THE CLASSICAL BOUSSINESQ HIERARCHY REVISITED

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Abstract. We develop a systematic approach to the classical Boussinesq (cBsq) hierarchy based on an elementary polynomial recursion formalism. Moreover, the gauge equivalence between the cBsq and AKNS hierarchies is studied in detail and used to provide an effortless derivation of algebro-geometric solutions and their theta function representations of the cBsq hierarchy.

1. Introduction

We develop an elementary algebraic approach to the classical Boussinesq (cBsq) hierarchy in close analogy to previous treatments of the KdV, AKNS, and Toda hierarchies (cf. [2], [9], [10], [12] and the references therein). The complete integrability of the classical Boussinesq system (and its closely related variants, also known as the Kaup-Boussinesq, Broer-Kaup, and classical Boussinesq-Burgers system),

$$u_t + \frac{1}{4}v_{xxx} + (uv)_x = 0, \quad v_t + vv_x - u_x = 0, \tag{1.1}$$

was originally established by Kaup [19], [20]. Various aspects of this system (and its variants) are studied, for instance, in [3], [4], [5], [8], [13], [14], [15], [18], [21], [23], [24], [25], [27], [30], [31], and [32].

Subsequently, the equivalence of the classical Boussinesq system (1.1) and the AKNS system,

$$p_t + \frac{i}{2}p_{xx} - ip^2q = 0, \quad q_t - \frac{i}{2}q_{xx} + ipq^2 = 0, \tag{1.2}$$

was established by Jaulent and Miodek [18] by means of the explicit transformation

$$u + \frac{i}{2}v_x = -pq, \quad v = i\frac{p_x}{p} \tag{1.3}$$

(cf. Section 4 for more details).

The principal purpose of this note is threefold. First, we develop the zero-curvature formalism of the cBsq hierarchy in Section 3 using a polynomial recursion formalism (independently of its connection with the AKNS hierarchy). Second, we provide a new and elementary proof of the gauge equivalence between the cBsq and AKNS hierarchies in Section 4. Finally, using this gauge equivalence, we derive the class of algebro-geometric solutions of the cBsq hierarchy in Section 5.

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The AKNS hierarchy and its class of algebro-geometric solutions, the fundamental ingredients for Sections 4 and 5, are briefly reviewed in Section 2.

2. The AKNS hierarchy

In this section we review the construction of the AKNS hierarchy and its algebro-geometric solutions following a recursive approach to the AKNS zero-curvature formalism developed in [11].

We start by recalling the recursive construction of the AKNS hierarchy. Suppose $p,q\colon \mathbb{C}\to\mathbb{C}_\infty$ with $\mathbb{C}_\infty=\mathbb{C}\cup\{\infty\}$, are meromorphic and introduce the matrix

$$U(z,x)=\begin{pmatrix}-iz & q(x)\\ p(x) & iz\end{pmatrix}. \tag{2.1}$$

Define $\{f_\ell(x)\}_{\ell\in\mathbb{N}_0}$, $\{g_\ell(x)\}_{\ell\in\mathbb{N}_0}$, and $\{h_\ell(x)\}_{\ell\in\mathbb{N}_0}$ recursively by

$$\begin{aligned}
&f_0(x)=-iq(x), \quad g_0(x)=1, \quad h_0(x)=ip(x),\\
&f_{\ell+1}(x)=\frac{i}{2}f_{\ell,x}(x)-iq(x)g_{\ell+1}(x),\\
&g_{\ell+1,x}(x)=p(x)f_\ell(x)+q(x)h_\ell(x),\\
&h_{\ell+1}(x)=-\frac{i}{2}h_{\ell,x}(x)+ip(x)g_{\ell+1}(x), \quad \ell\in\mathbb{N}_0.
\end{aligned} \tag{2.2}$$

Exlicitly, one finds

$$\begin{aligned}
&f_0=-iq, \quad f_1=\frac{1}{2}q_x+c_1(-iq),\\
&f_2=\frac{i}{4}q_{xx}-\frac{i}{2}pq^2+c_1\left(\frac{1}{2}q_x\right)+c_2(-iq),\\
&g_0=1, \quad g_1=c_1, \quad g_2=\frac{1}{2}pq+c_2,\\
&g_3=-\frac{i}{4}(p_xq-pq_x)+c_1\left(\frac{1}{2}pq\right)+c_3,\\
&h_0=ip, \quad h_1=\frac{1}{2}p_x+c_1(ip),\\
&h_2=-\frac{i}{4}p_{xx}+\frac{i}{2}p^2q+c_1\left(\frac{1}{2}p_x\right)+c_2(ip),\\
&\quad\text{etc.},
\end{aligned} \tag{2.3}$$

where $\{c_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ are integration constants.

Next, define the matrix $V_{n+1}(z,x)$ by

$$V_{n+1}(z,x)=i\begin{pmatrix}-G_{n+1}(z,x) & F_n(z,x)\\ -H_n(z,x) & G_{n+1}(z,x)\end{pmatrix}, \quad n\in\mathbb{N}_0, \tag{2.4}$$

where $F_n(z,x)$, $G_{n+1}(z,x)$, and $H_n(z,x)$ are polynomials in $z\in\mathbb{C}$ of the type,

$$\begin{aligned}
&F_n(z,x)=\sum_{\ell=0}^n f_{n-\ell}(x)z^\ell=-iq\prod_{j=1}^n(z-\mu_j(x)),\\
&G_{n+1}(z,x)=\sum_{\ell=0}^{n+1} g_{n+1-\ell}(x)z^\ell,
\end{aligned} \tag{2.5}$$

$$H_n(z,x)=\sum_{\ell=0}^{n} h_{n-\ell}(x)z^\ell = ip\prod_{j=1}^{n}(z-\nu_j(x)).$$

Using the recursion (2.2) one verifies

$$\begin{aligned} F_{n,x} &= -2izF_n + 2qG_{n+1},\\ G_{n+1,x} &= pF_n + qH_n, \\ H_{n,x} &= 2izH_n + 2pG_{n+1}, \end{aligned} \tag{2.6}$$

implying

$$(G_{n+1}^2 - F_nH_n)_x = 0, \tag{2.7}$$

and hence

$$G_{n+1}(z,x)^2 - F_n(z,x)H_n(z,x) = R_{2n+2}(z), \tag{2.8}$$

where $R_{2n+2}(z)$ is a monic polynomial of degree $2n+2$ with zeros $\{E_0,\dots,E_{2n+1}\}\subset\mathbb{C}$. Thus,

$$R_{2n+2}(z)=\prod_{m=0}^{2n+1}(z-E_m), \quad \{E_m\}_{m=0,\dots,2n+1}\subset\mathbb{C}. \tag{2.9}$$

In particular, there is a naturally associated hyperelliptic curve $\mathcal{K}_n$ of genus n obtained from the characteristic equation for V_{n+1},

$$\begin{aligned} \det\left(yI - iV_{n+1}(z,x)\right) &= y^2 - G_{n+1}(z,x)^2 + F_n(z,x)H_n(z,x) \\ &= y^2 - R_{2n+2}(z) = 0, \end{aligned} \tag{2.10}$$

with $I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. We compactify the curve by adding two points $P_{\infty_\pm}$. The compactified curve is still denoted by $\mathcal{K}_n$. A point P on the curve $\mathcal{K}_n\backslash\{P_{\infty_\pm}\}$ is written as $P=(z,y)$, where $y^2=R_{2n+2}(z)$. For precise definitions of detailed properties of the curve $\mathcal{K}_n$ we refer to Appendix A in [11].

The stationary AKNS hierarchy is obtained by enforcing the stationary zero-curvature relation

$$-V_{n+1,x} + [U,V_{n+1}] = 0, \tag{2.11}$$

which, using (2.2), (2.4), and (2.5), reduces to

$$\begin{aligned} &-V_{n+1,x}+[U,V_{n+1}] \\ &= \begin{pmatrix} iG_{n+1,x}-ipF_n-iqH_n & -iF_{n,x}+2zF_n+2iqG_{n+1} \\ iH_{n,x}+2zH_n-2ipG_{n+1} & -iG_{n+1,x}+ipF_n+iqH_n \end{pmatrix} &(2.12)\\ &= \begin{pmatrix} ig_{n+1,x}-ipf_n-iqh_n & -2f_{n+1} \\ -2h_{n+1} & -ig_{n+1,x}+ipf_n+iqh_n \end{pmatrix} = 0. &(2.13) \end{aligned}$$

Hence the stationary AKNS hierarchy is defined by

$$\begin{pmatrix} h_{n+1} \\ f_{n+1} \end{pmatrix} = 0, \quad n\in\mathbb{N}_0. \tag{2.14}$$

Explicitly, the first few equations read

$$\begin{aligned} n=0: &\quad \begin{pmatrix} -p_x + c_1(-2ip) \\ -q_x + c_1(2iq) \end{pmatrix} = 0, \\ n=1: &\quad \begin{pmatrix} \frac{i}{2}p_{xx} - ip^2q + c_1(-p_x) + c_2(-2ip) \\ -\frac{i}{2}q_{xx} + ipq^2 + c_1(-q_x) + c_2(2iq) \end{pmatrix} = 0, \end{aligned} \tag{2.15}$$

etc.

Next we introduce a deformation parameter $t_n \in \mathbb{C}$ in the functions p and q, that is, $p = p(x, t_n)$, $q = q(x, t_n)$. The time dependent zero-curvature relation then reads

$$U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0. \tag{2.16}$$

Employing (2.1), (2.12), and (2.13), one finds

$$\begin{aligned} &U_{t_n} - V_{n+1,x} + [U, V_{n+1}] \\ &\quad = \begin{pmatrix} iG_{n+1,x} - ipF_n - iqH_n & q_{t_n} - iF_{n,x} + 2zF_n + 2iqG_{n+1} \\ p_{t_n} + iH_{n,x} + 2zH_n - 2ipG_{n+1} & -iG_{n+1,x} + ipF_n + iqH_n \end{pmatrix} &(2.17)\\ &\quad = \begin{pmatrix} ig_{n+1,x} - ipf_n - iqh_n & q_{t_n} - 2f_{n+1} \\ p_{t_n} - 2h_{n+1} & -ig_{n+1,x} + ipf_n + iqh_n \end{pmatrix} = 0. &(2.18) \end{aligned}$$

Hence the AKNS hierarchy is defined by

$$\mathrm{AKNS}_n(p, q) = \begin{pmatrix} p_{t_n} - 2h_{n+1} \\ q_{t_n} - 2f_{n+1} \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{2.19}$$

Explicitly, the first few equations read

$$\begin{aligned} &\mathrm{AKNS}_0(p, q) = \begin{pmatrix} p_{t_0} - p_x + c_1(-2ip) \\ q_{t_0} - q_x + c_1(2iq) \end{pmatrix} = 0, \\ &\mathrm{AKNS}_1(p, q) = \begin{pmatrix} p_{t_1} + \frac{i}{2}p_{xx} - ip^2q + c_1(-p_x) + c_2(-2ip) \\ q_{t_1} - \frac{i}{2}q_{xx} + ipq^2 + c_1(-q_x) + c_2(2iq) \end{pmatrix} = 0, \\ &\qquad \text{etc.} \end{aligned} \tag{2.20}$$

We also recall a scale invariance of the AKNS hierarchy (see [11] for details). Suppose (p, q) satisfies one of the AKNS equations (2.19) for some $n \in \mathbb{N}_0$,

$$\mathrm{AKNS}_n(p, q) = 0 \tag{2.21}$$

and consider the scale transformation

$$(p(x, t_n), q(x, t_n)) \to (\breve{p}(x, t_n), \breve{q}(x, t_n)) = (Ap(x, t_n), A^{-1}q(x, t_n)), \quad A \in \mathbb{C}\backslash\{0\}. \tag{2.22}$$

Then

$$\mathrm{AKNS}_n(\breve{p}, \breve{q}) = 0. \tag{2.23}$$

The outlined recursive approach is not confined to the hierarchy of AKNS evolution equations. Analogous considerations apply to the KdV, Toda, Boussinesq hierarchy, etc. (see [2], [9], [10], [12] and the references therein).

Next we turn to the class of algebro-geometric solutions of the AKNS hierarchy. For brevity we present the formulas in the time-dependent setting only. The corresponding stationary formulas easily follow as a special case (cf. [11]). Let $(p^{(0)}, q^{(0)})$ be a solution of the nth stationary AKNS system, that is,

$$f_{n+1}(p^{(0)}, q^{(0)}) = h_{n+1}(p^{(0)}, q^{(0)}) = 0, \tag{2.24}$$

for a given set $\{c_j\}_{j=1,\dots,n+1} \subset \mathbb{C}$ of integration constants. Consider subsequently the rth time-dependent AKNS system for some fixed $r \in \mathbb{N}_0$, with integration constants $\{\tilde{c}_j\}_{j=1,\dots,r+1} \subset \mathbb{C}$. The corresponding quantities f_j, g_j, etc. (that is, with $c_1, \dots, c_{r+1}$ replaced by $\tilde{c}_1, \dots, \tilde{c}_{r+1}$), for this system will be denoted with a tilde, $\tilde{f}_j$, $\tilde{g}_j$, $\tilde{h}_j$, $\widetilde{V}_{r+1}$, etc. Thus, we are interested in the construction of solutions (p, q) of

$$\mathrm{AKNS}_r(p,q) = \begin{pmatrix} p_{t_r} - 2\tilde{h}_{r+1} \\ q_{t_r} - 2\tilde{f}_{r+1} \end{pmatrix} = 0, \quad (p,q)|_{t_r = t_{0,r}} = (p^{(0)}, q^{(0)}). \tag{2.25}$$

These algebro-geometric AKNS solutions are obtained by a careful analysis of two specific functions on $\mathcal{K}_n$. First one defines the meromorphic function ϕ on $\mathcal{K}_n$ by

$$\phi(P,x,t_r) = \frac{y(P) + G_{n+1}(z,x,t_r)}{F_n(z,x,t_r)} = \frac{-H_n(z,x,t_r)}{y(P) - G_{n+1}(z,x,t_r)}, \quad P = (z,y) \in \mathcal{K}_n. \tag{2.26}$$

The divisor of $\phi(P, x, t_n)$ is given by

$$(\phi(P,x,t_r)) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t_r)}. \tag{2.27}$$

Here

$$\underline{\hat{\mu}}(x,t_r) = (\hat{\mu}_1(x,t_r), \dots, \hat{\mu}_n(x,t_r)) \in \sigma^n \mathcal{K}_n, \\ \hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), G_{n+1}(\mu_j(x,t_r),x,t_r)), \quad j = 1,\dots,n, \tag{2.28}$$

and

$$\underline{\hat{\nu}}(x,t_r) = (\hat{\nu}_1(x,t_r), \dots, \hat{\nu}_n(x,t_r)) \in \sigma^n \mathcal{K}_n, \\ \hat{\nu}_j(x,t_r) = (\nu_j(x,t_r), -G_{n+1}(\nu_j(x,t_r),x,t_r)), \quad j = 1,\dots,n, \tag{2.29}$$

where $\sigma^n\mathcal{K}_n$ denotes the nth symmetric power of $\mathcal{K}_n$ and

$$\mathcal{D}_{Q_0\underline{Q}}(P) = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_n}, \\ \underline{Q} = (Q_1, \dots, Q_n) \in \sigma^n\mathcal{K}_n, \quad Q_0 \in \mathcal{K}_n, \tag{2.30}$$

and for any $Q \in \mathcal{K}_n$,

$$\mathcal{D}_Q \colon \mathcal{K}_n \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_n \backslash \{Q\}. \end{cases} \tag{2.31}$$

Secondly, we introduce the Baker–Akhiezer vector,

$$\Psi(P,x,x_0,t_r,t_{0,r}) = \begin{pmatrix} \psi_1(P,x,x_0,t_r,t_{0,r}) \\ \psi_2(P,x,x_0,t_r,t_{0,r}) \end{pmatrix}, \tag{2.32}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \exp\bigg(\int_{x_0}^{x} dx' \, (-iz + q(x',t_r)\phi(P,x',t_r)) \\ + i \int_{t_{0,r}}^{t_r} ds (\widetilde{F}_n(z,x_0,s)\phi(P,x_0,s) - \widetilde{G}_{n+1}(z,x_0,s)) \bigg),$$

$$\psi_2(P,x,x_0,t_r,t_{0,r}) = \phi(P,x,t_r)\psi_1(P,x,x_0,t_r,t_{0,r}), \\ P \in \mathcal{K}_n \backslash \{P_{\infty_\pm}\}, \, (x,t_r) \in \mathbb{C}^2.$$

The functions ϕ and ψ satisfy the following fundamental properties.

Theorem 2.1 (see [10], [11]). *Let $P = (z,y) \in \mathcal{K}_n \backslash \{P_{\infty_\pm}\}$ and $(z,x,x_0,t_r,t_{0,r}) \in \mathbb{C}^5$. Then $\phi(P,x,t_r)$ satisfies*

$$\phi_x(P,x,t_r) + q(x,t_r)\phi(P,x,t_r)^2 - 2iz\phi(P,x,t_r) = p(x,t_r). \tag{2.33}$$

$\Psi(\,\cdot\,,x,x_0,t_0,t_{0,r})$ *is meromorphic on* $\mathcal{K}_n\backslash\{P_{\infty_\pm}\}$ *and*

$$\Psi_x(P,x,x_0,t_r,t_{0,r}) = U(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{2.34}$$

$$iy(P)\Psi(P,x,x_0,t_r,t_{0,r}) = V_{n+1}(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{2.35}$$

$$\Psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \widetilde{V}_{r+1}(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}). \tag{2.36}$$

Further properties of ϕ and Ψ can be found in [11], Lemma 4.1.

In order to express the basic quantities in terms of Riemann's theta function associated with $\mathcal{K}_n$, we need to introduce differentials and some more notation in connection with the hyperelliptic curve $\mathcal{K}_n$. In the following we assume $\mathcal{K}_n$ to be nonsingular, that is,

$$E_m \neq E_{m'}, \quad m,m' = 0,\dots,2n+1,\, m\neq m'. \tag{2.37}$$

Given a canonical homology basis $\{a_j,b_j\}_{j=1,\dots,n}$ for $\mathcal{K}_n$ with intersection matrix $a_j\circ b_k = \delta_{j,k}$, one denotes by ω_j, $j=1,\dots,n$ a normalized basis of the space of holomorphic differentials on $\mathcal{K}_n$,

$$\int_{a_k}\omega_j = \delta_{j,k}, \quad j,k=1,\dots,n. \tag{2.38}$$

In addition, one considers a canonical dissection of $\mathcal{K}_n$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_n$ of the fundamental polygon

$$\partial\widehat{\mathcal{K}}_n = a_1b_1a_1^{-1}b_1^{-1}\dots a_n^{-1}b_n^{-1}. \tag{2.39}$$

Next, we choose, without loss of generality, the base point $P_0=(E_0,0)\in\mathcal{K}_n$ and denote by $\underline{A}_{P_0}$, $\underline{\alpha}_{P_0}$ the Abel maps

$$\begin{aligned}\underline{A}_{P_0}\colon \mathcal{K}_n\to J(\mathcal{K}_n), \quad P\mapsto \underline{A}_{P_0}(P) &= (A_{P_0,1}(P),\dots,A_{P_0,n}(P))\\ &= \left(\int_{P_0}^P\omega_1,\dots,\int_{P_0}^P\omega_n\right)(\mathrm{mod}\ L_n),\end{aligned} \tag{2.40}$$

and

$$\underline{\alpha}_{P_0}\colon\ \mathrm{Div}(\mathcal{K}_n)\to J(\mathcal{K}_n), \quad \mathcal{D}\mapsto\underline{\alpha}_{P_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_n}\mathcal{D}(P)\underline{A}_{P_0}(P), \tag{2.41}$$

where

$$L_n = \{\underline{z}\in\mathbb{C}^n \mid \underline{z} = \underline{N}+\tau\underline{M},\ \underline{N},\underline{M}\in\mathbb{Z}^n\} \tag{2.42}$$

denotes the period lattice, and

$$J(\mathcal{K}_n) = \mathbb{C}^n/L_n \tag{2.43}$$

the Jacobi variety. In addition we introduce

$$\begin{aligned}\underline{\widehat{A}}_{P_0}\colon \widehat{\mathcal{K}}_n\to\mathbb{C}^n, \quad P\mapsto\underline{\widehat{A}}_{P_0}(P) &= (\widehat{A}_{P_0,1}(P),\dots,\widehat{A}_{P_0,n}(P))\\ &= \left(\int_{P_0}^P\omega_1,\dots,\int_{P_0}^P\omega_n\right),\end{aligned} \tag{2.44}$$

$$\underline{\hat{\alpha}}_{P_0}\colon\ \mathrm{Div}(\mathcal{K}_n)\to\mathbb{C}^n, \quad \mathcal{D}\mapsto\underline{\hat{\alpha}}_{P_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_n}\mathcal{D}(P)\underline{\widehat{A}}_{P_0}(P), \tag{2.45}$$

and $\underline{\Xi}_{P_0} = (\Xi_{P_0,1},\dots,\Xi_{P_0,n})$, the vector of Riemann constants, by

$$\underline{\Xi}_{P_0} = \underline{\widehat{\Xi}}_{P_0}(\mathrm{mod}\ L_n),$$

$$\widehat{\Xi}_{P_0,j} = \frac{1}{2}\left(1 + \int_{b_j} \omega_j\right) - \sum_{\substack{\ell=1 \\ \ell\neq j}}^{n} \int_{a_\ell} \omega_\ell(P)\widehat{A}_{P_0,j}(P), \quad j = 1,\dots,n. \tag{2.46}$$

Next, consider the normal differential of the third kind $\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}$ with simple poles at P_{∞_+} and P_{∞_-}, corresponding residues $+1$ and -1, vanishing a-periods, being holomorphic otherwise on $\mathcal{K}_n$. Hence one obtains

$$\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \frac{\prod_{j=1}^{n}(\tilde{\pi} - \lambda_j)\, d\tilde{\pi}}{y}, \quad \omega^{(3)}_{P_{\infty_-},P_{\infty_+}} = -\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}, \tag{2.47}$$

$$\int_{a_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = 0, \quad j = 1,\dots,n, \tag{2.48}$$

$$\underline{U}^{(3)} = (U^{(3)}_1,\dots,U^{(3)}_n),$$

$$U^{(3)}_j = \frac{1}{2\pi i}\int_{b_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \widehat{A}_{P_{\infty_-},j}(P_{\infty_+}) = 2\widehat{A}_{P_0,j}(P_{\infty_+}), \quad j = 1,\dots,n, \tag{2.49}$$

$$\int_{P_0}^{P} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} \underset{\zeta\to 0}{=} \pm(\ln(\zeta) - \ln(\omega_0) + O(\zeta)),\ P = (\zeta^{-1}, y) \text{ near } P_{\infty_\pm}, \tag{2.50}$$

where the numbers $\{\lambda_j\}_{j=1,\dots,n}$ are determined by the normalization (2.48). The Abelian differentials of the second kind $\omega^{(2)}_{P_{\infty_\pm},0}$ are chosen such that

$$\omega^{(2)}_{\infty_\pm,0} \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))\, d\zeta \text{ near } P_{\infty_\pm}, \tag{2.51}$$

$$\int_{a_j} \omega^{(2)}_{P_{\infty_\pm},0} = 0, \quad j = 1,\dots,n, \tag{2.52}$$

$$\underline{U}^{(2)}_0 = (U^{(2)}_{0,1},\dots,U^{(2)}_{0,n}),\ U^{(2)}_{0,j} = \frac{1}{2\pi i}\int_{b_j} \Omega^{(2)}_0, \quad \Omega^{(2)}_0 = \omega^{(2)}_{P_{\infty_+},0} - \omega^{(2)}_{P_{\infty_-},0}, \tag{2.53}$$

$$\int_{P_0}^{P} \Omega^{(2)}_0 \underset{\zeta\to 0}{=} \mp(\zeta^{-1} + e_{0,0} + e_{0,1}\zeta + O(\zeta^2)),\ P = (\zeta^{-1}, y) \text{ near } P_{\infty_\pm}. \tag{2.54}$$

In addition we define Abelian differentials of the second kind $\omega^{(2)}_{P_{\infty_\pm},r}$ by

$$\omega^{(2)}_{P_{\infty_\pm},r} \underset{\zeta\to 0}{=} (\zeta^{-2-r} + O(1))\, d\zeta \text{ near } P_{\infty_\pm},\ r \in \mathbb{N}_0, \tag{2.55}$$

$$\int_{a_j} \omega^{(2)}_{P_{\infty_\pm},r} = 0, \quad j = 1,\dots,n, \tag{2.56}$$

$$\underline{\widetilde{U}}^{(2)}_r = (\widetilde{U}^{(2)}_{r,1},\dots,\widetilde{U}^{(2)}_{r,n}), \quad \widetilde{U}^{(2)}_{r,j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_r,$$

$$\widetilde{\Omega}^{(2)}_r = \sum_{q=0}^{r} (q+1)\tilde{c}_{r-q}(\omega^{(2)}_{P_{\infty_+},q} - \omega^{(2)}_{P_{\infty_-},q}), \tag{2.57}$$

$$\int_{P_0}^{P} \widetilde{\Omega}^{(2)}_r \underset{\zeta\to 0}{=} \mp\left(\sum_{q=0}^{r} \tilde{c}_{r-q}\zeta^{-1-q} + \tilde{e}_{r,0} + O(\zeta)\right), \quad P = (\zeta^{-1}, y) \text{ near } P_{\infty_\pm}, \tag{2.58}$$

with $\{\tilde c_\ell\}_{\ell=1,\dots,r}\subset\mathbb{C}$, $\tilde c_0=1$ denoting integration constants.

Finally, we abbreviate

$$\underline{z}(P,\underline{Q})=\widehat{\underline{A}}_{P_0}(P)-\underline{\hat\alpha}_{P_0}(\mathcal{D}_{\underline{Q}})-\widehat{\underline{\Xi}}_{P_0}, \tag{2.59}$$

$$\underline{z}_\pm(\underline{Q})=\underline{z}(P_{\infty_\pm},\underline{Q}),\quad \underline{Q}=(Q_1,\dots,Q_n). \tag{2.60}$$

Next, assume that (p,q) satisfies the rth time-dependent AKNS equation with initial data $(p^{(0)},q^{(0)})$ being solutions of the nth stationary AKNS equation at $t=t_{0,r}$, that is,

$$(p(x,t_{0,r}),q(x,t_{0,r}))=(p^{(0)}(x),q^{(0)}(x)),\quad x\in\mathbb{C}. \tag{2.61}$$

The principal aim is to derive explicit formulas for the solution (p,q) as well as the function ϕ and the Baker–Akhiezer function Ψ in terms of the Riemann theta function

$$\theta(\underline{z})=\sum_{\underline{m}\in\mathbb{Z}^n}\exp\big(2\pi i(\underline{m},\underline{z})+\pi i(\underline{m},\tau\underline{m})\big),\quad \underline{z}\in\mathbb{C}^n,$$

$$(\underline{u},\underline{v})=\sum_{j=1}^n\overline{u}_jv_j,\quad \tau=\bigg(\int_{b_j}\omega_k\bigg)_{j,k=1,\dots,n}. \tag{2.62}$$

This is the content of the next theorem.

Theorem 2.2 (see [10], [11]). *Let $P\in\mathcal{K}_n\backslash\{P_{\infty_\pm}\}$, $(x,x_0,t_r,t_{0,r})\in\mathbb{C}^4$, assume $\mathcal{K}_n$ to be nonsingular, and suppose $\mathcal{D}_{\underline{\hat\mu}(x,t)}$, or equivalently, $\mathcal{D}_{\underline{\hat\nu}(x,t)}$ to be nonspecial, that is, their index of speciality vanishes, $i(\mathcal{D}_{\underline{\hat\mu}(x,t)})=i(\mathcal{D}_{\underline{\hat\nu}(x,t)})=0$. Moreover, suppose that (p,q) satisfies $\mathrm{AKNS}_r(p,q)=0$ with $(p,q)|_{t_r=t_{0,r}}=(p^{(0)},q^{(0)})$, a solution of the nth stationary AKNS system. Let ϕ and Ψ be defined by* (2.26) *and* (2.32)*, respectively. Then*

$$\phi(P,x,t_r)=\frac{2i}{q(x_0,t_{0,r})\omega_0}\frac{\theta(\underline{z}_-(\underline{\hat\mu}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat\mu}(x_0,t_{0,r})))}\frac{\theta(\underline{z}_+(\underline{\hat\mu}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat\nu}(x,t_r)))}\frac{\theta(\underline{z}(P,\underline{\hat\nu}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat\mu}(x,t_r)))}\times$$
$$\times\exp\Big(\int_{P_0}^P\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}-2i(x-x_0)e_{0,0}-2i(t_r-t_{0,r})\tilde e_{r,0}\Big), \tag{2.63}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})=\frac{\theta(\underline{z}_+(\underline{\hat\mu}(x_0,t_{0,r})))}{\theta(\underline{z}(P,\underline{\hat\mu}(x_0,t_{0,r})))}\frac{\theta(\underline{z}(P,\underline{\hat\mu}(x,t_r)))}{\theta(\underline{z}_+(\underline{\hat\mu}(x,t_r)))}\times \tag{2.64}$$
$$\times\exp\bigg(i(x-x_0)\big(e_{0,0}+\int_{P_0}^P\Omega_0^{(2)}\big)+i(t_r-t_{0,r})\big(\tilde e_{r,0}+\int_{P_0}^P\widetilde\Omega_r^{(2)}\big)\bigg),$$

$$\psi_2(P,x,x_0,t_r,t_{0,r})=\frac{2i}{q(x_0,t_{0,r})\omega_0}\frac{\theta(\underline{z}_-(\underline{\hat\mu}(x_0,t_{0,r})))}{\theta(\underline{z}(P,\underline{\hat\mu}(x_0,t_{0,r})))}\frac{\theta(\underline{z}(P,\underline{\hat\nu}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat\nu}(x,t_r)))}\times$$
$$\times\exp\bigg(\int_{P_0}^P\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}+i(x-x_0)\big(-e_{0,0}+\int_{P_0}^P\Omega_0^{(2)}\big)$$
$$+i(t_r-t_{0,r})\big(-\tilde e_{r,0}+\int_{P_0}^P\widetilde\Omega_r^{(2)}\big)\bigg). \tag{2.65}$$

Furthermore, one derives

$$p(x,t_r)=p(x_0,t_{0,r})\frac{\theta(\underline{z}_-(\underline{\hat\nu}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat\nu}(x_0,t_{0,r})))}\frac{\theta(\underline{z}_+(\underline{\hat\nu}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat\nu}(x,t_r)))}\times$$
$$\times\exp(-2i(x-x_0)e_{0,0}-2i(t_r-t_{0,r})\tilde e_{r,0}), \tag{2.66}$$

$$q(x,t_r)=q(x_0,t_{0,r})\frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0,t_{0,r})))}\frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x,t_r)))}\times$$
$$\times\exp(2i(x-x_0)e_{0,0}+2i(t_r-t_{0,r})\tilde{e}_{r,0}), \tag{2.67}$$
$$p(x_0,t_{0,r})q(x_0,t_{0,r})=\frac{4}{\omega_0^2}\frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0,t_{0,r})))}\frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0,t_{0,r})))}, \tag{2.68}$$
$$p(x,t_r)q(x,t_r)=-e_{0,1}-\frac{\partial^2}{\partial x^2}\ln(\theta(\underline{z}_+(\underline{\hat{\mu}}(x,t_r)))), \tag{2.69}$$

and

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})=\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})})-i(x-x_0)\underline{U}_0^{(2)}-i(t_r-t_{0,r})\underline{\widetilde{U}}_r^{(2)}, \tag{2.70}$$
$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)})=\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})})-i(x-x_0)\underline{U}_0^{(2)}-i(t_r-t_{0,r})\underline{\widetilde{U}}_r^{(2)}. \tag{2.71}$$

Algebro-geometric solutions of the AKNS equations (i.e., the case $r=1$, $n\in\mathbb{N}$) have previously been derived by a variety of authors, see, for instance, [1], [6], [7], [16], [17], [22], [26], [28], [29] and the literature cited therein. The principal contribution of [11] to this circle of ideas is an effortless treatment of algebro-geometric solutions of the entire AKNS hierarchy (i.e., for $r,n\in\mathbb{N}$) using the elementary polynomial recursion formalism outlined in the first part of this section.

3. The classical Boussinesq hierarchy

In this section we follow the zero-curvature formalism introduced by Geng and Wu [8] for the cBsq hierarchy and adapt it to the recursion formalism outlined in Section 2. Fix[1] $\alpha\in\mathbb{C}\backslash\{0\}$, $\beta\in\mathbb{C}$, and define the matrix

$$\overline{U}(z,x)=\begin{pmatrix}-iz-\alpha v(x) & u(x)+\beta v_x(x)\\ -1 & iz+\alpha v(x)\end{pmatrix}. \tag{3.1}$$

Define recursively $\{\overline{f}_j(x)\}_{j\in\mathbb{N}_0}$, $\{\overline{g}_j(x)\}_{j\in\mathbb{N}_0}$, and $\{\overline{h}_j(x)\}_{j\in\mathbb{N}_0}$ by

$$\overline{f}_0(x)=-i(u(x)+\beta v_x(x)),\quad \overline{g}_0(x)=1,\quad \overline{h}_0(x)=-i, \tag{3.2a}$$
$$\overline{f}_{j+1}(x)=\frac{i}{2}\overline{f}_{j,x}(x)+i\alpha v(x)\overline{f}_j(x)-i(u(x)+\beta v_x(x))\overline{g}_{j+1}(x), \tag{3.2b}$$
$$\overline{g}_{j+1,x}(x)=(u(x)+\beta v_x(x))\overline{h}_j(x)-\overline{f}_j(x), \tag{3.2c}$$
$$\overline{h}_{j+1}(x)=-\frac{i}{2}\overline{h}_{j,x}(x)+i\alpha v(x)\overline{h}_j(x)-i\overline{g}_{j+1}(x),\quad j\in\mathbb{N}_0. \tag{3.2d}$$

Explicitly, the first few elements read

$$\begin{aligned}&\overline{f}_1=\frac{1}{2}(u+\beta v_x)_x+\alpha v(u+\beta v_x)+c_1(-i)(u+\beta v_x),\\ &\overline{g}_1=c_1,\quad \overline{g}_2=-\frac{1}{2}(u+\beta v_x)+c_2,\\ &\overline{h}_1=\alpha v-ic_1,\\ &\quad\text{etc.}\end{aligned} \tag{3.3}$$

where $\{c_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ are integration constants.

[1] The constants α and β remain fixed in the following and will not be emphasized in the notation.

Next, define

$$\overline{V}_{n+1}(z,x) = i \begin{pmatrix} -\overline{G}_{n+1}(z,x) & \overline{F}_n(z,x) \\ -\overline{H}_n(z,x) & \overline{G}_{n+1}(z,x) \end{pmatrix} \tag{3.4}$$

where $\overline{F}_n(z,x)$, $\overline{G}_{n+1}(z,x)$, and $\overline{H}_n(z,x)$ are polynomials in $z \in \mathbb{C}$,

$$\begin{aligned} \overline{F}_n(z,x) &= \sum_{\ell=0}^{n} \overline{f}_{n-\ell}(x) z^\ell = -i(u(x) + \beta v_x(x)) \prod_{j=1}^{n} (z - \overline{\mu}_j(x)), \\ \overline{G}_{n+1}(z,x) &= \sum_{\ell=0}^{n+1} \overline{g}_{n+1-\ell}(x) z^\ell, \\ \overline{H}_n(z,x) &= \sum_{\ell=0}^{n} \overline{h}_{n-\ell}(x) z^\ell = -i \prod_{j=1}^{n} (z - \overline{\nu}_j(x)). \end{aligned} \tag{3.5}$$

Using the recursion (3.2) one verifies

$$\begin{aligned} \overline{F}_{n,x} &= -2(iz + \alpha v)\overline{F}_n + 2(u + \beta v_x)\overline{G}_{n+1}, \\ \overline{G}_{n+1,x} &= (u + \beta v_x)\overline{H}_n - \overline{F}_n, \\ \overline{H}_{n,x} &= 2(iz + \alpha v)\overline{H}_n - 2\overline{G}_{n+1}, \end{aligned} \tag{3.6}$$

implying

$$(\overline{G}_{n+1}^2 - \overline{F}_n \overline{H}_n)_x = 0 \tag{3.7}$$

and hence

$$\overline{G}_{n+1}(z,x)^2 - \overline{F}_n(z,x)\overline{H}_n(z,x) = R_{2n+2}(z), \tag{3.8}$$

where $R_{2n+2}(z)$ is a monic polynomial of degree $2n+2$ with zeros $\{E_0, \dots, E_{2n+1}\} \subset \mathbb{C}$. Thus,

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}. \tag{3.9}$$

Again the corresponding hyperelliptic curve $\mathcal{K}_n$ of genus n is naturally obtained from the characteristic equation for $\overline{V}_{n+1}$,

$$\begin{aligned} \det\big(yI - i\overline{V}_{n+1}(z,x)\big) &= y^2 - \overline{G}_{n+1}(z,x)^2 + \overline{F}_n(z,x)\overline{H}_n(z,x) \\ &= y^2 - R_{2n+2}(z) = 0. \end{aligned} \tag{3.10}$$

The corresponding zero-curvature relation then reads, employing (3.4) and (3.5),

$$\begin{aligned} &-\overline{V}_{n+1,x} + [\overline{U}, \overline{V}_{n+1}] \\ &= i \begin{pmatrix} \overline{G}_{n+1,x} + \overline{F}_n - (u+\beta v_x)\overline{H}_n & -\overline{F}_{n,x} - 2(iz+\alpha v)\overline{F}_n + 2(u+\beta v_x)\overline{G}_{n+1} \\ \overline{H}_{n,x} - 2(iz+\alpha v)\overline{H}_n + 2\overline{G}_{n+1} & -\overline{G}_{n+1,x} - \overline{F}_n + (u+\beta v_x)\overline{H}_n \end{pmatrix} = 0. \end{aligned} \tag{3.11}$$

Using the recursion (3.2) to compute $\overline{f}_j$, $\overline{g}_j$, and $\overline{h}_j$ for $j = 0, \dots, n$, (3.11) equals

$$\begin{aligned} &-\overline{V}_{n+1,x} + [\overline{U}, \overline{V}_{n+1}] \\ &= i \begin{pmatrix} \overline{g}_{n+1,x} + \overline{f}_n - (u + \beta v_x)\overline{h}_n & -\overline{f}_{n,x} - 2\alpha v \overline{f}_n + 2(u + \beta v_x)\overline{g}_{n+1} \\ \overline{h}_{n,x} - 2\alpha v \overline{h}_n + 2\overline{g}_{n+1} & -\overline{g}_{n+1,x} - \overline{f}_n + (u + \beta v_x)\overline{h}_n \end{pmatrix} = 0. \end{aligned} \tag{3.12}$$

Next, let

$$\overline{g}_{n+1} = -\frac{1}{2}\overline{h}_{n,x} + \alpha v \overline{h}_n \tag{3.13}$$

(consistent with $\overline{h}_{n+1} = 0$ in (3.2d)). Inserting (3.13) into (3.12), we find that the stationary cBsq hierarchy is given by

$$\begin{pmatrix} \overline{f}_{n,x} + 2\alpha v\overline{f}_n + (u+\beta v_x)(\overline{h}_{n,x} - 2\alpha v\overline{h}_n) \\ -\frac{1}{2}\overline{h}_{n,xx} + \alpha(v\overline{h}_n)_x + \overline{f}_n - (u+\beta v_x)\overline{h}_n \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{3.14}$$

Remark 3.1. Observe that due to (3.13), the nth stationary cBsq system will contain only integration constants $c_1, \dots, c_n$ for $n \in \mathbb{N}$ coming from integrating (3.2c). Since we have $\overline{g}_{n+1,x} + \overline{f}_n - (u+\beta v_x)\overline{h}_n = 0$ from (3.12), our definition (3.13) is consistent with the definition of $\overline{g}_{n+1}$ given by the recursion (3.2c). However, no new integration constant is introduced.

The first few equations (after some simplifications) read

$$\begin{aligned} n=0: &\quad \begin{pmatrix} u_x \\ v_x \end{pmatrix} = 0, \\ n=1: &\quad \begin{pmatrix} (u+\beta v_x)_{xx} + 4\alpha(v(u+\beta v_x))_x - c_1 i(u+\beta v_x)_x \\ u_x + (\beta-\alpha)v_{xx} + 2\alpha^2(v^2)_x - 2ic_1\alpha v_x \end{pmatrix} = 0, \\ &\quad \text{etc.} \end{aligned} \tag{3.15}$$

In the special homogeneous case, the latter set of equations, the stationary classical Boussinesq system, can be rewritten in the more familiar form

$$u + (\beta-\alpha)v_x + 2\alpha^2 v^2 = 0, \quad (2\alpha\beta - \beta^2)v_{xxx} + (\alpha-\beta)u_{xx} + 4\alpha^2(uv)_x = 0. \tag{3.16}$$

Using the first equation in (3.16), the second can also be rewritten as $v_{xxx} - 12\alpha^2(v^2)_x = 0$.

To discuss the time-dependent hierarchy of classical Boussinesq systems we follow the AKNS case and introduce a deformation parameter $t_n \in \mathbb{C}$ in the functions u and v, that is, $u = u(x,t_n)$, $v = v(x,t_n)$. The time-dependent zero-curvature relation then reads

$$\overline{U}_{t_n} - \overline{V}_{n+1,x} + [\overline{U}, \overline{V}_{n+1}] = 0, \tag{3.17}$$

implying

$$\begin{aligned} 0 &= \overline{U}_{t_n} - \overline{V}_{n+1,x} + [\overline{U}, \overline{V}_{n+1}] \\ &= \begin{pmatrix} -\alpha v_{t_n} + i\overline{G}_{n+1,x} + i\overline{F}_n - i(u+\beta v_x)\overline{H}_n & (u+\beta v_x)_{t_n} - i\overline{F}_{n,x} - 2i(iz+\alpha v)\overline{F}_n + 2i(u+\beta v_x)\overline{G}_{n+1} \\ i\overline{H}_{n,x} - 2i(iz+\alpha v)\overline{H}_n + 2i\overline{G}_{n+1} & \alpha v_{t_n} - i\overline{G}_{n+1,x} - i\overline{F}_n + i(u+\beta v_x)\overline{H}_n \end{pmatrix}. \end{aligned} \tag{3.18}$$

Using now the recursion (3.2) to compute $\overline{f}_j$, $\overline{g}_j$, and $\overline{h}_j$ for $j = 0, \dots, n$, (3.18) reduces to

$$\begin{aligned} &\overline{U}_{t_n} - \overline{V}_{n+1,x} + [\overline{U}, \overline{V}_{n+1}] \\ &= \begin{pmatrix} -\alpha v_{t_n} + i\overline{g}_{n+1,x} + i\overline{f}_n - i(u+\beta v_x)\overline{h}_n & (u+\beta v_x)_{t_n} - i\overline{f}_{n,x} - 2i\alpha v\overline{f}_n + 2i(u+\beta v_x)\overline{g}_{n+1} \\ i\overline{h}_{n,x} - 2i\alpha v\overline{h}_n + 2i\overline{g}_{n+1} & \alpha v_{t_n} - i\overline{g}_{n+1,x} - i\overline{f}_n + i(u+\beta v_x)\overline{h}_n \end{pmatrix} = 0, \end{aligned} \tag{3.19}$$

or equivalently,

$$\alpha v_{t_n} - i\overline{g}_{n+1,x} - i\overline{f}_n + i(u+\beta v_x)\overline{h}_n = 0, \tag{3.20a}$$

$$(u+\beta v_x)_{t_n} - i\overline{f}_{n,x} - 2i\alpha v\overline{f}_n + 2i(u+\beta v_x)\overline{g}_{n+1} = 0, \tag{3.20b}$$

$$\overline{h}_{n,x} - 2\alpha v\overline{h}_n + 2\overline{g}_{n+1} = 0. \tag{3.20c}$$

Using $\overline{h}_{n,x} - 2\alpha v\overline{h}_n + 2\overline{g}_{n+1} = 0$ in order to eliminate $\overline{g}_{n+1}$ in (3.20a) and (3.20b), then yields the following expressions for the time-dependent classical Boussinesq hierarchy,

$$\begin{aligned}
&\alpha u_{t_n} - \frac{i}{2}\beta\overline{h}_{n,xxx} + i\alpha\beta v\overline{h}_{n,xx} - i\big((\beta+\alpha)u + \beta(\beta-\alpha)v_x\big)\overline{h}_{n,x} \\
&\quad - i\big(\beta(u+(\beta-\alpha)v_x)_x - 2\alpha^2(u+\beta v_x)v\big)\overline{h}_n + i(\beta-\alpha)\overline{f}_{n,x} - 2i\alpha^2 v\overline{f}_n = 0, \qquad (3.21)\\
&\alpha v_{t_n} + \frac{i}{2}\overline{h}_{n,xx} - i\alpha v\overline{h}_{n,x} + i(u+(\beta-\alpha)v_x)\overline{h}_n - i\overline{f}_n = 0, \quad n \in \mathbb{N}_0 .
\end{aligned}$$

For brevity, equations (3.21) will be denoted by

$$\mathrm{cBsq}_n(u,v) = 0, \quad n \in \mathbb{N}_0 . \tag{3.22}$$

Remark 3.2. Observe that $\overline{g}_{n+1}$ defined by (3.20c) does not satisfy (3.2c), but rather (3.20a). This is in contrast to the stationary case as well as the corresponding definitions for the AKNS hierarchy. As in the stationary case, the nth cBsq system contains n integration constants $c_1, \dots, c_n$ when $n \in \mathbb{N}$.

Explicitly, the first few equations read

$$\begin{aligned}
&\mathrm{cBsq}_0(u,v) = \begin{pmatrix} \alpha u_{t_0} - \alpha u_x \\ \alpha v_{t_0} - \alpha v_x \end{pmatrix} = 0, \qquad (3.23)\\
&\mathrm{cBsq}_1(u,v) = \begin{pmatrix} \alpha u_{t_1} - \frac{i}{2}\alpha\beta v_{xxx} + \frac{i}{2}(\beta-\alpha)(u+\beta v_x)_{xx} - 2i\alpha^2(uv)_x + c_1(-\alpha u_x) \\ \alpha v_{t_1} - \frac{i}{2}(u+\beta v_x)_x - 2i\alpha^2 vv_x + c_1(-\alpha v_x) \end{pmatrix} = 0, \\
&\text{etc.}
\end{aligned}$$

In the homogeneous case $\mathrm{cBsq}_1(u,v) = 0$ can be rewritten as

$$\begin{aligned}
\alpha u_{t_1} &= \frac{i}{2}\beta(2\alpha-\beta)v_{xxx} + \frac{i}{2}(\alpha-\beta)u_{xx} + 2i\alpha^2(uv)_x, \\
\alpha v_{t_1} &= \frac{1}{2}(\beta-\alpha)v_{xx} + 2i\alpha^2 vv_x + \frac{i}{2}u_x.
\end{aligned} \tag{3.24}$$

Finally, specializing to $\alpha = \beta$ one obtains the classical Boussinesq system

$$u_{t_1} = \frac{i}{2}\alpha v_{xxx} + 2i\alpha(uv)_x, \quad \alpha v_{t_1} = 2i\alpha^2 vv_x + \frac{i}{2}u_x. \tag{3.25}$$

In the next section we provide a new proof of the fact that the AKNS and the cBsq hierarchies are gauge equivalent by exibiting an explicit gauge transformation between them. In the last section this will be used to derive algebro-geometric solutions of the cBsq hierarchy.

4. The gauge equivalence of the cBsq and AKNS hierarchies

We start by briefly recalling the effect of gauge transformations on zero-curvature equations. Starting with the time-dependent equations

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.1}$$

whose compatibility relation $\Psi_{xt} = \Psi_{tx}$ yields the zero-curvature equation

$$U_t - V_x + [U,V] = 0, \tag{4.2}$$

we introduce the gauge transformation

$$\overline{\Psi} = S\Psi, \quad S \text{ invertible}. \tag{4.3}$$

Then one derives,

$$\overline{\Psi}_x = \overline{U}\,\overline{\Psi}, \quad \overline{\Psi}_t = \overline{V}\,\overline{\Psi}, \tag{4.4}$$

with

$$\overline{U} = S_x S^{-1} + SUS^{-1}, \quad \overline{V} = S_t S^{-1} + SVS^{-1} \tag{4.5}$$

and hence

$$\overline{U}_t - \overline{V}_x + [\overline{U}, \overline{V}] = 0. \tag{4.6}$$

The corresponding stationary formalism starts from

$$\Psi_x = U\Psi, \quad iy\Psi = V\Psi, \quad y \in \mathbb{C} \tag{4.7}$$

and

$$-V_x + [U, V] = 0. \tag{4.8}$$

The gauge transformation (4.3) then effects

$$\overline{\Psi}_x = \overline{U}\,\overline{\Psi}, \quad iy\overline{\Psi} = \overline{V}\,\overline{\Psi}, \tag{4.9}$$

with

$$\overline{U} = S_x S^{-1} + SUS^{-1}, \quad \overline{V} = SVS^{-1} \tag{4.10}$$

and hence

$$-\overline{V}_x + [\overline{U}, \overline{V}] = 0. \tag{4.11}$$

Introducing the particular choice

$$S = \begin{pmatrix} (-p)^{1/2} & 0 \\ 0 & 1/(-p)^{1/2} \end{pmatrix}, \quad p \text{ meromorphic on } \mathbb{C} \tag{4.12}$$

and applying it to (4.7)–(4.11) in the case of the stationary AKNS hierarchy, identifying (U, V) and (U, V_{n+1}), then yields the following result.

Theorem 4.1. *The stationary AKNS and cBsq hierarchies are gauge equivalent in the sense that*

$$\overline{U} = S_x S^{-1} + SUS^{-1}, \quad \overline{V}_{n+1} = SV_{n+1}S^{-1}, \tag{4.13}$$

with (U, V_{n+1}) *and* $(\overline{U}, \overline{V}_{n+1})$ *given by* (2.1), (2.4) *and* (3.1), (3.4), *respectively, and* S *defined by* (4.12) (*with* $p = p(x)$). *In particular, the pair* (u, v) *given by*

$$u(x) = -p(x)q(x) + \frac{\beta}{2\alpha}\left(\frac{p_x(x)}{p(x)}\right)_x, \quad v(x) = -\frac{1}{2\alpha}\frac{p_x(x)}{p(x)}, \tag{4.14}$$

satisfies the n*th stationary cBsq system if and only if the pair* (p, q), *given by*

$$\begin{aligned} p(x) &= \exp\left(-2\alpha \int^x dx'\, v(x')\right), \\ q(x) &= -(u(x) + \beta v_x(x)) \exp\left(2\alpha \int^x dx'\, v(x')\right), \end{aligned} \tag{4.15}$$

satisfies the n*th stationary AKNS system with identical sets of integration constants* $c_j \in \mathbb{C}$, $j = 1, \dots, n$ *for* $n \in \mathbb{N}$.

Proof. $\overline{U} = S_x S^{-1} + SUS^{-1}$ is easily seen to be equivalent to (4.14). Similarly, $\overline{V}_{n+1} = SV_{n+1}S^{-1}$ is equivalent to

$$\overline{F}_n = -pF_n, \quad \overline{G}_{n+1} = G_{n+1}, \quad \overline{H}_n = -\frac{1}{p}H_n. \tag{4.16}$$

Next suppose that (p,q) solves the nth stationary AKNS system, that is, equations (2.6) hold. Define $\overline{F}_n$, $\overline{G}_{n+1}$, and $\overline{H}_n$ by (4.16) and (u,v) by (4.14). Then clearly (3.6) is satisfied, proving that (u,v) satisfies the nth stationary cBsq system.

Conversely, starting with (u,v) solving the nth stationary cBsq system (3.6), we can define (p,q) and F_n, G_{n+1}, and H_n using (4.15) and (4.16), respectively. One then easily verifies that (2.6) holds, and thus (p,q) solves the nth stationary AKNS system. □

We note that the ambiguity inherent to (4.15), due to an arbitrary integration constant, corresponds to the scale invariance of the AKNS hierarchy as discussed in (2.21)–(2.23).

The time-dependent analog of Theorem 4.1 then reads as follows.

Theorem 4.2. *The time-dependent AKNS and cBsq hierarchies are gauge equivalent in the sense that*

$$\overline{U} = S_x S^{-1} + SUS^{-1}, \quad \overline{V}_{n+1} = S_{t_n} S^{-1} + SV_{n+1}S^{-1}, \tag{4.17}$$

with (U, V_{n+1}) *and* $(\overline{U}, \overline{V}_{n+1})$ *given by* (2.1), (2.4) *and* (3.1) *and* (3.4), *respectively, and S defined by* (4.12) *(with $p = p(x,t_n)$). In particular, the pair (u,v) given by*

$$u(x,t_n) = -p(x,t_n)q(x,t_n) + \frac{\beta}{2\alpha}\left(\frac{p_x(x,t_n)}{p(x,t_n)}\right)_x, \quad v(x,t_n) = -\frac{1}{2\alpha}\frac{p_x(x,t_n)}{p(x,t_n)}, \tag{4.18}$$

satisfies the nth cBsq system $\mathrm{cBsq}_n(u,v) = 0$ *if and only if the pair (p,q) given by*

$$\begin{aligned} p(x,t_n) &= \exp\left(-2\alpha \int^x dx'\, v(x',t_n)\right), \\ q(x,t_n) &= -(u(x,t_n) + \beta v_x(x,t_n)) \exp\left(2\alpha \int^x dx'\, v(x',t_n)\right), \end{aligned} \tag{4.19}$$

satisfies the nth AKNS system $\mathrm{AKNS}_n(p,q) = 0$ *with identical sets of integration constants $c_j \in \mathbb{C}$, $j = 1, \dots, n$ for $n \in \mathbb{N}$.*

Proof. $\overline{U} = S_x S^{-1} + SUS^{-1}$ is equivalent to (4.18) as noted in the proof of Theorem 4.1. By a direct calculation, $\overline{V}_{n+1} = S_{t_n} S^{-1} + SV_{n+1}S^{-1}$ is equivalent to

$$\overline{F}_n = -pF_n, \quad \overline{G}_{n+1} = G_{n+1} + \frac{i}{2}\frac{p_{t_n}}{p}, \quad \overline{H}_n = -\frac{1}{p}H_n. \tag{4.20}$$

Next assume that (p,q) solves the nth AKNS system, that is, equations (2.17) hold. Define $\overline{F}_n$, $\overline{G}_{n+1}$, and $\overline{H}_n$ by (4.20) and (u,v) by (4.18). Then clearly (3.11) is satisfied, proving that (u,v) satisfies the nth cBsq system.

Conversely, starting with (u,v) solving the nth cBsq system (3.11), we can define (p,q) and F_n, G_{n+1}, and H_n using (4.19) and (4.20), respectively. Again one verifies that (2.17) holds, and thus (p,q) solves the nth AKNS system. □

The equivalence of the cBsq and AKNS hierarchies, on the basis of the transformation (4.18) has first been noted by Jaulent and Miodek [18] and later by

Matveev and Yavor [27]. It has been further discussed and linked to Hirota's bilinear formalism by Sachs [30]. Our method of proof of Theorems 4.1 and 4.2, based on the polynomial recursion formalism developed in Section 3, to the best of our knowledge, is new.

5. Algebro-geometric solutions of the classical Boussinesq hierarchy

Finally we derive the theta function representation of algebro-geometric cBsq solutions utilizing the gauge equivalence of the cBsq and AKNS hierarchies.

Let $(u^{(0)}, v^{(0)})$ be a stationary solution of the nth classical Boussinesq system, that is,

$$\begin{aligned} &\overline{f}_{n,x} + 2\alpha v\overline{f}_n + (u+\beta v_x)(\overline{h}_{n,x} - 2\alpha v\overline{h}_n) = 0, \\ &\overline{f}_n - (u+\beta v_x)\overline{h}_n - \frac{1}{2}\overline{h}_{n,xx} + \alpha(v\overline{h}_n)_x = 0, \end{aligned} \tag{5.1}$$

for a given set of integration constants $\{c_j\}_{j=1,\dots,n} \subset \mathbb{C}$. Fix $r \in \mathbb{N}_0$ and corresponding integration constants $\{\tilde{c}_j\}_{j=1,\dots,r} \subset \mathbb{C}$. The aim in this section is to construct a solution (u,v) of

$$\mathrm{cBsq}_r(u,v) = 0, \quad (u,v)|_{t_r=t_{0,r}} = (u^{(0)}, v^{(0)}). \tag{5.2}$$

The function $\overline{\phi}$ and the Baker–Akhiezer function $\overline{\Psi}$ associated with the classical Boussinesq hierarchy can be obtained as follows.

Theorem 5.1. *Consider $P=(z,y) \in \mathcal{K}_n \backslash \{P_{\infty_\pm}\}$ and $(z,x,x_0,t_r,t_{0,r}) \in \mathbb{C}^5$. Let ϕ, Ψ, and S be given by* (2.26), (2.32), *and* (4.12), *respectively. Define*

$$\overline{\Psi} = \begin{pmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \end{pmatrix} = S\Psi, \quad \overline{\phi} = -\frac{\phi}{p}. \tag{5.3}$$

Then $\overline{\phi}(P,x,t_r)$ satisfies $\overline{\phi} = \overline{\psi}_2/\overline{\psi}_1$ and

$$\begin{aligned} &v(x,t_r)\overline{\phi}(P,x,t_r) - \frac{1}{2\alpha}\overline{\phi}_x(P,x,t_r) \\ &\quad + \frac{1}{2\alpha}(u(x,t_r) + \beta v_x(x,t_r))\overline{\phi}(P,x,t_r)^2 - \frac{iz}{\alpha}\overline{\phi}(P,x,t_r) - \frac{1}{2\alpha} = 0. \end{aligned} \tag{5.4}$$

$\overline{\Psi}(\,\cdot\,,x,x_0,t_0,t_{0,r})$ is meromorphic on $\mathcal{K}_n\backslash\{P_{\infty_\pm}\}$ and satisfies

$$\overline{\Psi}_x(P,x,x_0,t_r,t_{0,r}) = \overline{U}(z,x,t_r)\overline{\Psi}(P,x,x_0,t_r,t_{0,r}), \tag{5.5}$$

$$iy(P)\overline{\Psi}(P,x,x_0,t_r,t_{0,r}) = \overline{V}_{n+1}(z,x,t_r)\overline{\Psi}(P,x,x_0,t_r,t_{0,r}), \tag{5.6}$$

$$\overline{\Psi}_{t_r}(P,x,x_0,t_r,t_{0,r}) = \widetilde{\overline{V}}_{r+1}(z,x,t_r)\overline{\Psi}(P,x,x_0,t_r,t_{0,r}). \tag{5.7}$$

Proof. Immediate from Theorem 2.1 and (4.1)–(4.6). ☐

The explict representation of algebro-geometric solutions of the classical Boussinesq hierarchy in terms of the Riemann theta function associated with $\mathcal{K}_n$ then reads as follows (we use the notation employed in Theorem 2.2).

Theorem 5.2. *Let $P \in \mathcal{K}_n\backslash\{P_{\infty_\pm}\}$, $(x,x_0,t_r,t_{0,r}) \in \mathbb{C}^4$, assume $\mathcal{K}_n$ to be nonsingular, and suppose $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ to be nonspecial, that is, their index of speciality vanishes, $i(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = i(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = 0$. Moreover, suppose that (u,v) satisfies $\mathrm{cBsq}_r(u,v)=0$ with $(u,v)|_{t_r=t_{0,r}} = (u^{(0)}, v^{(0)})$, a solution of the nth* *stationary classical Boussinesq system. Then the theta function representation of* (u, v) *is given by*

$$\begin{aligned} u(x,t_r) &= e_{0,1} + \frac{\partial^2}{\partial x^2} \ln(\theta(\underline{z}_+(\underline{\hat{\mu}}(x,t_r)))) \\ &\quad + \frac{\beta}{2\alpha}\frac{\partial^2}{\partial x^2} \ln\left(\frac{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x,t_r)))} \right), \\ v(x,t_r) &= \frac{i}{\alpha} e_{0,0} - \frac{1}{2\alpha}\frac{\partial}{\partial x} \ln\left(\frac{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x,t_r)))} \right). \end{aligned} \tag{5.8}$$

Proof. Combine Theorem 2.2 and (4.18). ☐

Obviously one can derive formulas similar to (2.63)–(2.65) for the functions $\overline{\phi}$ and $\overline{\Psi}$ using the explicit relation (5.3). We leave the corresponding details to the reader.

Algebro-geometric solutions of the time-dependent classical Boussinesq system $\mathrm{cBsq}_1(u,v) = 0$ and their theta function representations were originally derived by Matveev and Yavor [27]. The case of real-valued solutions and additional reductions to elliptic solutions in the case of genus $n \leq 3$ were subsequently studied by Smirnov [31]. Theta function representations of algebro-geometric solutions of $\mathrm{cBsq}_r(u,v) = 0$ in the case $r \leq 3$ appeared in a recent preprint by Geng and Wu [8].

References

[1] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[2] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl, *Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchies*, Mem. Amer. Math. Soc. **135**, No. 641 (1998).

[3] P. A. Clarkson and D. K. Ludlow, *Symmetry reductions, exact solutions, and Painlevé analysis for a generalised Boussinesq equation*, J. Math. Anal. Appl. **186**, 132–155 (1994).

[4] R. Conte, M. Musette, and A. Pickering, *Factorization of the 'classical Boussinesq' system*, J. Phys. A **27**, 2831–2836 (1994).

[5] R. Conte, M. Musette, and A. Pickering, *The two-singular manifold method: II. Classical Boussinesq system*, J. Phys. A **28**, 179–187 (1995).

[6] B. A. Dubrovin, *Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties*, Funct. Anal. Appl. **11**, 265–277 (1977).

[7] B. A. Dubrovin, *Matrix finite-zone operators*, Revs. Sci. Tech. **23**, 20–50 (1983).

[8] X. Geng and Y. Wu, *The finite-band solution of the classical Boussinesq–Burgers equation*, preprint, 1998.

[9] F. Gesztesy and H. Holden, *Dubrovin equations and integrable systems on hyperelliptic curves*, preprint, 1998.

[10] F. Gesztesy and H. Holden, *Hierarchies of Soliton Equations and Their Algebro-Geometric Solutions*, monograph in preparation.

[11] F. Gesztesy and R. Ratnaseelan, *An alternative approach to algebro-geometric solutions of the AKNS hierarchy*, Rev. Math. Phys. **10**, 345–391 (1998).

[12] F. Gesztesy, R. Ratnaseelan, and G. Teschl, *The KdV hierarchy and associated trace formulas*, in *Recent Developments in Operator Theory and its Applications*, I. Gohberg, P. Lancaster, and P.N.Shivakumar (eds.), Operator Theory: Advances and Applications, Vol. 87, Birkhäuser, Basel, 1996, pp. 125–163.

[13] R. Hirota, *Classical Boussinesq equation is a reduction of the modified KP equation*, J. Phys. Soc. Japan **54**, 2409–2415 (1985).

[14] R. Hirota, *Solutions of the classical Boussinesq equation and the spherical Boussinesq equation: The Wronskian technique*, J. Phys. Soc. Japan **55**, 2137–2150 (1986).

[15] M. Ito, *Symmetries and conservation laws of the classical Boussinesq equation*, Phys. Lett. **104A**, 248–250 (1984).

[16] A. R. Its, *Inversion of hyperelliptic integrals and integration of nonlinear differential equations*, Vestnik Leningrad Univ. Math. **9**, 121–129 (1981).

[17] A. R. Its, *On connections between solitons and finite-gap solutions of the nonlinear Schrödinger equation*, Sel. Math. Sov. **5**, 29–43 (1986).

[18] M. Jaulent and I. Miodek, *Connection between Zakharov-Shabat and Schrödinger-type inverse-scattering transforms*, Lett. Nuovo Cim. **20**, 655–660 (1977).

[19] D. J. Kaup, *Finding eigenvalue problems for solving nonlinear evolution equations*, Progr. Theoret. Phys. **54**, 72–78 (1975).

[20] D. J. Kaup, *A higher-order water-wave equation and the method for solving it*, Progr. Theoret. Phys. **54**, 396–408 (1975).

[21] S. Kawamoto, *Exact explode-decay solitary wave solutions of the classical Boussinesq equation*, J. Phys. Soc. Japan **53**, 469–471 (1984).

[22] I. M. Krichever, *Integration of nonlinear equations by the methods of algebraic geometry*, Funct. Anal. Appl. **11**, 12–26 (1977).

[23] E. V. Krishnan, An exact solution of the classical Boussinesq equation, J. Phys. Soc. Japan **51**, 2391–2392 (1982).

[24] P. D. Lax, *Outline of a theory of the KdV equation*, in *Recent Mathematical Methods in Nonlinear Wave Propagation*, G. Boillat, C. M. Dafermos, P. D. Lax, and T. P. Liu (eds.), Lecture Notes in Mathematics, Vol. 1640, Springer, Berlin, 1996, pp. 70–102.

[25] Q.-M. Liu, X.-B. Hu, and Y. Li, *Rational solutions of the classical Boussinesq hierarchy*, J. Phys. A **23**, 585–591 (1990).

[26] Y.–C. Ma and M. J. Ablowitz, *The periodic cubic Schrödinger equation*, Stud. Appl. Math. **65**, 113–158 (1981).

[27] V. B. Matveev and M. I. Yavor, *Solutions presque périodiques et a N-solitons de l'équation hydrodynamique non linéaire de Kaup*, Ann. Inst. H. Poincaré, Phys. Théor. A **31**, 25–41 (1979).

[28] J. Mertsching, *Quasi periodic solutions of the nonlinear Schrödinger equation*, Fortschr. Phys. **85**, 519–536 (1987).

[29] E. Previato, *Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation*, Duke Math. J. **52**, 329–377 (1985).

[30] R. L. Sachs, *On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy*, Physica D **30**, 1–27 (1988).

[31] A. O. Smirnov, *Real finite-gap regular solutions of the Kaup-Boussinesq equation*, Theoret. Math. Phys. **66**, 19–31 (1986).

[32] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.

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