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Deconstructing Monopoles and Instantons

Giovanni Landi

Dipartimento di Scienze Matematiche, Università di Trieste
P.le Europa 1, I-34127, Trieste, Italy
and INFN, Sezione di Napoli, Napoli, Italy.

Abstract

We give a unifying description of the Dirac monopole on the 2-sphere S^2, of a graded monopole on a $(2,2)$-supersphere $S^{2,2}$ and of the BPST instanton on the 4-sphere S^4, by constructing a suitable global projector p via equivariant maps. This projector determines the projective modules of finite type of sections of the corresponding vector bundle. The canonical connection $\nabla = p \circ d$ is used to compute the topological charge which is found to be equal to -1 for the three cases. The transposed projector $q = p^t$ gives the value $+1$ for the charges. We also study the invariance under the action of suitable Lie groups.

This work is dedicated to Matteo.

1 Preliminaries and Introduction

It is well known since the early sixties that vector bundles can be though of as projective modules of finite type (finite projective modules 'for short'). The Serre-Swan's theorem [16] states that there is a complete equivalence between the category of (smooth) vector bundles over a (smooth) compact manifold and bundle maps, and the category of finite projective modules over commutative algebras and module morphisms. The space $\Gamma(M,E)$ of smooth sections of a vector bundle $E \to M$ over a compact manifold M is a finite projective module over the algebra $C(M)$ of (smooth) functions over M and any finite projective $C(M)$-module can be realized as the module of sections of some vector bundle over M. In fact, in [16] the correspondence is stated in the continuous category, meaning for topological manifolds and vector bundles and for functions and sections which are continuous. However, the equivalence can be extended to the smooth case [8]. This correspondence was already used in [11] to give an algebraic version of classical geometry, notably of the notions of connection and covariant derivative. But is as been with the advent of noncommutative geometry [7] that the equivalence has received a new emphasis and has been used, among several other things, to generalize the concept of vector bundles to noncommutative geometry and to construct noncommutative gauge and gravity theories. Furthermore, since the creation of noncommutative geometry, finite projective modules are increasingly being used among (mathematical)-physicists.

In this paper we present a finite-projective-module descriptions of the basic topologically non trivial gauge configurations, namely monopoles and instantons. Now, not all the associated projectors that we find are new. The Dirac monopole [9] projector is already present in [10] (in fact, the monopole projector is well know among physicists), while the BPST instanton [4] is present in the ADHM analysis [1], albeit in a local form. Our presentation is a global one which does not use any local chart or partition of unity and is based on a unifying description in terms of equivariant maps. We express the projectors in terms of a more fundamental object, a vector-valued function of basic equivariant maps. It is this reduction that motivated the word 'deconstructing' in our title.

For the time being, we present only the projectors carrying the lowest values of the charges, i.e. ± 1. Now, when the sphere S^2 is regarded as the complex projective space $\mathbb{C}P^1$ or the compactified plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ the monopole projector translates into the Bott projector (see for instance [18]). Thus, for the sphere S^4 and the supersphere $S^{2,2}$ the projector we construct could be considered as analogues of the Bott projector for S^2. The construction of global projectors for all values of the charges as well as for projective spaces will be the content of a paper in preparation. We refer to [12] for a friendly approach to modules of several kind (including finite projective). Throughout the paper we shall avoid writing explicitly the exterior product symbol for forms.

2 The General Construction

In this section we shall briefly describe the general scheme that will be used in the following for the monopoles and the instantons. All ingredients will be defined explicitly when needed later on. So let M be the sphere S^2, the supersphere $S^{2,2}$, or the sphere S^4. The symbol G will indicated the group $U(1)$, a supergroup $\mathcal{U}(1)$, or the group $Sp(1) \simeq SU(2)$

respectively. And $\pi : P \to M$ will be the corresponding G principal (super)fibration, with P the sphere S^3, the supergroup manifold $UOSP(1,2)$, or the sphere S^7. The symbol $\mathbb{F}$ will stand for the vectors spaces underlying the field of complex numbers $\mathbb{C}$, a complex Grassmann algebra C_L with L generators or the field of quaternions $\mathbb{H}$. We shall indicate with $\mathcal{B}_\mathbb{F}$ the algebra of $\mathbb{F}$-valued smooth functions on the total space P, while $\mathcal{A}_\mathbb{F}$ will be the algebra of $\mathbb{F}$-valued smooth functions on the base space M, thus $\mathcal{B}_\mathbb{F} =: C^\infty(P, \mathbb{F})$ and $\mathcal{A}_\mathbb{F} =: C^\infty(M, \mathbb{F})$.

On $\mathbb{F}$ there is a left action of the group G and $C^\infty_G(P, \mathbb{F})$ will denote the collection of corresponding equivariant maps:

$$\varphi : P \ \to \ \mathbb{F} \ , \quad \varphi(p \cdot w) = w^{-1} \cdot \varphi(p) \ , \tag{2.1}$$

with $\varphi \in C^\infty_G(P, \mathbb{F})$ and for any $p \in P$, $w \in G$; it is a right module over $\mathcal{A}_\mathbb{F}$. It is well known (see for instance [17]) that there is a module isomorphism between $C^\infty_G(P, \mathbb{F})$ and the right $\mathcal{A}_\mathbb{F}$-module of sections $\Gamma^\infty(M, E)$ of the associated vector bundle $E = P \times_G \mathbb{F}$ over M. In the spirit of Serre-Swan's theorem [16], the module $\Gamma^\infty(M, E)$ will be identified with the image in the trivial module $(\mathcal{A}_\mathbb{F})^N$ of a projector $p \in \mathbb{M}_N(\mathcal{A}_\mathbb{F})$, the latter being the algebra of $N \times N$ matrices with entries in $\mathcal{A}_\mathbb{F}$, i.e. $\Gamma^\infty(M, E) = p(\mathcal{A}_\mathbb{F})^N$. Since this projector is of rank 1 (over $\mathbb{F}$) it will be written as

$$p = |\psi\rangle \langle\psi| \ , \tag{2.2}$$

with

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \ , \tag{2.3}$$

a specific vector-valued function on P, thus a specific element of $(\mathcal{B}_\mathbb{F})^N$, the components being functions $\psi_i \in \mathcal{B}_\mathbb{F}$, $i = 1, \ldots, N$. The vector-valued function will be normalized,

$$\langle\psi|\psi\rangle = 1 \ , \tag{2.4}$$

a fact implying that p is a projector

$$p^2 = |\psi\rangle \langle\psi|\psi\rangle \langle\psi| = p \ , \quad p^\dagger = p \ , \tag{2.5}$$

(with † a suitable adjoint) of rank 1 over $\mathbb{F}$,

$$tr_\mathbb{F}\ p = \langle\psi|\psi\rangle = 1 \ . \tag{2.6}$$

The transformation rule of the vector-valued function $|\psi\rangle$ under the right action of an element $w \in G$ will be very simple, being indeed just component-wise multiplication,

$$|\psi\rangle \mapsto |\psi^w\rangle =: \begin{pmatrix} \psi_1 w \\ \vdots \\ \psi_N w \end{pmatrix} =: |\psi\rangle\, w \ . \tag{2.7}$$

As a consequence, the projector p will be invariant under the right action of G,

$$p \mapsto p^w = |\psi^w\rangle \langle\psi^w| = |\psi\rangle\, w w^\dagger \langle\psi| = p \ , \tag{2.8}$$

being $ww^\dagger = 1$. Thus, the entries of p are functions on the base space M, that is are elements of the algebra $\mathcal{A}_\mathbb{F}$ and $p \in \mathbb{M}_N(\mathcal{A}_\mathbb{F})$, as it should be.

To keep things distinct, we shall denote elements of $(\mathcal{A}_\mathbb{F})^N$ by the symbol

$$||f\rangle\rangle = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} , \tag{2.9}$$

with $f_1, \ldots, f_N$, elements of $\mathcal{A}_\mathbb{F}$. The module isomorphism between sections and equivariant maps will be explicitly given by,

$$\begin{aligned} \Gamma^\infty(M,E) \; &\leftrightarrow \; C^\infty_G(P,\mathbb{F}) \; , \\ \sigma = p\,||f\rangle\rangle \; &\leftrightarrow \; \varphi_\sigma = \langle\psi|\,|f\rangle\rangle =: \sum_{i=1}^N \overline{\psi}_i f_i \; . \end{aligned} \tag{2.10}$$

In fact, as we shall see, we shall first construct the equivariant maps, out of them the projector and then the sections of the associated bundle.

Having the projector, we can define a canonical connection (also called the Grassmann connection) on the module of sections by,

$$\begin{aligned} &\nabla =: p \circ d \; : \; \Gamma^\infty(M,E) \; \rightarrow \; \Gamma^\infty(M,E) \otimes_{\mathcal{A}_\mathbb{F}} \Omega^1(M,\mathbb{F}) \; , \\ &\nabla\sigma =: \nabla(p\,||f\rangle\rangle) = p(||df\rangle\rangle + dp\,||f\rangle\rangle) \; , \end{aligned} \tag{2.11}$$

where we have used the explicit identification $\Gamma^\infty(M,E) = p(\mathcal{A}_\mathbb{F})^N$. The corresponding connection 1-form $A_\nabla \in End_{\mathcal{B}_\mathbb{F}}(C^\infty_G(P,\mathbb{C})) \otimes_{\mathcal{B}_\mathbb{F}} \Omega^1(P,\mathbb{F})$ on the equivariant maps has a very simple expression in terms of the vector-valued function $|\psi\rangle$. Indeed, for any section $\sigma \in \Gamma^\infty(M,E)$, by using the isomorphism (2.10) and Leibniz rule we find,

$$\begin{aligned} \nabla\varphi_\sigma =: \varphi_{\nabla\sigma} \; &= \; \langle\psi|(||df\rangle\rangle + d(|\psi\rangle\,\langle\psi|)\,||f\rangle\rangle)\rangle \\ &= \; \langle\psi|(||df\rangle\rangle + |\psi\rangle\,\langle d\psi|\,|f\rangle\rangle) + |d\psi\rangle\,\langle\psi|\,|f\rangle\rangle\rangle \\ &= \; d\,\langle\psi|\,|f\rangle\rangle + \langle\psi|d\psi\rangle\,\langle\psi|\,|f\rangle\rangle \\ &= \; (d + \langle\psi|d\psi\rangle)\,\langle\psi|\,|f\rangle\rangle \\ &= \; (d + \langle\psi|d\psi\rangle)\varphi_\sigma \; , \end{aligned} \tag{2.12}$$

from which we get

$$A_\nabla = \langle\psi|d\psi\rangle \; . \tag{2.13}$$

This connection form is anti-hermitian, a consequence of the normalization $\langle\psi|\psi\rangle = 1$:

$$(A_\nabla)^\dagger =: \langle d\psi|\psi\rangle = -\,\langle\psi|d\psi\rangle = -A_\nabla \; . \tag{2.14}$$

As for the curvature of the connection (2.11), which is ∇^2, it is found to be,

$$\nabla^2 = p(dp)^2 \; . \tag{2.15}$$

We shall also need the operator $\nabla^2 \circ \nabla^2$ which is readily found to be

$$\nabla^2 \circ \nabla^2 = p(dp)^4 \; . \tag{2.16}$$

By using a suitable trace, these two operators give the first and the second Chern classes of the vector bundle (these are the only classes that we shall need) as [7],

$$C_1(p) =: -\frac{1}{2\pi i}\ tr(\nabla^2) = -\frac{1}{2\pi i}\ tr(p(dp)^2)\ ,$$
$$C_2(p) =: \frac{1}{2}(-\frac{1}{2\pi i})^2 tr(\nabla^2 \circ \nabla^2) = -\frac{1}{8\pi^2}\ tr(p(dp)^4)\ . \quad (2.17)$$

When integrated over M, they will give the corresponding Chern numbers,

$$c_1(p) = \int_M C_1(p)\ , \quad c_2(p) = \int_M C_2(p)\ . \quad (2.18)$$

On the ket-valued function (2.3) there will also be a global *left* action of a group of 'unitaries' $SU = \{s \mid ss^\dagger = 1\}$ (for the three cases considered this group will be $SU(2)$, $UOSP(1,2)$ and $Sp(2) \simeq Spin(5)$ respectively) which preserves the normalization,

$$|\psi\rangle \ \mapsto\ |\psi^s\rangle = s\,|\psi\rangle\ , \quad \langle\psi^s|\psi^s\rangle = 1\ . \quad (2.19)$$

The corresponding transformed projector

$$p^s = |\psi^s\rangle\langle\psi^s| = s\,|\psi\rangle\langle\psi|\,s^\dagger = sps^\dagger\ , \quad (2.20)$$

is equivalent to the starting one, the partial isometry being $v = sp$; indeed, $vv^\dagger = p^s$ and $v^\dagger v = p$. Furthermore, the connection 1-form is left invariant,

$$A_{\nabla^s} = \langle\psi^s|d\psi^s\rangle = \langle\psi|\,s^\dagger s\,|d\psi\rangle = A_\nabla\ . \quad (2.21)$$

To get new (in general gauge non-equivalent) connections one should act with group elements which do not preserve the normalization. Thus let g in some group GL (for the three cases we shall study this group will be $GL(2;\mathbb{C})$, $GL(1,2;C_L)$ and $GL(2;\mathbb{H})$ respectively) act on the ket-valued function (2.3) by

$$|\psi\rangle \ \mapsto\ |\psi^g\rangle = \frac{1}{[\langle\psi|\,g^\dagger g\,|\psi\rangle]^{\frac{1}{2}}}\ g\,|\psi\rangle\ . \quad (2.22)$$

The corresponding transformed projector

$$\begin{aligned} p^g &= |\psi^g\rangle\langle\psi^g| = \frac{1}{\langle\psi|\,g^\dagger g\,|\psi\rangle}\ g\,|\psi\rangle\langle\psi|\,g^\dagger\ , \\ &= \frac{1}{\langle\psi|\,g^\dagger g\,|\psi\rangle}\ gpg^\dagger \end{aligned} \quad (2.23)$$

is again equivalent to the starting one, the partial isometry being now,

$$v = \frac{1}{[\langle\psi|\,g^\dagger g\,|\psi\rangle]^{\frac{1}{2}}}\ gp\ . \quad (2.24)$$

Indeed,

$$\begin{aligned} vv^\dagger &= p^s\ , \\ v^\dagger v &= \frac{1}{\langle\psi|\,g^\dagger g\,|\psi\rangle}\ pg^\dagger gp = \frac{1}{\langle\psi|\,g^\dagger g\,|\psi\rangle}\ |\psi\rangle\langle\psi|\,g^\dagger g\,|\psi\rangle\langle\psi| \\ &= |\psi\rangle\langle\psi| = p\ . \end{aligned} \quad (2.25)$$

The associated connection 1-form is readily found to be

$$A_{\nabla^g} =: \langle \psi^g | d\psi^g \rangle = \frac{1}{2\,\langle \psi |\, g^\dagger g\, | \psi \rangle}\, \left[\langle \psi |\, g^\dagger g\, | d\psi \rangle - \langle d\psi |\, g^\dagger g\, | \psi \rangle \right] . \tag{2.26}$$

Thus, if $g \in SU$, we get back the previous invariance of connections (2.21), while for $g \in GL$ modulo SU we get new, gauge non-equivalent connections on the $\mathbb{F}$-line bundle over M determined by the projector p^g, line bundle which is (stable) isomorphic to the one determined by the projector p.

3 The Dirac Monopole

3.1 The $U(1)$ Bundle over S^2.

The $U(1)$ principal fibration $\pi : S^3 \to S^2$ over the two dimensional sphere is explicitly realized as follows. The total space is

$$S^3 = \{ (a, b) \in \mathbb{C}^2 \ , \ |a|^2 + |b|^2 = 1 \} \ . \tag{3.1}$$

with right $U(1)$-action

$$S^3 \times U(1) \ \to \ S^3 \ , \quad (a, b)w = (aw, bw); \tag{3.2}$$

Clearly $|aw|^2 + |bw|^2 = |a|^2 + |b|^2 = 1$. The bundle projection $\pi : S^3 \to S^2$, given by $\pi(a, b) =: (x_0, x_1, x_2)$, is

$$\begin{aligned} x_0 &= |a|^2 - |b|^2 = -1 + 2|a|^2 = 1 - 2|b|^2 \ , \\ x_1 &= a\bar{b} + b\bar{a} \ , \\ x_2 &= i(a\bar{b} - b\bar{a}) \ , \end{aligned} \tag{3.3}$$

and one checks that $\sum_{\mu=0}^{2} (x_\mu)^2 = (|a|^2 + |b|^2)^2 = 1$. The inversion of (3.3) gives the basic ($\mathbb{C}$-valued) invariant functions on S^3,

$$\begin{aligned} |a|^2 &= \frac{1}{2}(1 + x_0) \ , \\ |b|^2 &= \frac{1}{2}(1 - x_0) \ , \\ a\bar{b} &= \frac{1}{2}(x_1 - i x_2) \ , \end{aligned} \tag{3.4}$$

a generic invariant (polynomial) function on S^3 being any function of the previous variables. Later on we shall need the volume form of S^2 which turns out to be

$$d(vol(S^2)) = x_0 dx_1 dx_2 + x_1 dx_2 dx_0 + x_2 dx_0 dx_1 = 2i(da d\bar{a} + db d\bar{b}) \ . \tag{3.5}$$

3.2 The Bundle and the Projector for the Monopole

We need the rank 1 complex vector bundle associated with the defining left representation of $U(1)$ on $\mathbb{C}$. The latter is just left complex multiplication by $w \in U(1)$,

$$U(1) \times \mathbb{C} \ \to \ \mathbb{C} \ , \quad (w, c) \mapsto c' = wc \ . \tag{3.6}$$

The corresponding equivariant maps $\varphi : S^3 \to \mathbb{C}$ are of the form

$$\varphi(a, b) = \overline{a} f + \overline{b} g \ , \tag{3.7}$$

with f, g any two $\mathbb{C}$-valued functions which are invariant under the right action of $U(1)$ on S^3. Indeed,

$$\varphi((a, b)w) = \overline{aw} f + \overline{bw} g = w^{-1} \varphi((a, b)) \ . \tag{3.8}$$

We shall think of f, g as $\mathbb{C}$-valued functions on the base space S^2, namely elements of $\mathcal{A}_{\mathbb{C}} =: C^\infty(S^2, \mathbb{C})$. The space $C^\infty_{U(1)}(S^3, \mathbb{C})$ of equivariant maps is a right module over the (pull-back of) functions $\mathcal{A}_{\mathbb{C}}$.

Consider then, the vector-valued function,

$$|\psi\rangle =: \begin{pmatrix} a \\ b \end{pmatrix} \ , \tag{3.9}$$

which is normalized

$$\langle \psi | \psi \rangle = |a|^2 + |b|^2 = 1 \ . \tag{3.10}$$

By using it we can construct a projector in $\mathbb{M}_2(\mathcal{A}_{\mathbb{C}})$:

$$p =: |\psi\rangle \langle \psi| = \begin{pmatrix} |a|^2 & a\overline{b} \\ b\overline{a} & |b|^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + x_0 & x_1 - i x_2 \\ x_1 + i x_2 & 1 - x_0 \end{pmatrix} \ , \tag{3.11}$$

where we have used the definition (3.3) for the coordinates on S^2. It is clear that p is a projector,

$$p^2 =: |\psi\rangle \langle \psi | \psi \rangle \langle \psi| = |\psi\rangle \langle \psi| = p \ , \quad p^\dagger = p \ , \tag{3.12}$$

of rank 1 over $\mathbb{C}$,

$$trp = \langle \psi | \psi \rangle = 1 \ . \tag{3.13}$$

The $U(1)$-action (3.2) will transform the vector (3.9) multiplicatively,

$$|\psi\rangle \ \mapsto \ |\psi^w\rangle = \begin{pmatrix} aw \\ bw \end{pmatrix} = |\psi\rangle \, w \ , \quad \forall \ w \in U(1) \ . \tag{3.14}$$

As a consequence the projector p is invariant, a fact which is also evident from its explicit expression (3.11).

Thus, the right module of sections $\Gamma^\infty(S^2, E)$ of the associated bundle is identified with the image of p in $(\mathcal{A}_{\mathbb{C}})^2$ and the module isomorphism between sections and equivariant maps is given by,

$$\begin{aligned} \Gamma^\infty(S^2, E) \ &\leftrightarrow \ C^\infty_{U(1)}(S^3, \mathbb{C}) \ , \\ \sigma = p \begin{pmatrix} f \\ g \end{pmatrix} \ &\leftrightarrow \ \varphi_\sigma = \langle \psi | \begin{pmatrix} f \\ g \end{pmatrix} = \overline{a} f + \overline{b} g \ , \quad \forall \ f, g \in \mathcal{A}_{\mathbb{C}} \ . \end{aligned} \tag{3.15}$$

It is obvious that this map is a module isomorphism.

The canonical connection associated with the projector,

$$\nabla = p \circ d \ : \ \Gamma^\infty(S^2, E) \ \to \ \Gamma^\infty(S^2, E) \otimes_{\mathcal{A}_\mathbb{C}} \Omega^1(S^2, \mathbb{C}), \tag{3.16}$$

has curvature given by

$$\nabla^2 = p(dp)^2 = |\psi\rangle \langle d\psi | d\psi\rangle \langle \psi| \ . \tag{3.17}$$

The associated Chern 2-form is

$$C_1(p) =: -\frac{1}{2\pi i}\, tr(p(dp)^2) = -\frac{1}{2\pi i} \langle d\psi | d\psi\rangle = \frac{1}{2\pi i}(da d\overline{a} + db d\overline{b}) = -\frac{1}{4\pi} d(vol(S^2)) \ , \tag{3.18}$$

with corresponding first Chern number

$$c_1(p) = \int_{S^2} C_1(p) = -\frac{1}{4\pi}\int_{S^2} d(vol(S^2)) = -\frac{1}{4\pi} 4\pi = -1 \ . \tag{3.19}$$

By transposing the projector (3.11) we obtain an inequivalent projector,

$$q =: p^t = |\phi\rangle \langle \phi| \ , \tag{3.20}$$

with the transposed ket vector given by,

$$|\phi\rangle =: (\langle \psi|)^t = \begin{pmatrix} \overline{a} \\ b \end{pmatrix} \ . \tag{3.21}$$

We find that

$$q = \begin{pmatrix} |a|^2 & b\overline{a} \\ a\overline{b} & |b|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + x_0 & x_1 + i x_2 \\ x_1 - i x_2 & 1 - x_0 \end{pmatrix} \ . \tag{3.22}$$

That q is a projector ($q^2 = q$), of rank 1 ($trq = 1$) are both consequences of the normalization $\langle \phi | \phi\rangle = |a|^2 + |b|^2 = 1$. The projector q is obtained from p by exchanging $a \to \overline{a}$ and $b \to \overline{b}$ which amounts to the exchange $x_2 \to -x_2$. It is then clear that the corresponding Chern form and number are given by,

$$C_1(q) = -\frac{1}{2\pi i}(da d\overline{a} + db d\overline{b}) = \frac{1}{4\pi} d(vol(S^2)) \ , \tag{3.23}$$

$$c_1(q) = \int_{S^2} C_1(q) = \frac{1}{4\pi}\int_{S^2} d(vol(S^2)) = 1 \ . \tag{3.24}$$

Having different topological charges the projectors p and q are clearly inequivalent.

The connection 1-form (2.13) associated with the projector p is given by

$$A_\nabla = \langle \psi | d\psi\rangle = \overline{a} da + \overline{b} db \ . \tag{3.25}$$

This connection form is clearly anti-hermitian, so it is valued in $i\mathbb{R}$, the Lie algebra of $U(1)$. It coincides with the charge -1 monopole connection form [14, 17]. Furthermore, the invariance (2.21) states the invariance of (3.25) under left action of $SU(2)$. Gauge non-equivalent connections are obtained by the formula (2.26),

$$A_{\nabla^g} = \frac{1}{2\,\langle \psi|\, g^\dagger g\, |\psi\rangle} \left[\langle \psi|\, g^\dagger g\, |d\psi\rangle - \langle d\psi|\, g^\dagger g\, |\psi\rangle \right] \ , \quad |\psi\rangle =: \begin{pmatrix} a \\ b \end{pmatrix} \ , \tag{3.26}$$

with $g \in GL(2; \mathbb{C})$ modulo $SU(2)$. Similar considerations hold for the connection 1-form associated with the monopole projector q.

4 The Graded Monopole

4.1 The $\mathcal{U}(1)$ Bundle over $S^{2,2}$.

The $\mathcal{U}(1)$ principal fibration $\pi : UOSP(1,2) \to S^{2,2}$ over the $(2,2,)$-dimensional supersphere is explicitly realized as follows. The total space is the $(1,2,)$-dimensional supergroup $UOSP(1,2)$. Both $UOSP(1,2)$ and $\mathcal{U}(1)$ are described in more details in the Appendix. A generic element $s \in UOSP(1,2)$ can be parametrized as

$$s = \left(\begin{array}{ccc} 1 + \frac{1}{4}\eta\eta^\diamond & -\frac{1}{2}\eta & \frac{1}{2}\eta^\diamond \\ -\frac{1}{2}(a\eta^\diamond - b^\diamond\eta) & a(1 - \frac{1}{8}\eta\eta^\diamond) & -b^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \\ -\frac{1}{2}(b\eta^\diamond + a^\diamond\eta) & b(1 - \frac{1}{8}\eta\eta^\diamond) & a^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \end{array} \right) . \tag{4.1}$$

Here a, b and η are elements in a complexified Grassmann algebra $C_L = B_L \otimes_{\mathbb{R}} \mathbb{C}$ with L generators (B_L being obviously a real Grassmann algebra) with the restriction $a, b \in (C_L)_0$ and $\eta \in (C_L)_1$. Furthermore, elements of $UOSP(1,2)$ have superdeterminant equal to 1 and this give the condition

$$aa^\diamond + bb^\diamond = Sdet(s) = 1 \ . \tag{4.2}$$

We recall that the integer L is taken to be even and this assures [15] the existence of an even graded involution

$$\begin{aligned} {}^\diamond : C_L \ \to \ C_L \ , \qquad & |x^\diamond| = |x| \ , \quad \forall\, x \in (C_L)_{|x|} \ , \\ & (cx)^\diamond = \overline{c}x^\diamond \ , \quad \forall\, c \in \mathbb{C} \ , \ x \in C_L \ , \end{aligned} \tag{4.3}$$

which in addition verifies the properties

$$\begin{aligned} & (xy)^\diamond = x^\diamond \ y^\diamond \ , \quad \forall\, x, y \in C_L \ , \\ & x^{\diamond\diamond} = (-1)^{|x|}x \ , \quad \forall\, x \in (C_L)_{|x|} \ . \end{aligned} \tag{4.4}$$

The structure supergroup of the fibration is $\mathcal{U}(1)$, the Grassmann extension of $U(1)$. It can be realized as follows

$$\mathcal{U}(1) = \{w \in (C_L)_0 \mid ww^\diamond = 1\}. \tag{4.5}$$

Now, the Lie superalgebra of $UOSP(1,2)$ is generated by three even elements A_0, A_1, A_2 and two odd ones R_+, R_- whose matrix representation is given in (A.1). Then, by embedding $\mathcal{U}(1)$ in $UOSP(1,2)$ as

$$w \ \mapsto \ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^\diamond \end{array} \right) , \tag{4.6}$$

we may think of A_0 as the generator of $\mathcal{U}(1)$, i.e.

$$\mathcal{U}(1) \simeq \{exp(\lambda A_0) \mid \lambda \in (C_L)_0 \ , \ \lambda^\diamond = \lambda\}. \tag{4.7}$$

We let $\mathcal{U}(1)$ act on the right on $UOSP(1,2)$. If we parametrize any $s \in UOSP(1,2)$ by $s = s(a,b,\eta)$, then this action can be represented as follows,

$$UOSP(1,2) \times \mathcal{U}(1) \;\rightarrow\; UOSP(1,2) \;, \quad (s,w) \mapsto s \cdot w = s(aw, bw, \eta w) \;. \tag{4.8}$$

This action leaves unchanged the superdeterminant $Sdet(s \cdot w) = aw(aw)^\diamond + bw(bw)^\diamond = aa^\diamond + bb^\diamond = 1$.

The bundle projection

$$\begin{aligned} &\pi : UOSP(1,2) \;\rightarrow\; S^{2,2} =: UOSP(1,2)/\mathcal{U}(1) \;, \\ &\pi(a,b,\eta) = (x_0, x_1, x_2, \xi_+, \xi_-) \end{aligned} \tag{4.9}$$

can be given as the (co)-adjoint orbit through A_0. With $s^\dagger$ the adjoint of s as given in (A.9), one has that

$$\pi(s) =: s(\frac{2}{i}A_0)s^\dagger =: \sum_{k=0,1,2} x_k(\frac{2}{i}A_k) + \sum_{\alpha=+,-} \xi_\alpha(2R_\alpha) \;. \tag{4.10}$$

Explicitly,

$$\begin{aligned} x_0 &= (aa^\diamond - bb^\diamond)(1 - \frac{1}{4}\eta\eta^\diamond) = (-1 + 2aa^\diamond)(1 - \frac{1}{4}\eta\eta^\diamond) = (1 - 2bb^\diamond)(1 - \frac{1}{4}\eta\eta^\diamond) \;, \\ x_1 &= (a\overline{b} + b\overline{a})(1 - \frac{1}{4}\eta\eta^\diamond) \;, \\ x_2 &= i(a\overline{b} - b\overline{a})(1 - \frac{1}{4}\eta\eta^\diamond) \;, \\ \xi_- &= -\frac{1}{2}(a\eta^\diamond + \eta b^\diamond) \;, \\ \xi_+ &= \frac{1}{2}(\eta a^\diamond - b\eta^\diamond) \;. \end{aligned} \tag{4.11}$$

One sees directly that the x_k's are even, $x_k \in (C_L)_0$, and 'real', $x_k^\diamond = x_k$, while the ξ_α are odd, $\xi_\alpha \in (C_L)_1$, and such that ${\xi_-}^\diamond = \xi_+$ (and ${\xi_+}^\diamond = -\xi_-$). Furthermore,

$$\begin{aligned} \sum_{\mu=0}^{2}(x_\mu)^2 + 2\xi_-\xi_+ &= (aa^\diamond + bb^\diamond)^2(1 - \frac{1}{2}\eta\eta^\diamond) + \frac{1}{2}(aa^\diamond + bb^\diamond)\eta\eta^\diamond \\ &= 1 \;. \end{aligned} \tag{4.12}$$

Thus, the base space $S^{2,2}$ is a $(2,2)$-dimensional sphere in the superspace $B_L^{3,2}$. It turns out that $S^{2,2}$ is a De Witt supermanifold with *body* the usual sphere S^2 in $\mathbb{R}^3$ [3], a fact that we shall use later. The inversion of (4.11) gives the basic (C_L-valued) invariant functions on $UOSP(1,2)$. Firstly, notice that

$$\frac{1}{4}\eta\eta^\diamond = \xi_-\xi_+ \;. \tag{4.13}$$

Furthermore,

$$aa^\diamond = \frac{1}{2}[1 + x_0(1 + \xi_-\xi_+)] \;,$$

$$
\begin{aligned}
bb^\diamond &= \frac{1}{2}[1 - x_0(1 + \xi_-\xi_+)] \ , \\
ab^\diamond &= \frac{1}{2}(x_1 - ix_2)(1 + \xi_-\xi_+) \ , \\
\eta a^\diamond &= -(x_1 + ix_2)\xi_- + (1 + x_0)\xi_+ \ , \\
\eta b^\diamond &= (x_1 - ix_2)\xi_+ - (1 - x_0)\xi_- \ ,
\end{aligned} \tag{4.14}
$$

a generic invariant (polynomial) function on $UOSP(1,2)$ being any function of the previous variables.

4.2 The Bundle and the Projector for the Graded Monopole

We need the rank 1 (over C_L) vector bundle associated with the defining left representation of $\mathcal{U}(1)$ on C_L. The latter is just left complex multiplication by $w \in \mathcal{U}(1)$,

$$
\mathcal{U}(1) \times \mathbb{C} \ \rightarrow \ \mathbb{C} \ , \quad (w, c) \mapsto c' = wc \ . \tag{4.15}
$$

The corresponding equivariant maps $\varphi : UOSP(1,2) \rightarrow C_L$ will be written in the following form,

$$
\varphi(\eta, a, b) = \frac{1}{2}\eta^\diamond h + a^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) f + b^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) g \ , \tag{4.16}
$$

with h, f, g any three B_L-valued functions which are invariant under the right action of $\mathcal{U}(1)$ on $UOSP(1,2)$ (the reason for the choice of the additional invariant factor $(1-\frac{1}{8}\eta\eta^\diamond)$ will be given later). Indeed,

$$
\begin{aligned}
\varphi((\eta, a, b)w) &= \frac{1}{2}(\eta w)\eta^\diamond h + (aw)^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) f + (bw)^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) g \\
&= w^{-1}\varphi((\eta, a, b)) \ .
\end{aligned} \tag{4.17}
$$

We shall think of h, f, g as C_L-valued functions on the base space $S^{2,2}$, namely elements of the algebra of superfunctions $\mathcal{A}_{C_L} =: C^\infty(S^{2,2}, C_L)$ *. The space $C^\infty_{\mathcal{U}(1)}(S^{2,2}, C_L)$ of equivariant maps is a right module over the (pull-back of) superfunctions $\mathcal{A}_{C_L}$.

Next, let us consider the ket-valued superfunction,

$$
|\psi\rangle = \begin{pmatrix} -\frac{1}{2}\eta \\ \\ a(1 - \frac{1}{8}\eta\eta^\diamond) \\ \\ b(1 - \frac{1}{8}\eta\eta^\diamond) \end{pmatrix} \ , \tag{4.18}
$$

whose associated bra is given by

$$
\langle\psi| =: (\frac{1}{2}\eta^\diamond, \ a^\diamond(1 - \frac{1}{8}\eta\eta^\diamond), \ b^\diamond(1 - \frac{1}{8}\eta\eta^\diamond)) \ . \tag{4.19}
$$

They obey the relation

$$
\begin{aligned}
\langle\psi|\psi\rangle &= -\frac{1}{4}\eta^\diamond\eta + (aa^\diamond + bb^\diamond)(1 - \frac{1}{4}\eta\eta^\diamond) \\
&= 1 \ .
\end{aligned} \tag{4.20}
$$

*We refer to [2] for the definition of superfunctions where they are denoted with $\mathcal{G}$.

As a consequence, we get a projector in $\mathbb{M}_{2,1}(\mathcal{A}_{C_L})$,

$$p =: |\psi\rangle\langle\psi| = \begin{pmatrix} -\frac{1}{4}\eta\eta^\diamond & -\frac{1}{2}\eta a^\diamond & -\frac{1}{2}\eta b^\diamond \\ \\ \frac{1}{2}a\eta^\diamond & aa^\diamond(1-\frac{1}{4}\eta\eta^\diamond) & ab^\diamond(1-\frac{1}{4}\eta\eta^\diamond) \\ \\ \frac{1}{2}b\eta^\diamond & ba^\diamond(1-\frac{1}{4}\eta\eta^\diamond) & bb^\diamond(1-\frac{1}{4}\eta\eta^\diamond) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2\xi_+\xi_- \ ; & (x_1+ix_2)\xi_- - (1+x_0)\xi_+ \ ; & -(x_1-ix_2)\xi_+ + (1-x_0)\xi_- \\ \\ -(x_1-ix_2)\xi_+ - (1+x_0)\xi_- \ ; & 1+x_0+\xi_+\xi_- \ ; & x_1-ix_2 \\ \\ -(x_1+ix_2)\xi_- - (1-x_0)\xi_+ \ ; & x_1+ix_2 \ ; & 1-x_0+\xi_+\xi_- \end{pmatrix}, \quad (4.21)$$

where we have used the definition (4.11) for the coordinates on $S^{2,2}$. It is clear that p is a projector,

$$p^2 =: |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = p \ , \quad p^\dagger = p \ . \qquad (4.22)$$

And it is of (super-)rank 1,

$$Strp =: (-1)(-\frac{1}{4}\eta\eta^\diamond) + (aa^\diamond + bb^\diamond)(1-\frac{1}{4}\eta\eta^\diamond) = 1 \ , \qquad (4.23)$$

with Str denoting the supertrace of a supermatrix. The action (4.8) of $\mathcal{U}(1)$ will transform the vector (4.18) by

$$|\psi\rangle \ \mapsto \ |\psi^w\rangle = \begin{pmatrix} -\frac{1}{2}\eta w \\ \\ aw(1-\frac{1}{8}\eta\eta^\diamond) \\ \\ bw(1-\frac{1}{8}\eta\eta^\diamond) \end{pmatrix} = |\psi\rangle\, w \ , \quad \forall\ w \in \mathcal{U}(1) \ , \qquad (4.24)$$

and as a consequence the projector p is left invariant, a fact which is also evident from its explicit form (4.21).

Thus, the right module of sections $C^\infty(S^{2,2}, E)$ of the associated bundle is identified with the image of p in $(\mathcal{A}_{C_L})^3$ and the module isomorphism between sections and equivariant maps is given by,

$$\begin{aligned} C^\infty(S^{2,2}, E) \ &\leftrightarrow \ C^\infty_{\mathcal{U}(1)}(S^{2,2}, C_L) \ , \\ \sigma = p\begin{pmatrix} h \\ f \\ g \end{pmatrix} \ &\leftrightarrow \ \varphi_\sigma = \langle\psi|\begin{pmatrix} h \\ f \\ g \end{pmatrix} \\ &= \frac{1}{2}\eta^\diamond h + a^\diamond(1-\frac{1}{8}\eta\eta^\diamond)f + b^\diamond(1-\frac{1}{8}\eta\eta^\diamond)g \ , \end{aligned} \qquad (4.25)$$

for any $h, f, g \in \mathcal{A}_{C_L}$.

The canonical connection associated with p,

$$\nabla = p \circ d \ : \ C^\infty(S^{2,2}, E) \ \to \ C^\infty(S^{2,2}, E) \otimes_{\mathcal{A}_{C_L}} \Omega^1(S^{2,2}, \mathbb{C}), \qquad (4.26)$$

has curvature given by

$$\nabla^2 = p(dp)^2 = |\psi\rangle \langle d\psi|d\psi\rangle \langle\psi| \ . \tag{4.27}$$

The associated Chern 2-form is found to be

$$\begin{aligned} C_1(p) &=: -\frac{1}{2\pi i} \ Str(p(dp)^2) = -\frac{1}{2\pi i} \langle d\psi|d\psi\rangle \\ &= -\frac{1}{2\pi i}[-(dada^\diamond + dbdb^\diamond)(1 - \frac{1}{4}\eta\eta^\diamond) \\ &\qquad - \frac{1}{4}(ada^\diamond + bdb^\diamond)(d\eta\eta^\diamond + \eta d\eta^\diamond) - \frac{1}{4}d\eta d\eta^\diamond] \ , \\ &= -\frac{1}{2\pi i}[-(dada^\diamond + dbdb^\diamond) - \frac{1}{4}d(a\eta^\diamond)d(\eta a^\diamond) - \frac{1}{4}d(b\eta^\diamond)d(\eta b^\diamond)] \ . \end{aligned} \tag{4.28}$$

By using the coordinates on $S^{2,2}$ the previous 2-form results in

$$\begin{aligned} C_1(p) =: &-\frac{1}{4\pi}(x_0 dx_1 dx_2 + x_1 dx_2 dx_0 + x_2 dx_0 dx_1)(1 + 3\xi_-\xi_+) \\ &- \frac{1}{4\pi i}[(dx_1 - idx_2)\xi_+ d\xi_+ - (dx_1 + idx_2)\xi_- d\xi_- + dx_0(\xi_- d\xi_+ + \xi_+ d\xi_-) \\ &\qquad + (x_1 - ix_2)d\xi_+ d\xi_+ - (x_1 + ix_2)d\xi_- d\xi_- - 2x_0 d\xi_- d\xi_+] \ . \end{aligned} \tag{4.29}$$

Finally, to compute the corresponding first Chern number we need the Berezin integral over the supermanifold $S^{2,2}$. This is a rather simple task due to the fact that $S^{2,2}$ is a De Witt supermanifold over the two-dimensional sphere S^2 in $\mathbb{R}^3$. Indeed, with $\widetilde{\ } : \Omega^2(S^{2,2}) \to \Omega^2(S^2)$ the natural morphism of forms, the first Chern number given by the superform $C_1(p)$ is given by [6]

$$c_1(p) =: Ber_{S^{2,2}} C_1(p) =: \int_{S^2} \widetilde{C_1(p)} \ . \tag{4.30}$$

It is straightforward to find the projected form $\widetilde{C_1(p)}$. The bundle projection $\Phi : S^{2,2} \to S^2$ is explicitly realized in terms of the body map,

$$\Phi(x_0, x_1, x_2; \xi_-, \xi_+) \ \to \ (\sigma(x_0), \sigma(x_1), \sigma(x_2)) \ . \tag{4.31}$$

Recall that fermionic variables do not have body. On the other side, by denoting $\sigma_i = \sigma(x_i)$, $i = 0, 1, 2$, the σ_i's are cartesian coordinates for the sphere S^2 in $\mathbb{R}^3$ and obey the condition $(\sigma_0)^2 + (\sigma_1)^2 + (\sigma_2)^2 = 1$. The projected form $\widetilde{C_1(p)}$ is found to be

$$\widetilde{C_1(p)} = -\frac{1}{4\pi}(\sigma_0 d\sigma_1 d\sigma_2 + \sigma_1 d\sigma_2 d\sigma_0 + \sigma_2 d\sigma_0 d\sigma_1) = -\frac{1}{4\pi} vol(S^2) \ . \tag{4.32}$$

As a consequence

$$c_1(p) = Ber_{S^{2,2}} C_1(p) = -\frac{1}{4\pi} \int_{S^2} d(vol(S^2)) = -\frac{1}{4\pi} 4\pi = -1 \ . \tag{4.33}$$

By (super)transposing the projector (4.21) we obtain an inequivalent projector,

$$q =: p^{st} = |\phi\rangle \langle\phi| \ , \tag{4.34}$$

with the (super)transposed ket and bra vectors given by,

$$
|\phi\rangle =: (\langle\psi|)^{st} = \begin{pmatrix} \frac{1}{2}\eta^\diamond \\ \\ a^\diamond(1-\frac{1}{8}\eta\eta^\diamond) \\ \\ b^\diamond(1-\frac{1}{8}\eta\eta^\diamond) \end{pmatrix} ,
$$

$$
\langle\phi| =: (|\psi\rangle)^{st} = (\frac{1}{2}\eta,\ a(1-\frac{1}{8}\eta\eta^\diamond),\ b(1-\frac{1}{8}\eta\eta^\diamond)) \ . \tag{4.35}
$$

Explicitly, we find that

$$
q = \begin{pmatrix} -\frac{1}{4}\eta\eta^\diamond & \frac{1}{2}a\eta^\diamond & \frac{1}{2}b\eta^\diamond \\ \\ \frac{1}{2}a^\diamond\eta & aa^\diamond(1-\frac{1}{4}\eta\eta^\diamond) & ba^\diamond(1-\frac{1}{4}\eta\eta^\diamond) \\ \\ \frac{1}{2}b^\diamond\eta & ab^\diamond(1-\frac{1}{4}\eta\eta^\diamond) & bb^\diamond(1-\frac{1}{4}\eta\eta^\diamond) \end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix} 2\xi_+\xi_- \ ; & -(x_1-ix_2)\xi_+ - (1+x_0)\xi_- \ ; & -(x_1+ix_2)\xi_- - (1-x_0)\xi_+ \\ \\ -(x_1+ix_2)\xi_- + (1+x_0)\xi_+ \ ; & 1+x_0+\xi_+\xi_- \ ; & x_1+ix_2 \\ \\ (x_1-ix_2)\xi_+ - (1-x_0)\xi_- \ ; & x_1-ix_2 \ ; & 1-x_0+\xi_+\xi_- \end{pmatrix} \tag{4.36}
$$

That q is a projector ($q^2 = q$), of rank 1 ($Strq = 1$) are both consequences of the normalization

$$
\begin{aligned}
\langle\phi|\phi\rangle &= \frac{1}{4}\eta\eta^\diamond + (aa^\diamond + bb^\diamond)(1-\frac{1}{4}\eta\eta^\diamond) \\
&= 1 \ .
\end{aligned} \tag{4.37}
$$

We see that the transposed projector q is obtained from p by exchanging $a \leftrightarrow a^\diamond, b \leftrightarrow b^\diamond$ and $\eta \to -\eta^\diamond,\ \eta^\diamond \to \eta$. This amounts to the exchange $x_2 \to -x_2$ and $\xi_- \to -\xi_+,\ \xi_+ \to \xi_-$. It is than clear that the corresponding Chern form and number are given by,

$$
C_1(q) = -C_1(p) \ , \tag{4.38}
$$

$$
c_1(q) = -c_1(p) = 1 \ . \tag{4.39}
$$

Having different topological charges the projectors p and q are clearly inequivalent.

The connection 1-form (2.13) associated with the projector p is given by

$$
A_\nabla = \langle\psi|d\psi\rangle = (a^\diamond da + b^\diamond db)(1-\frac{1}{4}\eta\eta^\diamond) - \frac{1}{8}(\eta^\diamond d\eta + \eta d\eta^\diamond) \ . \tag{4.40}
$$

This connection form is clearly anti-hermitian, so it is valued in the Lie algebra of $\mathcal{U}(1)$. It coincides with the charge -1 graded monopole connection form found in [13]. Furthermore, the invariance (2.21) states the invariance of (4.40) under left action of $UOSP(1,2)$.

Gauge non-equivalent connections are obtained by the formula (2.26)

$$A_{\nabla^g} = \frac{1}{2\langle\psi|\, g^\dagger g\, |\psi\rangle} \left[\langle\psi|\, g^\dagger g\, |d\psi\rangle - \langle d\psi|\, g^\dagger g\, |\psi\rangle\right] , \quad |\psi\rangle = \begin{pmatrix} -\frac{1}{2}\eta \\ a(1-\frac{1}{8}\eta\eta^\diamond) \\ b(1-\frac{1}{8}\eta\eta^\diamond) \end{pmatrix} , \tag{4.41}$$

with $g \in GL(1,2;C_L)$ modulo $UOSP(1,2)$. Similar considerations hold for the connection 1-form associated with the monopole projector q.

5 The BPST Instanton

5.1 The $SU(2)$ Bundle over S^4.

The instanton could as well be called the quaternionic monopole due to the strict similarity obtained when one trades complex numbers with quaternions. So we shall start by giving a few fundamentals on quaternions $\mathbb{H}$. For their basis we shall use the symbols $(1, i, j, k)$ with relations $i^2 = -1, ij = k$, etc., so that any quaternion $q \in \mathbb{H}$ is written as

$$q = r_0 + r_1 i + r_2 j + r_3 k , \tag{5.1}$$

with r_0, r_1, r_2, r_3 real coefficients. The isomorphism $\mathbb{H} \simeq \mathbb{C}^2$ is realized as,

$$\begin{aligned} q &= v_1 + v_2 j , \quad v_1 = r_0 + r_1 i , \ v_2 = r_2 + r_3 i \\ \overline{q} &= \overline{v}_1 - j\overline{v}_2 , \quad vj = j\overline{v} , \qquad \forall\, v \in \mathbb{C} \end{aligned} \tag{5.2}$$

The quaternionic multiplication of $q = v_1 + v_2 j$ on the right by $w = w_1 + w_2 j$ is

$$q' =: qw = (v_1 w_1 - v_2 \overline{w}_2) + (v_1 w_2 + v_2 \overline{w}_1) j , \tag{5.3}$$

giving a right matrix multiplication on $\mathbb{C}^2$ via the isomorphism (5.2)

$$(v_1', v_2') = (v_1, v_2) \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} . \tag{5.4}$$

In particular, if $w \in Sp(1)$, namely $w\overline{w} = 1 = |w_1|^2 + |w_2|^2$, the action (5.4), provides the group isomorphism $Sp(1) \simeq SU(2)$,

$$w = w_1 + w_2 j \mapsto \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} . \tag{5.5}$$

The $SU(2) \simeq Sp(1)$ principal fibration $\pi : S^7 \to S^4$ over the four dimensional sphere is explicitly realized as follows. The total space is

$$S^7 = \{(a,b) \in \mathbb{H}^2 \ , \ |a|^2 + |b|^2 = 1\} . \tag{5.6}$$

with right action

$$S^7 \times Sp(1) \ \to \ S^7 , \quad (a,b)w = (aw, bw); \tag{5.7}$$

Clearly $|aw|^2 + |bw|^2 = |a|^2 + |b|^2 = 1$. The bundle projection $\pi : S^7 \to S^4$, $\pi(a,b) = (x_0, x_1, x_2, x_3, x_4)$ can be explicitly given as follows,

$$
\begin{aligned}
x_0 &= |a|^2 - |b|^2 = -1 + 2|a|^2 = 1 - 2|b|^2 \ , \\
\xi &= a\overline{b} - b\overline{a} =: x_1 i + x_2 j + x_3 k = -\overline{\xi} \ , \\
x_4 &= a\overline{b} + b\overline{a} \ .
\end{aligned} \tag{5.8}
$$

One checks that $\sum_{\mu=0}^{4}(x_\mu)^2 = (|a|^2 + |b|^2)^2 = 1$. The inversion of (5.8) gives the basic ($\mathbb{H}$-valued) invariant functions on S^7,

$$
\begin{aligned}
|a|^2 &= \frac{1}{2}(1 + x_0) \ , \\
|b|^2 &= \frac{1}{2}(1 - x_0) \ , \\
a\overline{b} &= \frac{1}{2}(x_4 + \xi) \ ,
\end{aligned} \tag{5.9}
$$

a generic invariant (polynomial) function on S^7 being any function of the previous variables.

5.2 The Bundle and the Projector for the Instanton

We need the rank 2 complex vector bundle associated with the defining left representation of $SU(2)$ on $\mathbb{C}^2$. We shall realize it as the rank 1 quaternionic vector bundle associated with the defining left representation of $Sp(1)$ on $\mathbb{H}$. For this we need a different identification $\mathbb{H} \simeq \mathbb{C}^2$ from the one in (5.2), (a left identification)

$$
l = l_1 - j l_2 \ , \quad l_1 = r_0 + r_1 i \ , \ l_2 = r_2 + r_3 i \ . \tag{5.10}
$$

The quaternionic multiplication of $l = l_1 - j l_2$ on the left by $w = w_1 + w_2 j \in Sp(1)$ is

$$
l' =: wl = (w_1 l_1 + w_2 l_2) - j(-\overline{w}_2 l_1 + \overline{w}_1 l_2) \ , \tag{5.11}
$$

giving the left matrix multiplication of $SU(2)$ on $\mathbb{C}^2$

$$
\begin{pmatrix} l'_1 \\ l'_2 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \ . \tag{5.12}
$$

The corresponding equivariant maps $\varphi : S^7 \to \mathbb{H}$ are of the form

$$
\varphi(a,b) = \overline{a} f + \overline{b} g \ , \tag{5.13}
$$

with f, g any two $\mathbb{H}$-valued functions which are invariant under the right action of $Sp(1)$ on S^7. Indeed,

$$
\varphi((a,b)w) = \overline{aw} f + \overline{bw} g = w^{-1}\varphi((a,b)) \ . \tag{5.14}
$$

We shall think of f, g as $\mathbb{H}$-valued functions on the base space S^4, namely elements of $\mathcal{A}_{\mathbb{H}} =: C^\infty(S^4, \mathbb{H})$. The space $C^\infty_{Sp(1)}(S^7, \mathbb{H})$ of equivariant maps is a right module over the (pull-back of) functions $\mathcal{A}_{\mathbb{H}}$.

Next, let us consider the ket-valued function,

$$|\psi\rangle =: \begin{pmatrix} a \\ b \end{pmatrix} , \tag{5.15}$$

which satisfies

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1 . \tag{5.16}$$

As before, we get a projector in $\mathbb{M}_2(\mathcal{A}_{\mathbb{H}})$ by,

$$p =: |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & a\overline{b} \\ b\overline{a} & |b|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+x_0 & x_4+\xi \\ x_4-\xi & 1-x_0 \end{pmatrix} , \tag{5.17}$$

where we have used the definition (5.8) for the coordinates on S^4. It is clear that p is a projector,

$$p^2 =: |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = p , \quad p^\dagger = p , \tag{5.18}$$

of rank 1 over $\mathbb{H}$,

$$trp = \langle\psi|\psi\rangle = 1 . \tag{5.19}$$

The $Sp(1)$-action (5.7) will transform the vector (5.15) multiplicatively,

$$|\psi\rangle \;\mapsto\; |\psi^w\rangle = \begin{pmatrix} aw \\ bw \end{pmatrix} = |\psi\rangle\, w , \quad \forall\, w \in Sp(1) , \tag{5.20}$$

while the projector p remains unchanged, a fact which is also obvious from the explicitly expression in (5.17).

Thus, the right module of sections $\Gamma^\infty(S^4, E)$ of the associated bundle is identified with the image of p in $(\mathcal{A}_{\mathbb{H}})^2$ and the module isomorphism between sections and equivariant maps is given by,

$$\begin{aligned} &\Gamma^\infty(S^4, E) \;\leftrightarrow\; C^\infty_{Sp(1)}(S^7, \mathbb{H}) , \\ &\sigma = p\begin{pmatrix} f \\ g \end{pmatrix} \;\leftrightarrow\; \varphi_\sigma = \langle\psi|\begin{pmatrix} f \\ g \end{pmatrix} = \overline{a}f + \overline{b}g \quad \forall\, f, g \in \mathcal{A}_{\mathbb{H}} . \end{aligned} \tag{5.21}$$

The canonical connection associated with the projector,

$$\nabla = p \circ d \;:\; \Gamma^\infty(S^4, E) \;\to\; \Gamma^\infty(S^4, E) \otimes_{\mathcal{A}_{\mathbb{H}}} \Omega^1(S^4, \mathbb{H}), \tag{5.22}$$

has curvature given by

$$\nabla^2 = p(dp)^2 = |\psi\rangle\langle\psi|d\psi\rangle\langle\psi|d\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi|d\psi\rangle\langle\psi| . \tag{5.23}$$

Notice that, contrary to what happens for the monopole, the first term in (5.23) does not vanish since $\langle\psi|d\psi\rangle$ is a quaternion-valued 1-form

The associated Chern 2-form and 4-form are given respectively by

$$\begin{aligned} C_1(p) &=: -\frac{1}{2\pi i}\; tr(p(dp)^2) , \\ C_2(p) &=: -\frac{1}{8\pi^2}\; [tr(p(dp)^4) - C_1(p)C_1(p)] . \end{aligned} \tag{5.24}$$

Now, in (5.24), the trace tr is really the tensor product of an ordinary matrix trace with a trace $tr_{\mathbb{H}}$ on $\mathbb{H}$. By ciclicity $tr_{\mathbb{H}}$ must vanish on the imaginary quaternions. Indeed, for instance, $tr_{\mathbb{H}}(i) = tr_{\mathbb{H}}(jk) = tr_{\mathbb{H}}(jk) = -tr_{\mathbb{H}}(i)$. Furthermore, we normalize it so that

$$tr_{\mathbb{H}}(1_{\mathbb{H}}) = 2 \; ; \tag{5.25}$$

this is motivated by the fact that a quaternion is a 2×2 matrix with complex entries.

It turns out that the 2-form $p(dp)^2$ is valued in the pure imaginary quaternions. As a consequence its trace vanishes so we may conclude that

$$C_1(p) = 0 \; . \tag{5.26}$$

As for the second Chern class, a straightforward calculation shows that,

$$\begin{aligned} C_2(p) &= -\frac{1}{32\pi^2}[(x_0 dx_4 - x_4 dx_0)(d\xi)^3 + 3dx_0 dx_4 \; \xi \; (d\xi)^2] \\ &= -\frac{3}{8\pi^2}[x_0 dx_1 dx_2 dx_3 dx_4 + x_0 dx_1 dx_2 dx_3 dx_4 \\ &\qquad + x_0 dx_1 dx_2 dx_3 dx_4 + x_0 dx_1 dx_2 dx_3 dx_4 + x_0 dx_1 dx_2 dx_3 dx_4] \\ &= -\frac{3}{8\pi^2} \; d(vol(S^4)) \; . \end{aligned} \tag{5.27}$$

The second Chern number is then given by

$$c_2(p) = \int_{S^4} C_2(p) = -\frac{3}{8\pi^2}\int_{S^4} d(vol(S^4)) = -\frac{3}{8\pi^2}\frac{8}{3}\pi^2 = -1 \; . \tag{5.28}$$

By transposing the projector (5.17) we obtain an inequivalent projector,

$$q =: p^t = \begin{pmatrix} |a|^2 & b\overline{a} \\ a\overline{b} & |b|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+x_0 & x_4+\xi \\ x_4-\xi & 1-x_0 \end{pmatrix} \; . \tag{5.29}$$

In order to express this projector as a ket-bra in the way used so far, we need to pay extra care to the noncommutativity of quaternions. Therefore, we introduce the *right* ket vector $|\phi\rangle_R$ to be the *row* defined by

$$|\phi\rangle_R =: (|\psi\rangle)^t = (a, b)_R \; . \tag{5.30}$$

Then, contrary to what we have been doing so far, we multiply from *right* to *left* in the expression,

$$|\phi\rangle_R \, {}_R\langle\phi| = (a, b)_R \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix}_R \; . \tag{5.31}$$

With this convention we get the expression (5.29). Thus, we can write

$$q = |\phi\rangle_R \, {}_R\langle\phi| \; , \tag{5.32}$$

and that q is a projector ($q^2 = q$), of rank 1 ($trq = 1$) are both consequences of the normalization ${}_R\langle\phi|\phi\rangle_R = |a|^2 + |b|^2 = 1$. It is worth noticing that the corresponding bundle of sections is realized as the *left* ($\mathcal{A}_{\mathbb{H}}$)-module $(\mathcal{A}_{\mathbb{H}})^2 q$. The transposed projector

q is obtained from p by exchanging $\xi \to -\xi$. It is than clear that the first Chern form still vanish, while, from the first equality in (5.27), the second Chern form and the corresponding number are given by,

$$\begin{aligned}
C_2(q) &= \frac{1}{32\pi^2}[(x_0dx_4 - x_4dx_0)(d\xi)^3 + 3dx_0dx_4\ \xi\ (d\xi)^2] \\
&= \frac{3}{8\pi^2}[x_0dx_1dx_2dx_3dx_4 + x_0dx_1dx_2dx_3dx_4 \\
&\qquad + x_0dx_1dx_2dx_3dx_4 + x_0dx_1dx_2dx_3dx_4 + x_0dx_1dx_2dx_3dx_4] \\
&= \frac{3}{8\pi^2}\ d(vol(S^4))\ . \qquad (5.33)
\end{aligned}$$

$$c_2(q) = \int_{S^4} C_2(q) = \frac{3}{8\pi^2}\int_{S^4} d(vol(S^4)) = \frac{3}{8\pi^2}\frac{8}{3}\pi^2 = 1\ . \qquad (5.34)$$

Having different topological charges the projectors p and q are clearly inequivalent.

As for the connection 1-form (2.13) associated with the projector p, it is given by

$$A_\nabla = \langle\psi|d\psi\rangle = \overline{a}da + \overline{b}db\ . \qquad (5.35)$$

This connection form is clearly anti-hermitian, so it is valued in the 'purely imaginary' quaternions which can be identified with the Lie algebra $sp(1) \simeq su(2)$. It coincides with the charge -1 instanton connection form [1, 17]. Furthermore, the invariance (2.21) states the invariance of (5.35) under left action of $Sp(2) \simeq Spin(5)$. Gauge non-equivalent connections are obtained by the formula (2.26),

$$A_{\nabla^g} = \frac{1}{2\,\langle\psi|\,g^\dagger g\,|\psi\rangle}\ [\langle\psi|\,g^\dagger g\,|d\psi\rangle - \langle d\psi|\,g^\dagger g\,|\psi\rangle]\ , \quad |\psi\rangle =: \begin{pmatrix} a \\ b \end{pmatrix}\ , \qquad (5.36)$$

with $g \in GL(2;\mathbb{H})$ modulo $Sp(2)$. However, not any transformed connection deserves to be called an instanton since, in general, it needs not be (anti)-self dual. Since the duality equations on S^4 are conformally invariant, it follows that only conformal transformations will convert the (anti)-instanton (5.35) in some other instanton. Now, it turns out that the proper (i.e. orientation preserving) conformal group of S^4 is $SL(2;\mathbb{H})$ whose action, when projected on S^4 reduces to fractional linear transformations of a homogeneous quaternionic variable (we recall that S^4 is naturally identifiable with the quaternionic projective space $\mathbb{H}P^1$) [1]. This action is the quaternionic analogue of the fact that the the proper conformal group of S^2 is $SL(2;\mathbb{C})$ whose action, when projected on S^2 reduces to fractional linear transformations of a homogeneous complex variable, the sphere S^2 being naturally identifiable with the complex projective space $\mathbb{C}P^1$. Thus, with $g \in SL(2;\mathbb{H})$ modulo $Sp(2)$, the connection 1-form in (5.36) represents an anti-instanton which is not gauge equivalent to the starting anti-instanton (5.35). Since $dim_{\mathbb{R}}(SL(2;\mathbb{H})) - dim_{\mathbb{R}}(Sp(2)) = 15 - 10 = 5$, we get a 5-parameter family of anti-instantons. Of course, the described procedure is nothing but the ADHM construction of (anti)-instantons [1]. Similar considerations hold for the connection 1-form associated with the instanton projector q.

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A The Supergroup $UOSP(1,2)$

We shall describe the basic facts about the supergroup $UOSP(1,2)$ that we need in this paper while referring to [5] for additional details. Some of our notation differ from the one used in [5].

With $B_L = (B_L)_0 + (B_L)_1$ we shall indicate a real Grassmann algebra with L generators. Let $osp(1,2)$ be the Lie B_L superalgebra of dimension $(3,2)$ with even generators A_0, A_1, A_2 and odd generator R_+, R_-, explicitly given in matrix representation by

$$A_0 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \quad A_1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \quad A_1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ ,$$
$$R_+ = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ , \quad R_- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{A.1}$$

Thus, a generic element $X \in osp(1,2)$ is written as $X = \sum_{k=0,1,2} a_k A_k + \sum_{\alpha=+,-} \eta_\alpha R_\alpha$ with $a_k \in (B_L)_0,\ \eta_\alpha \in (B_L)_1$.

If the integer L is taken to be even, on the complexification $C_L = B_L \otimes_{\mathbb{R}} \mathbb{C}$ there exists [15] an even graded involution $^\diamond : C_L \to C_L$ which satisfies the following properties,

$$(xy)^\diamond = x^\diamond\, y^\diamond \ , \quad \forall\, x, y \in C_L \ , \quad x^{\diamond\diamond} = (-1)^{|x|} x \ , \quad \forall\, x \in (C_L)_{|x|} \ . \tag{A.2}$$

Next one introduces the Lie C_L superalgebra $C_L \otimes_{\mathbb{R}} osp(1,2)$ and defines the superalgebra $uosp(1,2)$ to be the 'real' subalgebra made of elements of the form

$$X = \sum_{k=0,1,2} a_k A_k + \eta R_+ + \eta^\diamond R_- \ , \quad a_k \in (C_L)_0 \ , \ a_k^\diamond = a_k \ , \quad \eta \in (C_L)_1 \ . \tag{A.3}$$

Indeed, one introduces an adjoint operation † which is defined on the bases (A.1) as

$$A_i^\dagger = -A_i \ , \quad i = 0,1,2 \ ; \quad R_+^\dagger = -R_- \ , \ R_-^\dagger = R_+ \ , \tag{A.4}$$

and is extended to the whole of $C_L \otimes_{\mathbb{R}} osp(1,2)$ by using the involution $^\diamond$. Then, the superalgebra $uosp(1,2)$ is identified as the collection of 'anti-hermitian' elements

$$uosp(1,2) = \{X \in C_L \otimes_{\mathbb{R}} osp(1,2) \mid X^\dagger = -X\} \ . \tag{A.5}$$

The superalgebra $uosp(1,2)$ is the analogue of the compact real form of $C_L \otimes_{\mathbb{R}} osp(1,2)$.

Finally, the supergroup $UOSP(1,2)$ is defined to be the exponential map of $uosp(1,2)$,

$$UOSP(1,2) =: \{exp(X) \mid X \in uosp(1,2)\} \ . \tag{A.6}$$

A generic element $s \in UOSP(1,2)$ can be presented as the product of one-parameter subgroups,

$$\begin{aligned} & s = u\xi \ , \\ & u = exp(a_0 A_0) exp(a_1 A_1) exp(a_1 A_1) \ , \quad a_k^\diamond = a_k \in (C_L)_0 \ , \\ & \xi = exp(\eta R_+ + \eta^\diamond R_-) \ , \quad \eta \in (C_L)_0 \ . \end{aligned} \tag{A.7}$$

Explicitly,

$$s = \begin{pmatrix} 1 + \frac{1}{4}\eta\eta^\diamond & -\frac{1}{2}\eta & \frac{1}{2}\eta^\diamond \\ -\frac{1}{2}(a\eta^\diamond - b^\diamond\eta) & a(1 - \frac{1}{8}\eta\eta^\diamond) & -b^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \\ -\frac{1}{2}(b\eta^\diamond + a^\diamond\eta) & b(1 - \frac{1}{8}\eta\eta^\diamond) & a^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \end{pmatrix} . \tag{A.8}$$

By using (A.7) one also finds the adjoint of any element to be

$$\begin{aligned} s^\dagger &=: \xi^\dagger u^\dagger \\ &= \begin{pmatrix} 1 + \frac{1}{4}\eta\eta^\diamond & \frac{1}{2}(a^\diamond\eta + b\eta^\diamond) & \frac{1}{2}(b^\diamond\eta - a\eta^\diamond) \\ \frac{1}{2}\eta^\diamond & a^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) & b^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \\ \frac{1}{2}\eta & -b(1 - \frac{1}{8}\eta\eta^\diamond) & a(1 - \frac{1}{8}\eta\eta^\diamond) \end{pmatrix} . \end{aligned} \tag{A.9}$$

We have also used the one-parameter subgroup of $UOSP(1,2)$ generated by A_0,

$$\mathcal{U}(1) \simeq \{exp(\lambda A_0) \mid \lambda \in (C_L)_0 \ , \ \lambda^\diamond = \lambda\}. \tag{A.10}$$

A generic element $w \in \mathcal{U}(1)$ is written as

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i}{2}\lambda} & 0 \\ 0 & 0 & e^{-\frac{i}{2}\lambda} \end{pmatrix} . \tag{A.11}$$

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