# Adiabatic Evolution for Systems with Infinitely many Eigenvalue Crossings

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**Abstract**

We formulate an adiabatic theorem adapted to models that present an instantaneous eigenvalue experiencing an infinite number of crossings with the rest of the spectrum. We give an upper bound on the leading correction terms with respect to the adiabatic limit. The result requires only differentiability of the considered spectral projector, and some geometric hypothesis on the local behaviour of the eigenvalues at the crossings.

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# 1 Introduction

The availability of intense pulsed laser sources has opened a large field of possibilities to control atomic and molecular dynamical processes. One of the main theoretical tools to analyze these processes is adiabatic Floquet theory [GMDJ] and references therein. The general setup can be described as follows. One considers a molecule described by a Hamiltonian $H_0$ acting on a Hilbert space $\mathcal{H}$, in interaction with one radiation mode of frequency $\omega$. (The description of the interaction with several modes of different frequencies can be formulated along similar lines). Since the intensity of the field is quite large, the field is treated as a classical field. The Hamiltonian of the molecule perturbed by the electromagnetic field can be written for example as

$$H = H_0 + E\,M\,F(\omega\,t + \theta_0) \tag{1}$$

where $M$ is the dipole moment of the molecule and $E \in \mathbb{R}$ is the amplitude of the radiation field, $F$ is a $2\pi$-periodic function and $\theta_0$ the initial phase. We assume that $H_0$ has a discrete spectrum. In order to describe a laser pulse the amplitude is taken as a slowly varying time dependent function $E(\varepsilon\,t)$, where one takes e.g. $\varepsilon = 1/T_p$ with $T_p$ the duration of the pulse. A new technique that provides an efficient method for complete transfer of population is based on frequency chirping: within the pulse duration the frequency is also slowly modulated $\omega = \omega(\varepsilon\,t)$.

This model has thus two kinds of time-dependencies in the Hamiltonian: one that is periodic and another one that is slowly varying. The periodic part can be treated by Floquet methods, and the slowly varying part by adiabatic theory. Adiabatic Floquet theory is based on the following statement: Assume that in the Hamiltonian (1) the parameter $E$ and the frequency $\omega$ are made time dependent, $E(t)$, $\omega(t)$. Consider the propagator $U(t, t_0; \theta_0)$, solution of the Schrödinger equation

$$i\,\frac{\partial}{\partial t} U(t, t_0; \theta_0) = H(\omega\,t + \theta_0)\;U(t, t_0; \theta_0), \qquad U(t, t; \theta_0) = \mathbb{1} \tag{2}$$

acting on the Hilbert space $\mathcal{H}$. We consider an enlarged Hilbert space by tensoring $\mathcal{H}$ with the space of square integrable functions on the unit circle: $L^2(\mathbb{S}^1, \mathcal{H})$. The operator $U(t, t_0; \theta)$ can be lifted into the enlarged space, interpreting the $\theta$-dependence as a multiplication operator. We can then define

$$U_K(t, t_0) = e^{-t\omega(t)\partial}\,U(t, t_0; \theta)\,e^{t_0\omega(t_0)\partial}$$

where $\partial = \partial/\partial\theta$. The statement is that equation (2) is equivalent to

$$i\,\frac{\partial}{\partial t} U_K(t, t_0) = K(t)\,U_K(t, t_0) \tag{3}$$

with

$$K(t) = -i\,\varpi(t)\,\frac{\partial}{\partial\theta} + H_0 + E(t)\,M\,F(\theta)$$

and $\varpi(t)$ denotes an effective instantaneous frequency defined by $\varpi(t) = \omega(t) + t\,d\omega(t)/dt$. Assuming that the time dependence of $E(t)$, $\omega(t)$ is slow one can develop adiabatic techniques for the evolution of (3). When $K$ has pure point spectrum, the first ingredient are the instantaneous eigenvalues and eigenvectors. They always can be written and labelled in the form

$$\begin{aligned} \lambda_{j,k} &= \lambda_{j,0} + \varpi\, k, \qquad k \in \mathbb{Z} \\ \psi_{j,k}(\theta) &= \psi_{j,0}(\theta)\, e^{i\,k\,\theta} \end{aligned} \tag{4}$$

The index $j$ has the same cardinality as the dimension of the Hilbert space $\mathcal{H}$. Thus, even if we take simple models with finite dimensional $\mathcal{H}$, the Floquet spectrum has infinitely many eigenvalues. As functions of $E$ and $\varpi$, these eigenvalues may exhibit crossings, which the adiabatic approximation can accomodate in case there is a finite number of them, see [BF], [Hag]. The structure (4) of the eigenvalues is such that if we consider a slowly varying frequency $\varpi(t)$ that goes through 0, one can encounter situations in which a branch of instantaneous eigenvalues undergoes an infinite number of crossings with other branches. Let us stress here that this situation is not generic, as actual crossings of eigenvalues are more the exception than the rule. However, we give below a whole class of systems for which this situation is true. Note also that in case $\varpi(t)$ passes through 0, the domain of $K(t)$ becomes time dependent, so that technical issues regarding regularity of the evolution operator have to be addressed. This is done in the Appendix.

The goal of the present paper is to formulate an adiabatic theorem that can be applied to such situations with an estimate on the corrections to the adiabatic limit. Adiabatic Theorems without gap conditions are known to be true, see [AE], however, in general, no estimates on the error terms are available.

While this work was motivated by the physical situation described above and discussed below in the examples, our analysis of the adiabatic approximation is model independent and can be applied to more general situations.

# 2 Adiabatic Theorem

## 2.1 Context

The adiabatic approximation in Quantum Mechanics has a long history which we will not attempt to retrace here. We refer the reader to the recent surveys [AE2], [JP] and references therein. Let us simply recall here that the works following that of Born and Fock [BF] by Kato [Ka1], Nenciu [Nen] and Avron, Seiler, Yaffe [ASY] have lead to a formulation of the Adiabatic Theorem under the usual gap assuption that is general and where the error term is well controlled and of order $\varepsilon$. In case the gap assumption is modified, the situation is less explicit. In this section, we switch back to the notation $H(\varepsilon\, t)$ for the slowly varying time-dependent Hamiltonian. Assume $H(s)$ is smooth in $s \in [0,1]$ and there exists a spectral projector $P(s)$ of $H(s)$ which is strongly $C^2$ on $[0,1]$. Avron and Elgart have shown in [AE] that the adiabatic theorem holds under these conditions, provided $P(s)$ is of finite rank, independently of any spectral considerations. The limitation of this approach is that, in general, no estimate can be made on the rate at which the adiabatic regime is attained. In certain specific situations, an estimate on this rate is available. In the case where the spectral measure $\mu_\varphi$ is $\alpha$-Hölder continuous, with $\varphi = P'(s)\psi$, $\psi$ being the initial condition, the rate of convergence was shown in [AE] to be of order $\varepsilon^{\alpha/(2+\alpha)}$. A case where the spectrum of $H(s)$ is assumed to be dense pure point is dealt with in [AHS]. Another situation, considered in [Joy], where the gap hypothesis is not necessarily fulfilled occurs when $H(s) = H_0(s) + \varepsilon H_1(s)$, where the domain of $H_1(s)$ is smaller than that of $H_0(s)$. In both cases, the error term remains of order $\varepsilon$. In the present article, we consider another situation in which the usual gap assumption is modified and the error made in the adiabatic approximation can be estimated. We make the hypothesis that the spectral projector $P(s)$ is associated with an eigenvalue $\lambda(s)$, in the sense that $H(s)P(s) = \lambda(s)P(s)$, for all $s \in [0, 1]$. We assume that $\lambda(s)$ is isolated in the spectrum except at a series of times $\{o_k\}_{k\in\mathbb{N}}$ accumulating at $a \in ]0, 1[$ where it experiences crossings with the rest of the spectrum. Requiring some conditions on the local behaviour of the gap between $\lambda(s)$ and the rest of the spectrum near the crossing points $o_k$, we estimate the error term in the theorem without *a priori* knowledge on the nature of the rest of the spectrum. Note that for $s = o_k$ such that $\lambda(o_k)$ is not isolated in the spectrum, $P(o_k)$ does not represent the entire spectral projector associated with the eigenvalue $\lambda(o_k)$.

## 2.2 One crossing

Let us make more precise the regularity hypotheses under which we shall work. In order to deal with the application described above, we will assume the hamiltonian is unbounded. This causes technical difficulties motivating the part ii) of the hypothesis below which justifies our manipulations. We show in the appendix that this assumption is verified for our models. In case $H(s)$ is bounded, this part of the assumption is automatically verified.

$H0$) i) We assume that for all $s \in [0, 1]\backslash\{a\}$, $H(s)$ is a strongly $C^1$ self-adjoint operator defined on a dense domain $\mathcal{D}$ independent of $s$ in a separable Hilbert space $\mathcal{K}$, where $0 < a < 1$. Whereas $H(a)$ is bounded self-adjoint on $\mathcal{K}$. We also assume the existence of a spectral projector $P(s)$ of $H(s)$ which is strongly $C^2$ on $[0, 1]$ and such that $H(s)P(s) = P(s)H(s) = \lambda(s)P(s)$, for all $s \in [0, 1]$.

ii) Further assume that the unitary evolution operators $U(s) = U(s, 0)$ and $A(s) = A(s, 0)$ generated by $H(s)$, respectively $H(s) + \varepsilon i[P'(s), P(s)]$ (see (5), (6)) are well defined for all $s \in [0, 1]$ and possess the properties i) to v) listed in Theorem A.1.

Note that $P(s)$ needs not be finite dimensional and $\lambda$ is continuous.

We start by considering one crossing of $\lambda$ with the rest of the spectrum by revisiting the strategy proposed in [BF], making use of the general analysis presented in [ASY].

Let $g(s)$ be the gap between $\lambda(s)$ and the rest of the spectrum of $H(s)$: $g(s) = \text{dist}\,(\lambda(s), \sigma(s)\backslash\{\lambda(s)\}) \geq 0$, $s \in [0, 1]$. We also introduce the bounded, strongly $C^1$ operator $L(s) = i\,[P'(s), P(s)]$. We assume that $g^{-1}\{0\} = \{o\}$ and consider the strong differential equations on $\mathcal{D}$

$$i\varepsilon\, U'(s) = H(s)\, U(s)\,, \qquad\qquad U(0) = \mathbb{1} \tag{5}$$

$$i\varepsilon\, A'(s) = (H(s) + \varepsilon L(s))\, A(s)\,, \qquad A(0) = \mathbb{1}. \tag{6}$$

The unitary $A$ is the so called *adiabatic evolution* which possesses the well known intertwining relation $A(s)P(0) = P(s)A(s)$ [Ka1], [Kr]. Finally, let $W(s)$ be defined by $W(s) = A^{-1}(s)U(s)$. We have on $\mathcal{D}$

$$iW'(s) = -A^{-1}(s)L(s)A(s)\, W(s), \qquad W(0) = \mathbb{1}, \tag{7}$$

in the strong sense. To compare the adiabatic and actual evolutions, we need to compute the size of the difference of the unitary $W(s)$ at two times surrounding the crossing. This is the aim of the next result.

**Lemma 2.1** *Under the above assumptions, we have for any* $0 \le u_0 \le t < o < s \le u_1 \le 1$,

$$\|W(u_0) - W(u_1)\| \le C\, \left(\varepsilon|u_0 - t|/g_t^2 + \varepsilon|u_1 - s|/g_s^2 + \varepsilon/g_t + \varepsilon/g_s + |s-t|\right) \tag{8}$$

*where* $g_t = \inf_{u\in[u_0,t]} g(u)$, $g_s = \inf_{u\in[s,u_1]} g(u)$ *and the constant* $C$ *is uniform in* $u_0$, $u_1$, $s$ *and* $t$.

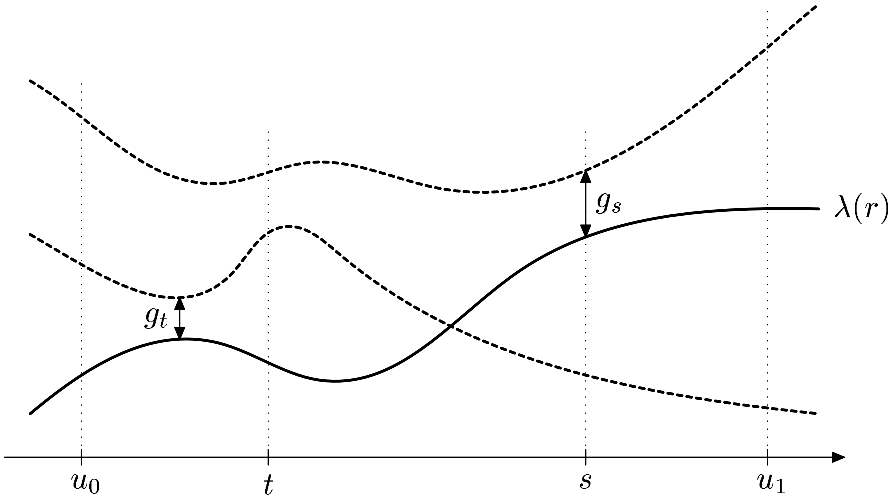


Figure 1: The various quantities defined in Lemma 2.1

**Remark:**
On the basis of the classical paper by Born and Fock, [BF], and the detailed analysis of crossings by Hagedorn [Hag], one would expect the corresponding estimate without the first two terms. However, such an estimate requires more detailed knowledge of the structure of spectrum, e.g. that the gap is given by the distance between two eigenvalues, than what we assume in our general setting.

Lemma 2.1 can be used to treat two standard situations:

*1)* If there is a gap $G$ between $\lambda(s)$ and the rest of the spectrum, this lemma implies that the adiabatic approximation holds with and error term bounded by $C\,\varepsilon/G^2$.

*2)* If one starts the evolution on a crossing point which splits like $s^\alpha$ near 0, we can use this lemma to show that the adiabatic approximation is valid with an error bounded by

$$\|U(1) - A(1)\| \leq C\, \varepsilon^{1/(1+2\alpha)}$$

if $\varepsilon$ is small enough. This is precisely the situation encountered at the beginning of the interaction of a laser pulse with frequency that is in resonance with the difference between two energy levels of the molecule [Hol, GJ].

To get this estimate, we can consider only half of the problem by letting aside all the terms containing a $t$ and setting $u_1 = 1$:

$$\|W(1) - W(0)\| \leq C\, \left(\varepsilon|1-s|/g_s^2 + \varepsilon/g_s + s\right) \tag{9}$$

This is indeed fully justified by the proof of the lemma (see below). Next, we have by hypothesis that $g(s) \geq g_s = G\, s^\alpha$ if $s$ is small. Introducing this behaviour in Equation (9), we obtain $\|W(1) - W(0)\| \leq C\, (\varepsilon/s^{2\alpha} + s)$. The result follows now by balancing the two contributions by choosing $s = s(\varepsilon) = \varepsilon^{1/(1+2\alpha)}$. Again, with more information on the spectrum, as in [BF, Hag], one should be able to improve the above estimate to order $\varepsilon^{1/(1+\alpha)}$.

**Proof of Lemma 2.1:**
The idea of the proof is to integrate Equation (7) over the interval $[u_0, u_1]$ and then to get "nice" estimates of the sizes on each subintervals $[u_0, t]$, $[t, s]$ and $[s, u_1]$. By integrating Equation (7), we get

$$\begin{aligned} i\,(W(u_1) - W(u_0)) = &-\int_{u_0}^{t} A^{-1}(u)L(u)A(u)\,W(u)\,du \\ &-\int_t^s A^{-1}(u)L(u)A(u)\,W(u)\,du - \int_s^{u_1} A^{-1}(u)L(u)A(u)\,W(u)\,du \end{aligned} \tag{10}$$

For the middle term, we simply use the properties of the operator norm and the fact that $A(u)$ and $W(u)$ are unitary to obtain:

$$\|W(s) - W(t)\| \leq \int_t^s \|L(u)\|\,du \leq \sup_{u\in[0,1]} \|L(u)\|\,|s-t|\,, \tag{11}$$

i.e. we do not care about the behaviour of $g(u)$ inside the subinterval $[t, s]$.
To estimate the first integral, let $Q(u) = \mathbb{1} - P(u)$. A simple computation, using $P(s)\,P'(s)\,P(s) \equiv 0$, shows that

$$P(u)\,L(u)\,P(u) = Q(u)\,L(u)\,Q(u) = 0\,, \tag{12}$$

and due to the intertwining property of $A(u)$, we can write

$$\begin{aligned} W(t) - W(u_0) = i\int_{u_0}^{t} \Big(&P(0)A^{-1}(u)L(u)A(u)Q(0) \\ &+ Q(0)A^{-1}(u)L(u)A(u)P(0)\Big)W(u)\,du \end{aligned} \tag{13}$$

Now, we need to extract an explicit $\varepsilon$ dependence from this equality in order to obtain the estimates stated in the lemma. To do this, we follow [ASY] and introduce the bounded operator $\mathcal{R}_L(u)$ defined by

$$\mathcal{R}_L(u) = \frac{1}{2i\pi}\oint_{\Gamma(u)} R(u,\lambda)L(u)R(u,\lambda)\,d\lambda$$

where $R(u,\lambda) = \Big(H(u)-\lambda\Big)^{-1}$ is the resolvent of $H(u)$ at $\lambda$ and where the loop $\Gamma(u)$ is a circle centered at $\lambda(u)$ of radius $g(u)/2$. It has the properties (see [ASY], [Joy])

$$[\mathcal{R}_L(u), H(u)] = [L(u), P(u)] \tag{14}$$
$$P(u)\mathcal{R}_L(u)P(u) = Q(u)\mathcal{R}_L(u)Q(u) = 0. \tag{15}$$

Standard arguments show that $\mathcal{R}_L(u)$ is strongly $C^1$ and that

$$\begin{aligned}\mathcal{R}_L'(u) = \frac{1}{2i\pi}\oint_{\Gamma(u)} \Big(&R(u,\lambda)L'(u)R(u,\lambda)\\ &- R(u,\lambda)H'(u)R(u,\lambda)L(u)R(u,\lambda)\\ &- R(u,\lambda)L(u)R(u,\lambda)H'(u)R(u,\lambda)\Big)d\lambda\,,\end{aligned} \tag{16}$$

where $H'(u)R(u,\lambda)$ is to be understood as the bounded operator

$$H'(u)R(u,\lambda) = H'(u)R(u,i)\,(\mathbb{1} + (\lambda - i)R(u,\lambda)). \tag{17}$$

Hence, we get the following estimates:

$$\|\mathcal{R}_L(u)\| \;\leq\; \frac{|\Gamma(u)|}{2\pi}\,\|L(u)\|\,(g(u)/2)^{-2} = 2\|L(u)\|/g(u). \tag{18}$$
$$\|\mathcal{R}_L'(u)\| \;\leq\; c\max\Big\{\|H'(u)R(u,i)\|\,\|L(u)\|, \|L'(u)\|\Big\}/g^2(u). \tag{19}$$

The main property of $\mathcal{R}_L(u)$ (see [ASY]) is that it satisfies for any $\psi \in \mathcal{D}$ the following equalities, as verified by means of (14):

$$\begin{aligned}P(0)A^{-1}(u)L(u)A(u)Q(0)\,\psi = &-i\varepsilon\,\frac{d}{du}\Big(P(0)A^{-1}(u)\mathcal{R}_L(u)A(u)Q(0)\,\psi\Big)\\ &+ i\varepsilon\,P(0)A^{-1}(u)\mathcal{R}_L'(u)A(u)Q(0)\,\psi\,.\end{aligned} \tag{20}$$

and

$$\begin{aligned}Q(0)A^{-1}(u)L(u)A(u)P(0)\,\psi = &\;i\varepsilon\,\frac{d}{du}\Big(Q(0)A^{-1}(u)\mathcal{R}_L(u)A(u)P(0)\,\psi\Big)\\ &- i\varepsilon\,Q(0)A^{-1}(u)\mathcal{R}_L'(u)A(u)P(0)\,\psi\,.\end{aligned} \tag{21}$$

These equations imply that $\int_{u_0}^t A^{-1}(u)L(u)A(u)W(u)du$ is proportional to $\varepsilon$. Indeed, Equalities (12) and the intertwining property of $A(u)$ show that the diagonal blocks

are 0.
Introducing Equalities (20) and (21) in Equation (13), we get

$$\begin{aligned} W(t) - W(u_0) = \quad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (22) \\ -\varepsilon \int_{u_0}^{t} \frac{d}{du}\Big(Q(0)A^{-1}(u)\mathcal{R}_L(u)A(u)P(0) - P(0)A^{-1}(u)\mathcal{R}_L(u)A(u)Q(0)\Big)W(u)du \\ -\varepsilon \int_{u_0}^{t} \Big(P(0)A^{-1}(u)\mathcal{R}'_L(u)A(u)Q(0) - Q(0)A^{-1}(u)\mathcal{R}'_L(u)A(u)P(0)\Big)W(u)du\,. \end{aligned}$$

Performing an integration by part in the first integral, using the differential equation (7) for $W(u)$ and taking into account that $A(u)$, $W(u)$ are unitary and $P(0)$, $Q(0)$ are projectors, gives us the following bound for the norm of the difference $W(t) - W(u_0)$:

$$\begin{aligned} \|W(t) - W(u_0)\| \le 2\varepsilon \Big( & \|\mathcal{R}_L(t)\| + \|\mathcal{R}_L(u_0)\| \\ & + \sup_{u\in[u_0,t]} \|\mathcal{R}_L(u)\|\,\|L(u)\|\,(t-u_0) + \sup_{u\in[u_0,t]} \|\mathcal{R}'_L(u)\|\,(t-u_0)\Big)\,. \end{aligned} \tag{23}$$

Next, we use first Estimates (18) and (19) and then the fact that $0 \le u_0 < t \le 1$ to obtain the desired bound:

$$\begin{aligned} \|W(t) - W(u_0)\| \;\le\; & \frac{8\varepsilon}{g_t} \sup_{u\in[u_0,t]} \|L(u)\| + \frac{4\varepsilon}{g_t} \sup_{u\in[u_0,t]} \|L(u)\|^2\,(t-u_0) \\ & \quad + c\,\frac{2\varepsilon}{g_t^2} \sup_{u\in[u_0,t]} \Big\{\|H'(u)R(u,i)\|\,\|L(u)\|, \|L'(u)\|\Big\}\,(t-u_0) \\ \le\; & 12\,\frac{\varepsilon}{g_t} \sup_{u\in[0,1]} \Big\{\|L(u)\|, \|L(u)\|^2\Big\} \qquad\qquad\qquad\qquad\qquad (24) \\ & \quad + 2\,\frac{\varepsilon|t-u_0|}{g_t^2} \sup_{u\in[0,1]} \Big\{\|H'(u)R(u,i)\|\,\|L(u)\|, \|L'(u)\|\Big\} \\ \le\; & c_2\Big(\frac{\varepsilon}{g_t} + \frac{\varepsilon|t-u_0|}{g_t^2}\Big)\,. \end{aligned}$$

Using the same kind of arguments, shows that on the subinterval $[s, u_1]$, we have

$$\|W(u_1) - W(s)\| \le c_2\Big(\frac{\varepsilon|s-u_1|}{g_s^2} + \frac{\varepsilon}{g_s}\Big)\,. \tag{25}$$

Combining Estimates (11), (24) and (25) gives the announced bound for $\|W(u_1) - W(u_0)\|$. □

## 2.3 Infinite number of crossings

We now have all the information required to proceed to the case of an infinite number of crossings. We make the following hypotheses describing what happens in the neighbourhood of each crossing.

**Spectral Hypotheses:** There exist two partitions $\{u_k^{\pm}\}_{k\in\mathbb{N}}$ of $[0,a)$ and $(a,1]$ respectively:

$$0 = u_0^- < \ldots < u_{k-1}^- < u_k^- \ldots \longrightarrow u_\infty^- = a = u_\infty^+ \longleftarrow \ldots u_k^+ < u_{k-1}^+ < \ldots < u_0^+ = 1$$

such that for each $k \in \mathbb{N}^*$, *H1)* one can find non empty open intervals $V_k^{\pm}$, which satisfy $V_k^- \subset [u_{k-1}^-, u_k^-]$, $V_k^+ \subset [u_k^+, u_{k-1}^+]$ and

$$\sup_{s\in V_k^{\pm}} g(s) \leq \inf_{t\in I_k^{\pm}} g(t) \tag{26}$$

where $I_k^- = [u_{k-1}^-, u_k^-]\backslash V_k^-$ and $I_k^+ = [u_k^+, u_{k-1}^+]\backslash V_k^+$.

*H2)* there are constants $G_{\pm}(k) > 0$ and a $k$-independent positive constant $\alpha$ such that for all $s \in V_k^{\pm}$:

$$G_{\pm}(k)\,|s - o_k^{\pm}|^{\alpha} \leq g(s)\,, \tag{27}$$

for some points $o_k^{\pm} \in V_k^{\pm}$.

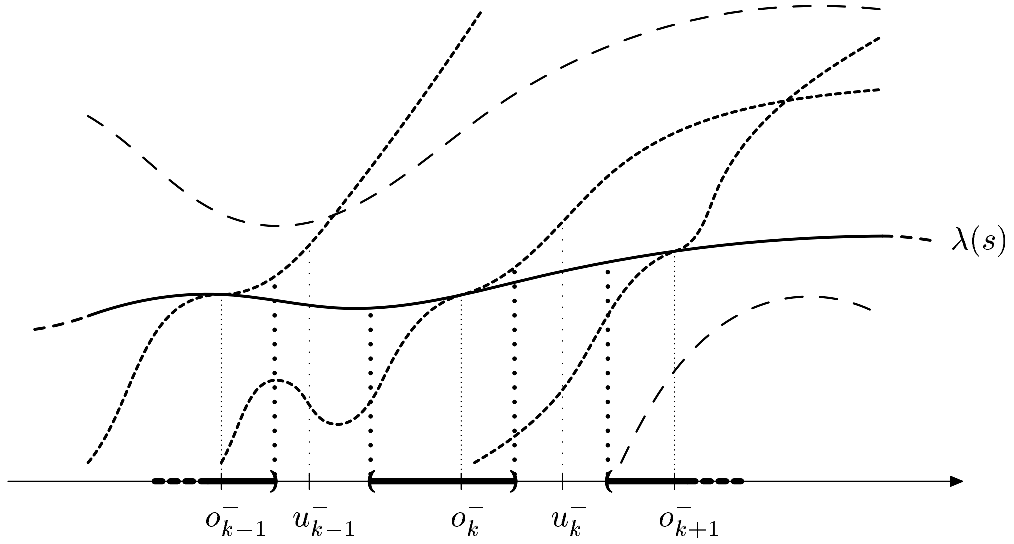


Figure 2: Illustration of the spectral hypotheses *H1–H2* on the interval $(0, a)$. The intervals $V_l^-$ are represented by $(\!\!\longrightarrow\!\!)$.

**Comments:**

*1)* These Spectral Hypotheses mean that the crossings are well separated and that they behave as power of order at most $\alpha$. Hypothesis *H1* tells us that outside the crossing regions ($V_k^{\pm}$) the gaps are relatively "large". This means that the only accumulation point of small gaps is $a$.

*2)* The choice of a constant exponent $\alpha$ is not as restrictive as it might look at first. Indeed, we are interested in an upper bound, so it is the greatest $\alpha$ that will determine the global behaviour.

*3)* In the applications, we will consider examples where $g^{-1}\{0\} = \{o_k^{\pm}\}$: the set of crossing points of $\lambda(s)$ with the rest of the spectrum. This implies $\alpha > 0$. But, the case of an infinite number of avoided crossings can be treated by taking $\alpha = 0$ in Hypothesis *H2*.

To obtain an estimate for the difference between the real evolution $U(1)$ and the adiabatic one $A(1)$, the idea is to apply Lemma 2.1 on a finite number of crossings and to take a simple integral bound (as in (11)) over the rest of the interval surrounding $a$. The choice of the number of crossings will be optimized with respect to $\varepsilon$ in order to get a simple form for the bound of the remainder term. To state the corresponding result, we need to introduce some notations. Let $\Delta_\pm(k) = \max\left\{|u_k^\pm - o_k^\pm|, |u_{k-1}^\pm - o_k^\pm|\right\}$ and $\tau_\pm(k) = \max\left\{\Delta_\pm(k)/G_\pm^2(k), \Delta_\pm^\alpha(k)/G_\pm(k)\right\}$. The functions $K \mapsto |u_K^\pm - a|/\sum_{k=1}^K \tau_\pm(k)^{1/(1+2\alpha)}$ are monotonically decreasing to zero, so, if $\varepsilon$ is small enough, we define $K_\pm(\varepsilon) \in \mathbb{N}^*$ as the greatest integer satisfying

$$\frac{|u_K^\pm - a|}{\sum_{k=1}^K \tau_\pm(k)^{1/(1+2\alpha)}} \geq \varepsilon^{1/(1+2\alpha)}\,. \tag{28}$$

This integer always exists if $\varepsilon$ is sufficiently small and, by construction, $K_\pm(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

**Theorem 2.1** *For $\varepsilon$ small enough, under H0 and the Spectral Hypotheses H1, H2 and provided that*

$$\varsigma\,(\varepsilon\,\tau_\pm(k))^{1/(1+2\alpha)} \leq |V_k^\pm|/2 \qquad \text{for all } 1 \leq k \leq K_\pm(\varepsilon)\,, \tag{29}$$

*for some constant $\varsigma > 0$, we have that*

$$U(1) = A(1) + O\left(\max\{|u^-_{K_-(\varepsilon)} - a|, |u^+_{K_+(\varepsilon)} - a|\}\right)\,.$$

*Hence, as $\lim_{\varepsilon\to 0} K_\pm(\varepsilon) = \infty$, $\|U(1) - A(1)\|$ goes to zero for $\varepsilon \to 0$ as fast as $\max\{|u^-_{K_-(\varepsilon)} - a|, |u^+_{K_+(\varepsilon)} - a|\}$.*

**Remarks:**
*1)* The theorem states that the error can be estimated provided we can compute the critical value $K_\pm(\varepsilon)$. Further considerations on the practical aspects of this computation are given in the next section.
*2)* Condition (29) implies that the size of the intervals $V_k^\pm$ cannot be too small with respect to $\varepsilon\,\tau_\pm(k)$.
*3)* While we shall apply the theorem in a situation where the spectrum is simple and pure point, the theorem remains valid under the sole existence of an eigenvalue separated from the rest of the spectrum by gaps with the properties stated in H1-H2, without any knowledge on the rest of the spectrum or restriction on the dimension of $P(s)$.
**Proof:**
In the sequel, we will denote by the same symbol $c$ all inessential constants. Let us consider the interval $[0; a)$. In order to simplify the notations, we will not write the sub/super-scripts $-$. Picking some $t, s \in V_k$ such that $t < o_k < s$ and $|t - o_k| = |s - o_k|$, we get

$$\begin{aligned}
\|W(u_k) - W(u_{k-1})\| &\leq c\,\left(\varepsilon|t - u_{k-1}|/g_t^2 + \varepsilon|s - u_k|/g_s^2 + \varepsilon/g_t + \varepsilon/g_s + |t - s|\right)\\
&\leq c\,\left(\varepsilon\,\frac{\Delta(k)}{G(k)^2}\,|t - o_k|^{-2\alpha} + \varepsilon\,\frac{1}{G(k)}\,|t - o_k|^{-\alpha} + |t - o_k|\right)\\
&\leq c\,\left(\varepsilon\,\frac{\Delta(k)}{G(k)^2}\,|t - o_k|^{-2\alpha} + \varepsilon\,\frac{\Delta^\alpha(k)}{G(k)}\,|t - o_k|^{-2\alpha} + |t - o_k|\right) \qquad (30)\\
&\leq c\,\left(\varepsilon\,\tau(k)\,|t - o_k|^{-2\alpha} + |t - o_k|\right) \qquad (31)
\end{aligned}$$

by the preceding section. Indeed, we have that $g_t = \inf_{u\in[u_{k-1},t]} g(u) = g(r_t)$ for some $r_t \in [u_{k-1}, t]$. Now, by Hypothesis H1, $r_t \in V_k$. Whence, we have that

$$g_t = g(r_t) \geq G(k)\,|r_t - o_k|^\alpha \geq G(k)\,|t - o_k|^\alpha$$

as $r_t \leq t \leq o_k$. Using the same kind of arguments, we can show that $g_s = \inf_{u\in[s,u_k]} g(u) \geq G(k)\,|s - o_k|^\alpha$. Finally to obtain the bound (30), it remains to notice that $|s-t| = |t-o_k|+|s-o_k| = 2\,|t-o_k|$ together with $|t-o_k|$, $|t-u_{k-1}| \leq \Delta(k)$ and $|s-o_k|$, $|s-u_k| \leq \Delta(k)$.

We now get an estimate by choosing $t = t(\varepsilon, k)$ in order to balance the two contributions appearing in the last term of Equation (30) above: for some constant $\varsigma > 0$, we set

$$\frac{\varsigma^{1+2\alpha}\,\varepsilon\,\tau(k)}{|t(\varepsilon,k) - o_k|^{2\alpha}} = |t(\varepsilon,k) - o_k|, \tag{32}$$

i.e.

$$|t(\varepsilon,k) - o_k| = \varsigma\left(\varepsilon\,\tau(k)\right)^{1/(1+2\alpha)}. \tag{33}$$

By definition, $t(\varepsilon,k) \in V_k$, hence, as $k$ will eventually be bounded from above by $K(\varepsilon)$, this imposes Condition (29) in the statement of the theorem. Replacing $t$ by $t(\varepsilon,k)$ in (30) and summing over $k$, we get for any $K \leq K(\varepsilon)$,

$$\|W(0) - W(u_K)\| \leq c\,(\varsigma + \varsigma^{-2\alpha}) \sum_{k=1}^{K} \left(\varepsilon\,\tau(k)\right)^{1/(1+2\alpha)}. \tag{34}$$

On the other hand, using the differential Equation (7), we obtain,

$$\|W(u_K) - W(a)\| \leq \int_{u_K}^{a} \|L(u)\|\,du \leq c\,|u_K - a|\,. \tag{35}$$

Again, we balance the two right hand sides in (34) and (35) by setting the integer $K = K(\varepsilon)$, which has been defined in Equation (28). Consequently,

$$\begin{aligned} \|W(0) - W(a)\| &\leq c\left((\varsigma + \varsigma^{-2\alpha})\,\varepsilon^{1/(1+2\alpha)} \sum_{k=1}^{K(\varepsilon)} \tau(k)^{1/(1+2\alpha)} + |u_{K(\varepsilon)} - a|\right) \\ &\leq c\,(\varsigma + \varsigma^{-2\alpha} + 1)\,|u_{K(\varepsilon)} - a| \equiv C(\varsigma)\,|u_{K(\varepsilon)} - a| \end{aligned} \tag{36}$$

where $C(\varsigma)$ is independent of $\varepsilon$. Proceeding similarly on $(a, 1]$ completes the proof.
□

**Remarks:**
*1)* The introduction of an adjustable constant $\varsigma$ is necessary in the following application to satisfy the hypothesis of the theorem.
*2)* In the step (30) we deliberately lost a little in the estimate by using $|t-o_k|^{-\alpha} \leq \Delta_k^\alpha |t-o_k|^{-2\alpha}$ in order to simplify the subsequent arguments. It is nevertheless possible to get slightly sharper results by not adopting this simplification, however the analysis gets more involved and less transparent. We simply note here that in the examples discussed below, this more careful analysis yields, for the generic situation, an error term of order $\epsilon^p$ with an exponent $p = 1/3$, instead of the value $p = 1/3 - \nu$, for any $\nu > 0$ obtained there.

# 3 Application

We can obtain more explicit estimates on the rest by considering some specific behaviour at the crossings.

Let us introduce the following notation: $F_k \sim f(k)$ means that there exist two constants $0 < c_1 < c_2 < \infty$ such that $c_1 f(k) \leq F_k \leq c_2 f(k)$ for $k \in \mathbb{N}^*$ large enough. We have the

**Proposition 3.1** *Assume the hypothesis of Theorem 2.1 and the following behaviour for the relevant quantities:*

$$\begin{aligned}
&|u_k^\pm - a| = C_1/k^\beta + C_2/k^{\beta+1} + o(1/k^{\beta+1}))\,, && \beta > 0, C_1 \neq 0\\
&G_\pm(k) \sim k^\gamma\,, && \\
&|V_k^\pm| \sim 1/k^\delta\,, && \delta > 0\,.
\end{aligned}$$

*We set* $\mu = \min\{\beta+1+2\gamma, \alpha(\beta+1)+\gamma\}$. *Then* $\|U(1) - A(1)\| = O(\varepsilon^p)$ *where the exponent* $p$ *is given by*

$$p = \begin{cases} \dfrac{1}{1+2\,\alpha} & \text{if } \mu > (1+2\alpha) \\ \dfrac{1}{1+2\,\alpha} - \nu \quad \forall \nu > 0 & \text{if } \mu = (1+2\alpha) \\ \dfrac{\beta}{(\beta+1)\,(1+2\alpha) - \mu} & \text{if } \mu < (1+2\alpha) \end{cases}$$

*provided that* $\delta$ *satisfy the following constraints:* $\beta+1 \leq \delta \leq \beta + \max\{1, \mu/(1+2\alpha)\}$.

**Proof:**
The idea of the proof is to explicit conditions on the different exponents ensuring the validity of Theorem 2.1. We will only consider the interval $[0, a)$, the same kind of arguments will apply on $(a, 1]$. Again, in order to simplify the notations we will let aside the sub/super-scripts $-$.

First, we have that $2\,\Delta(k) = u_k - u_{k-1} = C_1\beta/k^{\beta+1} + o(1/k^{\beta+1}) \sim 1/k^{\beta+1}$, which implies that

$$\delta \geq \beta + 1 > 0, \tag{37}$$

since $2\,\Delta(k) \geq |V_k| \sim 1/k^\delta$. Notice that the length of the $V_k$ can be rescaled by a uniform constant if $\delta = \beta + 1$.
Next, $\Delta(k)/G^2(k) \sim 1/k^{\beta+1+2\gamma}$ and $\Delta^\alpha(k)/G(k) \sim 1/k^{\alpha(\beta+1)+\gamma}$. So, if we denote by $\mu = \min\{\beta+1+2\gamma, \alpha(\beta+1)+\gamma\}$ then $\tau(k) = \max\{\Delta(k)/G^2(k), \Delta^\alpha(k)/G(k)\} \sim 1/k^\mu$ by increasing the overall constant in Theorem 2.1 if necessary. Whence,

$$\sum_{k=1}^{K} \tau(k)^{1/(1+2\alpha)} \sim \sum_{k=1}^{K} k^{-\mu/(1+2\alpha)} \sim \begin{cases} K^0 & \text{if } \mu > 1+2\alpha \\ \log K & \text{if } \mu = 1+2\alpha \\ K^{1-\mu/(1+2\alpha)} & \text{if } \mu < 1+2\alpha \end{cases} \tag{38}$$

and considering the definition of $K(\varepsilon)$ (see Eq. 28), we obtain

$$\varepsilon^{1/(1+2\alpha)} \sim \frac{|u_{K(\varepsilon)} - a|}{\sum_{k=1}^{K(\varepsilon)} \tau(k)^{1/(1+2\alpha)}} \sim \begin{cases} K(\varepsilon)^{-\beta} & \text{if } \mu > 1+2\alpha \\ K(\varepsilon)^{-\beta}/\log K(\varepsilon) & \text{if } \mu = 1+2\alpha \\ K(\varepsilon)^{-\beta-1+\mu/(1+2\alpha)} & \text{if } \mu < 1+2\alpha \end{cases} \tag{39}$$

Condition (29) stated in Theorem 2.1 reads

$$\varsigma \left(\varepsilon\, \tau(k)\right)^{1/(1+2\alpha)} \leq |V_k|/2 \tag{40}$$

for all $1 \leq k \leq K(\varepsilon)$. Notice that this condition is automatically satisfied if $\delta < \mu/(1+2\alpha)$. In general, it will be satisfied for a sufficiently small $\varsigma$, if

$$F(\varepsilon) \equiv \varepsilon^{1/(1+2\alpha)} K(\varepsilon)^{\delta-\mu/(1+2\alpha)} \tag{41}$$

remains bounded as $\varepsilon \to 0$. Using (39), we have

$$F(\varepsilon) \sim \begin{cases} K(\varepsilon)^{\delta-\beta-\mu/(1+2\alpha)} & \text{if } \mu > 1+2\alpha\,, \\ K(\varepsilon)^{\delta-1-\beta}/\log K(\varepsilon) & \text{if } \mu = 1+2\alpha\,, \\ K(\varepsilon)^{\delta-\beta-1} & \text{if } \mu < 1+2\alpha\,. \end{cases} \tag{42}$$

As $K(\varepsilon) \to \infty$ for $\varepsilon \to 0$, Equation (42) implies that $F(\varepsilon)$ will remain bounded if $\delta \leq \beta + \max\{1; \mu/(1+2\alpha)\}$.

Hence, using (36) and (39), we get that the Adiabatic Theorem 2.1 holds with a remainder term on $[0, a)$,

$$O\left(|u_{K_-(\varepsilon)} - a|\right) = O\left(K(\varepsilon)^{-\beta}\right) = O\left(\varepsilon^p\right)$$

where the exponent $p$ is given by

$$p = \begin{cases} \dfrac{1}{1+2\,\alpha} & \text{if } \mu > (1+2\alpha) \\ \dfrac{1}{1+2\,\alpha} - \nu \quad \forall \nu > 0 & \text{if } \mu = (1+2\alpha) \\ \dfrac{\beta}{(\beta+1)\,(1+2\alpha) - \mu} & \text{if } \mu < (1+2\alpha) \end{cases}$$

provided that $\beta+1 \leq \delta \leq \beta+\max\{1; \mu/(1+2\alpha)\}$. To determine $p$ in case $\mu = 1+2\alpha$ and $\delta = \beta+1$, we have used the estimate $\varepsilon^{-1/(1+2\alpha)} \sim K(\varepsilon)^{\beta} \log K(\varepsilon) < K(\varepsilon)^{\beta+\nu'}$ for all $\nu' > 0$. This ends the proof of the proposition. □

**Remark:**
We can strengthen the last comment of the preceding section with the following observation. In case $\alpha = \beta = \gamma = 1$ and $\delta = 2$, this last comment asserts that we have $p = 1/3$, instead of $p = 1/3 - \nu$, for all $\nu > 0$. Now, if in Lemma 2.1, the right member were missing the terms $\varepsilon|u_0 - t|/g_t^2 + \varepsilon|u_1 - s|/g_s^2$, as one would

expect with a little more information on the spectrum, an analysis similar to the one provided above leads to an error term of order $\varepsilon^{1/3}$. This makes it reasonable to expect that in such a situation the error actually is of that order, as it was the case in the corresponding analysis of one crossing performed in [BF], see [Hag]. Finally, it is shown in the examples below that the values $\alpha = \beta = \gamma = 1$ and $\delta = 2$ are generic in some sense.

## 4 Examples

We now consider a family of models for which the situation just described takes place as the effective frequency $\varpi$ takes the value zero. We start by considering the most general model for a two level system driven by a periodic field. The model can be characterized by choosing freely the eigenvalues $\lambda_{+,m} = \lambda_+ + m\,\omega$ and $\lambda_{-,k} = \lambda_- + k\,\omega$ and the corresponding eigenfunctions of the form:

$$\psi_{+,m}(\theta) = \begin{pmatrix} e^{i\,x(\theta)}\,\cos z(\theta) \\ e^{i\,y(\theta)}\,\sin z(\theta) \end{pmatrix} e^{i\,m\,\theta} \qquad \text{and} \qquad \psi_{-,k}(\theta) = \begin{pmatrix} -e^{-i\,y(\theta)}\,\sin z(\theta) \\ e^{-i\,x(\theta)}\,\cos z(\theta) \end{pmatrix} e^{i\,k\,\theta}\,, \tag{43}$$

in which the functions $x$, $y$ and $z$ are periodic modulo an integer multiple of $\theta$.

Defining the unitary matrix

$$\Upsilon(\theta) = \begin{pmatrix} e^{i\,x(\theta)}\,\cos z(\theta) & -e^{-i\,y(\theta)}\,\sin z(\theta) \\ e^{i\,y(\theta)}\,\sin z(\theta) & e^{-i\,x(\theta)}\,\cos z(\theta) \end{pmatrix}$$

the corresponding Floquet Hamiltonian can be written as (dropping the $\theta$ dependence in the notation)

$$K = -i\,\varpi\,\partial - i\,\varpi\,\Upsilon\,(\partial\Upsilon^{-1}) + \Upsilon\,D\,\Upsilon^{-1}$$

where $D = \mathrm{diag}(\lambda_+, \lambda_-)$. Using the notation $2\wp = x + y$, $2\vartheta = y - x$ and choosing, without loss of generality, $\lambda_+ = -\lambda_- = \lambda$, the Floquet Hamiltonian can be expressed as

$$\begin{aligned} K &= -i\,\varpi\partial \\ &\quad + \begin{pmatrix} \varpi\partial\vartheta + (\lambda - \varpi\partial\wp)\cos(2z) & \Big(-i\varpi\partial z + (\lambda - \varpi\partial\wp)\sin(2z)\Big)e^{-2i\vartheta} \\ \Big(i\varpi\partial z + (\lambda - \varpi\partial\wp)\sin(2z)\Big)e^{2i\vartheta} & -\varpi\partial\vartheta - (\lambda - \varpi\partial\wp)\cos(2z) \end{pmatrix} \end{aligned} \tag{44}$$

where $\partial f$ denotes the derivative with respect to $\theta$.

We will consider two different models with the same eigenvalues but with different eigenfunctions. We remark that since the adiabatic theorem depends only on the properties of the eigenvalues (and regularity properties of the projectors), it gives the same upper bound for the correction for all the models (44) with equal spectrum. But the actual deviation can be very different from this estimate, depending on the efficiency of the couplings.

We can choose for example the following eigenvalues:

$$\lambda_{\pm,k}(\varpi) = k\,\varpi \pm (\eta(\varpi) + \varpi)/2\,, \qquad \text{where } \eta(\varpi) = \sqrt{(\varpi - \omega_0)^2 + \Omega^2} \tag{45}$$

and $\omega_0$, $\Omega$ are constants. The first model is defined by choosing $x(\theta)=0$, $y(\theta)=\theta$, i.e. $2\vartheta(\theta)=2\wp(\theta)=\theta$ and $\cos(2\,z)=-(\varpi-\omega_0)/\eta(\varpi)$, $\sin(2\,z)=\Omega/\eta(\varpi)$, hence $z$ is independent of $\theta$. The corresponding Floquet Hamiltonian is given by

$$K_{\text{RWA}}(\theta)=-i\,\varpi\,\frac{\partial}{\partial\theta}+\frac{1}{2}\begin{pmatrix}\omega_0 & \Omega\,e^{-i\,\theta}\\ \Omega\,e^{i\,\theta} & -\omega_0\end{pmatrix}$$

which is the usual RWA model widely used in quantum optics.
The second model is defined by the choice $x(\theta)=-\varrho(\theta)/2$, $y(\theta)=\theta-\varrho(\theta)/2$, i.e. $2\vartheta(\theta)=\theta$, $2\wp(\theta)=\theta-\varrho(\theta)$ and the same $z$ as for the RWA case. This leads to

$$K_{\text{M}}(\theta)=K_{\text{RWA}}(\theta)+\frac{\varpi}{2\,\eta}\,\partial\varrho\,\begin{pmatrix}\omega_0-\varpi & \Omega\,e^{-i\,\theta}\\ \Omega\,e^{i\,\theta} & \varpi-\omega_0\end{pmatrix}$$

This Hamiltonian can thus be understood as a perturbation of the RWA model.

We consider now an extra smooth slow time dependence in the parameter $\varpi=\varpi(s)$ and in $\varrho=\varrho(\theta,s)$. This implies that the eigenvalues, the eigenvectors and the corresponding eigenprojectors are smooth functions of $s$, so that the regularity Hypothesis H0 i) is satisfied. We show in appendix that H0 ii) is satisfied as well for any choice of smooth functions $x,y,z$ and $\lambda$.

We assume, for simplicity, that $\varpi(s)=s$ (but any other smooth monotonic function of $s$ would equally do). We select the eigenvalue $\lambda(s)=\lambda_{+,0}(s)=(\eta(s)+s)/2$ and denote by $\psi$ the associated eigenvector. The only crossings that $\lambda$ experiences are with the $\lambda_{-,k+1}$'s and they take place at times $s$ such that

$$\eta(s)=k\,s,\qquad k\in\mathbb{Z}^*. \tag{46}$$

We remark however that these crossings can lead to corrections to adiabaticity, or not, depending on whether the corresponding eigenvectors are coupled. The non adiabatic coupling among the branches is measured by the following scalar product:

$$\begin{aligned}\langle\,\psi(s)\mid\partial_s\psi_{-,k+1}(s)\,\rangle&=-\frac{1}{2\pi}\int_0^{2\pi}e^{i\,(k+1)\,\theta-2\,i\,\wp(\theta,s)}\Big(z'(s)-i\,\sin(2\,z(s))\,\vartheta'(\theta,s)\Big)\,d\theta\\&=-\frac{z'(s)}{2\pi}\int_0^{2\pi}e^{i\,k\,\theta+i\,\varrho(\theta,s)}\,d\theta\end{aligned}$$

where the $'$ denotes the derivative with respect to $s$.

Recall that the couplings between the eigenstate $\psi(s)$ associated with the level $\lambda(s)$ and its orthogonal complement in the Hilbert space is given by the operator $L(s)=i\,[P'(s),P(s)]$, see (7), since the adiabatic evolution $A(s)$ follows the instantaneous eigenspaces. A direct computation of the matrix elements $\langle\,\psi_{-,k+1}(s)\mid L(s)\,\psi(s)\,\rangle$ with $P'(s)=\mid\psi'(s)\,\rangle\langle\psi(s)\mid+\mid\psi(s)\,\rangle\langle\,\psi'(s)\mid$ shows that the above scalar product is proportionnal to the couplings responsible for the non adiabatic transitions.
For the RWA model, as $\varrho=0$ the non adiabatic couplings are given by:

$$\langle\,\psi(s)\mid\partial_s\psi_{-,k+1}(s)\,\rangle=-z'(s)\,\delta_{k,0}$$

Thus, the level $\lambda(s)$ is *not* coupled to the infinitely many other levels it crosses. Hence we are lead in this case to an effective problem displaying no crossing, so that the error is of order $\varepsilon$ in this case.

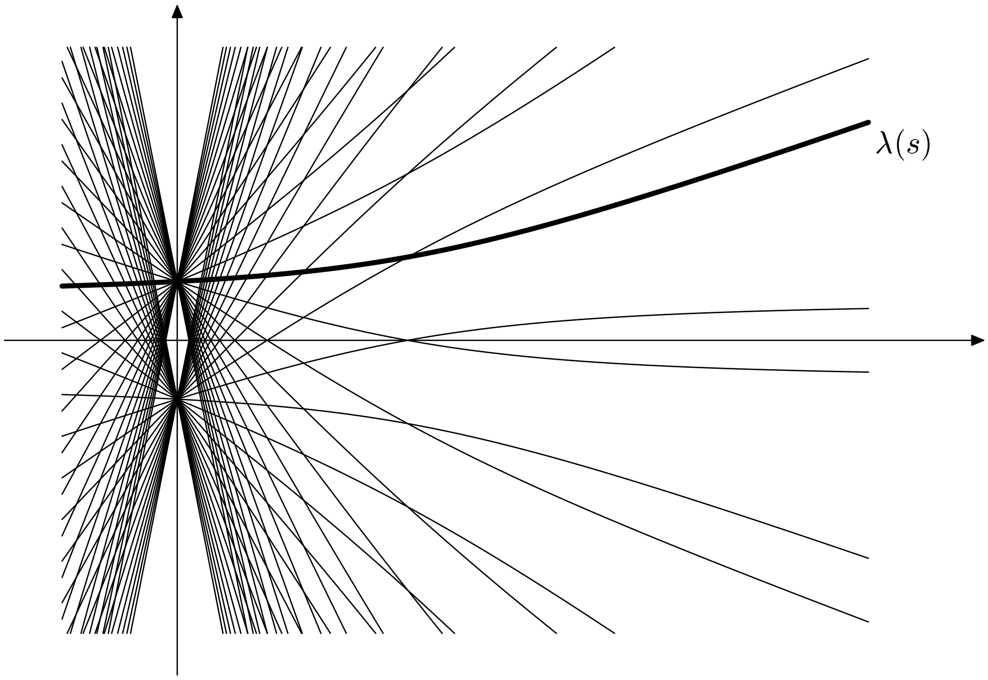


Figure 3: The first eigenvalues of the RWA and modified RWA models.

For the other model, we will obtain non zero couplings at all the crossings, if we choose $\varrho(\theta, s)$ such that $\exp(i\,\varrho(\theta, s))$ has infinitely many non zero Fourier components. For example, one can take $\varrho(\theta, s) = \rho(s)\,\sin(\theta)$ (in particular $\rho$ can be chosen constant). This coupling is then given by

$$\langle\, \psi(s) \mid \partial_s \psi_{-,k+1}(s)\,\rangle = (-1)^{k+1}\, z'(s)\, J_k(\rho(s))$$

where $J_k$ is a Bessel function.

We will now verify that the assumptions of Proposition 3.1 are satisfied. Let us focus on the interval $(0, S]$, for $S$ small enough. The interval $[-S, 0)$ can be treated similarly. Again to simplify the notations we will not explicit the + sub/superscripts.

In fact the following analysis is valid for all the models (44) under the assumption that the eigenvalues can be written as $\lambda_{\pm,m}(s) = m\, s \pm \aleph(s)/2$, where $\aleph$ is a $C^2$ function with bounded derivatives such that $\aleph(0) > 0$. In particular they are satisfied for the eigenvalues given in (45). The Hypotheses imply that the function $f_\zeta(s) = \aleph(s) - \zeta\, s$ is strictly decreasing for any $\zeta$ greater than, say, some $\zeta_0$. Under these conditions the following assertion shows that the crossings that $\lambda(s) = \aleph(s)/2$ experiences with the rest of the spectrum take place at times such that $\aleph(s) = k\, s$, $k \in \mathbb{N}$ large enough.

**Assertion 1** *For $\zeta \geq \zeta_0$, the function $f_\zeta(s) = \aleph(s) - \zeta\, s$ has a unique positive zero $o_\zeta$ and if $\zeta < \xi$ we have $o_\zeta > o_\xi$.*

From the expansion

$$f_\zeta(s) = \aleph(0) + (\aleph'(0) - \zeta)\, s + O(s^2)\,,$$

we obtain the behaviour of $o_\zeta$:

$$o_\zeta = \frac{\aleph(0)}{\zeta - \aleph'(0)} + O(1/\zeta^3). \tag{47}$$

We define the sequence $u_k > 0$ by the equation:

$$\aleph(u_k) - k\, u_k = (k+1)\, u_k - \aleph(u_k) \qquad \text{i.e. } \aleph(u_k) = (k+1/2)\, u_k\,. \tag{48}$$

Assertion 1 implies that $u_k < o_k < u_{k-1}$ and, from Equation (refzopos) and the fact that $u_k = o_{k+1/2}$, we obtain

$$u_k = \frac{\aleph(0)}{k + 1/2 - \aleph'(0)} + O(1/k^3). \tag{49}$$

Next, we have

**Assertion 2** *On the interval $[u_k, u_{k-1}]$, the spectral gap is given by*

$$g(s) = \mathrm{dist}\,(\lambda(s), \sigma(s)\backslash\{\lambda(s)\}) = |\aleph(s) - k\,s| \le u_{k-1}/2\,.$$

*More precisely, for $u_k \le s \le o_k$ we have that $g(s) = \aleph(s) - k\,s \le u_k/2$ and for $o_k \le s \le u_{k-1}$ we have that $g(s) = k\,s - \aleph(s) \le u_{k-1}/2$.*

This assertion is easily proven by considering the different cases.
We now prove that the Spectral Hypothesis H1–H2 are verified. Assertion 1 and Equation (49) show that the sequence $\{u_k\}$ is (for $k$ large enough) monotonically decreasing to $a = 0$. To define the intervals $V_k$, we choose any point $r_k$ in $(o_k, u_{k-1})$ such that $g(r_k) = k\,r_k - \aleph(r_k) \le u_k/2$ and set $V_k = (u_k, r_k)$. The $V_k$'s are disjoint and $I_k = \{u_k\} \cup [r_k, u_{k-1}]$. By definition of $V_k$, we have that $g(s) \le u_k/2 = g(u_k)$ and for $r_k \le s \le u_{k-1}$ the gap is given by $g(s) = k\,s - \aleph(s) \ge u_k/2$. Whence, Hypothesis H1 is satisfied. Finally to prove that H2 holds, we need to estimate the behaviour of $g(s)$ on $V_k$: the Mean Value Theorem implies that for each $s \in V_k \backslash \{o_k\}$, there is an $q_s$, in the interval joining $s$ and $o_k$, such that

$$g(s) = |\aleph(s) - k\,s| = |k - \aleph'(q_s)|\;|s - o_k| \sim k\,|s - o_k|\,,$$

which shows that H2 is satisfied with $\alpha = 1$ and $G(k) \sim k$.

It remains to check the conditions given in the statement of Proposition 3.1. We have

$$\begin{aligned} &|u_k - 0| = u_k = \aleph(0)/k + \aleph(0)(\aleph'(0) - 1/2)/k^2 + O(1/k^3) && \text{i.e. } \beta = 1 \\ &G(k) \sim k && \text{i.e. } \gamma = 1\,, \\ &|V_k| \sim 1/k^2 && \text{i.e. } \delta = 2\,. \end{aligned} \tag{50}$$

To get the estimate for $|V_k|$, we have used that $(u_k, o_k] \subset V_k \subset (u_k, u_{k-1}]$ and the expressions for $o_k$, and $u_k$ in Equations (47) and (49). This implies that, $\mu = \alpha\,(\beta+1) + \gamma = 1 + 2\,\alpha$ and $\delta = \beta + 1$. So, we can use the second case of Proposition 3.1 to prove that the adiabatic approximation holds for the models:

$$\|U(1) - A(1)\| \le c\,\varepsilon^p, \qquad \text{for any } p < \frac{1}{3}\,. \tag{51}$$

In keeping with the last remark of Section 2, we recall that a more careful analysis yields $p = 1/3$.

# A Appendix

In this appendix, we show that an operator on $L^2(S^1, \mathcal{H})$ of the form

$$K(s,\theta) = -i\,\varpi(s)\,\frac{\partial}{\partial\theta} + H(s,\theta)\,, \tag{52}$$

where $H(s,\theta)$ is a bounded operator in $\mathcal{H}$ such that $s \mapsto H(s,\theta)$ and $s \mapsto \partial/\partial\theta H(s,\theta)$ are norm continuous and $s \mapsto \varpi(s)$ is continuous, admits a strongly continuous unitary propagator $U(s) = U(s,0)$ with all expected regularity properties, even if there is a value $a$ for which $\varpi(a) = 0$. Notice that the assumptions on $H$ will be satisfied if, for example, $(s,\theta) \mapsto H(s,\theta)$ is strongly $C^1$.

The proof relies on a theorem of T. Kato [Ka2], which we will restate in a more suitable form for our purpose.

**Theorem A.1 (Kato)** *Let $\mathcal{K}$ and $\mathcal{D}$ be Hilbert spaces such that $\mathcal{D}$ is densely and continuously embedded in $\mathcal{K}$ and let $K(t)$, $0 \le t \le T$, be a family of self-adjoint operators in $\mathcal{K}$. Suppose that*

*1) $\mathcal{D} \subset \operatorname{dom} K(t)$ for all $0 \le t \le T$, whence the $K(t)$ are bounded operators from $\mathcal{D}$ to $\mathcal{K}$, and the application $t \mapsto K(t)$ is norm continuous from $\mathcal{D}$ to $\mathcal{K}$;*

*2) there exists a family of isomorphisms $S(t)$ from $\mathcal{D}$ to $\mathcal{K}$ which is strongly continuously differentiable and such that*

$$S(t)\,K(t)\,S(t)^{-1} = K(t) + B(t)$$

*where $B(t)$ is a strongly continuous bounded operator on $\mathcal{K}$.*

*Under those conditions, there exists a unique family of unitary operators $U(t,s)$ on $\mathcal{K}$ defined for $0 \le s,t \le T$ with the following properties:*

*i) $U(t,s)$ is strongly continuous on $\mathcal{K}$ in $s,t$ with $U(s,s) = \mathbb{1}$;*

*ii) $U(t,r) = U(t,s)\,U(s,r)$;*

*iii) $U(t,s)\,\mathcal{D} \subset \mathcal{D}$, $\|U(t,s)\|_{\mathcal{D}} \le N\,\mathrm{e}^{c\,|t-s|}$ and is strongly continuous on $\mathcal{D}$ in $s,t$ simultaneously;*

*iv) $\dfrac{d}{ds}U(t,s)\,\psi = i\,U(t,s)\,K(s)\,\psi$ for any $\psi \in \mathcal{D}$, for $0 \le s,t \le T$;*

*v) for each $\psi \in \mathcal{D}$ and fixed $s$, $\dfrac{d}{dt}U(t,s)\,\psi$ exists and is equal to $-i\,K(t)\,U(t,s)\,\psi$ and strongly continuous in $\mathcal{K}$ in $t$.*

To prove this theorem, we apply Theorem 6.1 in [Ka2] to the operator $A(t) = i\,K(t)$, which is stable with constants of stability $c = 0$ and $N = 1$ (see Definition 3.1 and Theorem 4.1 therein). The fact that $U(t,s)$ is unitary follows from the self-adjointness of $K(t)$, the construction of $U(t,s)$ by unitary approximants given in the proofs of Theorem 4.1 and 6.1 in [Ka2] and the invertibility of $U(t,s)$, which is a consequence of the fact that $A^\circ(t) = -i\,K(T-t)$ satisfies also the hypothesis of Theorem 6.1 in [Ka2]. See also Remark 5.3 therein.

We now prove that the family of self-adjoint operators defined by Equation (52) satisfies the hypothesis of Theorem A.1. To simplify the notation, we will not explicit the $\theta$-dependence and write $\partial$ for $\partial/\partial\theta$.

**Proof:**
For $\mathcal{D}$, we choose $\mathrm{dom}(-i\,w_*\partial)$ for some $w_* > 0$, and we notice that for any $t$ such that $\varpi(t) \neq 0$, we have that $\mathrm{dom}\,K(t) = \mathcal{D}$ and if $\varpi(t) = 0$, then $\mathrm{dom}\,K(t) = \mathcal{K}$. For the norm on $\mathcal{D}$, we choose the graph norm associated to $-i\,w_*\partial$:

$$\|\psi\|_{\mathcal{D}}{}^2 = \|\psi\|^2 + \|-i\,w_*\,\partial\,\psi\|^2 \geq \|\psi\|^2 .$$

Whence, $\mathcal{D}$ is a dense continuously embedded subspace of $\mathcal{K}$.
For any $s, t$ and any $\psi \in \mathcal{D}$, we have

$$\begin{aligned}\|(K(t)-K(s))\,\psi\|^2 &\leq 2\,\frac{|\varpi(t)-\varpi(s)|^2}{w_*^2}\,\|-i\,w_*\partial\,\psi\|^2 + 2\,\|(H(t)-H(s))\,\psi\|^2 \\ &\leq 2\,\max\left\{\frac{|\varpi(t)-\varpi(s)|^2}{w_*^2} ; \|H(t)-H(s)\|^2\right\}\,\|\psi\|_{\mathcal{D}}{}^2 .\end{aligned}$$

which shows the norm continuity of $K(t)$.

We set $S(t) = S = -i\,w_*\partial + i$. $S$ is an isomorphism between $\mathcal{D}$ and $\mathcal{K}$ which is strongly differentiable (by $t$ independence). It remains to show that $S$ satisfies Hypothesis 2) of Theorem A.1. For this, we first notice that for any $\psi \in \mathrm{dom}\,K(t)$, we have that $S^{-1}\psi \in \mathcal{D} \subset \mathrm{dom}\,K(t)$ and

$$\begin{aligned}K(t)\,S^{-1}\psi &= S^{-1}K(t)\,\psi + H(t)\,S^{-1}\psi - S^{-1}H(t)\,\psi \\ &= S^{-1}K(t)\,\psi + S^{-1}S\,H(t)\,S^{-1}\psi - S^{-1}H(t)\,\psi \\ &= S^{-1}\left(K(t) - i\,w_*\partial H(t)\,S^{-1}\right)\psi . \end{aligned} \tag{53}$$

Whence, for any $\psi \in \mathrm{dom}\,K(t)$, we have that the left-hand side of Equation (53) belongs to $\mathcal{D}$. So we can write,

$$S\,K(t)\,S^{-1}\psi = K(t)\,\psi - i\,w_*\partial H(t)\,S^{-1}\psi , \qquad \text{for all } \psi \in \mathrm{dom}\,K(t) .$$

Setting $B(t) = -i\,w_*\partial H(t)\,S^{-1}$, we have a strongly continuous bounded operator (by the assumptions on $H$) which satisfies $S\,K(t)\,S^{-1} \supset K(t) + B(t)$. To show the reverse inclusion, we can consider any $b \geq 2\,\sup_t \|B(t)\|$ which implies that $i\,b$ belongs to the resolvent set of both $K(t) + B(t)$ and $S\,K(t)\,S^{-1}$. It follows that $(K(t)+B(t)+i\,b)^{-1} \subset S\,(K(t)+i\,b)^{-1}\,S^{-1}$. But since the left hand side has domain $\mathcal{K}$, we must have equality between $K(t)+B(t)$ and $S\,K(t)\,S^{-1}$ instead of inclusion.
$\square$

In the examples of Section 4, both $H(s,\theta)$ defined through (44) by means of smooth fonctions $x, y, z, \lambda$ of $(s,\theta)$, and $H(s,\theta)+\varepsilon i[P'(s,\theta), P(s,\theta)]$ where $P(s,\theta) = |\psi(s,\theta)\rangle\langle\psi(s,\theta)|$ with $\psi(s,\theta)$ given by one of the vectors (43) satisfy the hypotheses of the Theorem. Hence assumption H0 ii) is satisfied for these models.

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