2\kappa + 1$. ■

Definition 4 *Let us consider a cube Λ in $\mathbb{Z}^3$. We say that Λ is crossed, or that there is a crossing in Λ, if and only if in the final configuration X_Λ of the dynamics restricted to Λ there is an occupied region joining two opposite faces of the cube Λ.*

We note that for any region A the following inclusion stands:

$$\{A \text{ is internally spanned}\} \subset \{\text{the smallest cube surrounding } A \text{ is crossed}\} \quad . \tag{12}$$

Hence, for any L and any κ such that $2\kappa + 1 < L$, we have:

$$R(L,p) \leq P(\exists l,\ \kappa \leq l < 2\kappa+1,\ \exists \Lambda_l \subset \Lambda_L,\ \Lambda_l \text{ is crossed}) \leq$$

$$(\kappa+1)\ L^3 \max_{\kappa \leq l < 2\kappa+1} P(\Lambda_l \text{ is crossed}) \tag{13}$$

Thus,

$$R(L,p) \leq L^3 \min_{1 \leq \kappa < (L-1)/2} (\kappa+1) \max_{\kappa \leq l < 2\kappa+1} P(\Lambda_l \text{ is crossed}) \quad . \tag{14}$$

We have reduced the estimate of $R(L,p)$ to the estimate of the probability that a cube Λ_l is crossed and by symmetry we can consider the case of a crossing along the first reticular direction (denoted by e_1 in the sequel):

$$P(\Lambda_l \text{ is crossed}) \leq 3\, P(\Lambda_l \text{ is crossed along } e_1) \quad . \tag{15}$$

In order to estimate the second hand term of equation (15) we reconduct to a two-dimensional situation by properly cutting the cube Λ_l in slices of thickness two and perpendicular to the first reticular direction. For the sake of definiteness we suppose $\Lambda_l := \{1, 2, ..., l\}^3$, l an even number and we define the slices

$$T_k := \{x \in \Lambda_l : \ x_1 = 2k-1 \text{ or } x_1 = 2k\}\,, \qquad 1 \leq k \leq \frac{l}{2} \quad . \tag{16}$$

We define a map s associating to each site in a slice the only nearest neighbor along the first reticular direction belonging to the same slice:

$$\forall x \in \Lambda_l \qquad s(x) := \begin{cases} (x_1+1, x_2, x_3) & \text{if } x_1 \text{ is odd} \\ (x_1-1, x_2, x_3) & \text{if } x_1 \text{ is even} \end{cases} \quad . \tag{17}$$

The process $X_{T_k,t}^{\Lambda_l \setminus T_k}$, restricted to the slice T_k and with all the sites in $\Lambda_l \setminus T_k$ occupied, dominates the original process in the same slice:

$$\forall k \in \{1, ..., \frac{l}{2}\} \quad \forall x \in T_k \quad \forall t \geq 0 \qquad X_t(x) \leq X_{T_k,t}^{\Lambda_l \setminus T_k}(x) \quad . \tag{18}$$

In each slice T_k for k in $\{1, ..., l/2\}$ we define a new process Y_t^k.

Definition 5 *We consider all the sites in $\Lambda_l \setminus T_k$ occupied and define the process Y_t^k on $\{0,1\}^{T_k}$ as follows: for any x in T_k*

- $Y_0^k(x) = \max(X_{T_k,0}^{\Lambda_l \setminus T_k}(x), X_{T_k,0}^{\Lambda_l \setminus T_k}(s(x))) = \max(X_0(x), X_0(s(x)))$
- *if $Y_t^k(x) = 1$ then $Y_{t+1}^k(x) = 1$*
- *if $Y_t^k(x) = 0$ and x has at least 3 occupied sites among its 6 nearest neighbors, then $Y_{t+1}^k(x) = 1$ and $Y_{t+1}^k(s(x)) = 1$*

- $Y^k_{t+1}(x) = 0$ *otherwise*

The mechanism to build $Y^k_{t+1}(x)$ is the one used for $X^{\Lambda_l \setminus T_k}_{T_k,t+1}$ followed by an additional step increasing the configuration. This mechanism ensures that for all t and any x in T_k one has $Y^k_t(x) = Y^k_t(s(x))$.

Finally we introduce a two-dimensional process that will be used to estimate (15).

Definition 6 *Let us associate to each slice T_k for k in $\{1, ..., l/2\}$ a two-dimensional $l \times l$ square $Q^k_l := \{1, 2, ..., l\}^2$; then on each square Q^k_l we define a process $Z_{Q^k_l,t}$ by*

$$\forall (x_2, x_3) \in Q^k_l \qquad Z_{Q^k_l,t}(x_2, x_3) := Y^k_t(2k-1, x_2, x_3) = Y^k_t(2k, x_2, x_3) \quad . \tag{19}$$

The processes $Z_{Q^k_l,t}$, $1 \le k \le l/2$, are independent and they are two-dimensional bootstrap percolation processes with parameter $\ell = 2$ and initial density $q = 2p - p^2$. Furthermore these processes dominate the original process on the slices in the sense:

$$\forall k \in \{1, ..., \frac{l}{2}\} \quad \forall x = (x_1, x_2, x_3) \in T_k \quad \forall t \ge 0 \qquad X_{T_k,t}(x) \le Z_{Q^k_l,t}(x_2, x_3) \quad . \tag{20}$$

The two-dimensional processes can be used to estimate the probability that a cubic region is crossed: we consider the $(l/2) \times l \times l$ parallelepiped $\mathcal{P}_l$ obtained by collecting the $l/2$ squares Q^k_l and we denote by $Z_{\mathcal{P}_l}$ the configuration on $\mathcal{P}_l$ defined as follows:

$$\forall x_1 \in \{1, ..., \frac{l}{2}\} \; \forall x_2, x_3 \in \{1, ..., l\} \qquad Z_{\mathcal{P}_l}(x_1, x_2, x_3) := Z_{Q^{x_1}_l}(x_2, x_3) \quad , \tag{21}$$

where $Z_{Q^k_l}$ with k in $\{1, ..., l/2\}$ is the final configuration of the process $Z_{Q^k_l,t}$. Finally we have the way to estimate (15):

$$P(\Lambda_l \text{ is crossed along } e_1) \le P(\text{in } Z_{\mathcal{P}_l} \text{ there is a crossing along } e_1) \quad . \tag{22}$$

Now we consider the two-dimensional bootstrap percolation with parameter $\ell = 2$ and initial density q. We define as well the concept of "being internally spanned" and we denote by $S(l)$ a square of side length l. We recall that the final configuration, for such a process, is a union of separated rectangular regions and we state a few results:

Lemma 7 (Aizenman - Lebowitz [3]) *For all $\kappa \geq 1$, a necessary condition for $S(l)$ to be internally spanned, where $\kappa \leq l$, is that it contains at least one rectangular region whose maximal side length is in the interval $[\kappa, 2\kappa+1]$ which is also internally spanned.*

Proof of Lemma 7 is given in [3].

Lemma 8 *Let A be a rectangular region of side lengths l_1 and l_2; suppose $l_1 \leq l_2$. For q small enough one has*

$$P(A \text{ is internally spanned}) \leq (4l_2 q)^{l_2/2} \quad . \tag{23}$$

Proof. If A is partitioned in $l_2/2$ disjoint slabs of width 2, a necessary condition for A to be internally spanned is that each slab contains initially an occupied site. Hence:

$$P(A \text{ is internally spanned}) \leq \left(1-(1-q)^{2l_1}\right)^{l_2/2} \leq$$
$$\exp\left(\frac{l_2}{2}\ \ln\left(1-\exp(2l_2\ \ln(1-q))\right)\right) \leq$$
$$\exp\left(\frac{l_2}{2}\ \ln\left(-2l_2\ \ln(1-q)\right)\right) \tag{24}$$

For q small enough, $\ln(1-q) \geq -2q$, whence

$$P(A \text{ is internally spanned}) \leq \exp\left(\frac{l_2}{2}\ \ln(4l_2 q)\right) = (4l_2 q)^{l_2/2} \quad . \ \blacksquare \tag{25}$$

Lemma 9 *For any l and any $\kappa \leq (l-1)/2$ let $\mathcal{E}$ be the event: $S(l)$ contains a rectangular region internally spanned whose maximal side length belongs to the interval $[\kappa, 2\kappa+1]$. For q small enough one has*

$$P(\mathcal{E}) \leq l^2\ (2\kappa+1)^2\ \exp\left(-\frac{\kappa}{2}\ \exp(-4(2\kappa+1)q)\right) \quad . \tag{26}$$

Proof. We suppose q small enough to have $\ln(1-q) \geq -2q$ and we bound the probability of the event $\mathcal{E}$ as follows.

$$P(\mathcal{E}) \leq l^2\ (2\kappa+1)^2 \max_{\kappa \leq l_2 \leq 2\kappa+1} \exp\left(\frac{l_2}{2}\ \ln(1-\exp(2l_2\ \ln(1-q)))\right) \leq$$

$$l^2\ (2\kappa+1)^2\ \exp\left(\frac{\kappa}{2}\ \ln(1-\exp(2(2\kappa+1)\ \ln(1-q)))\right) \leq$$

$$l^2\ (2\kappa+1)^2\ \exp\left(\frac{\kappa}{2}\ \ln(1-\exp(-4(2\kappa+1)q))\right) \leq$$

$$l^2\ (2\kappa+1)^2\ \exp\left(-\frac{\kappa}{2}\ \exp(-4(2\kappa+1)q)\right)\quad . \ \blacksquare \tag{27}$$

Now we come back to the proof of the upper bound. Let us consider $\alpha > 0$, we notice that $p \leq q \leq 2p$. We denote by $\underline{1}$ a configuration in a square Q_l^h with all the sites occupied. We still increase the configuration $Z_{Q_l^k}$ by setting $Z_{Q_l^k} = \underline{1}$ in case $Z_{Q_l^k}$ contains at least one rectangular region of maximal side length larger than α/q. Supposing $l \geq \alpha/q$ and q small enough so that $\alpha/q > 3$, by applying Lemma 9 with $\kappa = \alpha/3q$ one has

$$P(Z_{Q_l^k} = \underline{1}) \leq l^2\ \left(\frac{2\alpha}{3q}+1\right)^2\ \exp\left(-\frac{\alpha}{6q}\ \exp\left(-4\left(\frac{2\alpha}{3q}+1\right)q\right)\right) \leq$$

$$\frac{l^2\alpha^2}{q^2}\ \exp\left(-\frac{\alpha}{6q}\ \exp(-4\alpha)\right) \tag{28}$$

We suppose that α is small enough to have $\exp(-4\alpha) \geq 1/2$; thus,

$$P(Z_{Q_l^k} = \underline{1}) \leq \frac{l^2\alpha^2}{q^2}\ \exp\left(-\frac{\alpha}{12q}\right)\quad . \tag{29}$$

Let M be the number of indices k such that one has $Z_{Q_l^k} = \underline{1}$ and let $k(1), ..., k(M)$ be these indices arranged in increasing order. Let $\mathcal{E}_1$ be the event $\{$there is a crossing along e_1 in $Z_{\mathcal{P}_l}\}$. We decompose this event as follows:

$$P(\mathcal{E}_1) = P(\mathcal{E}_1, M=0) + \sum_{m=1}^{l/2}\ \sum_{i_1<\cdots<i_m} P(\mathcal{E}_1,\ M=m, k(1)=i_1, ..., \ k(m)=i_m)\quad . \tag{30}$$

Let $i < j$ be two indices in $\{1, ..., l/2\}$; by $\mathcal{E}(i,j)$ we denote the following event: there exists a sequence of H disjoint rectangular regions $(R_h,\ 1 \leq h \leq H)$ in $Z_{\mathcal{P}_l}$ such that

- R_1 is included in Q_l^i, R_H is included in Q_l^j;
- the regions R_h with $2 \leq h \leq H-1$ are included in $\bigcup_{i<h<j} Q_l^h$;

- the maximal side length of all these regions is strictly less than α/q: we denote by r_h the maximal side length of R_h;
- for each h, $1 \leq h \leq H-1$, a site of R_h is the neighbor of a site of R_{h+1};
- all the sites of these regions are occupied in $Z_{\mathcal{P}_l}$.

We remark that H is free, however it has to be larger than $j-i+1$. Moreover the sequence of rectangles (R_h, $2 \leq h \leq H-1$) can go back and forth between the squares Q_l^{i+1} and Q_l^{j-1}. We make the convention that for any i the events $\mathcal{E}(i,i)$ and $\mathcal{E}(i,i-1)$ are the full events. We have the following estimate:

$$P(\mathcal{E}_1,\ M=m, k(1)=i_1,...., \ k(m)=i_m) \leq \ P(M=m,\ k(1)=i_1,....,\ k(m)=i_m,$$

$$\mathcal{E}(1,i_1-1),\ \mathcal{E}(i_1+1,i_2-1),...,\ \mathcal{E}(i_{m-1}+1,i_m-1),\ \mathcal{E}(i_m+1,\frac{l}{2}))=$$

$$\prod_{h=1}^{m+1} P(\mathcal{E}(i_{h-1}+1,i_h-1))\ P(Z_{Q_l^{i_1}}=\underline{1})\cdots P(Z_{Q_l^{i_m}}=\underline{1}) \leq$$

$$\left(\frac{l^2\alpha^2}{q^2}\ \exp\left(-\frac{\alpha}{12q}\right)\right)^m \prod_{h=1}^{m+1} P(\mathcal{E}(i_{h-1}+1,i_h-1)) \tag{31}$$

where we have set $i_0=0$ and $i_{m+1}=\frac{l}{2}+1$.

In order to estimate $P(\mathcal{E}(i,j))$ with $i<j$, we consider a fixed sequence $r_1,...,r_H$ in $\{1,...,\alpha/q-1\}$; the number of sequences $R_1,...,R_H$ of rectangles with maximal sides $r_1,...,r_H$ and satisfying the above requirements is smaller than

$$l^2 r_1 \times r_1^2 r_2 \times r_2^2 r_3 \times ... \times r_{H-1}^2 r_H \leq l^2\ (r_1 r_2 \cdots r_H)^3 \quad . \tag{32}$$

Notice that several rectangles of the sequence $R_1,...,R_H$ can belong to the same slice. However the rectangles are disjoint and the events that they are internally spanned depend only on the dynamics restricted to the rectangles, hence these events are independent, so that

$$P(R_1,...,R_H \text{ are occupied in } Z_{\mathcal{P}_l}) \leq P(R_1,...,R_H \text{ internally spanned}) \leq$$

$$P(R_1 \text{ internally spanned}) \cdots P(R_H \text{ internally spanned}) \leq (4r_1 q)^{r_1/2} \cdots (4r_H q)^{r_H/2}, \tag{33}$$

where in the last inequality we have used the Lemma 8. Thus

$$P(\mathcal{E}(i,j)) = \sum_{H \geq j-i+1} P(\exists R_1, ..., R_H \text{ realizing } \mathcal{E}(i,j)) \leq$$

$$\sum_{H \geq j-i+1} \sum_{r_1,...,r_H < \alpha/q} l^2 \ (r_1 \cdots r_H)^3 \ (4r_1 q)^{r_1/2} \cdots (4r_H q)^{r_H/2} =$$

$$\sum_{H \geq j-i+1} l^2 \left(\sum_{1 \leq r < \alpha/q} r^3 \ (4rq)^{r/2} \right)^H \tag{34}$$

We estimate the sum $\sum_{1 \leq r < \alpha/q} r^3 \ (4rq)^{r/2}$ as follows:

$$\sum_{1 \leq r < \alpha/q} r^3 \ (4rq)^{r/2} = \sum_{1 \leq r < 8} r^3 \ (4rq)^{r/2} \ + \sum_{9 \leq r < \alpha/q} r^3 \ (4rq)^{r/2} \leq$$

$$8^3 (32q)^{1/2} + \left(\frac{\alpha}{q}\right)^4 \max_{9 \leq r < \alpha/q} (4rq)^{r/2} \quad . \tag{35}$$

Let $f(r) = (4rq)^{r/2}$; for α small $f(r)$ is decreasing on $[9, \alpha/q[$, whence

$$\sum_{1 \leq r < \alpha/q} r^3 \ (4rq)^{r/2} \leq 8^3 (32q)^{1/2} + \alpha^4 36^{9/2} q^{1/2} \leq b_0 q^{1/2} \tag{36}$$

where b_0 is a constant not depending on α. Thus,

$$P(\mathcal{E}(i,j)) \leq \sum_{H \geq j-i+1} l^2 (b_0 \sqrt{q})^H \tag{37}$$

Finally, for q small enough, so that $b_0 \sqrt{q} < 1/2$, one has

$$P(\mathcal{E}(i,j)) \leq 2l^2 (b_0 \sqrt{q})^{j-i+1} = 2l^2 \exp((j-i+1)(\ln b_0 + \frac{1}{2} \ln q)) \quad . \tag{38}$$

Coming back to inequality (31)

$$P(\mathcal{E}_1, \ M = m, k(1) = i_1, ..., \ k(m) = i_m) \leq$$

$$\left(\frac{l^2 \alpha^2}{q^2} \exp\left(-\frac{\alpha}{12q}\right) \right)^m (2l^2)^{m+1} \exp\left[\left(\frac{l}{2} - m\right) \left(\ln b_0 + \frac{1}{2} \ln q\right) \right] \tag{39}$$

and

$$P(\mathcal{E}_1) = P(\mathcal{E}_1, M = 0) + \sum_{m=1}^{l/2} \sum_{i_1 < \cdots < i_m} P(\mathcal{E}_1,\ M = m, k(1) = i_1, ..., \ k(m) = i_m) \leq$$

$$\sum_{m=0}^{l/2} \left(\frac{l}{2}\right)^m \left(\frac{l^2\alpha^2}{q^2} \exp\left(-\frac{\alpha}{12q}\right)\right)^m (2l^2)^{m+1} \exp\left[\left(\frac{l}{2} - m\right)\left(\ln b_0 + \frac{1}{2}\ln q\right)\right] \leq$$

$$2l^2 \exp\left[\frac{l}{2}\left(\ln b_0 + \frac{1}{2}\ln q\right)\right] \sum_{m=0}^{l/2} \left[\frac{l^5\alpha^2}{q^2} \exp\left(-\frac{\alpha}{12q}\right) \exp\left[-\left(\ln b_0 + \frac{1}{2}\ln q\right)\right]\right]^m . \tag{40}$$

This is our estimate of the probability of a crossing in the case $l \geq \alpha/q$. On the other hand, in the case $l < \alpha/q$ we estimate the probability of a crossing by considering directly the event $\mathcal{E}(1, l/2)$:

$$P(\Lambda_l \text{ is crossed along } e_1) \leq P(\mathcal{E}(1, \frac{l}{2})) \leq 2l^2 \exp\left[\frac{l}{2}\left(\ln b_0 + \frac{1}{2}\ln q\right)\right] . \tag{41}$$

We note that it would have not been possible to use the same strategy in the case $l \geq \alpha/q$ because the probability of having a very large internally spanned rectangle does not vanish.

Supposing that $l \leq \exp(\alpha/120q)$ one has

$$\frac{l^5\alpha^2}{q^2} \exp\left(-\frac{\alpha}{12q}\right) \exp\left[-\left(\ln b_0 + \frac{1}{2}\ln q\right)\right] \leq \frac{\alpha^2}{q^2} \exp\left(-\frac{\alpha}{24q} - \ln b_0 - \frac{1}{2}\ln q\right) . \tag{42}$$

Now, if q is sufficiently small so that the right hand term is smaller than 1 and $\ln b_0 \leq -(1/4)\ln q$, under the hypothesis $2 \leq l \leq \exp(\alpha/120q)$, from (15), (40), (41) and (42) one has

$$P(\Lambda_l \text{ is crossed}) \leq 6l^2 \exp\left[\frac{l}{2}\left(\ln b_0 + \frac{1}{2}\ln q\right)\right]\left(\frac{l}{2} + 1\right) \leq 6l^3 \exp\left(\frac{l}{8}\ln q\right) . \tag{43}$$

Hence, there exists $\alpha > 0$ such that for p sufficiently small and $2 \leq l \leq \exp(\alpha/240p)$

$$P(\Lambda_l \text{ is crossed}) \leq 6l^3 \exp\left(\frac{l}{8}\ln(2p)\right) . \tag{44}$$

Finally, we can use equations (14) and (44) to estimate the probability $R(L, p)$ and to complete the proof of Theorem 2. First we consider the case $L \leq \exp(\alpha_0/p)$ with $\alpha_0 = \alpha/240$ and we write

$$R(L,p) \leq L^3 \min_{1 \leq \kappa < (L-1)/2} (\kappa + 1) \times 6(2\kappa + 1)^3 \exp\left(\frac{\kappa}{8}\ln(2p)\right) \leq 6L^7 \exp\left(\frac{L}{24}\ln(2p)\right) \tag{45}$$

and we remark that the right hand term goes to zero in the limit $p \to 0$ and $L \to \infty$. On the other hand, in the case $L > \exp(\alpha_0/p)$ one can restrict the minimum to $2\kappa + 1 \leq \exp(\alpha_0/p)$ and write

$$R(L,p) \leq 6L^3 \exp\left(\frac{4\alpha_0}{p}\right) \exp\left[\frac{1}{24}\exp\left(\frac{\alpha_0}{p}\right)\ln(2p)\right] \quad . \tag{46}$$

From the estimate above it is clear that there exists a positive constant c_- such that if L is less than $\exp\exp(c_-/p)$ then $R(L,p)$ goes to 0 in the limit $L \to \infty$ and $p \to 0$. This completes the proof of Theorem 2. ■

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