Determination Of a Third Order Operator From Two Of Its Spectra

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Abstract

We consider a complex third order differential operator on a bounded interval with boundary conditions presenting a mixed aspect of the Dirichlet and the periodic problems. It is proved that two spectra are sufficient to determine the operator. This result is valid under readily verifiable hypotheses applying simultaneously to the two spectra.

1 Introduction and result.

Consider the third order differential operator

$$L_{p,q} = iD^3 + iDq + iqD + p$$

defined on $[0,1]$ with the boundary conditions

$$(DP^+) \quad \begin{cases} y(1) = 0 \\ y'(1) = y'(0) \\ y''(0) = 0 \end{cases} .$$

Here $(p,q) \in L^2_{\mathbf{R}}[0,1] \times H^1_{\mathbf{R}}[0,1]$. When $q(0) = 0$ this operator is self-adjoint in $L^2_{\mathbf{C}}[0,1]$ with the scalar product $(f,g) = \int_0^1 f \ \overline{g} \, dx$ and has a discrete spectrum. We denote by $\mu^+(p,q) = \left(\mu_j^+(p,q)\right)_{j\in\mathbf{Z}}$ the increasing sequence of eigenvalues. Each eigenvalue is real and is of multiplicity of at most *two*.

The same considerations hold when $L_{p,q}$ is associated with the boundary conditions

$$(DP^-) \quad \begin{cases} y(1) = 0 \\ y'(1) = -y'(0) \\ y''(0) = 0 \end{cases} .$$

Then we denote by $\mu^-(p,q) = \left(\mu_j^-(p,q)\right)_{j\in\mathbf{Z}}$ the increasing sequence of eigenvalues, each eigenvalue being real and of multiplicity of at most *two*.
Let $\lambda^+ = (\lambda_j^+)_{j\in\mathbf{Z}}$ and $\lambda^- = (\lambda_j^-)_{j\in\mathbf{Z}}$ be two sequences of real numbers. We write $\lambda^+ \cap \lambda^- = \phi$ if and only if $\lambda_i^+ \neq \lambda_j^-$, $\forall (i,j) \in \mathbf{Z}\times\mathbf{Z}$.
The main result of the paper is as following.

Theorem 1 *Let $(p,q) \in L^2_R \times H^1_R$ with $q(0) = 0$. Suppose $\mu^+(p,q) \cap \mu^-(p,q) = \phi$. Then*
(i) all eigenvalues $\mu_j^+(p,q)$ and $\mu_j^-(p,q)$ are of multiplicity one.
(ii) the set $(\mu^+(p,q), \mu^-(p,q), q(1))$ determines uniquely (p,q) .

Part (i) provides a sufficient condition for having a simple spectra for both DP^+ and DP^- boundary conditions: each eigenvalue for the DP^+ boundary conditions is not an eigenvalue for the DP^- boundary conditions. It is proved in Theorem 7.

Given μ^+ (resp. μ^-), the additional knowledge of μ^- (resp. μ^+) gives the multiplicity number (one) of all the μ_j^+ (resp. μ_j^-) when $\mu^+ \cap \mu^- = \phi$.

Part (ii) says the following. Let $(p,q) \in L^2_R \times H^1_R$ and $(\tilde{p},\tilde{q}) \in L^2_R \times H^1_R$, suppose $\mu^+(p,q) \cap \mu^-(p,q) = \phi$, if $\mu^+(p,q) = \mu^+(\tilde{p},\tilde{q})$, $\quad \mu^-(p,q) = \mu^-(\tilde{p},\tilde{q})$, $q(1) = \tilde{q}(1)$, $q(0) = \tilde{q}(0) = 0$ then $p = \tilde{p}$ $\quad$ *a.e.* in $[0,1]$ and $q = \tilde{q}$ in $[0,1]$.

Theorem 1 is not valid, replacing the DP^+ and DP^- boundary conditions with the boundary conditions $y(0) = y(1) = 0,\ y'(1) = y'(0)$ and $y(0) = y(1) = 0,\ y'(1) = -y'(0)$. It easy to check that $L_{p(x),q(x)}$ and $L_{p(1-x),q(1-x)}$ associated with one of these boundary conditions give the same spectrum.

Let us mention here that the indexation of the $\mu_j^(p,q)$'s follows from *the Counting Lemma* of Section 3. Furthermore we have $\mu^\pm(p,q) = \mu^\pm(r,s) \Longleftrightarrow \mu_j^\pm(p,q) = \mu_j^\pm(r,s) \quad \forall j \in \mathbf{Z}$. *The Counting Lemma* yields also that only a finite number of eigenvalues are of multiplicity two, with or without the hypothesis $\mu^+(p,q) \cap \mu^-(p,q) = \phi$.

For the second order case $-y'' + qy = \lambda y$ with boundary conditions of the type $\cos\alpha y(0) + \sin\alpha y'(0) = 0$, $\cos\beta y(1) + \sin\beta y'(1) = 0$, the determination of q from two of its spectra has been extensively studied, giving numerous results through various methods. These are mainly based on Volterra operator transformations and contour integrations, using spectral functions or norming constants (see, for example, [Bo], [Ma], [Ge-Le], [Le-Ga], [Da-Tru]). The problem is solved for first order differential systems in [Am]. The fourth order case is also largely studied, see [Ca-Pe-Sc] for recent results and references. Two sequences of spectral data are not sufficient to determine the two coefficients of a self-adjoint fourth order operator with separated boundary conditions. This can be seen from the exact solution reconstructive method given in [McLa1] and [McLa2]. Let us mention [Gl] and [Ba] where conditions are given for three sequences to be spectral data for a fourth order self-adjoint boundary value problem with separated boundary conditions.

For the third order operator the unique result in this direction is given by Leibenzon[Le] in 1966. The equation $y''' + p_1 y + p_2 y = \lambda y$ is associated to three boundary conditions of the type $y(a) + \alpha y'(a) + \beta y''(a) = 0$, $a = 0$ or 1. This is not a self adjoint problem. It is proved that p_1 and p_2 are determined by two spectra with in addition the so-called matrix-functions. It is supposed as in theorem 1 that the two sets of eigenvalues do not intersect. It is also assumed that some other weight numbers exist. The result is actually generalized for n^{th} order differential equations, $n > 2$.

In theorem 1 the two boundary conditions are of a different kind to those of Leibenzon. They allow the determination of the operator from only two spectra and use only one of the assumptions of Leibenzon. The proof of theorem 1(ii) is mainly based on the methods of Pöschel and Trubowitz[Pö-Tr], a contour integration and a counting lemma.

A first generalization of the boundary conditions DP^+ or DP^- should be $y(1) = 0$, $y'(1) = e^{i\phi} y'(0)$, $y''(0) = 0$ for some $\phi \in \mathbf{R}/2\pi\mathbf{Z}$ and one may expect that two sequences of spectral data for two different ϕ determine the operator together with the boundary conditions. This is an problem.

The counting lemma gives a rough asymptotic expansions of the $\mu_j^{\pm}(p,q)$'s. Namely, $\mu_j^{+}(p,q)-(2j\pi)^3$ and $\mu_j^{-}(p,q)-(2(j+1)\pi)^3$ are bounded as $|j| \to \infty$. More precise asymptotic expansions are given in the following theorem.

Theorem 2 *Suppose $(p,q) \in L^2_R \times H^1_R$ with $q(0)=0$ and let $[q]=\int_0^1 q(t)\,dt$.*

$$\mu_j^{+}(p,q) = (2j\pi)^3 - 2j\pi[q] + O(1)$$

and

$$\mu_j^{-}(p,q) = ((2j+1)\pi)^3 - (2j+1)\pi[q] + O(1),$$

as $|j| \to \infty$.

Let us mention the work of McKean[McKe] devoted to the Boussinesq equation but giving numerous results in inverse spectral theory for the non self adjoint operator $D^3 + qD + Dq + p$, for the periodic case and also for the conditions $y(0) = y(1) = y(2)$, p and q being sufficiently small.

This article is organized as follows. In section 2 the estimates used in sections 3 and 4 are established. Then in section 3 , Theorem 1(i) is proved. Next the Counting Lemma is derived. This involves the construction of $Z_1(1,\cdot,p,q)$ (resp. $Y_1(1,\cdot,p,q)$), the analytic extensions of the function vanishing at the $\mu_j^{+}(p,q)$'s (resp. the $\mu_j^{-}(p,q)$'s). Theorem 1 (*ii*) is proved in section 4. Finally, the proof of theorem 2 is given in section 5.

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2 Notations, estimates, analyticity.

Consider $(p,q) \in L^2_R \times H^1_R$ and let $\lambda \in \mathbf{C}$. The purpose of this section is to give estimates and analyticity results for the fundamental basis of the solutions to $L_{p,q}y = \lambda y$.

We will often use the abbreviated notation $' = \partial/\partial x$.

The functions $y_1(x,\lambda,p,q)$, $y_2(x,\lambda,p,q)$, $y_3(x,\lambda,p,q)$ are defined as the unique solutions to $L_{p,q}y(x) = \lambda y(x)$, for *a.e.* $x \in [0,1]$, satisfying the initial conditions

$$\begin{pmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{pmatrix} = Identity.$$

In particular, every solution to $L_{p,q}y = \lambda y$ can be expressed as
$y(x) = a\, y_1(x,\lambda,p,q) + b\, y_2(x,\lambda,p,q) + c\, y_3(x,\lambda,p,q)$,
where (a,b,c)= $((y(0), y'(0), y''(0))$.

Since there is no second order derivative in the definition of $L_{p,q}$ the Wronskian of y_1, y_2 and y_3 is independent of x. Therefore, for a.e. $x \in [0,1]$

$$\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix} = 1. \tag{1}$$

In particular

$$\forall x_0 \in [0,1], \quad (y(x_0), y'(x_0), y''(x_0)) \neq (0,0,0). \tag{2}$$

Let $\omega = e^{\frac{2i\pi}{3}}$ and $k^3 = \lambda$. When $(p,q) \equiv (0,0)$ it is easy to obtain

$$\begin{aligned} y_1(x,\lambda,0,0) &= \frac{1}{3}\left(e^{ikx} + e^{i\omega kx} + e^{i\omega^2 kx}\right) \\ y_2(x,\lambda,0,0) &= \frac{1}{3ik}\left(e^{ikx} + \omega^2 e^{i\omega kx} + \omega e^{i\omega^2 kx}\right) \\ y_3(x,\lambda,0,0) &= \frac{1}{3(ik)^2}\left(e^{ikx} + \omega e^{i\omega kx} + \omega^2 e^{i\omega^2 kx}\right) \end{aligned} \tag{3}$$

Note that $\lambda = 0$ is a removable singularity for $y_2(x,\lambda,0,0)$ and $y_3(x,\lambda,0,0)$.

Lemma 1

$$\begin{aligned}
y_1(x,\lambda,0,0) &= \frac{1}{3}(4\cos(\frac{kx}{2})\cos(\frac{\omega kx}{2})\cos(\frac{\omega^2 kx}{2})-1\\
&\quad -4i\sin(\frac{kx}{2})\sin(\frac{\omega kx}{2})\sin(\frac{\omega^2 kx}{2}))\\
y_2(x,\lambda,0,0) &= \frac{1}{3ik}(4\cos(\frac{kx}{2})\cos(\frac{\omega kx}{2}-\frac{\pi}{3})\cos(\frac{\omega^2 kx}{2}+\frac{\pi}{3})-1\\
&\quad -4i\sin(\frac{kx}{2})\sin(\frac{\omega kx}{2}-\frac{\pi}{3})\sin(\frac{\omega^2 kx}{2}+\frac{\pi}{3})) \qquad (4)\\
y_3(x,\lambda,0,0) &= \frac{1}{3(ik)^2}(4\cos(\frac{kx}{2})\cos(\frac{\omega kx}{2}+\frac{\pi}{3})\cos(\frac{\omega^2 kx}{2}-\frac{\pi}{3})-1\\
&\quad -4i\sin(\frac{kx}{2})\sin(\frac{\omega kx}{2}+\frac{\pi}{3})\sin(\frac{\omega^2 kx}{2}-\frac{\pi}{3}))
\end{aligned}$$

Lemma 1 is elementary but fundamental in achieving all estimates in the *whole* complex plane. Combining (4) and the inequality $|\sin z| \leq e^{|\Im m z|}$, $z \in \mathbf{C}$, the functions y_1, y_2 and y_3 for $(p,q) \equiv (0,0)$ can be estimated as follows

Lemma 2 *On* $[0,1] \times \mathbf{C}$

$$\begin{aligned}
|y_1(x,\lambda,0,0)| &\leq 3e^{\left(|\Im \mathrm{m}\frac{k}{2}|+|\Im \mathrm{m}\frac{\omega k}{2}|+|\Im \mathrm{m}\frac{\omega^2 k}{2}|\right)x}\\
|y_2(x,\lambda,0,0)| &\leq \frac{3}{|k|}e^{\left(|\Im \mathrm{m}\frac{k}{2}|+|\Im \mathrm{m}\frac{\omega k}{2}|+|\Im \mathrm{m}\frac{\omega^2 k}{2}|\right)x} \qquad (5)\\
|y_3(x,\lambda,0,0)| &\leq \frac{3}{|k|^2}e^{\left(|\Im \mathrm{m}\frac{k}{2}|+|\Im \mathrm{m}\frac{\omega k}{2}|+|\Im \mathrm{m}\frac{\omega^2 k}{2}|\right)x}.
\end{aligned}$$

Let us define for $x \in [0,1]$ and $\lambda \in \mathbf{C}$

$$\Xi(x,\lambda) = e^{\left(|\Im m\frac{k}{2}|+|\Im m\frac{\omega k}{2}|+|\Im m\frac{\omega^2 k}{2}|\right)x} \qquad (6)$$

We also have

Lemma 3 *Let $j = 1, 2, 3$ and suppose $n \leq j-1$. On $[0,1] \times \mathbf{C}$*

$$|\frac{\partial^n}{\partial x^n} y_j(x,\lambda,0,0)| \leq 3\Xi(x,\lambda). \tag{7}$$

Proof : Clearly, for $x \in [0,1]$ and $\lambda \in \mathbf{C}$

$$y_2(x,\lambda,0,0) = \int_0^x y_1(t,\lambda,0,0)\,dt, \quad y_3(x,\lambda,0,0) = \int_0^x y_2(t,\lambda,0,0)\,dt. \tag{8}$$

Equalities in (8) show successively with Lemma 2 that

$$|y_2(x,\lambda,0,0)| \leq \Xi(x,\lambda), \quad |y_3(x,\lambda,0,0)| \leq \Xi(x,\lambda). \tag{9}$$

Besides

$$y_3'(x,\lambda,0,0) = y_2(x,\lambda,0,0), \quad y_3'(x,\lambda,0,0) = y_2'(x,\lambda,0,0) = y_1'(x,\lambda,0,0). \tag{10}$$

Then (9) and (10) finish the proof. □

The $L^2_R \times H^1_R$ norm is chosen for convenience as

$$\|(p,q)\|^2_{L^2_R \times H^1_R} = 2\|q\|^2_{L^2_R} + \|q'\|^2_{L^2_R} + \|p\|^2_{L^2_R}.$$

Theorem 3 *For $(x,\lambda,p,q) \in [0,1] \times \mathbf{C} \times L^2_R \times H^1_R$ with $q(0) = 0$*

$$|y_j(x,\lambda,p,q)| \leq \frac{3}{|k|^{j-1}} \Xi(x,\lambda) e^{\sqrt{x}\|(p,q)\|_{L^2_R \times H^1_R}}, \quad j = 1,2,3. \tag{11}$$

Proof : For $j = 1, 2, 3$ we have the integral equation

$$\begin{aligned} y_j(x,\lambda,p,q) &= y_j(x,\lambda,0,0) + \int_0^x y_3(x-t,\lambda,0,0)(-2q(t)y_j'(t,\lambda,p,q) \\ &\quad +(-q'(t) + ip(t))y_j(t,\lambda,p,q))\,dt \end{aligned} \tag{12}$$

Integrating (12) by parts gives

$$\begin{aligned} y_j(x,\lambda,p,q) &= y_j(x,\lambda,0,0) + \int_0^x \Big(-2y_3'(x-t,\lambda,0,0)q(t) \\ &\quad + y_3(x-t,\lambda,0,0)(q'(t)+ip(t))\Big) y_j(t,\lambda,p,q)\,dt \\ &\quad - \left[2y_3(x-t,\lambda,0,0)q(t)y_j(t,\lambda,p,q)\right]\big|_{t=0}^{t=x}. \end{aligned} \tag{13}$$

The last term in (13) is zero since $q(0)=0$.
Following Picard's iteration we write

$$y_j(x,\lambda,p,q) = y_j(x,\lambda,0,0) + \sum_{n\geq 1} c_n^j(x,\lambda,p,q) \tag{14}$$

where

$$\begin{aligned} c_0^j(x,\lambda,p,q) &= y_j(x,\lambda,0,0) \\ c_n^j(x,\lambda,p,q) &= \int_0^x \Big(-2y_3'(x-t,\lambda,0,0)q(t) + y_3(x-t,\lambda,0,0)(q'(t) \\ &\quad + ip(t)\Big) c_{n-1}^j(t,\lambda,p,q)\,dt. \end{aligned} \tag{15}$$

¿From (15) we have for each $n \geq 1$

$$\begin{aligned} c_n^j(x,\lambda,p,q) &\leq \int_{t_1\leq t_2\leq\cdots\leq t_{n+1}=x} y_j(t_1,\lambda,0,0) \prod_{m=1}^{n} \Big(-2y_3'(t_{m+1}-t_m,\lambda,0,0) \\ &\quad q(t_m) + y_3(t_{m+1}-t_m,\lambda,0,0)(q'(t_m)+ip(t_m))\Big)\,dt_1\cdots dt_n. \end{aligned} \tag{16}$$

¿From Lemma 3

$$|y_3(x,\lambda,0,0)| \leq 3\Xi(x,\lambda), \quad |y_3'(x,\lambda,0,0)| \leq 3\Xi(x,\lambda) \quad (x,\lambda)\in[0,1]\times\mathbf{C}. \tag{17}$$

Then (16) gives using (17) and $|y_j(t_1,\lambda,0,0)| \leq 3|k|^{1-j}\Xi(x,\lambda)$

$$
\begin{aligned}
|c_n^j(x,\lambda,p,q)| &\leq 3\frac{\Xi(x,\lambda)}{n!|k|^{j-1}}\left(\int_0^x 2|q(t)|+|q'(t)+ip(t)|\,dt\right)^n \\
&\leq 3\frac{\Xi(x,\lambda)}{n!|k|^{j-1}}\left(\sqrt{x}\|(p,q)\|_{L_R^2\times H_R^1}\right)^n, \quad n\geq 1. \qquad (18)
\end{aligned}
$$

Inequalities of Lemma 3 and (18) together with (14) finish the proof. □

Theorem 4 *Let $j=1,2,3$. For $(x,\lambda,p,q)\in[0,1]\times\mathbf{C}\times L_R^2\times H_R^1$ with $q(0)=0$*

$$
|y_j(x,\lambda,p,q)-y_j(x,\lambda,0,0)|\leq\frac{3}{|k|^j}\Xi(x,\lambda)e^{\sqrt{x}\|(p,q)\|_{L_R^2\times H_R^1}}. \qquad (19)
$$

Proof : It is a repetition of the proof of Theorem 3 except that in (16) we replace once in the product, say for $m=1$, the inequalities (17) by the inequalities

$$
|y_3(x,\lambda,0,0)|\leq\frac{3}{|k|}\Xi(x,\lambda), \quad |y_3'(x,\lambda,0,0)|\leq\frac{3}{|k|}\Xi(x,\lambda) \quad (x,\lambda)\in[0,1]\times\mathbf{C}. \qquad (20)
$$

Inequalities (20) can be easily derived in a similar way as those in Lemma 3. □

Theorem 5 *Let $j=1,2,3$. For each $x\in[0,1]$ and each $(p,q)\in L_R^2\times H_R^1$ with $q(0)=0$*

$$
y_j(x,\lambda,p,q), \quad y_j'(x,\lambda,p,q), \quad y_j''(x,\lambda,p,q) \qquad (21)
$$

are entire functions of λ.

Remark. It can be proved that these functions are also analytic functions of (λ,p,q).

Proof : Fix $j=1,2,3$. The coefficient $c_n^j(x,\lambda,p,q)$, n^{th} term in the power series of $y_j(x,\lambda,p,q)$ is an entire function of λ. The convergence in (14) is

locally uniform in λ by (18). Then $y_j(x,\lambda,p,q)$ is an entire function of λ according to Weierstrass's theorem.
The first derivative with respect to x of the integral equation (13) is

$$\begin{aligned} y_j'(x,\lambda,p,q) &= y_j'(x,\lambda,0,0) + \int_0^x \Big(-2y_3''(x-t,\lambda,0,0)q(t) + \\ &\quad y_3'(x-t,\lambda,0,0)(q'(t)+ip(t))\Big) y_j(t,\lambda,p,q)\,dt. \end{aligned} \tag{22}$$

The analyticity of $y_j(x,\lambda,p,q)$ gives the analyticity of $y_j'(x,\lambda,p,q)$. Similarly

$$\begin{aligned} y_j''(x,\lambda,p,q) &= y_j''(x,\lambda,0,0) + \int_0^x \Big(-2y_3'''(x-t,\lambda,0,0)q(t) + \\ &\quad y_3''(x-t,\lambda,0,0)(q'(t)+ip(t))\Big) y_j(t,\lambda,p,q)\,dt - 2q(x) \end{aligned} \tag{23}$$

proves the analyticity result of $y_j''(x,\lambda,p,q)$.

The estimates for the derivatives of $y_j(x,\lambda,p,q)$ are given in

Theorem 6 *Let $j=1,2,3$. For $(x,\lambda,p,q) \in [0,1]\times \mathbf{C} \times L_R^2 \times H_R^1$ with $q(0)=0$*

$$|y_j'(x,\lambda,p,q) - y_j'(x,\lambda,0,0)| \le 3\frac{\|(p,q)\|_{L_R^2\times H_R^1}}{|k|^{j-1}}\Xi(x,\lambda)e^{\sqrt{x}\|(p,q)\|_{L_R^2\times H_R^1}} \tag{24}$$

and

$$\begin{aligned} |y_j''(x,\lambda,p,q) + 2q(x) - y_j''(x,\lambda,0,0)| &\le 3\frac{\|(p,q)\|_{L_R^2\times H_R^1}}{|k|^{j-2}}\Xi(x,\lambda) \\ &\quad \times e^{\sqrt{x}\|(p,q)\|_{L_R^2\times H_R^1}}. \end{aligned} \tag{25}$$

The proof of (24) and (25) is a consequence of (22) and (23) respectively.□

3 Proof of Theorem 1(i) and Counting Lemma.

We begin this section by proving Theorem 1(i). It is theorem 7(iii).

Theorem 7 *Let* $(p,q) \in L^2_R \times H^1_R$ *satisfying* $q(0) = 0$.
(i) $\mu^+(p,q) \cap \mu^-(p,q) = \phi$ *if and only if* $y_1(1,\lambda,p,q)$ *has no zero in* $\mathbf{C}$.

(ii) Any $\lambda \in \mathbf{R}$ *is a simple eigenvalue of* $L_{p,q}$ *associated with the* DP^+ *boundary conditions if and only if*

$$\left\{\begin{array}{l} \Im\mathrm{m}\, y_1(1,\lambda,p,q) = 0 \\ \Re\mathrm{e}\, y_1(1,\lambda,p,q) \neq 0 \end{array}\right. \quad or \quad \left\{\begin{array}{l} \Im\mathrm{m}\, y_1(1,\lambda,p,q) = 0 \\ \Re\mathrm{e}\, y_1''(1,\lambda,p,q) \neq 0 \end{array}\right. .$$

Any $\lambda \in \mathbf{R}$ *is a simple eigenvalue of* $L_{p,q}$ *associated with the* DP^- *boundary conditions if and only if*

$$\left\{\begin{array}{l} \Re\mathrm{e}\, y_1(1,\lambda,p,q) = 0 \\ \Im\mathrm{m}\, y_1(1,\lambda,p,q) \neq 0 \end{array}\right. \quad or \quad \left\{\begin{array}{l} \Re\mathrm{e}\, y_1(1,\lambda,p,q) = 0 \\ \Im\mathrm{m}\, y_1''(1,\lambda,p,q) \neq 0 \end{array}\right. .$$

(iii) If $\mu^+(p,q) \cap \mu^-(p,q) = \phi$ *then* $\mu^+(p,q)$ *and* $\mu^-(p,q)$ *are both simple spectra.*

The proof uses the following preliminary observations.

Let $[y,z] = y'z - yz'$.

Lemma 4 *Let* $\lambda \in \mathbf{C}$ *and suppose* $(p,q) \in L^2_R \times H^1_R$. *Then*

$$\begin{aligned} [y_3(x,\lambda,p,q), y_2(x,\lambda,p,q)] &= \bar{y}_3(x,\bar{\lambda},p,q) \\ [y_3(x,\lambda,p,q), y_1(x,\lambda,p,q)] &= \bar{y}_2(x,\bar{\lambda},p,q) \\ [y_2(x,\lambda,p,q), y_1(x,\lambda,p,q)] &= \bar{y}_1(x,\bar{\lambda},p,q) - 2q(0)\bar{y}_3(x,\bar{\lambda},p,q). \end{aligned} \tag{26}$$

This result is given by McKean[McK].

Proof : Check that the left and right hand side of each identity in (26) satisfy $L_{-p,q}y = -\lambda y$ and the same initial conditions at $x = 0$. □

Let

$$M^{\pm}(\lambda,p,q) = \begin{pmatrix} y_1(1,\lambda,p,q) & y_2(1,\lambda,p,q) \\ y_1'(1,\lambda,p,q) & y_2'(1,\lambda,p,q) \mp 1 \end{pmatrix}. \tag{27}$$

Lemma 5 *Fix (p,q) in $L^2_R \times H^1_R$ and suppose $q(0)=0$.*
For all $\lambda \in \mathbf{R}$ we have

$$(i) \quad \left\{ \begin{array}{l} y_1(1,\lambda,p,q) = 0 \\ \Re\mathrm{e}\, y_1''(1,\lambda,p,q) = 0 \end{array} \right. \Longleftrightarrow M^+(\lambda,p,q) = 0. \tag{28}$$

$$(ii) \quad \left\{ \begin{array}{l} y_1(1,\lambda,p,q) = 0 \\ \Im\mathrm{m}\, y_1''(1,\lambda,p,q) = 0 \end{array} \right. \Longleftrightarrow M^-(\lambda,p,q) = 0. \tag{29}$$

Proof : Suppose $q(0)=0$ and fix $\lambda \in \mathbf{R}$. We omit (p,q) from the notation for brevity. ¿From $\bar{y}_1(1,\lambda) = y_2'(1,\lambda)y_1(1,\lambda) - y_2(1,\lambda)y_1'(1,\lambda)$ we have $y_1(1,\lambda)=0 \Rightarrow y_1'(1,\lambda)=0$ or $y_2(1,\lambda)=0$. The equality $\bar{y}_2(1,\lambda) = y_3'(1,\lambda)y_1(1,\lambda) - y_3(1,\lambda)y_1'(1,\lambda)$ gives $y_1(1,\lambda) = y_1'(1,\lambda) = 0 \Rightarrow y_2(1,\lambda)=0$.
Therefore

$$y_1(1,\lambda) = 0 \Rightarrow y_2(1,\lambda) = 0. \tag{30}$$

Similarly, $\bar{y}_1'(1,\lambda) = y_2''(1,\lambda)y_1(1,\lambda) - y_2(1,\lambda)y_1''(1,\lambda)$ yields $y_1(1,\lambda) = y_2(1,\lambda) = 0 \Rightarrow y_1'(1,\lambda) = 0$. Thus, with (30) we obtain

$$y_1(1,\lambda) = 0 \Leftrightarrow y_1(1,\lambda) = y_1'(1,\lambda) = y_2(1,\lambda) = 0.$$

Since $y_1''(1,\lambda) \neq 0$ when $y_1(1,\lambda) = y_1'(1,\lambda) = 0$ the proof is finished using $\bar{y}_1''(1,\lambda) = y_2''(1,\lambda)y_1'(1,\lambda) - y_2'(1,\lambda)y_1''(1,\lambda)$. □

Proof of Theorem 7: Looking for a solution $y(x)$ to $L_{p,q}y = \lambda y$, for *a.e.* $x \in [0,1]$ and satisfying the $DP^{\pm}$ boundary conditions, we write

$$y(x) = a\, y_1(x,\lambda,p,q) + b\, y_2(x,\lambda,p,q) + c\, y_3(x,\lambda,p,q)$$

with $(a,b,c) \in \mathbf{C}^3 \backslash 0$.
Clearly

$$\left\{ \begin{array}{l} a\, y_1(1,\lambda,p,q) + b\, y_2(1,\lambda,p,q) + c\, y_3(1,\lambda,p,q) = 0 \\ a\, y_1'(1,\lambda,p,q) + b\, y_2'(1,\lambda,p,q) + c\, y_3'(1,\lambda,p,q) = \pm b \\ c = 0, \end{array} \right. \tag{31}$$

that is to say,

$$M^{\pm}(\lambda,p,q)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (a,b)\neq(0,0).$$

Then any real λ is an eigenvalue of $L_{p,q}$ associated with the $DP^{\pm}$ boundary conditions if and only if $dim\, Ker\, M^{\pm}(\lambda,p,q) \geq 1$, *i.e.*, if and only if

$$det\, M^{\pm}(\lambda,p,q) = 0. \tag{32}$$

Using Lemma 4 we have $det\, M^{\pm}(\lambda,p,q) = \bar{y}_1(1,\lambda,p,q) \mp y_1(1,\lambda,p,q)$. Therefore

$$\Im m y_1(1,\lambda,p,q) = 0 \text{ (resp. } \Re e y_1(1,\lambda,p,q) = 0) \tag{33}$$

gives the eigenvalues for the DP^+ boundary conditions (resp. for the DP^- boundary conditions). This proves (i).

Any real λ is a *simple* eigenvalue $L_{p,q}$ associated with the $DP^{\pm}$ boundary conditions if and only if $dim\, Ker\, M^{\pm}(\lambda,p,q) = 1$, *i.e.*, if and only if

$$det\, M^{\pm}(\lambda,p,q) = 0 \quad \text{and} \quad M^{\pm}(\lambda,p,q)\neq\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{34}$$

This proves (ii) with Lemma 5. Then (iii) follows from (i) and (ii). □

When λ is a *simple* eigenvalue, an immediate consequence of (27) and (34) is that either the two expressions

$$\begin{aligned} &y_2(1,\lambda,p,q)y_1(x,\lambda,p,q) - y_1(1,\lambda,p,q)y_2(x,\lambda,p,q) \\ &\text{and} \\ &(y_2'(1,\lambda,p,q) \mp 1)y_1(x,\lambda,p,q) - y_1'(1,\lambda,p,q)y_2(x,\lambda,p,q) \end{aligned} \tag{35}$$

are equivalent up to a multiplicative constant$\neq 0$ or one (and only one) of the two expressions is identically zero. This provides the eigenfunction (up to a multiplicative constant$\neq 0$) corresponding to λ.

When λ is a *double* eigenvalue, $y_1(\cdot,\lambda,p,q)$ and $y_2(\cdot,\lambda,p,q)$ are two linearly independent eigenfunctions corresponding to λ.

Now we construct explicitly the analytic extensions in $\mathbf{C}$ of $\lambda \mapsto \Im m y_1(1,\lambda,p,q)$ and $\lambda \mapsto \Re e y_1(1,\lambda,p,q)$.

For this, just define the operator

$$l_{p,q} = \begin{pmatrix} p & -D^3 - Dq - qD \\ D^3 + Dq + qD & p \end{pmatrix}$$

on the *real* valued function space $\mathrm{L}^2_{\mathbf{R}}[0,1] \times \mathrm{L}^2_{\mathbf{R}}[0,1]$ and consider the eigenvalue problem

$$l_{p,q} \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix} = \lambda \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix}, \qquad \text{for } a.e.\ x \in [0,1]$$

with either the dl^+ or the dl^- boundary conditions defined by

$$(dl^{\pm}) \qquad \begin{cases} Y(1) = Z(1) = 0 \\ Y'(1) = \pm Y'(0), \quad Z'(1) = \pm Z'(0) \\ Y''(0) = Z''(0) = 0. \end{cases}$$

The operator $l_{p,q}$ with the dl^+ or dl^- boundary conditions is self adjoint on $\mathrm{L}^2_{\mathbf{R}}[0,1] \times \mathrm{L}^2_{\mathbf{R}}[0,1]$ with the inner scalar product

$$\left\langle \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix}, \begin{pmatrix} \tilde{Y}(x) \\ \tilde{Z}(x) \end{pmatrix} \right\rangle = \int_0^1 Y(x)\tilde{Y}(x) + Z(x)\tilde{Z}(x)\, dx.$$

Write $y = Y + iZ$, Y and Z being real valued functions, and observe that $L_{p,q}y = \lambda y$ is equivalent to $l_{p,q} \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix} = \lambda \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix}$ for *real* λ. It is different for all complex λ because of right hand sides.

The spectrum of $L_{p,q}$ with the DL^+ (resp. DL^-) boundary conditions is then the spectrum of $l_{p,q}$ with the dl^+ (resp. dl^-) boundary conditions.

For each $\lambda \in \mathbf{C}$, $\begin{pmatrix} Y_1(x) \\ Z_1(x) \end{pmatrix}$ denotes the solution to

$$l_{p,q} \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix} = \lambda \begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix}$$

satisfying the initial condition

$$(Y_1+iZ_1)(0,\lambda,p,q) = 1 \quad (Y_1'+iZ_1')(0,\lambda,p,q) = 0 \quad (Y_1''+iZ_1'')(0,\lambda,p,q) = 0.$$

We have

$$y_1(x,\lambda,p,q) = Y_1(x,\lambda,p,q) + iZ_1(x,\lambda,p,q) \quad \text{for } a.e\ x \in [0,1],\ \forall\lambda \in \mathbf{R}. \tag{36}$$

Again (36) is not satisfied for all $\lambda \in \mathbf{C}$.
Moreover, it is clear that

$$\begin{aligned} \left\{\mu_j^+(p,q), \quad j \in \mathbf{Z}\right\} &= \left\{\lambda \in \mathbf{C}, Z_1(1,\lambda,p,q) = 0\right\}, \\ \left\{\mu_j^-(p,q), \quad j \in \mathbf{Z}\right\} &= \left\{\lambda \in \mathbf{C}, Y_1(1,\lambda,p,q) = 0\right\} \end{aligned} \tag{37}$$

The same considerations (integral equation and Picard's iteration) as those for $y_1(x,\lambda,p,q)$ in Section 2 show that $Y_1(1,\lambda,p,q)$ and $Z_1(1,\lambda,p,q)$ are entire functions of λ (see [Am-Gu, appendix] for a very similar treatment with a first order system). They are the analytic extensions in $\mathbf{C}$ of $\lambda \mapsto \Im m y_1(1,\lambda,p,q)$ and $\lambda \mapsto \Re e y_1(1,\lambda,p,q)$. Moreover, under the same assumptions of Theorem 4, it can be seen that

$$\begin{aligned} &|Y_1(x,\lambda,p,q) - Y_1(x,\lambda,0,0)| \le \frac{3}{|k|}\Xi(x,\lambda)e^{\sqrt{x}\|(p,q)\|_{L^2_R\times H^1_R}} \\ &\text{and} \\ &|Z_1(x,\lambda,p,q) - Z_1(x,\lambda,0,0)| \le \frac{3}{|k|}\Xi(x,\lambda)e^{\sqrt{x}\|(p,q)\|_{L^2_R\times H^1_R}}, \end{aligned} \tag{38}$$

$\forall\lambda \in \mathbf{C}$.

The final part of this section is concerned with the proof of the Counting Lemma.

Let

$$\begin{aligned} \Omega_j^- &= \left\{z \in \mathbf{C},\ |\omega^{j-1}z - (2j+1)\pi| \ge \frac{\pi}{2}\right\} \\ \Omega_j^+ &= \left\{z \in \mathbf{C},\ |\omega^{j-1}z - 2j\pi| \ge \frac{\pi}{2}\right\} \end{aligned} \tag{39}$$

for $j = 1, 2, 3$ and define

$$\Omega^{\pm} = \bigcap_{j=1}^{3} \Omega_j^{\pm}. \tag{40}$$

With these notations we have

Lemma 6 (THE COUNTING LEMMA) *Suppose* $(p,q) \in L^2_{\mathbf{C}} \times H^1_{\mathbf{C}}$ *with* $q(0) = 0$.
(i) *Let the integer* J *satisfy* $(2J+1)\pi > 48e^{\|(p,q)\|}$. *Then* $Z_1(1,\lambda,p,q)$ *has exactly* $2J+1$ *roots, counted with multiplicity in the open k-disc*

$$\{k \in \mathbf{C},\ |k| < (2J+1)\pi\}$$

and exactly one simple root noted $\mu_j^+(p,q)$ *in each open k-disc* $\{k \in \mathbf{C},\ |k - 2j\pi| < \pi\}$ *for* $|j| > J$.
(ii) *Let the integer* J *satisfy* $2J\pi > \min(\ln 32, 96e^{\|(p,q)\|})$. *Then* $Y_1(1,\lambda,p,q)$ *has exactly* $2J$ *roots, counted with multiplicity in the open k-disc*

$$\{k \in \mathbf{C},\ |k| < 2J\pi\}$$

and exactly one simple root noted $\mu_j^-(p,q)$ *in each open k-disc* $\{k \in \mathbf{C},\ |k - (2j+1)\pi| < \pi\}$ *for* $|j| \geq J$.

The proof of Lemma 6 involves the following inequalities.

Lemma 7

$$\begin{aligned} &(i)\ 96|Y_1(1,\lambda,0,0)| \geq \Xi(1,\lambda) \quad \forall k \in \Omega^- \setminus \{|k| \leq \ln 32\}\,. \\ &(ii)\ 48|Z_1(1,\lambda,0,0)| \geq \Xi(1,\lambda) \quad \forall k \in \Omega^+. \end{aligned} \tag{41}$$

The definition of Ξ is given in (6).

Proof : (i) Similarly as in Lemma 1 it can be seen that

$$Y_1(1,\lambda,0,0) = \frac{1}{3}\left(4\cos(\frac{k}{2})\cos(\frac{\omega k}{2})\cos(\frac{\omega^2 k}{2}) - 1\right),\ \forall \lambda \in \mathbf{C}.$$

Then

$$|Y_1(1,\lambda,0,0)|^2 = 16|\Lambda|^2 - 8\Re e\,\Lambda + 1$$

where

$$\Lambda = \cos(\frac{k}{2})\cos(\frac{\omega k}{2})\cos(\frac{\omega^2 k}{2}).$$

Therefore, for any $0 < \varepsilon < 4$

$$9|Y_1(1,\lambda,0,0)|^2 \geq \varepsilon^2|\Lambda|^2 \tag{42}$$

if

$$(16-\varepsilon^2)|\Lambda|^2 - 8|\Lambda| + 1 \geq 0,$$

that is to say, if

$$|\Lambda| \geq (4-\varepsilon)^{-1}. \tag{43}$$

Inequality $e^{|\Im m z|} \leq 4|\sin z|$ in $\{z \in \mathbf{C},\ |z| \geq \pi/4\}$ ([Pö-Tr, lemma 2.1]) yields

$$\begin{aligned}
|\cos(\frac{k}{2})| &\geq \frac{1}{4}e^{|\Im m\frac{k}{2}|} \geq \frac{1}{4}, \quad \text{for} \quad k \in \Omega_1^- \\
|\cos(\frac{\omega k}{2})| &\geq \frac{1}{4}e^{|\Im m\omega\frac{k}{2}|} \geq \frac{1}{4}, \quad \text{for} \quad k \in \Omega_2^- \\
|\cos(\frac{\omega^2 k}{2})| &\geq \frac{1}{4}e^{|\Im m\omega^2\frac{k}{2}|} \geq \frac{1}{4}, \quad \text{for} \quad k \in \Omega_3^-
\end{aligned} \tag{44}$$

¿From (44) we have

$$|\Lambda| \geq \frac{1}{4^3}e^{|\Im m\omega\frac{k}{2}|} \quad \text{and} \quad |\Lambda| \geq \frac{1}{4^3}e^{|\Im m\omega^2\frac{k}{2}|}, \quad \forall k \in \Omega^-.$$

Consequently

$$|\Lambda| \geq \frac{1}{2}$$

$$\text{if} \quad k \in \left(\left\{|\Im m\omega\frac{k}{2}| \geq \ln 32\right\} \cup \left\{|\Im m\omega^2\frac{k}{2}| \geq \ln 32\right\}\right) \cap \Omega^-. \tag{45}$$

Therefore (43) is satisfied for $\varepsilon = 2$ and for $k \in \Omega^- \setminus \{|k| \leq \ln 32\}$.
Then

$$|Y_1(1,\lambda,0,0)| \geq \frac{2}{3}|\Lambda|, \quad \forall k \in \Omega^- \setminus \{|k| \leq \ln 32\}, \tag{46}$$

and using again (44) we have

$$|\Lambda| \geq \frac{1}{4^3}\Xi(1,\lambda), \quad for \quad k \in \Omega^-. \tag{47}$$

Inequalities (46) and (47) prove (i).

The proof of (ii) is immediate since

$$
\begin{aligned}
|Z_1(1,\lambda,0,0)| &= \frac{1}{3}\left|4\sin(\frac{k}{2})\sin(\frac{\omega k}{2})\sin(\frac{\omega^2 k}{2})\right|, \quad \forall k\in\mathbf{C} \\
&\geq \frac{1}{48}e^{\left(|\Im m\frac{k}{2}|+|\Im m\frac{\omega k}{2}|+|\Im m\frac{\omega^2 k}{2}|\right)x}, \quad \forall k\in\Omega^+.
\end{aligned}
$$

□

Proof of Lemma 6: (i) Combining (38) and (41) we obtain

$$
|Z_1(1,\lambda,p,q)-Z_1(1,\lambda,0,0)|<|Z_1(1,\lambda,0,0)|,
$$

$\forall\, k\in\Omega^+$ satisfying $|k|\geq 48e^{\|(p,q)\|}$.
Consequently, Rouché's theorem can be applied on the k-circles $|k|=(2J+1)$ and $|k-2j\pi|=\pi$ for $|j|>J$. Then $Z_1(1,\lambda,p,q)$ and $Z_1(1,\lambda,0,0)=\frac{4}{3}\sin(\frac{kx}{2})\sin(\frac{\omega kx}{2})\sin(\frac{\omega^2 kx}{2})$ have the same number of zeros, counted with multiplicity, in the corresponding open k-discs, the zeros of $Z_1(1,\lambda,0,0)$ being the $(2j\pi)^3$'s, $j\in\mathbf{Z}$.

(ii) Similarly, (38) and (41) give

$$
|Y_1(1,\lambda,p,q)-Y_1(1,\lambda,0,0)|<|Y_1(1,\lambda,0,0)|,
$$

$\forall\, k\in\Omega^-$ satisfying $|k|\geq\max(\ln 32, 96e^{\|(p,q)\|})$. Then $Y_1(1,\lambda,p,q)$ and $Y_1(1,\lambda,0,0)$ have the same number of zeros in the open k-discs defined in lemma $6(ii)$, the zeros of $Y_1(1,\lambda,0,0)$ being the $((2j+1)\pi)^3$'s, $j\in\mathbf{Z}$, up to an irrelevant additive $O(e^{-c|k|})$ term, c being a positive numerical constant.

□

4 Proof of Theorem 1 (ii).

Let us mention first that the indexing of the $\mu_j^{\mp}(p,q)$'s following Lemma 6 has the property

$$
\mu^{\pm}(p_1,q_1)=\mu^{\pm}(p_2,q_2)\Longleftrightarrow\mu_j^{\pm}(p_1,q_1)=\mu_j^{\pm}(p_2,q_2), \quad \forall j\in\mathbf{Z}.
$$

Define

$$y(x, \lambda, p, q) = y_2(1, \lambda, p, q) y_1(x, \lambda, p, q) - y_1(1, \lambda, p, q) y_2(x, \lambda, p, q), \tag{48}$$

for all $\lambda \in \mathbf{C}$ and *a.e.* $x \in [0, 1]$.
Let us recall that under the assumptions of Theorem 1, $y(\cdot, \mu_j^{\pm}, p, q)$ is the eigenfunction corresponding to the eigenvalue $\mu_j^{\pm}(p, q)$.

We will often use the abbreviated notation $\cdot = \partial/\partial\lambda$.
We have

Lemma 8 *Let $(p, q) \in L^2_R \times H^1_R$. Suppose $q(0) = 0$. Then*

$$\int_0^1 |y(x, \mu_j^+(p, q), p, q)|^2\, dx = -2(Y_1 \dot{Z}_1)(1, \mu_j^+(p, q), p, q),$$
$$\int_0^1 |y(x, \mu_j^-(p, q), p, q)|^2\, dx = 2(Z_1 \dot{Y}_1)(1, \mu_j^-(p, q), p, q), \tag{49}$$

for all $j \in \mathbf{Z}$.

Remark. When $\mu^+(p, q) \cap \mu^-(p, q) = \phi$ the geometric multiplicity of each $\mu_j^+(p, q)$ (resp. $\mu_j^-(p, q)$) is one. Besides, from Lemma 8, its algebraic multiplicity, defined as its order as a root in $Z_1(1, \lambda, p, q)$ (resp. $Y_1(1, \lambda, p, q)$), is also one.

Proof: Let $\lambda \in \mathbf{C}$. We have

$$i\dot{y}''' + 2iq\dot{y}' + (iq' + p)\dot{y} = \lambda\dot{y} + y \tag{50}$$

and

$$i\bar{y}''' + 2iq\bar{y}' + (iq' - p)\bar{y} = -\bar{\lambda}\bar{y}. \tag{51}$$

Combining (50) and (51) we have for $\lambda \in \mathbf{R}$

$$\begin{aligned} |y|^2 &= 2i(q\dot{y}\bar{y})' + i\dot{y}'''\bar{y} + i\dot{y}\bar{y}''' \\ &= i(\dot{y}\bar{y}'' - \dot{y}'\bar{y}' + \dot{y}''\bar{y} + 2q\dot{y}\bar{y})'. \end{aligned} \tag{52}$$

Suppose $\lambda = \mu_j^{\pm}$. ¿From (52), since $q(0) = 0$ and $y(0) = 0$, we obtain

$$\int_0^1 |y|^2\, dx = i[\dot{y}\bar{y}'' - \dot{y}'\bar{y}' + \dot{y}''\bar{y}]|_{x=0}^{x=1}. \tag{53}$$

Besides, it is easy to compute

$$\begin{aligned}
&\dot{y}(1)=0, \quad \dot{y}''(0)=0, \quad \dot{y}'(0)=-\dot{y}_1(1),\\
&\bar{y}'(0)=-\bar{y}_1(1), \quad \bar{y}'(1)=-y_1(1), \quad \dot{y}'(1)=-\dot{\bar{y}}_1(1),
\end{aligned} \tag{54}$$

where all functions in (54) are evaluated at $\mu_j^{\pm}(p,q),\ j\in\mathbf{Z}$.
Finally, since

$$\begin{aligned}
y_1(1,\lambda,p,q)&=\bar{y}_1(1,\lambda,p,q) \quad \text{for } \lambda=\mu_j^+(p,q),\\
y_1(1,\lambda,p,q)&=-\bar{y}_1(1,\lambda,p,q) \quad \text{for } \lambda=\mu_j^-(p,q),
\end{aligned} \tag{55}$$

we have from (52), (53) and (54)

$$\int_0^1 |y(x,\lambda,p,q)|^2\,dx = -iy_1(1,\lambda,p,q)\frac{\partial}{\partial\lambda}\left(y_1(1,\lambda,p,q)\pm\bar{y}_1(1,\lambda,p,q)\right)$$

at $\lambda=\mu_j^{\pm}(p,q),\ j\in\mathbf{Z}$.

The proof is finished since $\Re e y_1(1,\cdot,p,q)$ coincides with $Y_1(1,\cdot,p,q)$ and $\Im m y_1(1,\cdot,p,q)$ coincides with $Z_1(1,\cdot,p,q)$ on the real axis. □

For $g\in L^2_{\mathbf{R}}[0,1]$, g^* is defined by $g^*(x)=g(1-x)$, for *a.e.* $x\in[0,1]$.

Lemma 9 *Let $(p,q)\in L^2_R\times H^1_R$, $\lambda\in\mathbf{C}$ and suppose $q(0)=0$. Then*

$$\begin{pmatrix} y_1 & y_2 & y_3\\ y_1' & y_2' & y_3'\\ y_1'' & y_2'' & y_3'' \end{pmatrix}(1,\lambda,p^*,q^*)=\begin{pmatrix} y_3'' & y_3' & y_3\\ y_2'' & y_2' & y_2\\ y_1'' & y_1' & y_1 \end{pmatrix}(1,\lambda,p,q). \tag{56}$$

Remark: The equality $y_3(1,\lambda,p,q)=y_3(1,\lambda,p^*,q^*)$ may be explained. The eigenvalues corresponding to the boundary conditions $y(0)=y(1)=0$, $y'(0)=e^{i\phi}y'(1)$ with $\phi\in[0,2\pi)$ are the zeros of $\cos(\phi/2)\Re e y_3(1,\lambda,p,q)-\sin(\phi/2)\Im m y_3(1,\lambda,p,q)$. Take $\phi=0$. One should expect, using a Weierstrass product for the analytic extension of $\Re e y_3(1,\lambda,p,q)$, that theses eigenvalues determine $\Re e y_3(1,\lambda,p,q)$. The sequence of eigenvalues corresponding to $\phi=\pi$ should determine $\Im m y_3(1,\lambda,p,q)$. These two sequences of eigenvalues do not change replacing (p,q) by (p^*,q^*), the upshot being the symmetry of the boundary conditions.

Proof: Since $y_j(1-x,\bar{\lambda},p^*,q^*)$ solves $L_{p,q}y_j=\lambda y_j$, $j=1,2,3$, then

$$\begin{pmatrix} \bar{y}_1(1-x,\bar{\lambda},p^*,q^*) \\ \bar{y}_2(1-x,\bar{\lambda},p^*,q^*) \\ \bar{y}_3(1-x,\bar{\lambda},p^*,q^*) \end{pmatrix} = N \begin{pmatrix} y_1(x,\lambda,p,q) \\ y_2(x,\lambda,p,q) \\ y_3(x,\lambda,p,q) \end{pmatrix} \tag{57}$$

where N is a 3×3 matrix with complex coefficients independent of x.

At $x=1$, (57) gives

$$N \begin{pmatrix} y_1 & -y_1' & y_1'' \\ y_2 & -y_2' & y_2'' \\ y_3 & -y_3' & y_3'' \end{pmatrix} (1,\lambda,p,q) = Identity.$$

Thus $det\, N=-1$,

$$N = \begin{pmatrix} y_3''y_2'-y_3'y_2'' & y_1''y_3'-y_1'y_3'' & y_2''y_1'-y_2'y_1'' \\ y_3''y_2-y_3y_2'' & y_1''y_3-y_1y_3'' & y_2''y_1-y_2y_1'' \\ y_3'y_2-y_3y_2' & y_1'y_3-y_1y_3' & y_2'y_1-y_2y_1' \end{pmatrix} (1,\lambda,p,q)$$

and

$$N = \begin{pmatrix} \bar{y}_3'' & -\bar{y}_2'' & \bar{y}_1'' \\ \bar{y}_3' & -\bar{y}_2' & \bar{y}_1' \\ \bar{y}_3 & -\bar{y}_2 & \bar{y}_1 \end{pmatrix} (1,\bar{\lambda},p,q) \tag{58}$$

using (26) and $q(0)=0$. The Lemma follows from (58) together with (57) and its derivatives evaluated at $x=0$. □

Next we consider the $DP_*^{\pm}$ boundary conditions

$$(DP_*^{\pm}) \quad \begin{cases} z(0)=0 \\ z'(1)=\pm z'(0) \\ z''(1)=0 \end{cases}.$$

Using the map $(p,q)\mapsto(p^*,q^*)$, Lemma 9 and section 2, it is easy to see that the increasing sequence $\left(\nu_j^{\pm}(p,q)\right)_{j\in\mathbf{Z}}$ of eigenvalues of $L_{p,q}$ with the $DL_*^{\pm}$ boundary conditions are the roots of $y_3''(1,\lambda,p,q)\pm\bar{y}_3''(1,\lambda,p,q)$. Lemma 6 holds for the $\nu_j^{\pm}(p,q)$'s with the self evident changes of notations. In particular, $\nu_j^{\pm}(p,q)$ is defined.
Furthermore

$$\nu_j^{\pm}(p,q)=\mu_j^{\pm}(p^*,q^*), \quad \forall\, j\in\mathbf{Z}.$$

Each $\nu_j^{\pm}(p,q)$ is of multiplicity one if $y_3''(1,\nu_j^{\pm}(p,q),p,q) \neq 0$.

Then, define for $a.e\ x \in [0,1]$ and for $\lambda \in \mathbf{C}$

$$z(x,\lambda,p,q) = y_3''(1,\lambda,p,q)y_2(x,\lambda,p,q) - y_2''(1,\lambda,p,q)y_3(x,\lambda,p,q) \tag{59}$$

For every $j \in \mathbf{Z}$, $z(\cdot,\nu_j^{\pm}(p,q),p,q)$ is the eigenfunction associated to the eigenvalue $\nu_j^{\pm}(p,q)$ if $y_3''(1,\nu_j^{\pm}(p,q),p,q) \neq 0$.

The $DP_*^{\pm}$ boundary conditions will be considered only for the operator L_{p^*,q^*}. Then observe that

$$y_3''(1,\lambda,p^*,q^*) \neq 0 \Longleftrightarrow y_1(1,\lambda,p,q) \neq 0, \quad \forall\lambda \in \mathbf{C}.$$

Of course we can replace above, the statement $\forall\lambda \in \mathbf{C}$ with $\forall\lambda = \mu_j^{\pm}(p,q)$, $j \in \mathbf{Z}$. All these conditions are also equivalent to the basic one $\mu^+(p,q) \cap \mu^-(p,q) = \phi$.

Now fix $(p,q) \in L_R^2 \times H_R^1$ and $(\tilde{p},\tilde{q}) \in L_R^2 \times H_R^1$ with $q(0) = \tilde{q}(0) = 0$. Suppose

$$\mu^+(p,q) \cap \mu^-(p,q) = \phi. \tag{60}$$

Also suppose that,

$$\mu_j^+(p,q) = \mu_j^+(\tilde{p},\tilde{q}) \quad \text{and} \quad \mu_j^-(p,q) = \mu_j^-(\tilde{p},\tilde{q}). \tag{61}$$

For all $\lambda \in \mathbf{C}$, define

$$f(\lambda) = \frac{\left[\dfrac{y(x,\lambda,p,q)}{y_1(1,\lambda,p,q)} - \dfrac{y(x,\lambda,\tilde{p},\tilde{q})}{y_1(1,\lambda,\tilde{p},\tilde{q})}\right]\left[\dfrac{z(1-x,\lambda,p^*,q^*)}{y_3''(1,\lambda,p^*,q^*)} - \dfrac{z(1-x,\lambda,\tilde{p}^*,\tilde{q}^*)}{y_3''(1,\lambda,\tilde{p}^*,\tilde{q}^*)}\right]}{Y_1(1,\lambda,p,q)Z_1(1,\lambda,p,q)}. \tag{62}$$

The function $f(\lambda)$ is meromorphic and has simple poles at $\mu_j^+(p,q)$ and $\mu_j^-(p,q)$, $j \in \mathbf{Z}$.

For $\lambda = \mu_j^{\pm}(p,q) = \nu_j^{\pm}(p^*,q^*)$, $y(x,\lambda,p,q)$ and $\overline{z(1-x,\lambda,p^*,q^*)}$ are eigenfunctions of $L_{p,q}$ for the $DL^{\pm}$ boundary conditions. Then for some complex number $c_k^{\pm}(p,q)$

$$\frac{y(x,\mu_j^{\pm}(p,q),p,q)}{y_1(1,\mu_j^{\pm}(p,q),p,q)} = c_j^{\pm}(p,q)\,\frac{\overline{z(1-x,\nu_j^{\pm}(p^*,q^*),p^*,q^*)}}{\overline{y_3''(1,\nu_j^{\pm}(p^*,q^*),p^*,q^*)}}. \tag{63}$$

Differentiate (63) with respect to x, then take $x=1$, use $y'(1,\mu_j^{\pm}(p,q),p,q) = -\overline{y_1(1,\mu_j^{\pm}(p,q),p,q)}$ and $z'(0,\nu_j^{\pm}(p^*,q^*),p^*,q^*) = y_3''(1,\nu_j^{\pm}(p^*,q^*),p^*,q^*)$ to obtain

$$c_j^{\pm}(p,q) = \frac{\overline{y_1(1,\mu_j^{\pm}(p,q),p,q)}}{y_1(1,\mu_j^{\pm}(p,q),p,q)}. \tag{64}$$

Then (64) gives

$$c_j^{+}(p,q) = 1 \qquad c_j^{-}(p,q) = -1$$

for all $j \in \mathbf{Z}$.

The computation of the residues of f is as follows. From (63)

$$Res\left(f,\mu_j^{+}(p,q)\right) = \frac{1}{(Y_1\dot{Z}_1)(1,\mu_j^{+}(p,q),p,q)}\left|\frac{y(x,\mu_j^{+}(p,q),p,q)}{y_1(1,\mu_j^{+}(p,q),p,q)} - \frac{y(x,\mu_j^{+}(\tilde{p},\tilde{q}),\tilde{p},\tilde{q})}{y_1(1,\mu_j^{+}(\tilde{p},\tilde{q}),\tilde{p},\tilde{q})}\right|^2 \tag{65}$$

and

$$Res\left(f,\mu_j^{-}(p,q)\right) = -\frac{1}{(Z_1\dot{Y}_1)(1,\mu_j^{-}(p,q),p,q)}\left|\frac{y(x,\mu_j^{-}(p,q),p,q)}{y_1(1,\mu_j^{-}(p,q),p,q)} - \frac{y(x,\mu_j^{-}(\tilde{p},\tilde{q}),\tilde{p},\tilde{q})}{y_1(1,\mu_j^{-}(\tilde{p},\tilde{q}),\tilde{p},\tilde{q})}\right|^2 \tag{66}$$

for all $j \in \mathbf{Z}$.

Therefore (65), (66) and Lemma 8 give

$$\forall\, j \in \mathbf{Z}, \quad Res\left(f,\mu^{+}(p,q)\right) \le 0, \ \textit{and}\ Res\left(f,\mu^{-}(p,q)\right) \le 0. \tag{67}$$

Beside the numerator $n(\lambda)$ of f is bounded from above using

$$\frac{y(x,\lambda,p,q)}{y_1(1,\lambda,p,q)} = \frac{y(x,\lambda,0,0)}{y_1(1,\lambda,0,0)} + O\left(\frac{e^{\left(|\Im m \frac{k}{2}|+|\Im m \frac{\omega k}{2}|+|\Im m \frac{\omega^2 k}{2}|\right)x}}{|k|^2}\right) \tag{68}$$

and

$$\frac{z(x,\lambda,p,q)}{y_3''(1,\lambda,p,q)}+2q(1)\frac{y_3(x,\lambda,0,0)-y_2(x,\lambda,0,0)}{y_3''(x,\lambda,0,0)}=$$

$$\frac{z(x,\lambda,0,0)}{y_3''(1,\lambda,0,0)}+O\left(\frac{e^{\left(|\Im m\frac{k}{2}|+|\Im m\frac{\omega k}{2}|+|\Im m\frac{\omega^2 k}{2}|\right)x}}{|k|^2}\right). \tag{69}$$

Identities (68) and (69) are derived using (24) and (25).

Observe that $y_1(1,\lambda,0,0)$ and $y_3''(1,\lambda,0,0)$ never vanish on $\mathbf{C}$.

¿From (68) and (69), if $q(1)=\tilde{q}(1)$ we have $|n(\lambda)|=O\left(\frac{\Xi(x,\lambda)}{|k|^2}\right)O\left(\frac{\Xi(1-x,\lambda)}{|k|^2}\right)$, thus

$$|n(\lambda)|=O\left(\frac{\Xi(1,\lambda)}{|k|^4}\right). \tag{70}$$

The denominator $d(\lambda)$ of $f(\lambda)$ is easily bounded from below as follows.

Combining (38) and (57) we have

$$|Y_1(1,\lambda,p,q)|\geq\frac{1}{192}\Xi(1,\lambda),\text{for } k\in\Omega^-\text{ with } |k|>192e^{\|(p,q)\|_{L^2_R\times H^1_R}}$$

and

$$|Z_1(1,\lambda,p,q)|\geq\frac{1}{96}\Xi(1,\lambda),\ \text{ for } k\in\Omega^+\text{ with } |k|>96e^{\|(p,q)\|_{L^2_R\times H^1_R}}.$$

Therefore

$$|d(\lambda)|>C\Xi(1,\lambda),\ \text{ for } k\in\Omega^+\cap\Omega^-\text{ with } |k|>192e^{\|(p,q)\|_{L^2_R\times H^1_R}} \tag{71}$$

where C is a fixed numerical constant.

Finally (70) and (71) prove

$$f(\lambda)=O\left(\frac{1}{|k|^4}\right),\ \text{ for } k\in\Omega^+\cap\Omega^-\text{ with } |k|>192e^{\|(p,q)\|_{L^2_R\times H^1_R}} \tag{72}$$

We now conclude that the sum of the residues is zero using the classical integration contour, or equivalently (see [Pö-Tr], lemma 3.2) by proving that $\sup_{|\lambda|=r_n} |f(\lambda)| = o(1/r_n)$ as $n \to +\infty$, for an unbounded sequence of positive real number r_n. ¿From (72) this condition is verified by the function f defined in (62) with $r_n = (n + \frac{1}{2})^3\pi^3$, n sufficiently large.

Consequently, using (67) all residues of f are zero. In particular, there exists a function ϕ satisfying $L_{p,q}\phi = \mu_0^+(p,q)\phi$ and $L_{\tilde{p},\tilde{q}}\phi = \mu_0^+(p,q)\phi$. Then

$$2i(q - \tilde{q})\phi' + \left(i(q - \tilde{q})' + (p - \tilde{p})\right)\phi = 0. \qquad (73)$$

Multiply (73) by $\overline{\phi}$ and subtract ϕ multiplied by the complex conjugate of (73) to have

$$\left((q - \tilde{q})|\phi|^2\right)' = 0 \qquad (74)$$

Similarly, multiply (73) by $\overline{\phi}$ and add to ϕ multiplied by the complex conjugate of (73) to obtain

$$(p - \tilde{p})|\phi|^2 = 0. \qquad (75)$$

Using (2), $q(0) = \tilde{q}(0) = 0$ and the continuity of q and $\tilde{q}$, the proof follows (75) and (74). □

5 Proof of Theorem 2.

We now prove the asymptotic behavior of $\mu_j^+(p,q)$ given in theorem 2. The proof of the asymptotic behavior of $\mu_j^-(p,q)$ is similar and we will omit it.

¿From (14), (16), (36), we have

$$0 = Z_1(1, \mu_j^+(p,q), p, q) =$$

$$Z_1(1, \mu_j^+(p,q), 0, 0) + \Im m\, c_j^1(1, \mu_j^+(p,q), p, q) + O\left(\frac{\Xi(1,k_j)}{|k_j|^2}\right), \qquad (76)$$

where $\mu_j^+(p,q) = k_j^3$. Using (15)

$$c_j^1(1, \mu_j^+(p,q), p, q) =$$

$$-2\int_0^1 y_1(t, \mu_j^+(p,q), 0, 0) y_3'(1-t, \mu_j^+(p,q), 0, 0) q(t)\, dt + O\left(\frac{\Xi(1,k_j)}{|k_j|^2}\right). \qquad (77)$$

Besides, for $p = q \equiv 0$, $\lambda \in \mathbf{C}$ and $x, t \in [0,1]$, it is easy to check using (3) that

$$3y_1(t)y_3'(x-t) = y_2(x) + y_2\left(x - t(1-\omega)\right) + y_2\left(x - t(1-\omega^2)\right). \qquad (78)$$

Integrating by part (77) using $q \in H^1$ and (78) at $x = 1$, we obtain

$$c_j^1(1, \mu_j^+(p,q), p, q) = -\frac{2}{3}[q]y_2(1, \mu_j^+(p,q), p, q) + O\left(\frac{\Xi(1,k_j)}{|k_j|^2}\right).$$

Then

$$\Im m c_j^1(1, \mu_j^+(p,q), p, q) =$$

$$\frac{2}{9k_j}[q]\left(\cos k_j + e^{-\frac{\sqrt{3}}{2}k_j}\cos\left(\frac{k_j}{2} + \frac{2\pi}{3}\right) + e^{\frac{\sqrt{3}}{2}k_j}\cos\left(\frac{k_j}{2} - \frac{2\pi}{3}\right)\right) + O\left(\frac{\Xi(1,k_j)}{|k_j|^2}\right). \qquad (79)$$

Besides, using lemma 1

$$\begin{aligned} Z_1(1, \mu_j^+(p,q), 0, 0) &= -\tfrac{4}{3}\sin\tfrac{k_j}{2}|\sin\omega\tfrac{k_j}{2}|^2 \\ &\sim -\tfrac{1}{3}\sin\tfrac{k_j}{2}e^{\frac{\sqrt{3}}{2}|k_j|}. \end{aligned} \qquad (80)$$

The counting lemma gives $k_j = 2(\pi j + \delta_j)$ with $|\delta_j| < \pi$.

Therefore, using (79), (80) and

$$O\left(\Xi(1,k_j)\right) = O\left(e^{\frac{\sqrt{3}}{2}|k_j|}\right)$$

we obtain from (76)

$$\sin\delta_j = \frac{2}{3k_j}[q]\cos\left(\pm\delta_j - \frac{2\pi}{3}\right) + O\left(\frac{1}{|k_j|^2}\right). \qquad (81)$$

The irrelevant sign before δ_j in (81) depends on the sign of k_j. Then $\delta_j \to 0$ as $j \to +\infty$. Furthermore

$$\delta_j = -\frac{[q]}{3k_j} + O\left(\frac{1}{|k_j|^2}\right). \qquad (82)$$

which completes the proof. □

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