On the Virial Theorem in Quantum Mechanics

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Abstract

We review the various assumptions under which abstract versions of the quantum mechanical virial theorem have been proved. We point our a relationship between the virial theorem for a pair of operators H, A and the regularity properties of the map $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}\mathrm{e}^{\mathrm{i}sA}$. We give an example showing that the statement of the virial theorem in [CFKS] is incorrect.

The virial theorem in Quantum Mechanics

The virial relation is the statement that if H, A are two selfadjoint operators on a Hilbert space $\mathcal{H}$, the expectation value of the commutator $[H, \mathrm{i}A]$ vanishes on eigenvectors of H :

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]\mathbb{1}_{\{\lambda\}}(H) = 0. \tag{1}$$

The virial relation is a very important part of Mourre's positive commutator method. In fact, combined with a positive commutator estimate, one can use the virial relation to obtain the local finiteness of point spectrum, (or even the absence of point spectrum). Moreover for Hamiltonians having a multiparticle structure, it is an essential tool to prove the positive commutator estimate itself (see eg [Mo], [PSS], [FH]).

If H, A are both unbounded operators, some care has to be taken with the definition of the commutator $[H, \mathrm{i}A]$ which a priori is only defined as a quadratic form on $\mathcal{D}(H)\cap\mathcal{D}(A)$. A rather weak assumption under which (1) can be formulated without ambiguity is the following one:

there exists a subspace $\mathcal{S} \subset \mathcal{D}(H)\cap\mathcal{D}(A)$ dense in $\mathcal{D}(H^n)$ for some $n \in \mathbb{N}$, such that

$$|(Hu, Au) - (Au, Hu)| \leq C(\|H^n u\|^2 + \|u\|^2),\ u \in \mathcal{S}. \tag{2}$$

The quadratic form $[H, \mathrm{i}A]$ extends then uniquely from $\mathcal{S}$ to $\mathcal{D}(H^n)$ which means that the left hand side of (1) has an unambiguous meaning.

The obstacle to a direct proof of (1) is of course that an eigenvector of H needs not be in $\mathcal{D}(A)$. Actually the counterexample that we will construct below shows that the virial relation does not hold under assumption (2).

To overcome this, additional assumptions on H and A are needed. To our knowledge, three different types of assumptions have been used in the literature to prove the virial theorem in an abstract setting.

- In [Mo, Prop. II.4], (1) is proved under the following assumptions:

$$(M)\quad \begin{array}{l} i)\ \mathcal{D}(H)\cap\mathcal{D}(A) \text{ is dense in } \mathcal{D}(H),\\ ii)\ \mathrm{e}^{\mathrm{i}sA} \text{ preserves } \mathcal{D}(H) \text{ and for each } u\in\mathcal{D}(H)\ \sup_{|s|\le 1}\|H\mathrm{e}^{\mathrm{i}sA}u\|<\infty,\\ iii)\ \text{the quadratic form } [H,\mathrm{i}A] \text{ on } \mathcal{D}(H)\cap\mathcal{D}(A) \text{ is bounded below, closeable,}\\ \quad\ \text{and it extends as a bounded operator from } \mathcal{D}(H) \text{ to } \mathcal{H}. \end{array}$$

In fact the condition " $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H)$" implies $i)$ and the second part of $ii)$, see [ABG, Prop. 3.2.5]. Moreover, it was noticed in [PSS] that Mourre's proof works without change under a condition weaker than $iii)$. So the assumptions which are really needed for the validity of Mourre's proof are:

$$(M')\quad \begin{array}{l} i)\ \mathrm{e}^{\mathrm{i}sA} \text{ preserves } \mathcal{D}(H),\\ ii)\ |(Hu,Au)-(Au,Hu)|\le C(\|Hu\|^2+\|u\|^2),\ u\in\mathcal{D}(H)\cap\mathcal{D}(A). \end{array}$$

- In [ABG, Prop. 7.2.10], (1) is proved if H is of class $C^1(A)$ i.e. if

$$(ABG)\quad \begin{array}{l} \exists z\in\mathbb{C}\backslash\sigma(H) \text{ such that}\\ \mathbb{R}\ni s\mapsto \mathrm{e}^{\mathrm{i}sA}R_z\mathrm{e}^{-\mathrm{i}sA} \text{ is } C^1 \text{ for the strong topology of } B(\mathcal{H}). \end{array}$$

We have used the notation $R_z=(z-H)^{-1}$. Two equivalent characterizations of the $C^1(A)$ property in terms of commutators are:

$$(ABG')\quad \begin{array}{l} \exists z\in\mathbb{C}\backslash\sigma(H) \text{ such that}\\ |(Au,R_zu)-(R_z^*u,Au)|\le C\|u\|^2,\ u\in\mathcal{D}(A), \end{array}$$

and:

$$(ABG'')\quad \begin{array}{l} i)\exists z\in\mathbb{C}\backslash\sigma(H) \text{ such that } R_z\mathcal{D}(A)\subset\mathcal{D}(A), R_z^*\mathcal{D}(A)\subset\mathcal{D}(A),\\ ii)|(Hu,Au)-(Au,Hu)|\le C(\|Hu\|^2+\|u\|^2),\ u\in\mathcal{D}(H)\cap\mathcal{D}(A). \end{array}$$

- Finally in [CFKS, Thm. 4.6], (1) is proved under the following assumptions:

$$(CFKS)\quad \begin{array}{l} i)\ \mathcal{D}(H)\cap\mathcal{D}(A) \text{ is dense in } \mathcal{D}(H),\\ ii)\ |(Hu,Au)-(Au,Hu)|\le C(\|Hu\|^2+\|u\|^2),\ u\in\mathcal{D}(H)\cap\mathcal{D}(A),\\ iii)\ \exists\, H_0, \text{ selfadjoint such that } \mathcal{D}(H)=\mathcal{D}(H_0),\ [H_0,\mathrm{i}A] \text{ extends as a bounded}\\ \quad\ \text{operator from } \mathcal{D}(H_0) \text{ to } \mathcal{H}, \text{ and } \mathcal{D}(A)\cap\mathcal{D}(H_0A) \text{ is a core for } H_0. \end{array}$$

Since $\mathcal{D}(H_0A)=\{u\in\mathcal{D}(A)|Au\in\mathcal{D}(H_0)\}\subset\mathcal{D}(A)$ one can suspect that there is a misprint in the last condition and that it should be replaced by the stronger version: $\mathcal{D}(H_0)\cap\mathcal{D}(H_0A)$ is a core for H_0. Anyway, such a change does not invalidate the discussion below.

It is easy to verify that *(M)* implies that $\mathrm{e}^{\mathrm{i}sA}R_z\mathrm{e}^{-\mathrm{i}sA}$ is in $B(\mathcal{H},\mathcal{D}(H))$ and that

$$\mathbb{R}\ni s\mapsto \mathrm{e}^{\mathrm{i}sA}R_z\mathrm{e}^{-\mathrm{i}sA} \text{ is } C^1 \text{ for the strong topology of } B(\mathcal{H},\mathcal{D}(H)).$$

and hence *(M)* implies *(ABG)*. The relation between (M') and *(ABG)* is even more straightforward: if $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H)$ then (M') is equivalent to *(ABG)* (see Theorem 6.3.4 in [ABG]).

If $H\in C^1(A)$ then $(Au,R_zu)-(R_z^*u,Au)$ is the quadratic form of a bounded operator $[A,R_z]_0\in B(\mathcal{H})$ (cf. (ABG')). From (ABG'') it follows then that $\mathcal{D}(H)\cap\mathcal{D}(A)$ is a core of H and that the quadratic form $(Hu,Au)-(Au,Hu)$ is continuous for the topology of $\mathcal{D}(H)$, hence it extends uniquely to a continuous quadratic form $[H,A]_0$ on $\mathcal{D}(H)$. Identifying $\mathcal{D}(H)\subset\mathcal{H}\subset\mathcal{D}(H)^*$ in the usual way $[H,A]_0$ becomes a continuous operator $\mathcal{D}(H)\longrightarrow\mathcal{D}(H)^*$ and then one has (see [ABG, Thm. 6.2.10])

$$[A,R_z]_0=R_z[H,A]_0R_z. \tag{3}$$

We shall prove in an appendix that $\mathcal{D}(H)$ is preserved by $\mathrm{e}^{\mathrm{i}sA}$ if $[H,A]_0\mathcal{D}(H)\subset\mathcal{H}$. In other terms, if (ABG) holds and $[H,A]_0\mathcal{D}(H)\subset\mathcal{H}$ then (M) is satisfied.

That *(ABG)* is more general than (M') can be seen from the following example: consider in $L^2(\mathbb{R})$ the operator H of multiplication by a real rational function (which may have poles, e.g. take $H(x)=1/x$) and let $A=-\mathrm{i}d/dx$; then clearly $H\in C^1(A)$ but $\mathrm{e}^{\mathrm{i}sA}$ and $(A+\mathrm{i}\lambda)^{-1}$ do not leave the domain of H invariant.

In conditions *(M)* and *(ABG)* assumptions either on the action of $\mathrm{e}^{\mathrm{i}sA}$ on $\mathcal{D}(H)$ or on the action of $(z-H)^{-1}$ on $\mathcal{D}(A)$ are made. No comparable assumptions are made in condition *(CFKS)*, which probably explains why it was more widely used in the literature (see [H] and references therein). In fact, the version of the Mourre theory that appears in [CFKS] would be a significant improvement of the other ones from the point of view of the applications to operators on manifolds, for example, because the objects $\mathrm{e}^{\mathrm{i}sA},(z-H)^{-1},(z-H_0)^{-1}$ are only implicitly defined and quite complicated in such situations. However reading the proof (in particular the proof of [CFKS, Lemma 4.5]) one can see that the assumption that $(z-H_0)^{-1}$ preserves $D(A)$ is implicitly used, to justify the identity (3) (with H replaced by H_0). We give below an example showing that the virial relation does not hold if one only assumes *(CFKS)* (or a slightly stronger version of it). In particular, we show that the relation $(A+\mathrm{i}\lambda)^{-1}\mathcal{D}(H)\subset\mathcal{D}(H)$, which plays a crucial role in the argument from [CFKS], is not true under their conditions.

Let us introduce the following definition concerning multicommutators: we set $\mathrm{ad}_A^0H=H$. For $k\geq 0$, if ad_A^kH is a bounded operator from $\mathcal{D}(H)$ to $\mathcal{H}$ and the quadratic form $[\mathrm{ad}_A^kH,A]$ on $\mathcal{D}(H)\cap\mathcal{D}(A)$ extends as a bounded operator from $\mathcal{D}(H)$ into $\mathcal{H}$ we denote it by $\mathrm{ad}_A^{k+1}H$.

Theorem 1 *There exists a pair H,A of sefadjoint operators on a Hilbert space $\mathcal{H}$ such that:*
i) H,A satisfy (CFKS),
ii) the multicommutators ad_A^kH extend as bounded operators from $\mathcal{D}(H)$ to $\mathcal{H}$ for all $k\in\mathbb{N}$,
iii) the pair H,A satisfies a Mourre estimate away from 0: for each I interval in $\mathbb{R}\backslash\{0\}$ there exist $c>0$, K compact such that

$$\mathbb{1}_I(H)[H,\mathrm{i}A]\mathbb{1}_I(H)\geq c\mathbb{1}_I(H)+K,$$

iv) the virial relation does not hold for H,A: there exists $\lambda\in\sigma_{\mathrm{pp}}(H)$ such that

$$\mathbb{1}_{\{\lambda\}}(H)[H,\mathrm{i}A]\mathbb{1}_{\{\lambda\}}(H)\neq 0.$$

Thm. 1 is a consequence of Thm. 2 below, which establishes a link between the virial relation and the $C^1(A)$ property.

Let H_0 be a positive selfadjoint operator on a Hilbert space $\mathcal{H}$. For $\phi \in \mathcal{H}$ we consider the rank one perturbation of H_0

$$H_\phi := H_0 - |\phi >< \phi|,$$

which is selfadjoint with $\mathcal{D}(H_\phi) = \mathcal{D}(H_0)$. Note that $\lambda < 0$ is an eigenvalue of H_ϕ if and only if $(\phi, (H_0 - \lambda)^{-1}\phi) = 1$ and $\mathrm{Ker}(H_\phi - \lambda)$ is generated by $(H_0 - \lambda)^{-1}\phi$.

Let A be another selfadjoint operator on $\mathcal{H}$ such that

$$\begin{aligned}
&\mathcal{D}(H_0) \cap \mathcal{D}(A) \text{ is dense in } \mathcal{D}(H_0),\\
&\text{the quadratic form } [H_0, A] \text{ on } \mathcal{D}(H_0) \cap \mathcal{D}(A) \text{ is bounded for the topology of } \mathcal{D}(H_0).
\end{aligned} \tag{4}$$

Theorem 2 *Assume that H_0 is positive and H_0, A satisfy (4). Assume that the virial relation holds for H_ϕ, A for each $\phi \in \mathcal{D}(A^\infty)$. Then H_0 is of class $C^1(A)$.*

Proof. Let $\phi \in \mathcal{D}(A^\infty)$. If the virial relation holds for H_ϕ, A we have for $u = (H_0 - \lambda)^{-1}\phi$ and $(\phi, u) = (\phi, (H_0 - \lambda)^{-1}\phi) = 1$:

$$\begin{aligned}
0 &= (u, [H_0, A]_0 u) + (u, A\phi)(\phi, u) - (u, \phi)(A\phi, u)\\
&= ((H_0 - \lambda)^{-1}\phi, [H_0, A]_0 (H_0 - \lambda)^{-1}\phi) + ((H_0 - \lambda)^{-1}\phi, A\phi) - (A\phi, (H_0 - \lambda)^{-1}\phi).
\end{aligned}$$

Using (4), this implies that

$$|((H_0 - \lambda)^{-1}\phi, A\phi) - (A\phi, (H_0 - \lambda)^{-1}\phi)| \leq C\|\phi\|^2, \ \forall \phi \in \mathcal{D}(A^\infty).$$

Since $\mathcal{D}(A^\infty)$ is dense in $\mathcal{D}(A)$, this implies (ABG') and hence that H_0 is of class $C^1(A)$. $\square$

If we assume the following condition which is stronger than (4):

$$\begin{aligned}
&\mathcal{D}(H_0) \cap \mathcal{D}(A) \text{ is dense in } \mathcal{D}(H_0),\\
&[H_0, A] \text{ extends to a bounded operator } [H_0, A]_0 : \mathcal{D}(H_0) \longrightarrow \mathcal{H},\\
&\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A) \text{ is dense in } \mathcal{D}(H_0),
\end{aligned} \tag{5}$$

then for $\phi \in \mathcal{D}(A)$ we have:

$$[H_\phi, A] = [H_0, A] - [|\phi >< \phi|, A] = [H_0, A]_0 + |A\phi >< \phi| - |\phi >< A\phi|,$$

and hence the pair H_ϕ, A satisfies then *(CFKS)*.

Note that if in addition to (5) we assume that the multicommutators $\mathrm{ad}_A^k H_0$ are bounded operators on $\mathcal{D}(H_0)$ then for $\phi \in \mathcal{D}(A^\infty) = \cap_{p \in \mathbb{N}} \mathcal{D}(A^p)$ the multicommutators $\mathrm{ad}_A^k H_\phi$ have the same property.

By Thm. 2 to construct the pair H, A in Thm. 1, it suffices to find a pair H_0, A satisfying (5) such that H_0 is not of class $C^1(A)$.

Let $\mathcal{H} = L^2(\mathbb{R}, dx)$, q the operator of multiplication by x in $\mathcal{H}$ and p the self-adjoint operator in $\mathcal{H}$ associated to $-\mathrm{i}d/dx$.

We will consider the operators

$$H_0 = e^{\omega q}, \ A = e^{\omega p} - p, \tag{6}$$

which are selfadjoint operators on their natural domains given by the functional calculus. We note that $\mathcal{D}(A) = \mathcal{D}(p) \cap \mathcal{D}(e^{\omega p})$. Noting also that $\mathcal{D}(\mathrm{e}^{\alpha p}) \subset \mathcal{D}(\mathrm{e}^{\omega p})$ if $0 < \alpha < \omega$ and using Fatou lemma we see that the domain of $\mathrm{e}^{\omega p}$ can be described as follows: a function $f \in L^2(\mathbb{R})$ belongs to $\mathcal{D}(\mathrm{e}^{\omega p})$ if and only if f has an analytic extension to the strip $\{x + \mathrm{i}y| - \omega < y < 0\}$ and $\|f(\cdot + \mathrm{i}y)\|_{L^2} \leq$ const. Then $\lim_{y\to\omega} f(x + \mathrm{i}y) \equiv f(x + \mathrm{i}\omega)$ exists in L^2 and one has $(\mathrm{e}^{\omega p} f)(x) = f(x - \mathrm{i}\omega)$.

The operators $\mathrm{e}^{\omega p}, \mathrm{e}^{\omega q}$ were considered by Fuglede in [Fu] in order to show that the Heisenberg form of the canonical commutation relations is not equivalent to the Weyl form.

From the Weyl form of the canonical commutation relations $\mathrm{e}^{\mathrm{i}\alpha p}\mathrm{e}^{\mathrm{i}\beta q} = \mathrm{e}^{\mathrm{i}\alpha\beta}\mathrm{e}^{\mathrm{i}\beta q}\mathrm{e}^{\mathrm{i}\alpha p}$ it follows, by formally taking $\alpha = \beta = -\mathrm{i}\omega$ with $\omega = (2\pi)^{1/2}$, that $\mathrm{e}^{\omega p}\mathrm{e}^{\omega q} = \mathrm{e}^{\omega q}\mathrm{e}^{\omega p}$. This commutation property will certainly hold on a large domain (we give below the details of the proof) although the operators $\mathrm{e}^{\omega p}$ and $\mathrm{e}^{\omega q}$ do not commute, which is the reason why H_0 is not of class $C^1(A)$.

Lemma 1 *Let H_0, A be the pair defined in (6). Then*
i) H_0, A satisfy (5),
ii) the multicommutators $\mathrm{ad}_A^k H_0$ are bounded operators from $\mathcal{D}(H_0)$ into $\mathcal{H}$ for all $k \in \mathbb{N}$,
iii) on $\mathcal{D}(H_0) \cap \mathcal{D}(A)$ we have $[H_0, \mathrm{i}A] = \omega H_0$,
iv) H_0 is not of class $C^1(A)$.

Proof of Thm. 1. By Thm. 2 and Lemma 1, there exists $\phi \in \mathcal{D}(A^\infty)$ such that for $H = H_\phi$ properties *i)*, *ii)* and *iv)* of Thm. 1 are satisfied. Property *iii)* follows from Lemma 1 *iii)* and the fact that $H - H_0, [H, A] - [H_0, A]$ are compact operators. □

Proof of Lemma 1. Let us consider the sequence of operators $\mathrm{e}^{-q^2/n}$. Clearly $\mathrm{e}^{-q^2/n}$ tends strongly to $\mathbb{1}$ in the spaces $\mathcal{H}$ and $\mathcal{D}(\mathrm{e}^{\omega q})$. Let us verify that the same is true in $\mathcal{D}(\mathrm{e}^{\omega p})$. In fact using the Fourier transformation, we see that $\mathrm{e}^{\omega p}\mathrm{e}^{-q^2/n} = \mathrm{e}^{-(q-\mathrm{i}\omega)^2/n}\mathrm{e}^{\omega p}$, in particular $\mathrm{e}^{-q^2/n}$ preserves $\mathcal{D}(\mathrm{e}^{\omega p})$. This easily implies that $\mathrm{e}^{-q^2/n}$ tends strongly to $\mathbb{1}$ in $\mathcal{D}(\mathrm{e}^{\omega p})$. Similarly we have $p\mathrm{e}^{-q^2/n} = \mathrm{e}^{-q^2/n}p - 2\mathrm{i}\mathrm{e}^{-q^2/n}q/n$, which shows that $\mathrm{e}^{-q^2/n}$ tends strongly to $\mathbb{1}$ in $\mathcal{D}(p)$ and hence in $\mathcal{D}(e^{\omega p} - p)$.

After conjugation by Fourier transformation, we see that the same results hold for the operator $\mathrm{e}^{-p^2/n}$. Let now

$$T_n = \mathrm{e}^{-q^2/n}\mathrm{e}^{-p^2/n}.$$

We deduce from the above observations that

$$\text{s-}\lim_{n\to+\infty} T_n = \mathbb{1}, \text{ in the spaces }, \mathcal{D}(H_0), \mathcal{D}(A), \mathcal{D}(H_0) \cap \mathcal{D}(A). \tag{7}$$

where $\mathcal{D}(H_0) \cap \mathcal{D}(A)$ is equipped with the intersection topology. Since T_n maps $\mathcal{H}$ into $\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A)$, we see that the first and third conditions of (5) are satisfied.

Let us now check the second condition of (5). We claim that

$$[H_0, \mathrm{i}A] = \omega H_0, \text{ on } \mathcal{D}(H_0) \cap \mathcal{D}(A). \tag{8}$$

In fact let $u \in \mathcal{D}(H_0) \cap \mathcal{D}(A)$, and $u_n = T_n u$. By (7) it suffices to check that $(Au_n, H_0 u_n) - (H_0 u_n, Au_n) = \mathrm{i}\omega(u_n, H_0 u_n)$ for each n. Since $Au_n \in \mathcal{D}(H_0)$ and $H_0 u_n \in \mathcal{D}(A)$, we have

$$(Au_n, H_0 u_n) - (H_0 u_n, Au_n) = (u_n, AH_0 u_n - H_0 A u_n).$$

But u_n is an entire function, decreasing faster than any exponential on each line $Imz = Cst$. Hence we have

$$\begin{aligned} AH_0 u_n(x) &= \mathrm{e}^{\omega(x-\mathrm{i}\omega)} u_n(x-\mathrm{i}\omega) + \mathrm{i}\tfrac{d}{dx}(\mathrm{e}^{\omega x} u_n(x)) \\ &= e^{\omega x}(u_n(x-\mathrm{i}\omega) + \mathrm{i}\tfrac{d}{dx} u_n(x)) + \mathrm{i}\omega e^{\omega x} u_n(x) \\ &= H_0 A u_n(x) + \mathrm{i}\omega H_0 u_n(x), \end{aligned}$$

since $\omega^2 = 2\pi$. This proves (8) and hence the second condition of (5). Moreover it follows from (8) that the multicommutators $\mathrm{ad}_A^k H_0$ are bounded on $\mathcal{D}(H_0)$.

Let us now prove that H_0 is not of class $C^1(A)$. Assume the contrary. Then $(H_0+1)^{-1}$ would send $\mathcal{D}(A)$ into itself. The function $u(x) = \mathrm{e}^{-x^2}$ belongs to $\mathcal{D}(A)$ and $(H_0+1)^{-1}u$ equals $(\mathrm{e}^{\omega x}+1)^{-1}\mathrm{e}^{-x^2}$. This function has a pole at $z = -\mathrm{i}\omega/2$ and hence is not in $\mathcal{D}(A)$. This gives a contradiction and hence H_0 is not of class $C^1(A)$. □

Appendix

The following result is of some independent interest.

Lemma 2 *Let A, H be self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $H \in C^1(A)$ and $[A,H]_0\mathcal{D}(H) \subset \mathcal{H}$. Then $\mathrm{e}^{\mathrm{i}sA}\mathcal{D}(H) \subset \mathcal{D}(H)$ for all real s.*

Proof. For any bounded operator S of class $C^1(A)$ the commutator $[S,A]$ extends to a bounded operator in $\mathcal{H}$ denoted $[S,A]_0$, and one has

$$S\mathrm{e}^{\mathrm{i}tA} = \mathrm{e}^{\mathrm{i}tA}S + \int_0^t \mathrm{e}^{\mathrm{i}(t-s)A}[S,\mathrm{i}A]_0\mathrm{e}^{\mathrm{i}sA}ds.$$

So if $t > 0, u \in \mathcal{H}$:

$$\|S\mathrm{e}^{\mathrm{i}tA}u\| \le \|Su\| + \int_0^t \|[S,A]_0\mathrm{e}^{\mathrm{i}sA}u\|ds.$$

We shall take

$$S = H_\varepsilon = H(1+\mathrm{i}\varepsilon H)^{-1} = -\mathrm{i}/\varepsilon + (\mathrm{i}/\varepsilon)R^\varepsilon$$

where $R^\varepsilon = (1+\mathrm{i}\varepsilon H)^{-1}$. We set $T = [A,H]_0(H+\mathrm{i})^{-1} \in B(\mathcal{H})$ and we use [ABG, Thm. 6.2.10]; then

$$[A,H_\varepsilon]_0 = R^\varepsilon T(H+\mathrm{i})R^\varepsilon = R^\varepsilon T H_\varepsilon + \mathrm{i}R^\varepsilon T R^\varepsilon.$$

Since $\|R^\varepsilon\| \le 1$ we obtain

$$\|H_\varepsilon\mathrm{e}^{\mathrm{i}tA}u\| \le \|H_\varepsilon u\| + t\|T\|\|u\| + \|T\|\int_0^t \|H_\varepsilon\mathrm{e}^{\mathrm{i}sA}u\|ds.$$

From the Gronwall lemma it follows that for each $t_0 > 0$ there is a constant C such that $\|H_\varepsilon\mathrm{e}^{\mathrm{i}tA}u\| \le C(\|H_\varepsilon u\| + \|u\|)$ for all $\varepsilon > 0, 0 \le t \le t_0, u \in \mathcal{H}$. Now it suffices to apply Fatou lemma.□

As a final remark we shall prove a version of the virial theorem. Let A, H be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $\mathrm{e}^{\mathrm{i}sA}\mathcal{D}(|H|^\sigma) \subset \mathcal{D}(|H|^\sigma)$ for some real number $\sigma \ge 1/2$ and all s (then the domain of $|H|^\tau$ will also be invariant if $0 \le \tau \le \sigma$). Set $\mathcal{K} = \mathcal{D}(|H|^\sigma)$ and identify $\mathcal{K} \subset \mathcal{H} \subset \mathcal{K}^*$. Then the group induced by $\mathrm{e}^{\mathrm{i}sA}$ in $\mathcal{K}$ is strongly continuous hence the space $\mathcal{D}(A;\mathcal{K}) = \{u \in \mathcal{K}\cap\mathcal{D}(A) | Au \in \mathcal{K}\}$ is dense in $\mathcal{K}$. So the sesquilinear form $(Au, Hu) - (Hu, Au)$

is well defined on the dense linear subspace $\mathcal{D}(A;\mathcal{K})$ of $\mathcal{K}$ (one needs this restricted subspace only if $\sigma < 1$; e.g. if $\sigma = 1/2$ then one does not have anything better than $H\mathcal{K} \subset \mathcal{K}^*$).

Assume, moreover, that the preceding sesquilinear form is continuous for the topology of $\mathcal{K}$ and denote by $[A, H]_0$ the operator in $B(\mathcal{K}, \mathcal{K}^*)$ associated to it. If we set $A_\varepsilon = (\mathrm{e}^{\mathrm{i}\varepsilon A} - 1)/(\mathrm{i}\varepsilon)^{-1}$ then it is easily seen that

$$[H, A_\varepsilon] = \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{e}^{\mathrm{i}(\varepsilon - s)A}[H, \mathrm{i}A]_0 \mathrm{e}^{\mathrm{i}sA} ds$$

holds in the strong operator topology of $B(\mathcal{K}, \mathcal{K}^*)$. In particular we see that $[H, A_\varepsilon]$ converges strongly in $B(\mathcal{K}, \mathcal{K}^*)$ to $[H, \mathrm{i}A]_0$. This clearly implies the virial theorem, because the eigenvectors of H belong to $\mathcal{K}$.

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