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Exactly soluble model with two kinds of Bose-Einstein condensations

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Abstract

We present a simple exactly soluble model which manifests two kinds of Bose-Einstein condensations. They occurs in two stages: for intermediate densities (negative chemical potentials $\mu_0 < \mu < 0$) one has a *non-conventional* Bose-condensation in the mode $k = 0$ due to a diagonal perturbation of the perfect Bose-gas, whereas for $\mu = 0$ and large densities $\rho > \rho_c^0(\theta)$ one gets a *conventional (generalized)* Bose-condensation corresponding to the standard mechanism of saturation.

Keywords : conventional/non-conventional Bose-condensations, generalized (non-extensive) Bose condensation, Weakly Imperfect Bose Gas

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1 Introduction

Recent study of the Bogoliubov model [1] of a Weakly Imperfect Bose-Gas (WIBG) shows that, under certain conditions on the interaction, it manifests *two kinds* of Bose condensations [2, 3, 4]. They occurs in two stages. For an interval $[\mu_0, 0]$ of *negative* chemical potentials up to $\mu = 0$, one has a macroscopic occupation $\rho_0(\mu)$ of the mode with momentum $k = 0$ due to an *effective attraction* of bosons in this mode [4], i.e. a kind of *dynamical* Bose condensation. Since the WIBG exists only for $\mu \leq 0$ and $\rho_0(\mu)$ as well as a total density $\rho(\mu)$ attain the maximal values at $\mu = 0$, for $\rho > \rho(\mu = 0)$ one gets a *conventional*(*generalized* [5]) Bose condensation due to the well-known saturation mechanism (cf. the ideal Bose gas (IBG)). One has to note that this kind of condensation occurs in spite of an *effective two-bosons repulsion* in the WIBG for $k \neq 0$ [4].

The possibility of a *two-stage* condensation was discussed at the first time in the framework of a Pair Hamiltonian Model (PHM), see e.g. [6, 7]. There it was a condensation in the one-particle and two-particle states due to off-diagonal interaction terms.

On the other hand, [8] and [9] demonstrate how subtle is the Bose condensation, even if (in contrast with the WIBG or PHM) one takes into account only a diagonal part of the non-ideal Bose gas interaction. In particular, they show that in so-called Perturbed Meanfield model with a Gaussien interaction kernel there is no Bose condensation for negative chemical potentials; condensation appears for $\mu \in (0, \mu_+)$ and then again disappears for $\mu \in (\mu_+, \widetilde{\mu}_+)$! Here $\widetilde{\mu}_+ > 2\mu_+$.

The aim of this paper is to present an exactly soluble model of Bose-gas with *diagonal* interactions which displays a two-stage Bose-Einstein condensation close to that in the WIBG:
(a) the model exists (the grand-canonical partition function converges) only for $\mu \leq 0$;
(b) there is $\mu_0 < 0$ such that there is *no condensation* for $\mu < \mu_0$;
(c) a macroscopic occupation $\rho_0(\theta, \mu)$ of the mode $k = 0$ starts at $\mu = \mu_0$ due to an instability implied by the interaction (a *dynamical* Bose condensation);
(d) since $\rho_0(\theta, \mu)$ and the total density of particles $\rho(\theta, \mu)$ attain their maxima at $\mu = 0$, a *conventional* Bose condensation occurs for densities $\rho > \rho(\theta, \mu = 0)$, but in a generalized sense, i.e. not directly in the mode $k = 0$;
(e) this *generalized* Bose condensation takes place even if there is a *repulsive interaction* between bosons for $k \neq 0$ similar to the effective two-boson repulsion in the WIBG for these modes.

In section 2 we define our model. We study the two-stage Bose condensation when there is no repulsion for $k \neq 0$ in section 3. In section 4, we show that this two-stage condensation is robust in the face of switching on a two-boson repulsion à la WIBG. We reserve section 5 for concluding remarks and discussions.

2 The Model

2.1 We consider a system of bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^d$ of volume $V = |\Lambda|$ centered at the origin. Assuming periodic boundary conditions, the Hamiltonian of the model is given by

$$H_\Lambda = T_\Lambda + U_\Lambda^0 + U_\Lambda = H_\Lambda^0 + U_\Lambda, \tag{1}$$

where (formal operator series)

$$T_\Lambda = \sum_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k a_k^* a_k, \ \ \varepsilon_{k\neq 0} = \hbar^2 k^2/2m,$$

$$U_\Lambda^0 = \varepsilon_0 a_0^* a_0 + \tfrac{g_0}{V} a_0^* a_0^* a_0 a_0, \ \ \varepsilon_0 \equiv \varepsilon_{k=0} \in \mathbb{R}^1, \ \ g_0 > 0, \tag{2}$$

$$U_\Lambda = \tfrac{1}{V} \sum_{k\in\Lambda^*, k\neq 0} g_k a_k^* a_k^* a_k a_k, \ \ g_k > 0,$$

correspond to the kinetic-energy and to diagonal interactions in modes $k = 0$ and $k \neq 0$. Here $\left\{a_k^\#\right\}_{k\in\Lambda^*}$ are the boson creation/annihilation operators in the boson Fock space $\mathcal{F}_\Lambda$ over $L^2(\Lambda)$ corresponding to the second quantization in the box $\Lambda = \mathop{\times}\limits_{\alpha=1}^{d} L$ with dual

$$\Lambda^* = \left\{k \in \mathbb{R}^d : \ k_\alpha = \frac{2\pi n_\alpha}{L}, \ n_\alpha = 0, \pm 1, \pm 2, ..., \ \alpha = 1, 2, 3..., d\right\}.$$

Notice that the interaction U_Λ^0 in the mode $k = 0$ consists of repulsion ($g_0 > 0$) and excitation energy ε_0 which we choose below to ensure a macroscopic occupation (*dynamical* Bose condensation) of this mode for some interval of negative chemical potentials. The interaction U_Λ describes a pure elastic (momentum transfer equals to zero) repulsive interaction of bosons in the modes $k \neq 0$. We return to discussion of the choice of the interaction U_Λ in section 4.
2.2 Below we consider our model in the grand-canonical ensemble (β, μ), where β is the inverse temperature, i.e. $\beta = \theta^{-1} > 0$ and $\mu \in \mathbb{R}^1$ is the chemical potential of this ensemble. Let

$$p_\Lambda(\beta, \mu) \equiv p_\Lambda[H_\Lambda] \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda} e^{-\beta(H_\Lambda - \mu N_\Lambda)} \tag{3}$$

be the grand-canonical pressure corresponding to H_Λ, where $N_\Lambda = \sum_{k\in\Lambda^*} a_k^* a_k$ is the particle-number operator.

Theorem 2.1 *Let $0 < g_k \leq G_+$ for $k \in \Lambda^*$. Then*
(i) the domain of stability of H_Λ, i.e.

$$Q \equiv \left\{\left(\theta \geq 0, \mu \in \mathbb{R}^1\right) : \ \lim_\Lambda p_\Lambda(\beta, \mu) < +\infty\right\},$$

is equal to $Q = \{\theta \geq 0\} \times \{\mu \leq 0\}$ and
(ii) in the thermodynamic limit one gets

$$p(\beta, \mu) \equiv \lim_\Lambda p_\Lambda(\beta, \mu) = \lim_\Lambda p_\Lambda\left[H_\Lambda^0\right] = p^I(\beta, \mu) - \inf_{\rho\geq 0}\left[(\varepsilon_0 - \mu)\rho + g_0\rho^2\right], \tag{4}$$

for $(\theta, \mu) \in Q$, where $p^I(\beta, \mu) \equiv \lim_\Lambda p_\Lambda[T_\Lambda]$ is the pressure of the ideal Bose gas (IBG).

Proof. (*i*) By explicit calculation, we have

$$p_\Lambda(\beta,\mu) = \frac{1}{\beta V}\sum_{k\in\Lambda^*}\ln\sum_{n=0}^{+\infty} e^{-\beta\left[\left(\varepsilon_k-\mu-\frac{g_k}{V}\right)n+\frac{g_k}{V}n^2\right]}. \tag{5}$$

Since $g_k > 0$, the pressure $p_\Lambda(\beta,\mu)$ is bounded for *any* $\mu \in \mathbb{R}^1$. For each mode $k \in \Lambda^*$ we consider the function

$$h_{k\Lambda}(\mu,\rho) \equiv \left(\varepsilon_k - \mu - \frac{g_k}{V}\right)\rho + g_k\rho^2.$$

Let $\mu > 0$. Then for $0 < \varepsilon < \mu$ one gets the estimates

$$p_\Lambda(\beta,\mu) \geq \frac{1}{\beta V}\sum_{\{k\in\Lambda^*:\varepsilon_k-\mu\leq-\varepsilon\}}\ln\sum_{n=0}^{+\infty} e^{-\beta V h_{k\Lambda}(\mu,n/V)}. \tag{6}$$

Since for the set $\{k \in \Lambda^*\backslash\{0\} : \varepsilon_k - \mu \leq -\varepsilon\}$ we have

$$\inf_{\rho\geq 0} h_{k\Lambda}(\mu,\rho) \leq -\frac{\varepsilon^2}{4G_+}, \tag{7}$$

the sum in (6) can be estimated from below by

$$A_\Lambda = \frac{1}{\beta V}\sum_{\{k\in\Lambda^*\backslash\{0\}:\varepsilon_k-\mu\leq-\varepsilon\}}\beta V\frac{\varepsilon^2}{4G_+} + \frac{\left(\varepsilon_0-\mu-\frac{g_0}{V}\right)^2}{4g_0}. \tag{8}$$

Note that $\lim_\Lambda A_\Lambda = +\infty$ which can be combined with (6) to deduce

$$\lim_\Lambda p_\Lambda(\beta,\mu) = +\infty \tag{9}$$

for $\mu > 0$. On the one hand, since $U_\Lambda \geq 0$ (2), one obviously obtains

$$H_\Lambda \geq H_\Lambda^0,$$

with H_Λ^0 defined by (1). Hence

$$p_\Lambda(\beta,\mu) \leq p_\Lambda^0(\beta,\mu), \tag{10}$$

where $p_\Lambda^0(\beta,\mu) \equiv p_\Lambda[H_\Lambda^0]$ is the pressure associated with the Hamiltonian H_Λ^0. Notice that

$$p_\Lambda^0(\beta,\mu) = \widetilde{p}_\Lambda^I(\beta,\mu) + \widehat{p}_\Lambda^0(\beta,\mu), \tag{11}$$

where

$$\widetilde{p}_\Lambda^I(\beta,\mu) \equiv \frac{1}{\beta V}\sum_{k\in\Lambda^*,k\neq 0}\ln\left(1-e^{-\beta(\varepsilon_k-\mu)}\right)^{-1} \tag{12}$$

is the pressure of the IBG with *excluded* mode $k = 0$ and

$$\widehat{p}_\Lambda^0(\beta,\mu) \equiv \frac{1}{\beta V}\ln\sum_{n_0=0}^{+\infty} e^{-\beta V h_{0\Lambda}(\mu,n_0/V)} \tag{13}$$

is the pressure induced by the mode $k=0$. By

$$h_{0\Lambda}\left(\mu,\frac{n_0}{V}\right)-\inf_{n\geq 0}h_{0\Lambda}\left(\mu,\frac{n_0}{V}\right)\leq g_0\left(\frac{n_0}{V}\right)^2,$$

one gets from (13) that

$$\lim_{\Lambda}\widehat{p}_{\Lambda}^{0}\left(\beta,\mu\right)=-\inf_{\rho\geq 0}\left[\left(\varepsilon_0-\mu\right)\rho+g_0\rho^2\right]<+\infty, \tag{14}$$

for $\mu\in\mathbb{R}^1$. The thermodynamic limit in (11) gives for $\mu\leq 0$:

$$p^0\left(\beta,\mu\right)\equiv\lim_{\Lambda}p_{\Lambda}^{0}\left(\beta,\mu\right)=p^I\left(\beta,\mu\right)-\inf_{\rho\geq 0}\left[\left(\varepsilon_0-\mu\right)\rho+g_0\rho^2\right]<+\infty, \tag{15}$$

where $p^I\left(\beta,\mu\right)$ is the pressure of the IBG. Hence from (9), (10) and (15), we find the domain of stability $Q=\{\theta\geq 0\}\times\{\mu\leq 0\}$ of the model (1).
(ii) By (5) we have for $\mu\leq 0$ the estimate

$$p_{\Lambda}\left(\beta,\mu\right)-\widehat{p}_{\Lambda}^{0}\left(\beta,\mu\right)\geq\frac{1}{\beta V}\sum_{k\in\Lambda^*,k\neq 0}\ln\sum_{n=0}^{\ln V}e^{-\beta\left[\left(\varepsilon_k-\mu\right)n-\frac{g_k}{V}\left(n^2-n\right)\right]}. \tag{16}$$

Since $n^2-n\leq\left(\ln V\right)^2$, (16) implies

$$\begin{aligned}p_{\Lambda}\left(\beta,\mu\right)-\widehat{p}_{\Lambda}^{0}\left(\beta,\mu\right)\;&\geq\;\frac{1}{\beta V}\sum_{k\in\Lambda^*,k\neq 0}\ln\frac{1-e^{-\beta\left(\varepsilon_k-\mu\right)\ln V}}{1-e^{-\beta\left(\varepsilon_k-\mu\right)}}\\&\quad+\frac{\left(\ln V\right)^2}{V^2}\sum_{k\in\Lambda^*,k\neq 0}g_k.\end{aligned} \tag{17}$$

where by $g_k\leq G_+$ one obviously gets

$$\lim_{\Lambda}\frac{\left(\ln V\right)^2}{V^2}\sum_{k\in\Lambda^*,k\neq 0}g_k=0, \tag{18}$$

and

$$\lim_{\Lambda}\frac{1}{\beta V}\sum_{k\in\Lambda^*,k\neq 0}\ln\frac{1-e^{-\beta\left(\varepsilon_k-\mu\right)\ln V}}{1-e^{-\beta\left(\varepsilon_k-\mu\right)}}=p^I\left(\beta,\mu\right). \tag{19}$$

Combining (10)-(15) and (17)-(19) we find the thermodynamic limit (4) for $\mu\leq 0$. ■

Corollary 2.2 *Let* $0<g_k\left(V\right)\leq\gamma_k V^{\alpha_k}$ *for* $k\in\Lambda^*\backslash\{0\}$, *with* $\alpha_k\leq\alpha_+<1$ *and* $0<\gamma_k\leq\gamma_+$. *Then the domain of stability of* H_Λ *is equal to* $Q=\{\theta\geq 0\}\times\{\mu\leq 0\}$ *and in the thermodynamic limit one gets (4) for* $(\theta,\mu)\in Q$.

Proof. The first assertion is due to (6)-(8), for $G_+=\gamma_+V^{\alpha_+}$, and the estimates (10) and (15). The second assertion follows from the estimate from above (10) and the estimate from below (17). ■

Remark 2.3 *Notice that in contrast to the IBG the pressure (12) is bounded (uniformly in V) for $\mu \leq 0$, i.e., for $\mu = 0$ included. Therefore, by (13) and (14) the pressure $p^0_\Lambda(\beta,\mu)$ (11) is bounded for $\mu \leq 0$, whereas the pressure $p_\Lambda(\beta,\mu)$ (5) is bounded for $\mu \in \mathbb{R}^1$ and $\lim_\Lambda p_\Lambda(\beta,\mu > 0) = +\infty$.*

Remark 2.4 *We keep $\varepsilon_0 \in \mathbb{R}^1$ as a parameter of our models H_Λ and H^0_Λ. Notice that for $\varepsilon_0 \geq 0$, Theorem 2.1 gives*

$$p(\beta,\mu) = p^I(\beta,\mu), \quad (\theta,\mu) \in Q.$$

Let domain D_{ε_0} be defined by

$$D_{\varepsilon_0} \equiv \left\{(\theta,\mu) \in Q : p^I(\beta,\mu) < p(\beta,\mu)\right\}. \tag{20}$$

Then Theorem 2.1 says that ε_0 must be negative and

$$D_{\varepsilon_0} = \{(\theta,\mu) \in Q : \varepsilon_0 < \mu\} \neq \emptyset. \tag{21}$$

Notice that in contrast to the pressure of the IBG, for $\varepsilon_0 < 0$ one gets

$$\lim_{\beta\to+\infty} p(\beta,\mu) = -\inf_{\rho\geq 0}\left[(\varepsilon_0-\mu)\rho + g_0\rho^2\right] > 0 \tag{22}$$

if $(\theta,\mu) \in D_{\varepsilon_0}$. Below we shall show that the domain D_{ε_0} corresponds to the existence of a *non-conventional* Bose condensation in the mode $k = 0$ for the model H_Λ as well as for H^0_Λ (cf. (1)). At the boundary $\mu = 0$ (for $d \geq 3$) we encounter another kind of condensation: a *conventional* (generalized) Bose condensation in the vinicity of $k = 0$.

3 Two-stage Bose condensation

First we consider the model H^0_Λ defined by (1) and (2), i.e. the model without repulsion between bosons for $k \neq 0$. Since the pressure $p^0(\beta,\mu)$ corresponding to H^0_Λ is known explicitly (15), in this section we highlight two kinds of Bose condensation which can occur in this model for $\varepsilon_0 < 0$.

3.1 Let $\rho^0_\Lambda(\beta,\mu)$ denote the total density in the model H^0_Λ, i.e.

$$\rho^0_\Lambda(\beta,\mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H^0_\Lambda} = \partial_\mu p^0_\Lambda(\beta,\mu), \tag{23}$$

where $\langle - \rangle_{H^0_\Lambda}$ represents the grand-canonical Gibbs state for the Hamiltonian H^0_Λ.

Theorem 3.1 *Let $\varepsilon_0 < 0$ and $\rho^0(\beta,\mu) \equiv \lim_\Lambda \rho^0_\Lambda(\beta,\mu)$. Then*
(i) for $(\theta,\mu) \in Q \backslash D_{\varepsilon_0}$ we have

$$\rho^0(\beta,\mu) = \rho^I(\beta,\mu), \tag{24}$$

where $\rho^I(\beta,\mu)$ is the total density of the IBG with $\rho^I(\beta,\mu=0)$ defined by taking the limit $\mu \to 0^-$;

(ii) for $(\theta, \mu) \in D_{\varepsilon_0}$ we get

$$\rho^0(\beta, \mu) = \rho^I(\beta, \mu) + \rho_0(\mu), \tag{25}$$

with

$$\rho_0(\mu) = \frac{\mu - \varepsilon_0}{2g_0} > 0. \tag{26}$$

Proof. By (11)-(13) one calculates (23):

$$\rho^0_\Lambda(\beta, \mu) = \widetilde{\rho}^I_\Lambda(\beta, \mu) + \left\langle \frac{n_0}{V} \right\rangle_{H^0_\Lambda}, \tag{27}$$

where

$$\widetilde{\rho}^I_\Lambda(\beta, \mu) \equiv \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \tag{28}$$

is the total density of the IBG with *excluded* mode $k = 0$. Since

$$\left\langle \frac{n_0}{V} \right\rangle_{H^0_\Lambda} = -\partial_{\varepsilon_0} p^0_\Lambda(\beta, \mu), \tag{29}$$

and $\{p^0_\Lambda(\beta, \mu)\}_\Lambda$ are convex functions of $\varepsilon_0 \in \mathbb{R}^1$, then combining (29) with the Griffiths lemma [10] for the limit (15), we obtain for $\mu \leq 0$ and $\varepsilon_0 < 0$ (cf. (21)):

$$\begin{aligned} \rho_0(\theta, \mu) &\equiv \lim_\Lambda \left\langle \frac{n_0}{V} \right\rangle_{H^0_\Lambda} = -\partial_{\varepsilon_0} p^0(\beta, \mu) \\ &= \left\{ \begin{array}{c} 0, \text{ for } (\theta, \mu) \in Q \backslash D_{\varepsilon_0} \\ \left[\frac{\mu - \varepsilon_0}{2g_0}\right], \text{ for } (\theta, \mu) \in D_{\varepsilon_0} \end{array} \right\}. \end{aligned} \tag{30}$$

Notice that

$$\rho^I(\beta, \mu) \equiv \lim_\Lambda \widetilde{\rho}^I_\Lambda(\beta, \mu). \tag{31}$$

Then from (27), (28) and (30), (31), one deduces (24)-(26). ■
Hence by Theorem 3.1, the domain D_{ε_0} corresponds to

$$D_{\varepsilon_0} = \left\{ (\theta, \mu) \in Q : \ \rho_0(\theta, \mu) \equiv \lim_\Lambda \left\langle \frac{n_0}{V} \right\rangle_{H^0_\Lambda} > 0 \right\} \neq \{\emptyset\}. \tag{32}$$

Therefore this kind of Bose condensation (the macroscopic occupation of the mode $k = 0$) is entirely due to the interaction U^0_Λ. It does not depend of the temperature and thus for $\varepsilon_0 < 0$, $\mu \in]\varepsilon_0, 0]$, one has by (26) and (30) that

$$\rho_0(\theta, \mu) = \rho_0(\mu) > 0 \tag{33}$$

for any temperature θ. Moreover, we have a saturation of the particle density in the mode $k = 0$:

$$\sup_{\mu \leq 0} \rho_0(\theta, \mu) = -\frac{\varepsilon_0}{2g_0}. \tag{34}$$

Behaviour of this first kind of condensate is illustrated by figure 1.

Remark 3.2 *Since for IBG, one has (see (28), (31))*

$$\sup_{\mu\leq 0}\rho^{I}(\beta,\mu)=\rho^{I}(\beta,\mu=0)\equiv\rho_{c}^{I}(\theta)<+\infty \tag{35}$$

for $d\geq 3$ (cf. figure 2), we get by (34) and (35) that the total particle density (25) is bounded:

$$\sup_{\mu\leq 0}\rho^{0}(\beta,\mu)\equiv\rho_{c}^{0}(\theta)=\rho_{c}^{I}(\theta)-\frac{\varepsilon_{0}}{2g_{0}}. \tag{36}$$

3.2 Now, since there is a critical density $\rho_{c}^{0}(\theta)$ one can pose a question (similar to the one for IBG) about thermodynamic behaviour of the model H_{Λ}^{0} for densities $\rho>\rho_{c}^{0}(\theta)$.

Theorem 3.3 *For $\rho>\rho_{c}^{0}(\theta)$ ($d\geq 3$) and any $\varepsilon_{0}\in\mathbb{R}^{1}$, one has*

$$\rho_{0}^{*}(\theta,\rho)\equiv\lim_{\Lambda}\sum_{\left\{k\in\Lambda^{*}:\|k\|=\frac{2\pi}{L}\right\}}\left\langle\frac{n_{k}}{V}\right\rangle_{H_{\Lambda}^{0}}=\rho-\rho_{c}^{0}(\theta)>0. \tag{37}$$

Proof. First, we remark that, by (27)-(28), we have

$$\lim_{\mu\to\varepsilon_{\Lambda,1}}\rho_{\Lambda}^{0}(\beta,\mu)=+\infty, \tag{38}$$

where $\varepsilon_{\Lambda,1}\equiv\frac{\hbar^{2}}{2m}\left(\frac{2\pi}{L}\right)^{2}$. Hence for any total density ρ there is a unique solution $\mu_{\Lambda}^{0}(\beta,\rho)$ of the equation $\rho=\rho_{\Lambda}^{0}(\beta,\mu)$ and we have for $\rho>\rho_{c}^{0}(\theta)$

$$\lim_{\Lambda}\mu_{\Lambda}^{0}(\beta,\rho)=0. \tag{39}$$

Second, by (23) we get the representation:

$$\rho_{\Lambda}^{0}(\beta,\mu)=\left\langle\frac{n_{0}}{V}\right\rangle_{H_{\Lambda}^{0}}+\sum_{\left\{k\in\Lambda^{*}:\|k\|=\frac{2\pi}{L}\right\}}\left\langle\frac{n_{k}}{V}\right\rangle_{H_{\Lambda}^{0}}+\sum_{\left\{k\in\Lambda^{*}:\|k\|>\frac{2\pi}{L}\right\}}\left\langle\frac{n_{k}}{V}\right\rangle_{H_{\Lambda}^{0}}, \tag{40}$$

where for $k\in\Lambda^{*}, k\neq 0$ one has

$$\langle n_{k}\rangle_{H_{\Lambda}^{0}}=\frac{1}{e^{\beta(\varepsilon_{k}-\mu)}-1}. \tag{41}$$

Hence, by (36) the solution $\mu_{\Lambda}^{0}(\beta,\rho)$ (39) has for $V\to\infty$ the asymptotics:

$$\varepsilon_{\Lambda,1}-\mu_{\Lambda}^{0}(\beta,\rho)=\frac{1}{V}\frac{d}{\beta\left(\rho-\rho_{c}^{0}(\theta)\right)}+o\left(\frac{1}{V}\right),$$

which ensures (37). ■

Therefore, the Bose condensation $\rho_{0}^{*}(\theta,\rho)$ (37) is:

(a) a *conventional* one because it appears (as in the IBG) due to the *saturation* of the total density $\rho^{0}(\beta,\mu)$ at $\mu=0$, see (36);

(b) a *generalized* one, since it occurs in the first $d \geq 3$ modes with $\|k\| = 2\pi/L \neq 0$.
We can resume the behaviour of the model H_Λ^0 as follows:
Let $\varepsilon_0 \geq 0$. Then by (11) and (15) we have $p^0(\beta,\mu) = p^I(\beta,\mu)$, i.e. by (4), (20) one gets $D_{\varepsilon_0} = \{\emptyset\}$ and by (32), (36) one has $\rho_c^0(\theta) = \rho_c^I(\theta)$. Hence the condensation in the model H_Λ^0 coincides with the one in the IBG with *excluded* mode $k = 0$. It is a *conventional* generalized Bose condensation (37) in the sense of [5] which occurs for $\rho > \rho_c^I(\theta)$, i.e. for $\mu = 0$. This is the case of the one-stage condensation.
If $\varepsilon_0 < 0$. Then $D_{\varepsilon_0} \neq \{\emptyset\}$ and Theorems 3.1 and 3.3 say that the Bose condensation starts at $k = 0$ for $\varepsilon_0 < \mu \leq 0$ as a *non-conventional* one. This stage lasts till $\mu = 0$, when for $\rho > \rho_c^0(\theta)$ a second stage corresponding to a *conventional* (generalized) Bose condensation comes into existence.

4 Robustness of the two-stage Bose condensation

The aim of this section is to show a robustness of the two kinds of Bose condensations manifested by the model H_Λ^0 in two stages (for $\varepsilon_0 < 0$) face the direct repulsion

$$\frac{1}{V} \sum_{k \in \Lambda^*, k \neq 0} g_k a_k^{*^2} a_k^2, \tag{42}$$

for $k \neq 0$, $g_k > 0$ giving rise in the model H_Λ (1).
4.1 Let us consider first $\rho_\Lambda(\beta,\mu)$, the total density of particles in the model H_Λ defined by

$$\rho_\Lambda(\beta,\mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda} = \left\langle \frac{n_0}{V} \right\rangle_{H_\Lambda} + \frac{1}{V} \sum_{k \in \Lambda^*, k \neq 0} \left\langle \frac{n_k}{V} \right\rangle_{H_\Lambda} = \partial_\mu p_\Lambda(\beta,\mu), \tag{43}$$

and the corresponding thermodynamic limit $\rho(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda(\beta,\mu)$.

Theorem 4.1 *Let $\varepsilon_0 < 0$ and $0 < g_k(V) \leq \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \setminus \{0\}$, with $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. Then we have:*
(i) for $(\theta, \mu \leq \varepsilon_0)$

$$\rho(\beta,\mu) = \rho^I(\beta,\mu); \tag{44}$$

(ii) for $(\theta, \varepsilon_0 < \mu < 0)$

$$\rho(\beta,\mu) = \rho^I(\beta,\mu) + \rho_0(\mu), \tag{45}$$

with $\rho_0(\mu)$ defined by (26).

Proof. Since pressures $\{p_\Lambda(\beta,\mu)\}_\Lambda$ are convex functions of $\mu < 0$, then combining (43) and Theorem 2.1 with the Griffiths lemma [10], we obtain

$$\rho(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda(\beta,\mu) = \partial_\mu p(\beta,\mu), \tag{46}$$

for $\mu < 0$. By the same token, $\{p_\Lambda(\beta,\mu)\}_\Lambda$ are convex functions of $\varepsilon_0 \in \mathbb{R}^1$. Then again by the Griffiths lemma [10] one gets for $\mu \leq 0$

$$\lim_\Lambda \left\langle \frac{n_0}{V} \right\rangle_{H_\Lambda} = -\partial_{\varepsilon_0} p(\beta,\mu) = \rho_0(\theta,\mu), \tag{47}$$

where the last equality is due to (4) and (30). Therefore (44) and (45) follow from (43), (46), (47) by Corollary 2.2 and Remark 2.4. ■
Notice that for $\varepsilon_0 < 0$, $\mu \in]\varepsilon_0, 0]$, $\rho_0(\mu)$ in (45) coincides with the same *non-conventional* Bose condensation in the mode $k = 0$ (cf. (47)), as in the model H^0_Λ (cf. (26)).

Remark 4.2 *Since by Theorem 4.1,* $\rho(\beta,\mu) = \rho^0(\beta,\mu)$ *(cf. (24)-(25) and (44), (45)), by Remark 3.2 one gets*

$$\rho_c(\theta) = \lim_{\mu\to 0^-} \rho(\beta,\mu) = \rho_c^0(\theta) = \rho_c^I(\theta) - \frac{\varepsilon_0}{2g_0} < +\infty \tag{48}$$

for $d \geq 3$ *(cf. figure 2); while for* $d \leq 2$,

$$\lim_{\mu\to 0^-} \rho(\beta,\mu) = +\infty.$$

4.2 Notice that from the convexity of $p_\Lambda(\beta,\mu)$

$$\frac{p_\Lambda(\beta,\mu) - p_\Lambda(\beta,\mu=0)}{\mu} \leq \partial_\mu p_\Lambda(\beta,\mu) = \rho_\Lambda(\beta,\mu). \tag{49}$$

Then by Remark 2.3 one gets from (49) that for any $\mu > 0$:

$$\lim_\Lambda \rho_\Lambda(\beta,\mu) = +\infty, \tag{50}$$

whereas for $\mu \leq 0$ the limit (50) is bounded by $\rho_c(\theta)$ see (45), (48). In particular, this means that for any total density ρ there is Λ_0 such that for all $\Lambda \supset \Lambda_0$ one has a unique $\mu_\Lambda(\beta,\rho)$ such that $\rho = \rho_\Lambda(\beta,\mu_\Lambda(\beta,\rho))$ and

$$\mu(\beta,\rho) \equiv \lim_\Lambda \mu_\Lambda(\beta,\rho) = \left\{ \begin{array}{l} < 0;\ \rho < \rho_c^0(\theta) \\ = 0;\ \rho \geq \rho_c^0(\theta) \end{array} \right\}. \tag{51}$$

In order to characterize the thermodynamic properties of the model (1) for a fixed total particle density, we need the following lemmas.

Lemma 4.3 *Let* $0 < g_k(V) \leq \gamma_k V^{\alpha_k}$ *for* $k \in \Lambda^* \backslash \{0\}$, *with* $\alpha_k \leq \alpha_+ < 1$ *and* $0 < \gamma_k \leq \gamma_+$. *Then for any* $\varepsilon_0 \in \mathbb{R}^1$ *and* $\|k\| \geq \delta > 0$ *we have the estimate:*

$$\langle n_k \rangle_{H_\Lambda} \leq \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{2g_k}{V}\right)} - 1}, \tag{52}$$

for V *sufficiently large.*

Proof. By the correlation inequalities for the Gibbs state $\omega_\Lambda(-) \equiv \langle - \rangle_{H_\Lambda}$ (see [11, 12]):

$$\beta\omega_\Lambda\left(X^*\left[H_\Lambda(\mu), X\right]\right) \geq \omega_\Lambda(X^*X)\ln\frac{\omega_\Lambda(X^*X)}{\omega_\Lambda(XX^*)}, \tag{53}$$

where X is an observable from the domain of the commutator $[H_\Lambda(\mu), .]$ with $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$, we can deduce

$$\beta\omega_\Lambda\left(a_k^*\left[H_\Lambda(\mu), a_k\right]\right) \geq \omega_\Lambda(n_k)\ln\frac{\omega_\Lambda(n_k)}{\omega_\Lambda(n_k)+1}, \tag{54}$$

for $X = a_k$ $(k \neq 0)$. Since for $k \neq 0$,

$$\left[H_\Lambda(\mu), a_k\right] = (\mu - \varepsilon_k)\, a_k - \frac{2g_k(V)}{V}a_k^* a_k^2,$$

from (54) one finds for $\mu = \mu_\Lambda(\beta, \rho)$

$$\beta\omega_\Lambda\left(\left(\mu_\Lambda(\beta,\rho) + \frac{2g_k(V)}{V} - \varepsilon_k\right)n_k - \frac{2g_k(V)}{V}n_k^2\right) \geq \omega_\Lambda(n_k)\ln\frac{\omega_\Lambda(n_k)}{\omega_\Lambda(n_k)+1},$$

which gives:

$$\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{2g_k(V)}{V}\right)\omega_\Lambda(n_k) \leq \omega_\Lambda(n_k)\ln\frac{\omega_\Lambda(n_k)+1}{\omega_\Lambda(n_k)}. \tag{55}$$

Now the rest of the proof is essentially due to solution for $x \geq 0$ of the inequality (55):

$$b_k \leq \ln\frac{x+1}{x}, \tag{56}$$

where

$$x \equiv \omega_\Lambda(n_k),\ b_k \equiv \beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{2g_k(V)}{V}\right). \tag{57}$$

Since for $\|k\| \geq \delta > 0$ and V sufficiently large (cf. (51)) we have by (57) that $b_k > 0$, the inequality (56) implies (52) for those Λ. ■

Lemma 4.4 *Let the conditions be as above. Then for any* $\varepsilon_0 \in \mathbb{R}^1$ *and* $\|k\| \geq \delta > 0$, *there are* $\tau_\Lambda(k)$ *such that*

$$\langle n_k\rangle_{H_\Lambda} \geq \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{g_k}{V}\right)} - 1} - \frac{\tau_\Lambda(k)}{V^{1-\alpha_+}}, \tag{58}$$

where $\tau_\Lambda(k)$ *is defined by*

$$\tau_\Lambda(k) \equiv \beta\gamma_+\left\{7\left[\frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1}\right]^3 + 3\left[\frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1}\right]^2\right\}. \tag{59}$$

Proof. Since $\|k\| \geq \delta > 0$, i.e. for V sufficiently large

$$\varepsilon_k - \mu_\Lambda(\beta, \rho) - \frac{g_k}{V} \geq 0, \tag{60}$$

we have

$$\begin{aligned} \langle n_k \rangle_{H_\Lambda} &= \frac{\sum_{n_k=0}^{+\infty} n_k e^{-\beta\left[\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{g_k}{V}\right) n_k + \frac{g_k}{V} n_k^2\right]}}{\sum_{n_k=0}^{+\infty} e^{-\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{g_k}{V}\right) n_k}} \\ &\quad \times \frac{\sum_{n_k=0}^{+\infty} e^{-\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{g_k}{V}\right) n_k}}{\sum_{n_k=0}^{+\infty} e^{-\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{g_k}{V}\right) n_k} e^{-\beta \frac{g_k}{V} n_k^2}}. \end{aligned} \tag{61}$$

¿From $e^{-x} \geq 1 - x$ and (60) one deduces

$$\langle n_k \rangle_{H_\Lambda} \geq \left\langle n_k \left(1 - \frac{\beta\gamma_+}{V^{1-\alpha_+}} n_k^2\right)\right\rangle_{T_\Lambda}. \tag{62}$$

Since

$$\langle n_k \rangle_{T_\Lambda} = \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{g_k}{V}\right)} - 1}$$

and since by the Wick theorem [12] one has

$$\left\langle n_k^3 \right\rangle_{T_\Lambda} = 7 \langle n_k \rangle_{T_\Lambda}^3 + 3 \langle n_k \rangle_{T_\Lambda}^2,$$

from (51) we deduce (58) with $\tau_\Lambda(k)$ defined by (59). ■

Now we can formulate the following Theorem.

Theorem 4.5 *Let $\rho > \rho_c^0(\theta)$ $(d \geq 3)$ and $0 < g_- \leq g_k(V) \leq \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \setminus \{0\}$, with $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. Then for any $\varepsilon_0 \in \mathbb{R}^1$ we have:*
(i)

$$\lim_\Lambda \left\langle \frac{n_k}{V} \right\rangle_{H_\Lambda} = 0,\ k \in \Lambda^* \setminus \{0\}, \tag{63}$$

i.e. no macroscopic occupation of modes $k \neq 0$;
(ii) a generalized (non-extensive) Bose condensation:

$$\lim_{\delta \to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^* :\ 0 < \|k\| < \delta\}} \langle n_k \rangle_{H_\Lambda} = \rho - \rho_c^0(\theta) > 0 \tag{64}$$

for those modes which for $\varepsilon_0 < 0$ coexists with the (non-conventional) condensation in the mode $k = 0$ (see Remark 4.2).

Proof. (i) Consider functions defined for $\nu \in \mathbb{R}^1$ by

$$p_\Lambda(\beta,\mu;\nu) \equiv p_\Lambda\left[H_\Lambda(\nu) \equiv H_\Lambda - \nu a_k^* a_k\right], \ k \neq 0. \tag{65}$$

Since $g_k(V) > 0$ the function (65) is well-defined and

$$\partial_\nu p_\Lambda(\beta,\mu;\nu) = \frac{\langle n_k\rangle_{H_\Lambda(\nu)}}{V}. \tag{66}$$

By the same line of reasoning as in Theorem 2.1 (for the mode $k=0$) we can show that (cf. (4))

$$\lim_\Lambda p_\Lambda(\beta,\mu_\Lambda(\beta,\rho);\nu) = p(\beta,\mu(\beta,\rho)) + \widetilde{p}(\mu(\beta,\rho);\nu), \tag{67}$$

with $\mu_\Lambda(\beta,\rho)$ defined by the equation $\rho = \rho_\Lambda(\beta,\mu_\Lambda(\beta,\rho))$ and

$$\widetilde{p}(\mu;\nu) = -\inf_{\rho\geq 0}\left\{\left[\varepsilon_k - (\mu+\nu)\right]\rho + g_k\rho^2\right\}, \tag{68}$$

where $g_k \equiv \lim_\Lambda g_k(V) > 0$ and $\mu(\beta,\rho) = \lim_\Lambda \mu_\Lambda(\beta,\rho)$ (51). Since $\{p_\Lambda(\beta,\mu;\nu)\}_\Lambda$ are convex functions of $\nu \in \mathbb{R}^1$, by the Griffiths lemma [10], we obtain from (66) and (67) that

$$\lim_\Lambda \frac{\langle n_k\rangle_{H_\Lambda(\nu)}}{V} = \partial_\nu \widetilde{p}(\mu;\nu). \tag{69}$$

Notice that by (68) we have $\widetilde{p}(\mu \leq 0; \nu = 0) = 0$ for any $k \neq 0$. Therefore by (51), (66), (67) and (69) for $\nu = 0$ one gets

$$\lim_\Lambda \frac{\langle n_k\rangle_{H_\Lambda}}{V} = 0, \ k \in \Lambda^*\backslash\{0\}, \tag{70}$$

i.e. there is no macroscopic occupation of modes $k \neq 0$ even for $\rho > \rho_c^0(\theta)$, when $\mu(\beta,\rho) = \lim_\Lambda \mu_\Lambda(\beta,\rho) = 0$, (51).

(ii) Since the total density ρ is fixed, we have

$$\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|<\delta\}} \langle n_k\rangle_{H_\Lambda} = \rho - \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geq\delta\}} \langle n_k\rangle_{H_\Lambda}. \tag{71}$$

Applying Lemma 4.3 to (71), one obtains the estimates

$$\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|<\delta\}} \langle n_k\rangle_{H_\Lambda} \geq \rho - \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geq\delta\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda(\beta,\rho) - \frac{2g_k}{V}\right)} - 1},$$

which in the thermodynamic limit for $\rho > \rho_c^0(\theta)$ gets the form

$$\lim_\Lambda \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|<\delta\}} \langle n_k\rangle_{H_\Lambda} \geq \rho - \frac{1}{(2\pi)^3}\int_{\|k\|\geq\delta} \frac{d^3k}{e^{\beta\varepsilon_k} - 1}. \tag{72}$$

Using now Lemma 4.4 we find the estimate

$$\begin{aligned}\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|<\delta\}}\langle n_k\rangle_{H_\Lambda} &\leq \rho-\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geq\delta\}}\frac{1}{e^{\beta\left(\varepsilon_k-\mu_\Lambda(\beta,\rho)-\frac{g_k}{V}\right)}-1}\\ &\quad+\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geq\delta\}}\frac{\tau_\Lambda(k)}{V^{1-\alpha_+}}.\end{aligned}$$

which gives in the thermodynamic limit for $\rho>\rho_c^0(\theta)$:

$$\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|<\delta\}}\langle n_k\rangle_{H_\Lambda}\leq\rho-\frac{1}{(2\pi)^3}\int\limits_{\|k\|\geq\delta}\frac{d^3k}{e^{\beta\varepsilon_k}-1}. \tag{73}$$

Hence, (64) follows from (47), (48), (72) and (73)by taking the limit $\delta\to0^+$. Since by (70) no levels with $k\neq0$ are macroscopically occupied, the condensation (64) is called non-extensive [13, 14]. ■

Remark 4.6 *Notice that from (68) and (69) we get (70) under conditions of Lemma 4.3, i.e. $\|k\|\geq\delta>0$ and $g_k>0$. If one relaxes the first condition to $k\in\Lambda^*\setminus\{0\}$ (which allows to consider in (70), e.g. $\|k\|=2\pi/L$ etc) then we need a stronger condition $g_k(V)\geq g_->0$ on the repulsive interaction.*

Corollary 4.7 *Since the estimate (52) is valid even for $\inf_{k\in\Lambda^*\setminus\{0\}}g_k(V)=0$, one gets the non-extensive Bose-condensation, i.e. (63) and (64), if the condition of the Theorem 4.5 is relaxed to $\inf_{\|k\|<\delta_0,V}g_k(V)>0$ for some $\delta_0>0$.*

We can resume the results of this section as follows. By Theorem 4.1, we get the robustness of the *first* kind of the Bose condensations in the mode $k=0$ with respect to the interaction (42). In fact it *coincides* with the one in the model H_Λ^0, see (45) and Remark 4.2. In the both model this *non-conventional* condensation occurs only for $\varepsilon_0<0$. In spite of the similarity can be extended to the thermodynamic equivalence between the two models H_Λ^0 and H_Λ (Theorem 2.1)

$$p(\beta,\mu)=p^0(\beta,\mu), \tag{74}$$

the *second* kind of the Bose condensations which occurs in these models are quite different. Both of them are *conventional*, i.e. due to the saturation of the total particle density ρ. Moreover they start at the same critical density: $\rho\geq\rho_c(\theta)=\rho_c^0(\theta)$, see Remark 4.2. Since condensation occurs for $k\neq0$, both of them are generalized [13, 14]. The difference comes from the fact that in the model H_Λ^0 one has macroscopic occupation of one (next to $k=0$) single particle level, i.e. generalized condensation of the type I, when only a finite number of levels are microscopically occupied [13, 14]. But in the model H_Λ we have a generalized condensation of the type III: non-extensive condensation, i.e. no levels ($k\neq0$) macroscopically occupied [13, 14]. For $\varepsilon_0<0$ it *coexists* with the type I (non-conventional) condensation in the mode $k=0$, see (64) where $\rho_c^0(\theta)=\rho_c^I(\theta)+\rho_0(\theta,\mu=0)$ and the last term is the type I condensate, cf. (30), (47), (48).

Notice that a particular choice of $g_k(V)$ has no influence on the thermodynamic properties of the model H_Λ. In fact we have only to control that for $k \in \Lambda^* \backslash \{0\}$, there is $\gamma_k \geq 0$ and $\alpha_k \leq \alpha_+ < 1$ such that $0 < g_k(V) \leq \gamma_k V^{\alpha_k}$. To ensure the *non-extensive condensation* we need a stronger condition $0 < g_- \leq g_{k\neq 0}(V) \leq \gamma_k V^{\alpha_k}$.

5 Concluding remarks

The present paper exposes a model of Bose-gas (1) whose thermodynamic behaviour for $\varepsilon_0 < 0$ is very close to thermodynamic behaviour of the Bogoliubov Hamiltonian for the WIBG studied in [2, 3, 4]:
(*i*) There is a subset $D_{\varepsilon_0} = \{\theta \geq 0\} \times \{\varepsilon_0 < \mu \leq 0\}$ of the stability domain $Q = \{\theta \geq 0\} \times \{\mu \leq 0\}$ in which a *non-conventional* Bose-Einstein condensation in mode $k = 0$ occurs (the first stage). Similar to the WIBG this kind of Bose condensation corresponds to domain where the model pressure is larger than the pressure of the IBG, cf. (20) and (32). Notice that this condensation in the mode $k = 0$ is due to the term $(\varepsilon_0 a_0^* a_0)$ which mimics (for $\varepsilon_0 < 0$) an *attraction* by an external potential [15]. In fact this makes difference between the model (1) and the WIBG since in the latter the attraction between particles in mode $k = 0$ is an *effective interaction* due to the non-diagonal term of the WIBG interaction [4].
(*ii*) Similar to (1) the total density $\rho^B(\beta, \mu)$ in the WIBG is bounded for $\mu \to 0^-$ and for densities $\rho > \rho_c^B(\theta) \equiv \rho^B(\beta, \mu = 0)$, the WIBG manifests a *conventional (generalized* of the type I [13, 14]) Bose condensation in the first mode above $k = 0$, i.e.

$$\lim_{\Lambda} \sum_{\left\{k\in\Lambda^*:\|k\|=\frac{2\pi}{L}\right\}} \left\langle \frac{n_k}{V} \right\rangle_{H_\Lambda^B} = \rho - \rho_c^B(\theta),$$

due to the mechanism of saturation (the second stage of condensation). In this sense the WIBG is closer to our model H_Λ^0, i.e. to the case $g_{k\neq 0} = 0$.

One of the main results of this paper concerns a robustness of the thermodynamic properties of the model H_Λ^0 with respect to the repulsive diagonal terms of interaction (2), even if the coupling constant g_k tends to infinity in thermodynamic limit as V^{α_k}, $\alpha_k \leq \alpha_+ < 1$. In particular, the first (*non-conventional*) kind of Bose condensation in mode $k = 0$ occurs for H_Λ in the same domain D_{ε_0}. The second (*conventional*) kind of Bose condensation also persists but the repulsive interaction (2) for $k \neq 0$ can change the nature of this Bose condensation. If $g_{k\neq 0}(V) \geq g_- > 0$ or $\inf_{\|k\|<\delta_0, V} g_k(V) > 0$ in the band $\delta_0 > 0$, then by Theorem 4.5 and Corollary 4.7 there is no macroscopic occupation of any mode $k \neq 0$ (as it is for the mode $k = 0$) but a *generalized* non-extensive Bose condensation of type III [13, 14].

This observation shows that there is no contradiction between the *conventional* Bose condensation in the WIBG and the effective repulsion between particles outside the mode $k = 0$ induced by non-diagonal terms of interaction [4]. A separate question is to know a real nature of this Bose condensation, e.g. whether there is a macroscopic occupation of the mode k above $k = 0$. Indeed, the thermodynamic properties of the WIBG are more complicated than the models H_Λ^0 or H_Λ. For example, in spite of effective repulsion between particles for all modes outside $k=0$ [4] it is not strong enough (cf. Remark 2.3): the pressure for the Bogoliubov Hamiltonian $p_\Lambda^B(\beta,\mu)$ gets *infinite* for $\mu>\varepsilon_{\Lambda,1}$ even for a *finite* volume V [16]. This makes the WIBG more similar to the model H_Λ^0 than to the perturbed model H_Λ for $g_{k\neq 0}(V)\geq g_->0$, since for the latter the pressure in a *finite* volume V is defined for all $\mu\in\mathbb{R}^1$, see Remark 2.3. For details of the above two stages of Bose condensation in the WIBG see [17].

The fact that the gentle repulsive interaction U_Λ (2) can produce a non-extensive Bose-Einstein condensation has been discovered recently in the paper [18]. This is done in the context of the Imperfect Bose Gas (ImBG) model [19, 20]:

$$H_\Lambda^{ImBG}=\sum_{k\in\Lambda^*}\varepsilon_k a_k^* a_k+\frac{\lambda}{V}N_\Lambda^2,\ \lambda>0, \tag{75}$$

perturbed by the repulsive diagonal interaction

$$\widetilde{U}_\Lambda=\frac{\lambda}{2V}\sum_{k\in\Lambda^*}n_k^2. \tag{76}$$

Notice that the ImBG (75) manifests the type I condensation [19], i.e. the macroscopic occupation of only the ground-state level $k=0$. The main result of [18] is that in spite of the ImBG-pressure is not changed by the perturbation (76) (cf. our Theorem 2.1 for $\varepsilon_0>0$) and the Bose condensation starts at the same parameters as for the ImBG, it is spread over a macroscopic number of levels in the vinicity of $k=0$. Therefore, in contrast to our model (1), where for $\varepsilon_0<0$ one gets the *coexistence* of the two kinds of Bose condensations (the type I and the type III, see Theorem 4.5), the repulsion (76) ensures only *one* kind of condensate, namely the type III (i.e. non-extensive) generalized Bose condensation. In fact in the both models the repulsions U_Λ (2) and $\widetilde{U}_\Lambda$ (76) transfer a type I condensate into a type III one.

A remained open question concerns the difference between equilibrium states of unperturbed and perturbed models, see discussion in [18]. Since in the both cases (1) and (75), (76) the perturbations change the nature of the Bose condensate, we conjecture that this difference should be non trivial, see e.g. [21].

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FIGURE CAPTIONS

Figure 1. Illustration of the Bose condensate density $\rho_0(\theta, \mu)$ as a function of the chemical potential μ for the model H_Λ^0.

Figure 2. Illustration of the total density as a function of the chemical potential μ at fixed temperature $\theta = \beta^{-1}$ for:
(1) the model H_Λ^0, i.e. the graph of $\rho^0(\beta, \mu)$ $(\rho_c^0(\theta) \equiv \rho^0(\theta^{-1}, \mu = 0))$;
(2) the model T_Λ (IBG), i.e. the graph of $\rho^I(\beta, \mu)$ $(\rho_c^I(\theta) \equiv \rho^I(\theta^{-1}, \mu = 0))$.