Local quantum constraints

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Abstract

We analyze the situation of a local quantum field theory with constraints which are also local. In particular we find "weak" Haag–Kastler axioms which will ensure that the final constrained theory satisfies the usual Haag–Kastler axioms. We develop in detail Gupta–Bleuler electromagnetism as an example of such a theory which satisfies the "weak" Haag–Kastler axioms but not the usual ones.

We conclude the analysis by comparing the final physical algebra produced by a system of local constrainings to the one obtained from a single global constraining and also consider the issue of reduction by stages. For the usual spectral condition on the generators of the translation group, we also find a "weak" version, and show that the Gupta–Bleuler example satisfies it, using a representation of the physical algebra coming from the usual indefinite metric Fock representation of the original algebra.

1 Introduction

In many quantum field theories there are constraints consisting of local expressions of the quantum fields, generally written as a selection condition for the physical subspace $\mathcal{H}^{(p)}$. In the physics literature the selection condition usually takes the form:

$$\mathcal{H}^{(p)} := \{\psi \mid \chi(x)\,\psi = 0 \quad \forall\, x \in \mathbb{R}^4\}$$

where χ is some operator–valued distribution (so more accurately $\chi(x)\,\psi = 0$ should be written as $\chi(f)\,\psi = 0$ for all test functions f). Since the constraints χ are constructed from the smeared quantum fields, one expects them to have the same locality structure in space–time as the quantum fields. The question then arises as to how locality and constraining intertwines. This will be at the focus of our interest in this paper.

To properly study locality questions, we shall use algebraic quantum field theory, a well–developed theory built on a net of C*–algebras satisfying the Haag–Kastler axioms [14, 13], but we shall assume in addition a local net of constraints (to be defined in Section 3). To impose these constraints at the algebraic level, we use the method developed by Grundling and Hurst [8], and this can be done either in each local algebra separately or globally in the full field algebra. We will compare the results of these two different routes, and will find conditions on the local net of constraints to ensure that the net of algebras obtained after constraining satisfies the Haag–Kastler axioms. In fact one can weaken the Haag–Kastler axioms on the original system, providing that after constraining the final net obtained satisfies these axioms. We characterise precisely what these "weak" Haag–Kastler axioms are. In our subsequent example (Gupta–Bleuler electromagnetism) we find that this weakening is crucial, since the original constraints violate the causality axiom. In our example we will avoid the usual indefinite metric representations, but will obtain by C*–algebraic means both the correct physical algebra, and the same (positive energy) representation than the one produced via the indefinite metric.

Since local constraints are usually generators of gauge transformations of the second kind, the theory developed here can be considered as complementing the deep Doplicher–Haag–Roberts analysis of systems with gauge transformations of the first kind [5]. Our axioms will be slightly different (weakened Haag–Kastler axioms), and we will work with an abstract net of C*–algebras, whereas the DHR analysis is done concretely in a positive energy representation.

The architecture of the paper is as follows. In Section 2 we collect general facts of the constraint procedure of [8] which we will need in the subsequent sections. There is some new material in this section, since we need to extend the previous method to cover the current situation. In Section 3 we introduce our basic object, a "system of local quantum constraints" as well as the "weak Haag–Kastler axioms" and prove that after local constraining of such a system, we obtain a system satisfying the Haag–Kastler axioms. Section 4 consists of our example, Gupta–Bleuler electromagnetism (C*–algebra version adapted from [6]) and we verify all the weak Haag–Kastler axioms for it. We concretely characterize the net of constrained algebras, but it turns out that in order to obtain a simple global algebra we need to do a second stage of constraining (traditionally thought of as imposing the Maxwell equations, but here it is slightly stronger than that). We also verify the weak Haag–Kastler axioms for this second stage of constraints. In Section 5 we consider miscellaneous topics raised by the previous Sections. First, for a system of local constraints, we consider the relation between the algebras obtained from a single global constraining, and the inductive limit of the algebras found from local constrainings. We show that for the Gupta–Bleuler example these two algebras are the same. Secondly, we develop the theory of constraint reduction by stages (i.e. impose the constraints sequentially along an increasing chain of subsets instead of all at once). Here we find a criterion when the final algebra obtained from an n–step reduction procedure will be the same as the one obtained from enforcing all constraints at once. We then show that the two–step reduction procedure of the Gupta–Bleuler example satisfies this criterion. Thirdly, we consider the spectral condition (on the generators of translations) which occur in Haag–Kastler theory, and find a "weak" version of it which will guarantee that the final constrained theory satisfies the usual spectral condition. We show that the Gupta–Bleuler example satisfies it, by demonstrating that from the indefinite metric in the heuristic theory we can define a (positive metric) representation of the constrained algebra which satisfies the spectral condition.

2 Kinematics for Quantum Constraints.

In this section we collect the mathematical structures associated with the kinematics of quantum constraints which we intend to use in the subsequent sections. This consists of a small generalisation of the T–procedure of Grundling and Hurst [8, 6] as well as some new material. All new results will be proven here and for other proofs we refer to the literature.

In heuristic physics a set of constraints is a set $\{A_i \mid i \in I\}$ (with I an index set) of operators on some Hilbert space together with a selection condition for the subspace of physical vectors:

$$\mathcal{H}^{(p)} := \overline{\Big\{\psi \mid A_i\psi = 0 \quad \forall\, i \in I\Big\}}.$$

The set of traditional observables is then the commutant $\{A_i \mid i \in I\}'$ which one can enlarge to the set of all observables which preserve $\mathcal{H}^{(p)}$. The final constrained system is the restriction of this algebra to the subspace $\mathcal{H}^{(p)}$. On abstraction of such a system into C*–algebra terms, one starts with a unital C*–algebra $\mathcal{F}$ (the *field algebra*) containing all physical relevant observables. This is an abstract C*–algebra, i.e. we ignore the initial representation in which the system may be defined. We need to decide in what form the constraints should appear in $\mathcal{F}$ as a subset $\mathcal{C}$. We have the following possibilities:

- If all A_i are bounded we can identify $\mathcal{C}$ directly with $\{A_i \mid i \in I\} \subset \mathcal{F}$.
- If the A_i are unbounded but essentially selfadjoint, we can take $\mathcal{C} := \{U - \mathbb{1} \mid U \in \mathcal{U}\} =: \mathcal{U} - \mathbb{1}$, where the set of unitaries $\mathcal{U} \subset \mathcal{F}$ is identified with $\{\exp(it\bar{A}_j) \mid t \in \mathbb{R},\ \ j \in I\}$. This is the form in which constraints were analyzed in [8], and also the form which we will use here in the following sections.
- If the A_i are unbounded and normal, we can identify $\mathcal{C}$ with $\{f(A_j) \mid j \in I\}$ where f is a bounded real valued Borel function with $f^{-1}(0) = \{0\}$.
- If the A_i are unbounded, closable and not normal, then we can replace each A_i by the essentially selfadjoint operator $A_i^* A_i$ which is justified because for any closed operator A we have $\mathrm{Ker}\, A = \mathrm{Ker}\, A^*A$, reducing this case to the one for essentially selfadjoint constraints.

Finally, notice that we can replace any constraint set $\mathcal{C}$ as above, by one which satisfies $\mathcal{C}^* = \mathcal{C}$ as a set and which selects the same physical subspace, using the fact that $\mathrm{Ker}\, A = \mathrm{Ker}\, A^*A$.

2.1 Imposing constraints: The T–procedure.

Motivated from above, our starting point is:

2.1 Definition *A* **quantum system with constraints** *is a pair* $(\mathcal{F},\ \mathcal{C})$ *where the* **field algebra** $\mathcal{F}$ *is a unital* C*–*algebra containing the* **constraint set** $\mathcal{C} = \mathcal{C}^*$, *on which we impose a* **constraint condition** *through selection of the physical state space (consisting of* **Dirac states***) by:*

$$\mathfrak{S}_D := \Big\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \pi_\omega(C)\Omega_\omega = 0 \quad \forall\, C \in \mathcal{C}\Big\},$$

where $\mathfrak{S}(\mathcal{F})$ *denotes the state space of* $\mathcal{F}$*, and* $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ *denotes the GNS–data of* ω*. The case of* **unitary constraints** *means that* $\mathcal{C} = \mathcal{U} - \mathbb{1}$, $\mathcal{U} \subset \mathcal{F}_u$*, and here we will also use the notation* $(\mathcal{F}, \mathcal{U})$.

Thus in the GNS-representation of each Dirac state, the GNS cyclic vector Ω_ω satisfies the physical selection condition above. The assumption is that all physical information is contained in the pair $(\mathcal{F}, \mathfrak{S}_D)$.

For the case of unitary constraints we have the following equivalent characterizations of the Dirac states (cf. [8, Theorem 2.19 (ii)]):

$$\begin{aligned}\mathfrak{S}_D &= \Big\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(U) = 1 \quad \forall\, U \in \mathcal{U}\Big\} && (1)\\ &= \Big\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(FU) = \omega(F) = \omega(UF) \quad \forall\, F \in \mathcal{F},\ U \in \mathcal{U}\Big\}. && (2)\end{aligned}$$

Moreover, the set $\{\alpha_U := \mathrm{Ad}(U) \mid U \in \mathcal{U}\}$ of automorphisms of $\mathcal{F}$ leaves every Dirac state invariant, i.e. we have $\omega \circ \alpha_U = \omega$ for all $\omega \in \mathfrak{S}_D$, $U \in \mathcal{U}$.

For a general constraint set $\mathcal{C}$, observe that we have:

$$\begin{aligned}\mathfrak{S}_D &= \Big\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(C^*C) = 0 \quad \forall\, C \in \mathcal{C}\Big\}\\ &= \Big\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \mathcal{C} \subseteq N_\omega\Big\} \;=\; \mathcal{N}^\perp \cap \mathfrak{S}(\mathcal{F})\,.\end{aligned}$$

Here $\mathcal{N} := \cap \{N_\omega \mid \omega \in \mathfrak{S}_D\}$, where $N_\omega := \{F \in \mathcal{F} \mid \omega(F^*F) = 0\}$ is the left kernel of ω and the superscript $\perp$ denotes the annihilator of the corresponding subset in the dual of $\mathcal{F}$. The equality $\mathcal{N} = [\mathcal{F}\mathcal{C}]$ (where we use the notation $[\cdot]$ for the closed linear space generated by its argument), follows from the fact that every closed left ideal is the intersection of the left kernels which contains it (cf. 3.13.5 in [22]). Thus $\mathcal{N}$ is the left ideal generated by $\mathcal{C}$. Since $\mathcal{C}$ is selfadjoint and contained in $\mathcal{N}$ we conclude $\mathcal{C} \subset \mathrm{C}^*(\mathcal{C}) \subset \mathcal{N} \cap \mathcal{N}^* = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$, where $\mathrm{C}^*(\cdot)$ denotes the C*–algebra in $\mathcal{F}$ generated by its argument.

2.2 Theorem *Now for the Dirac states we have:*

(i) $\mathfrak{S}_D \neq \emptyset$ *iff* $\mathbb{1} \notin \mathrm{C}^*(\mathcal{C})$ *iff* $\mathbb{1} \notin \mathcal{N} \cap \mathcal{N}^* =: \mathcal{D}$.

(ii) $\omega \in \mathfrak{S}_D$ *iff* $\pi_\omega(\mathcal{D})\Omega_\omega = 0$.

(iii) *An extreme Dirac state is pure.*

Proof: (i) The first equivalence is proven in Theorem 2.7 of [8]. If $\mathbb{1} \in \mathcal{D} \subset \mathcal{N}$, then $\omega(\mathcal{N}) \neq 0$ for all states ω, i.e. $\mathfrak{S}_D = \emptyset$. If $\mathbb{1} \notin \mathcal{D}$, then $\mathbb{1} \notin \mathcal{N}$ so $\mathcal{N}$ is a proper closed left ideal and hence by 3.10.7 in [22] $\mathfrak{S}_D \neq \emptyset$.

(ii) If $\omega \in \mathfrak{S}_D$, then $\mathcal{D} \subset \mathcal{N} \subset N_\omega$, hence $\pi_\omega(\mathcal{D})\Omega_\omega = 0$. Conversely, since $\mathcal{C} \subset \mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$ we have that $\pi_\omega(\mathcal{D})\Omega_\omega = 0$ implies $\pi_\omega(\mathcal{C})\Omega_\omega = 0$ hence $\omega \in \mathfrak{S}_D$.

(iii) Denote the quasi-state space of $\mathcal{F}$ by Q [22]. We can write the set of Dirac states as

$$\mathfrak{S}_D = \mathfrak{S}(\mathcal{F}) \cap \{\phi \in Q \mid \phi(L) = 0 \quad \forall\, L \in \mathcal{N}\}\,.$$

Since $\mathcal{N}$ is a left ideal, if it is in Ker ω, it must be in N_ω. By Theorem 3.10.7 in [22] the set $Q_0 := \{\phi \in Q \mid \phi(L) = 0 \quad \forall\, L \in \mathcal{N}\}$ is a weak* closed face in Q. Now if we can decompose a Dirac state ω, since it is in Q_0, so are its components by the facial property of Q_0. These components are multiples of Dirac states, dominated by ω so ω cannot be extreme. Thus extreme Dirac states must be pure. ■

We will call a constraint set $\mathcal{C}$ *first class* if $\mathbb{1} \notin \mathrm{C}^*(\mathcal{C})$, and this is the nontriviality assumption which we henceforth make [11, Section 3].

Now define

$$\mathcal{O} := \{F \in \mathcal{F} \mid [F,\, D] := FD - DF \in \mathcal{D} \quad \forall\, D \in \mathcal{D}\}.$$

Then $\mathcal{O}$ is the C*–algebraic analogue of Dirac's observables (the weak commutant of the constraints) [3].

2.3 Theorem *With the preceding notation we have:*

(i) $\mathcal{D} = \mathcal{N} \cap \mathcal{N}^*$ *is the unique maximal* C*–*algebra in* $\cap \{\mathrm{Ker}\,\omega \mid \omega \in \mathfrak{S}_D\}$. *Moreover* $\mathcal{D}$ *is a hereditary* C*–*subalgebra of* $\mathcal{F}$.

(ii) $\mathcal{O} = M_{\mathcal{F}}(\mathcal{D}) := \{F \in \mathcal{F} \mid FD \in \mathcal{D} \ni DF \quad \forall\, D \in \mathcal{D}\}$, *i.e. it is the relative multiplier algebra of* $\mathcal{D}$ *in* $\mathcal{F}$.

(iii) $\mathcal{O} = \{F \in \mathcal{F} \mid \ [F, \mathcal{C}] \subset \mathcal{D}\}$.

(iv) $\mathcal{D} = [\mathcal{O}\mathcal{C}] = [\mathcal{C}\mathcal{O}]$.

(v) *For the case of unitary constraints, i.e.* $\mathcal{C} = \mathcal{U} - \mathbb{1}$, *we have* $\mathcal{U} \subset \mathcal{O}$ *and* $\mathcal{O} = \{F \in \mathcal{F} \mid \alpha_U(F) - F \in \mathcal{D} \quad \forall\, U \in \mathcal{U}\}$ *where* $\alpha_U := \mathrm{Ad}\, U$.

Proof: (i) The proof of Theorem 2.13 in [8] is still valid for the current more general constraints. To see that $\mathcal{D}$ is hereditary, use Theorem 3.2.1 in [21] and the fact that $\mathcal{N} = [\mathcal{F}\mathcal{C}]$ is a closed left ideal of $\mathcal{F}$.

(ii) Since $\mathcal{D}$ is a two–sided ideal for $M_\mathcal{F}(\mathcal{D})$ it is obvious that $M_\mathcal{F}(\mathcal{D}) \subset \mathcal{O}$. Conversely, consider $B \in \mathcal{O}$, then for any $D \in \mathcal{D}$, we have $BD = DB + D' \in \mathcal{N}$ with D' some element of $\mathcal{D}$, where we used $\mathcal{F}\mathcal{D} = \mathcal{F}(\mathcal{N} \cap \mathcal{N}^*) \subset \mathcal{N}$. Similarly we see that $DB \in \mathcal{N}^*$. But then $\mathcal{N} \ni BD = DB + D' \in \mathcal{N}^*$, so $BD \in \mathcal{N} \cap \mathcal{N}^* = \mathcal{D}$. Likewise $DB \in \mathcal{D}$ and so $B \in M_\mathcal{F}(\mathcal{D})$.

(iii) Since $\mathcal{C} \subset \mathcal{D}$ we see from the definition of $\mathcal{O}$ that $F \in \mathcal{O}$ implies that $[F,\, \mathcal{C}] \subset \mathcal{D}$. Conversely, let $[F,\, \mathcal{C}] \subset \mathcal{D}$ for some $F \in \mathcal{F}$. Now $F[\mathcal{F}\mathcal{C}] \subset [\mathcal{F}\mathcal{C}]$ and $F[\mathcal{C}\mathcal{F}] = [F\mathcal{C}\mathcal{F}] \subset [(\mathcal{C}F + \mathcal{D})\mathcal{F}] \subset [\mathcal{C}\mathcal{F}]$ because $\mathcal{C}F + \mathcal{D} \subset \mathcal{C}F + [\mathcal{C}\mathcal{F}] \subset [\mathcal{C}\mathcal{F}]$. Thus $F\mathcal{D} = F([\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]) \subset [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}] = \mathcal{D}$. Similarly $\mathcal{D}F \subset \mathcal{D}$, and thus by (ii) we see $F \in \mathcal{O}$.

(iv) $\mathcal{D} \subset \mathcal{O}$ so by (i) it is the unique maximal C*–algebra annihilated by all the states $\omega \in \mathfrak{S}_D(\mathcal{O}) = \mathfrak{S}_D \restriction \mathcal{O}$ (since $\mathcal{C} \subset \mathcal{O}$). Thus $\mathcal{D} = [\mathcal{O}\mathcal{C}] \cap [\mathcal{C}\mathcal{O}]$. But $\mathcal{C} \subset \mathcal{D}$, so by (ii) $[\mathcal{O}\mathcal{C}] \subset \mathcal{D} \subset [\mathcal{O}\mathcal{C}]$ and so $\mathcal{D} = [\mathcal{O}\mathcal{C}] = [\mathcal{C}\mathcal{O}]$.

(v) $\mathcal{U} \subset \mathcal{O}$ because $\mathcal{U} - \mathbb{1} \subset \mathcal{D} \subset \mathcal{O} \ni \mathbb{1}$, and so $[F,\, \mathcal{C}] \subset \mathcal{D}$ implies $[U - \mathbb{1},\, F]U^{-1} = \alpha_U(F) - F \in \mathcal{D}$ for $U \in \mathcal{U}$. The converse is similar. ■

Thus $\mathcal{D}$ is a closed two-sided ideal of $\mathcal{O}$ and it is proper when $\mathfrak{S}_D \neq \emptyset$ (which we assume here by $\mathbb{1} \notin C^*(\mathcal{C})$). Below we will see that the traditional observables $\mathcal{C}' \subset \mathcal{O}$, where $\mathcal{C}'$ denotes the relative commutant of $\mathcal{C}$ in $\mathcal{F}$.

Define the *maximal* C*–*algebra of physical observables* as

$$\mathcal{R} := \mathcal{O}/\mathcal{D}.$$

The factoring procedure is the actual step of imposing constraints. We require that after the T–procedure all physical information is contained in the pair $(\mathcal{R}, \mathfrak{S}(\mathcal{R}))$, where $\mathfrak{S}(\mathcal{R})$ denotes the set of states on $\mathcal{R}$. Now, it is possible that $\mathcal{R}$ may not be simple [9, Section 2], and this would not be acceptable for a physical algebra. So, using physical arguments, one would in practice choose a C*–subalgebra $\mathcal{O}_c \subset \mathcal{O}$ containing the traditional observables $\mathcal{C}'$ such that

$$\mathcal{R}_c := \mathcal{O}_c/(\mathcal{D} \cap \mathcal{O}_c) \subset \mathcal{R}\,,$$

is simple. The following result justifies the choice of $\mathcal{R}$ as the algebra of physical observables.

2.4 Theorem *There exists a w*–continuous isometric bijection between the Dirac states on $\mathcal{O}$ and the states on $\mathcal{R}$.*

2.2 Hereditary structure.

Next we wish to gain further understanding of these algebras by exploiting the hereditary property of $\mathcal{D}$. Denote by π_u the universal representation of $\mathcal{F}$ on the universal Hilbert space $\mathcal{H}_u$ [22, Section 3.7]. $\mathcal{F}''$ is the strong closure of $\pi_u(\mathcal{F})$ and since π_u is faithful we make the usual identification of $\mathcal{F}$ with a subalgebra of $\mathcal{F}''$, i.e. generally omit explicit indication of π_u. If $\omega \in \mathfrak{S}(\mathcal{F})$, we will use the same symbol for the unique extension of ω from $\mathcal{F}$ to $\mathcal{F}''$.

2.5 Theorem *For a constrained system $(\mathcal{F}, \mathcal{C})$ there exists a projection*[1] *$P \in \mathcal{F}''$ such that $\mathcal{D} = P\mathcal{F}''P \cap \mathcal{F}$. $P \in \mathcal{F}''$ such that*

(i) $\mathcal{N} = \mathcal{F}''\,P \cap \mathcal{F}$,

(ii) $\mathcal{D} = P\,\mathcal{F}''\,P \cap \mathcal{F}$ *and*

[1] Such a projection is called *open* in [22].

(iii) $\mathfrak{S}_D = \{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(P) = 0\}$.

Proof: From Theorem 2.3 (i) $\mathcal{D}$ is a hereditary C*–subalgebra of $\mathcal{F}$ and by 3.11.10 and 3.11.9 in [22] there exists a projection $P \in \mathcal{F}''$ such that $\mathcal{D} = P\mathcal{F}''P \cap \mathcal{F}$. Further by the proof of Theorem 2.2 (iii) as well as 3.10.7 and 3.11.9 in [22] we obtain that $\mathcal{N} = \mathcal{F}''\,P \cap \mathcal{F}$ and

$$\mathfrak{S}_D = \{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(P) = 0\}\,,$$

which concludes this proof. ■

2.6 Theorem *Let P be the open projection in Theorem 2.5. Then:*

(i) $\mathcal{O} = \{A \in \mathcal{F} \mid PA(\mathbb{1} - P) = 0 = (\mathbb{1} - P)AP\} = P' \cap \mathcal{F}$, *and*

(ii) $\mathcal{C}' \cap \mathcal{F} \subset \mathcal{O}$.

Proof: (i) Recall that $\mathcal{O} = M_{\mathcal{F}}(\mathcal{D})$, and let $A \in \mathcal{F}$ and $D \in \mathcal{D}$. Then by Theorem 2.5 there exists an $F \in \mathcal{F}''$ such that $D = PFP$ and so

$$\begin{aligned} AD &= (PAP + (\mathbb{1} - P)AP + PA(\mathbb{1} - P) + (\mathbb{1} - P)A(\mathbb{1} - P))\,PFP \\ &= PAPFP + (\mathbb{1} - P)APFP \\ &= PAPD + (\mathbb{1} - P)APD\,. \end{aligned}$$

Therefore using Theorem 2.5 again we have $AD \in \mathcal{D}$ for all $D \in \mathcal{D}$ iff $(\mathbb{1} - P)APD = 0$ for all $D \in \mathcal{D}$. But from 3.11.9 in [22] P is in the strong closure of $\mathcal{D}$ in $\mathcal{F}''$ so that $AD \in \mathcal{D}\ \forall\, D \in \mathcal{D}$ iff $(\mathbb{1} - P)AP = 0$. Taking adjoints we get also the condition $PA(\mathbb{1} - P) = 0$ iff $\mathcal{D}A \subset \mathcal{D}$.

(ii) Let $D \in \mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$ and $A \in \mathcal{C}' \cap \mathcal{F}$. Then $AD \in [\mathcal{F}\mathcal{C}] \cap [A\,\mathcal{C}\mathcal{F}] = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\,A\,\mathcal{F}] \subset \mathcal{D}$. Similarly, $DA \in \mathcal{D}$ so that by definition we have $A \in \mathcal{O}$. ■

What these two last theorems mean, is that with respect to the decomposition

$$\mathcal{H}_u = P\,\mathcal{H}_u \oplus (\mathbb{1} - P)\,\mathcal{H}_u$$

we may rewrite

$$\begin{aligned} \mathcal{D} &= \left\{ F \in \mathcal{F} \mid F = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\,, \quad D \in P\mathcal{F}P \right\} \\ \text{and} \qquad \mathcal{O} &= \left\{ F \in \mathcal{F} \mid F = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\,, \quad A \in P\mathcal{F}P, \quad B \in (\mathbb{1} - P)\mathcal{F}(\mathbb{1} - P) \right\}. \end{aligned}$$

It is clear that in general $\mathcal{O}$ can be much greater than the traditional observables $\mathcal{C}' \cap \mathcal{F}$. Next we show how to identify the final algebra of physical observables $\mathcal{R}$ with a subalgebra of $\mathcal{F}''$.

2.7 Theorem *For P as above we have:*

$$\mathcal{R} \cong \left\{ F \in \mathcal{F} \mid F = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \right\} = (\mathbb{1} - P)\,(P' \cap \mathcal{F}) \subset \mathcal{F}''.$$

Proof: The homomorphism Φ: $\mathcal{O} \to (\mathbb{1} - P)\,(P' \cap \mathcal{F})$ defined by $\Phi(A) := (\mathbb{1} - P)A$, $A \in \mathcal{O} = P' \cap \mathcal{F}$, will establish an isomorphism with $\mathcal{R} := \mathcal{O}/\mathcal{D}$ if we can show that $\operatorname{Ker}\Phi = \mathcal{D}$, i.e. $(\mathbb{1} - P)A = 0$ iff $A \in \mathcal{D}$. Clearly, if $A \in \mathcal{D} = P\,\mathcal{F}''\,P \cap \mathcal{F}$ (cf. Theorem 2.5), then $(\mathbb{1} - P)A = 0$. Conversely, assume that $A \in \mathcal{O} = P' \cap \mathcal{F}$ satisfies $(\mathbb{1} - P)A = 0$, i.e. $A = PA$. Then, $A \in P\,\mathcal{F}'' \cap \mathcal{F}$ and so since $A \in P' \cap \mathcal{F}$, we have $A \in P\,\mathcal{F}''\,P \cap \mathcal{F} = \mathcal{D}$, which ends the proof. ■

2.8 Remark (i) With the preceding result we may interpret the projection P in Theorem 2.5 as replacing the set $\mathcal{C}$ if we are willing to enlarge $\mathcal{F}$ to $\mathrm{C}^*(\mathcal{F} \cup \{P\})$. This can be partially justified by the fact that

$$\mathcal{R} \cong (\mathbb{1} - P)\,(P' \cap \mathcal{F}) \subset \mathrm{C}^*(\mathcal{F} \cup \{P\})\,.$$

(ii) The projection P can also be used to make contact with the original heuristic picture. Given a Dirac state $\omega \in \mathfrak{S}_D$ we see from Theorem 2.5 (iii) that $\mathbb{1} - \pi_\omega(P)$ is the projection onto the physical subspace

$$\mathcal{H}_\omega^{(p)} := \{\psi \in \mathcal{H}_\omega \mid \pi_\omega(\mathcal{C})\psi = 0\}.$$

Since $\pi_\omega(\mathcal{O}) \subset \pi_\omega(P)' \cap \pi_\omega(\mathcal{F})$, then $\pi_\omega(\mathcal{O})$ is a subalgebra of the algebra of observables in the field algebra $\pi_\omega(\mathcal{F})$ which preserves the physical subspace. (In fact $\mathcal{O} = \{F \in \mathcal{F} \mid \pi_\omega(F)\mathcal{H}_\omega^{(p)} \subseteq \mathcal{H}_\omega^{(p)} \quad \forall\, \omega \in \mathfrak{S}(\mathcal{F})\}$). If π_ω is faithful, then $\pi_\omega(\mathcal{O})$ contains the traditional observables $\pi_\omega(\mathcal{C})' \cap \pi_\omega(\mathcal{F})$. Now restricting $\pi_\omega(\mathcal{O})$ to the subspace $\mathcal{H}_\omega^{(p)} = (\mathbb{1} - \pi_\omega(P))\,\mathcal{H}_\omega$ we have for the final constrained system:

$$\pi_\omega(\mathcal{O})\restriction\mathcal{H}_\omega^{(p)} = (\mathbb{1} - \pi_\omega(P))\pi_\omega(\mathcal{O}) = \pi_\omega((\mathbb{1} - P)(P' \cap \mathcal{F})),$$

which by Theorem 2.7 produces a representation of $\mathcal{R}$.

2.3 Equivalent constraints.

We will need in Section 3 the notion of equivalent constraint sets.

2.9 Definition *Two first class constraint sets $\mathcal{C}_1$, $\mathcal{C}_2$ for the field algebra $\mathcal{F}$ are called* **equivalent** *if they select the same Dirac states, i.e. if for any state $\omega \in \mathfrak{S}(\mathcal{F})$ we have*

$$\mathcal{C}_1 \subseteq N_\omega \quad \text{iff} \quad \mathcal{C}_2 \subseteq N_\omega .$$

In this case we denote $\mathcal{C}_1 \sim \mathcal{C}_2$, and for unitary constraints situation $\mathcal{C}_i = \mathcal{U}_i - \mathbb{1}$, $i = 1, 2$, we also write $\mathcal{U}_1 \sim \mathcal{U}_2$.

2.10 Remark (i) It is clear that the preceding definition introduces an equivalence relation on the family of first class constraint sets for $\mathcal{F}$. Denote by $\mathcal{D}_i = [\mathcal{F}\mathcal{C}_i] \cap [\mathcal{C}_i\mathcal{F}]$, $i = 1, 2$, the C*–algebras of Theorem 2.2 (i). Now $\mathcal{C}_1 \sim \mathcal{C}_2$ iff $\mathcal{D}_1 = \mathcal{D}_2$ by Theorems 2.3 (i) and 2.2 (ii), therefore the corresponding observables $\mathcal{O}_i$ and physical observables $\mathcal{R}_i$ given by the T–procedure will also coincide. This justifies calling these constraint sets equivalent – the replacement of $\mathcal{C}_1$ by $\mathcal{C}_2$ leaves the physics unchanged. Also note that $\mathcal{C} \sim [\mathcal{C}] \sim \mathrm{C}^*(\mathcal{C})$. If $\mathcal{C}_1 \sim \mathcal{C}_2$, one can have that $\mathcal{C}_1' \neq \mathcal{C}_2'$, i.e. the traditional observables is more sensitive to the choice of constraints than $\mathcal{O}$.

(ii) Whilst the definition of equivalence $\mathcal{C}_1 \sim \mathcal{C}_2$ as stated, depends on $\mathcal{F}$, it depends in fact only on the subalgebra $\mathrm{C}^*(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{\mathbb{1}\}) =: \mathcal{A}$. This is because the extension (resp. restriction) of a Dirac state from (resp. to) a unital C*–subalgebra containing the constraints, is again a Dirac state. Explicitly, the condition: $\omega(C^*C) = 0$ for all $C \in \mathcal{C}_1$ iff $\omega(C^*C) = 0$ for all $C \in \mathcal{C}_2$, clearly depends only on the behaviour of ω on $\mathcal{A}$.

Next we give an algebraic characterization of equivalent constraints, and introduce a maximal constraint set associated to an equivalence class of constraint sets. In the case of unitary constraints, $\mathcal{C}_i = \mathcal{U}_i - \mathbb{1}$, we obtain a unitary group in $\mathcal{F}$.

2.11 Theorem *Let $\mathcal{C}_i$, $i = 1, 2$, be two first class constraint sets for $\mathcal{F}$, with associated algebras $\mathcal{D}_i$ as above. Then*

(i) $\mathcal{C}_1 \sim \mathcal{C}_2 \quad$ *iff* $\quad \mathcal{C}_1 - \mathcal{C}_2 \subset \mathcal{D}_1 \cap \mathcal{D}_2$.
In the case when $\mathcal{C}_i = \mathcal{U}_i - \mathbb{1}$, we have $\mathcal{U}_1 \sim \mathcal{U}_2 \quad$ iff $\quad \mathcal{U}_1 - \mathcal{U}_2 \subset \mathcal{D}_1 \cap \mathcal{D}_2$.

(ii) *The maximal constraint set which is equivalent to $\mathcal{C}_1$ is $\mathcal{D}_1$. In the case when $\mathcal{C}_1 = \mathcal{U}_1 - \mathbb{1}$, the set of unitaries*

$$\mathcal{U}_m := \bigcup_{\mathcal{U} \sim \mathcal{U}_1} \mathcal{U} \subset \mathcal{F}_u \,,$$

is the maximal set of unitaries equivalent to $\mathcal{U}_1$, and it is a group with respect to multiplication in $\mathcal{F}$.

Proof: (i) Suppose that $\mathcal{C}_1 \sim \mathcal{C}_2$ so that by the Remark 2.10 (i), $\mathcal{D}_1 = \mathcal{D}_2$. Then, we have that $\mathcal{C}_1 \subset \mathcal{D}_1 = \mathcal{D}_2 \supset \mathcal{C}_2$, and hence $\mathcal{C}_1 - \mathcal{C}_2 \subset \mathcal{D}_1 = \mathcal{D}_1 \cap \mathcal{D}_2$.

Conversely, assume $\mathcal{C}_1 - \mathcal{C}_2 \subset \mathcal{D}_1 \cap \mathcal{D}_2$. If $\omega \in \mathfrak{S}(\mathcal{F})$ satisfies $\pi_\omega(\mathcal{C}_1)\Omega_\omega = 0$, then by assumption, $\pi_\omega(\mathcal{C}_1)\Omega_\omega - \pi_\omega(\mathcal{C}_2)\Omega_\omega \subseteq \pi_\omega(\mathcal{D}_1 \cap \mathcal{D}_2)\Omega_\omega \subset \pi_\omega(\mathcal{D}_1)\Omega_\omega = 0$ using Theorem 2.2 (ii). Thus $\pi_\omega(\mathcal{C}_2)\Omega_\omega = 0$, i.e. $\mathcal{C}_1 \subset N_\omega$ implies that $\mathcal{C}_2 \subset N_\omega$. Interchanging the roles of $\mathcal{C}_1$ and $\mathcal{C}_2$, we conclude that $\mathcal{C}_1 \sim \mathcal{C}_2$. The second claim for $\mathcal{C}_i = \mathcal{U}_i - \mathbb{1}$ follows from $\mathcal{C}_1 - \mathcal{C}_2 = \mathcal{U}_1 - \mathcal{U}_2$.

(ii) That $\mathcal{D}_1 \sim \mathcal{C}_1$ is just the content of Theorem 2.2 (ii). That it is maximal follows from the implication $\mathcal{C}_2 \sim \mathcal{C}_1 \;\Rightarrow\; \mathcal{C}_2 \subseteq \mathcal{D}_2 = \mathcal{D}_1$.

For unitary constraints, since $\mathcal{U}_m := \bigcup_{\mathcal{U} \sim \mathcal{U}_1} \mathcal{U}$ it follows from part (i) that $\mathcal{U}_1 - \mathcal{U}_m \subset \mathcal{D}_1$. Further $\mathcal{U}_1 \subset \mathcal{U}_m$ implies also $\mathcal{D}_1 \subset \mathcal{D}_m$, so that $\mathcal{U}_1 - \mathcal{U}_m \subset \mathcal{D}_1 \cap \mathcal{D}_m = \mathcal{D}_1$ and $\mathcal{U}_m \sim \mathcal{U}_1$. By construction it is also clear that $\mathcal{U}_m$ is the maximal unitary constraint set in $\mathcal{F}$ equivalent to $\mathcal{U}_1$. We only have to prove that $\mathcal{U}_m$ is a group. Let $\mathcal{U}_0$ be the group generated in $\mathcal{F}$ by $\mathcal{U}_m$. If $\omega \in \mathfrak{S}(\mathcal{F})$ satisfies $\omega(\mathcal{U}_m) = 1$, we have $1 = \overline{\omega(U)} = \omega(U^*) = \omega(U^{-1})$, $U \in \mathcal{U}_m$, and also $\omega(UV) = \omega(U) = 1$, $U, V \in \mathcal{U}_m$, i.e. $\omega(\mathcal{U}_0) = 1$. Thus $\mathcal{U}_0 \sim \mathcal{U}_m \sim \mathcal{U}$ and maximality implies $\mathcal{U}_0 = \mathcal{U}_m$. Hence $\mathcal{U}_m$ is a group. ■

2.12 Remark Observe that for a given unitary constraint system $(\mathcal{F}, \mathcal{U})$ we have that $\mathcal{U}_m = \{U \in \mathcal{F}_u \mid \omega(U) = 1 \quad \forall\, \omega \in \mathfrak{S}_D\} = \{U \in \mathcal{O}_u \mid \omega(U) = 1 \quad \forall\, \omega \in \mathfrak{S}_D\}$ since $\mathcal{U}_m \subset \mathcal{O}$, cf. Theorem 2.3 (v).

Recall from Remark 2.8(i) that in the larger algebra $\mathrm{C}^*(\mathcal{F} \cup \{P\})$ we have a single constraint P which is equivalent to $\mathcal{C}$. Next we show that for a large class of first class constraint systems $(\mathcal{F}, \mathcal{C})$ we can find a single constraint in $\mathcal{F}$ which is equivalent to $\mathcal{C}$, and hence can replace it.

2.13 Theorem *If $[\mathcal{C}]$ is separable, there exists a positive element $C \in \mathcal{D}_+$ such that $\{C\} \sim \mathcal{C}$.*

Proof: Let $\{C_n\}_{n=1}^\infty$ be a denumerable basis of $[\mathcal{C}]$ such that $\|C_n\| < 1$, $n \in \mathbb{N}$. Then

$$\mathfrak{S}_D = \Big\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(C_n^* C_n) = 0 \quad \forall\, n \in \mathbb{N}\Big\}\,.$$

Define

$$C := \sum_{n=1}^\infty \frac{C_n^* C_n}{2^n} \in \mathcal{D}_+\,.$$

Then $\omega(C) = 0$ iff $\omega(C_n^* C_n) = 0$ for all $n \in \mathbb{N}$, which proves that $\mathfrak{S}_D = \{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(C) = 0\}$. Thus $C^{\frac{1}{2}} \in \mathcal{N}$, but since for any positive operator A we have $\mathrm{Ker}\, A = \mathrm{Ker}\, A^n$, for all $n \in \mathbb{N}$, we conclude $\mathfrak{S}_D = \{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(C^2) = 0\}$, so $\{C\} \sim \mathcal{C}$. ■

2.14 Remark Note that from [21, p. 85] the preceding statement is not true if the separability condition is relaxed.

2.15 Theorem *Let $(\mathcal{F},\,\mathcal{C})$ be a first class constraint system, then there is a set of unitaries $\mathcal{U}\subset\mathcal{F}_u$ such that $\mathcal{C}\sim\mathcal{U}-\mathbb{1}$ and $\mathcal{U}=\mathcal{U}^*$.*

Proof: Define the unitaries $\mathcal{U}:=\{\exp(itD)\mid t\in\mathbb{R},\ D\in\mathcal{D}_+\}$. Then for $\omega\in\mathfrak{S}(\mathcal{F})$ we have that $1=\omega(\exp\,itD)=1+\sum\limits_{k=1}^{\infty}(it)^k\omega(D^k)/k!$ for all $t\in\mathbb{R}$, $D\in\mathcal{D}_+$ iff $\omega(D)=0$ for all $D\in\mathcal{D}_+$ iff $\omega(\mathcal{D})=0$ iff $\omega\in\mathfrak{S}_D$. Thus $\mathcal{U}-\mathbb{1}\ \sim\ \mathcal{C}$. It is obvious that $\mathcal{U}=\mathcal{U}^*$. ■

Hence no constraint system is excluded by the assumption of unitary constraints. Moreover, by Theorem 2.11 there is a canonical unitary group $\mathcal{U}_m$ associated with each first class constraint system, and hence a group of inner automorphisms $\mathrm{Ad}\,\mathcal{U}_m$, which one can take as a gauge group in the absence of any further physical restrictions.

2.4 Outer constraints.

For Gupta–Bleuler electromagnetism, which is our example in Section 4 we will need the following variant where the constraints are defined through a group action which is not necessarily inner. One assumes, following [6] that:

- There is a distinguished group action $\beta\colon\ \mathcal{G}\to\mathrm{Aut}\,\mathcal{F}$ on the field algebra $\mathcal{F}$, and all physical information is contained in $\mathcal{F}$ and its set of invariant states:

$$\mathfrak{S}^{\mathcal{G}}(\mathcal{F}):=\{\omega\in\mathfrak{S}(\mathcal{F})\mid\omega(\beta_g(A))=\omega(A)\quad\forall\,g\in\mathcal{G},\ A\in\mathcal{F}\}\,.$$

If $\mathcal{G}$ is locally compact, we can construct the (abstract) multiplier algebra of the crossed product $\mathcal{F}_e=M(\mathcal{G}\underset{\beta}{\times}\mathcal{F})$ and otherwise we will just take the discrete crossed product. In either case we obtain a C*–algebra $\mathcal{F}_e\supset\mathcal{F}$ which contains unitaries U_g for all $g\in\mathcal{G}$ that implement $\beta\colon\ \mathcal{G}\to\mathrm{Aut}\,\mathcal{F}$. Then this situation is reduced to the previous one by the following theorem [6, Section 3]:

2.16 Theorem *$\mathfrak{S}^{\mathcal{G}}(\mathcal{F})$ is precisely the restriction to $\mathcal{F}$ of the Dirac states on $\mathcal{F}_e$ with respect to $\mathcal{C}=U_{\mathcal{G}}-\mathbb{1}$, i.e. $\mathfrak{S}^{\mathcal{G}}(\mathcal{F})=\mathfrak{S}_D(\mathcal{F}_e)\restriction\mathcal{F}$ where*

$$\mathfrak{S}_D(\mathcal{F}_e):=\{\omega\in\mathfrak{S}(\mathcal{F}_e)\mid\omega(U_g)=1\quad\forall\,g\in\mathcal{G}\}\ .$$

Hence we can apply the T–procedure to $U_{\mathcal{G}}-\mathbb{1}$ in $\mathcal{F}_e$, and intersect the resulting algebraic structures with $\mathcal{F}$. This is called the *outer* constraint situation.

2.5 Bosonic constraints.

For free bosons, one takes for $\mathcal{F}$ the C*–algebra of the CCRs, which we now define following Manuceau [19, 20]. Let $\mathfrak{X}$ be a linear space and B a (possibly degenerate) symplectic form on it. Denote by $\Delta(\mathfrak{X},\,B)$ the linear space of complex–valued functions on $\mathfrak{X}$ with finite support. It has as linear basis the set $\{\delta_f\mid f\in\mathfrak{X}\}$, where

$$\delta_f(h):=\left\{\begin{array}{ll}1 & \text{if}\ \ f=h\\ 0 & \text{if}\ \ f\neq h.\end{array}\right.$$

Make $\Delta(\mathfrak{X},\,B)$ into a *–algebra, by defining the following product $\delta_f\cdot\delta_h:=e^{\frac{i}{2}B(f,\,h)}\,\delta_{f+h}$ and involution $(\delta_f)^*:=\delta_{-f}$, where $f,\ h\in S$ and the identity is δ_0. Let $\Delta_1(\mathfrak{X},\,B)$ be the closure of $\Delta(\mathfrak{X},\, B)$ w.r.t. the norm $\left\| \sum\limits_{i=1}^{m} \alpha_i \delta_{f_i} \right\|_1 := \sum\limits_{i=1}^{m} |\alpha_i|$, $\alpha_i \in \mathbb{C}$, then the CCR–algebra $\overline{\Delta(\mathfrak{X},\, B)}$ is defined as the enveloping C*–algebra of the the Banach *-algebra $\Delta_1(\mathfrak{X},\, B)$. That is, it is the closure with respect to the enveloping C*-norm:

$$\|A\| := \sup_{\omega \in \mathfrak{S}(\Delta_1(\mathfrak{X},\, B))} \sqrt{\omega(A^*A)}\,.$$

It is well–known (cf. [20]) that:

2.17 Theorem *$\overline{\Delta(\mathfrak{X},\, B)}$ is simple iff B is nondegenerate.*

An important state on $\overline{\Delta(\mathfrak{X},\, B)}$ is the central state defined by

$$\omega_0\Big(\delta_f\Big) := \begin{cases} 1 & \text{if } \ f = 0 \\ 0 & \text{otherwise}\,. \end{cases} \tag{3}$$

Using it, we make the following useful observations. The relation between the norms on $\Delta(\mathfrak{X},\, B)$ is

$$\|F\|_2 := \Big(\sum_{i=1}^{n} |\lambda_i|^2\Big)^{1/2} = \omega_0(F^*F)^{1/2} \leq \|F\| \leq \|F\|_1 \qquad \text{for} \qquad F = \sum_{i=1}^{n} \lambda_i \delta_{f_i}$$

and hence $\Delta_1(\mathfrak{X},\, B) \subset \overline{\Delta(\mathfrak{X},\, B)} \subset \ell^2(\mathfrak{X})$, so we can write an $A \in \overline{\Delta(\mathfrak{X},\, B)}$ as $A = \sum\limits_{i=1}^{\infty} \lambda_i \delta_{f_i}$ where $\{\lambda_i\} \in \ell^2$ and $f_i = f_j$ iff $i = j$. Let $A_n := \sum\limits_{i=1}^{M_n} \gamma_i^{(n)}\, \delta_{f_i^{(n)}} \subset \Delta(\mathfrak{X},\, B)$ converge to $A \in \overline{\Delta(\mathfrak{X},\, B)}$ in C*–norm then since the family of $f_i^{(n)}$ is denumerable we can arrange it into a single sequence f_i and thus write $A_n := \sum\limits_{i=1}^{N_n} \lambda_i^{(n)}\, \delta_{f_i}$. We shall frequently use this way of writing a Cauchy sequence in $\Delta(\mathfrak{X},\, B)$.

Now to define a constrained system corresponding to linear selfadjoint constraints in $\mathcal{F} = \overline{\Delta(\mathfrak{X},\, B)}$, we choose a set $\mathcal{C} = \mathcal{U} - \mathbb{1}$ where $\mathcal{U} = \{\delta_f \mid f \in \mathfrak{s}\}$ and $\mathfrak{s} \subset \mathfrak{X}$ is a subspace corresponding to the "test functions" of the heuristic constraints.

2.18 Theorem *Define the symplectic commutant $\mathfrak{s}' := \{f \in \mathfrak{X} \mid B(f,\, \mathfrak{s}) = 0\}$, then $\mathcal{C} = \mathcal{U} - \mathbb{1}$ is first class iff $\mathfrak{s} \subseteq \mathfrak{s}'$.*

For the proof, see Lemma 6.1 in [11].

Above we saw that for the observable algebra we sometimes need to choose a smaller algebra $\mathcal{O}_c \subset \mathcal{O}$ in order to ensure that the physical algebra $\mathcal{R}_c$ is simple. For bosonic constraints with nondegenerate B, such an algebra is $\mathcal{O}_c = \mathrm{C}^*(\delta_{\mathfrak{s}'}) = \mathcal{C}'$ (in which case $\mathcal{D} \cap \mathcal{O}_c = \mathrm{C}^*(\delta_{\mathfrak{s}'}) \cdot \mathrm{C}^*(\delta_{\mathfrak{s}} - \mathbb{1})$), which is what was chosen in [8, 11]. However, we now show that with this choice we have in fact $\mathcal{R}_c = \mathcal{R}$, i.e. we obtain the same physical algebra than with the full T–procedure, so nothing was lost by this choice of $\mathcal{O}_c$.

2.19 Theorem *Given nondegenerate $(\mathfrak{X},\, B)$ and $\mathfrak{s} \subset \mathfrak{X}$ as above, where $\mathfrak{s} \subset \mathfrak{s}'$ and $\mathfrak{s} = \mathfrak{s}''$, then*

$$\mathcal{O} = \mathrm{C}^*(\delta_{\mathfrak{s}'} \cup \mathcal{D}) = \Big[\mathrm{C}^*(\delta_{\mathfrak{s}'}) \cup \mathcal{D}\Big]\,.$$

Proof: The proof of this is new but long, so we put it in the Appendix. ∎

2.20 Theorem *Consider a nondegenerate symplectic space $(\mathfrak{X}, B)$ and a first class set $\mathfrak{s} \subset \mathfrak{X}$. Denote by $\widetilde{B}$ the lift of B to the factor space $\mathfrak{s}'/\mathfrak{s}$. Then we have the following isomorphism:*

$$\mathrm{C}^*(\delta_{\mathfrak{s}'}) \;/\; \mathrm{C}^*(\delta_{\mathfrak{s}} - \mathbb{1})\, \mathrm{C}^*(\delta_{\mathfrak{s}'}) \cong \overline{\Delta(\mathfrak{s}'/\mathfrak{s}\,,\, \widetilde{B})}\,.$$

In particular, if $\mathfrak{s} = \mathfrak{s}''$, then $(\mathfrak{s}'/\mathfrak{s}\,,\, \widetilde{B})$ is nondegenerate, so the above CCR–algebra is simple, and using Theorem 2.19 we have

$$\mathcal{R} \cong \overline{\Delta(\mathfrak{s}'/\mathfrak{s}\,,\, \widetilde{B})}\,.$$

For proofs and further details see [6, Theorem 5.2 and 5.3, as well as Corollary 5.4 and 5.5]. The surprise is that for linear bosonic constrained systems, the choice of traditional observables $\mathcal{O}_c = \mathcal{C}'$ produces the same physical algebra $\mathcal{R}$ than the T–procedure, which is not true in general.

A typical pathology which occurs for bosonic constraints, is that the Dirac states are not regular, i.e. the one parameter groups $\mathbb{R} \ni t \mapsto \pi_\omega(\delta_{tf})$ for $\omega \in \mathfrak{S}_D$ will not be strong operator continuous for all $f \in \mathfrak{X}$, and so the corresponding generators (which are the smeared quantum fields in many models of bosonic fields), will not exist for some $f \in \mathfrak{X}$, cf. [10]. The resolution of this, is that the pathology only occurs on nonphysical elements, i.e. on $\delta_f \notin \mathcal{O}$, with the result that a Dirac state when restricted to $\mathcal{O}$ and lifted to $\mathcal{R}$ (i.e. taken through the bijection in Theorem 2.4) can be regular again on the physical algebra $\mathcal{R}$. This is also obvious from Theorem 2.20, since a nontrivial $\mathcal{R}$ clearly has regular states. Thus for the physical algebra, quantum fields can exist.

2.6 Inclusion structures

The following results on inclusion relations between constraint systems will be necessary for the analysis of the isotony property of a net of local algebras [14]. Henceforth we assume unitary first class constraints $\mathcal{C} = \mathcal{U} - \mathbb{1}$, indicated by a pair $(\mathcal{F}, \mathcal{U})$ (cf. Theorem 2.15).

2.21 Definition *A first class constrained system $(\mathcal{F}, \mathcal{U})$ is said to be* **included** *in another one $(\mathcal{F}_e, \mathcal{U}_e)$, if the* C*–*algebras $\mathcal{F} \subset \mathcal{F}_e$ have a common identity and $\mathcal{U} = \mathcal{U}_e \cap \mathcal{F}$. We denote this by $(\mathcal{F}, \mathcal{U}) \subseteq (\mathcal{F}_e, \mathcal{U}_e)$.*

For the rest of this subsection we will assume that $(\mathcal{F}, \mathcal{U}) \subseteq (\mathcal{F}_e, \mathcal{U}_e)$. From the T–procedure sketched above we obtain the corresponding quadruples

$$(\mathfrak{S}_D,\, \mathcal{D},\, \mathcal{O},\, \mathcal{R}) \qquad \text{and} \qquad (\mathfrak{S}_{De},\, \mathcal{D}_e,\, \mathcal{O}_e,\, \mathcal{R}_e)\,.$$

2.22 Lemma *Suppose that $\mathcal{D} = \mathcal{D}_e \cap \mathcal{O}$ and $\mathcal{O} \subset \mathcal{O}_e$, then there is *–isomorphism from $\mathcal{R}$ to a* C*–*subalgebra of $\mathcal{R}_e$, which maps the identity of $\mathcal{R}$ to the identity of $\mathcal{R}_e$.*

Proof: From

$$\mathcal{R}_e = \mathcal{O}_e/\mathcal{D}_e = \Big(\mathcal{O}/\mathcal{D}_e\Big) \cup \Big((\mathcal{O}_e \setminus \mathcal{O})/\mathcal{D}_e\Big)\,,$$

it is enough to show that $\mathcal{O}/\mathcal{D}_e \cong \mathcal{R} = \mathcal{O}/\mathcal{D}$. Now, a $\mathcal{D}_e$–equivalence class consists of $A, B \in \mathcal{O}$ such that $A - B \in \mathcal{D}_e$ and therefore $A - B \in \mathcal{D}_e \cap \mathcal{O} = \mathcal{D}$. This implies $\mathcal{O}/\mathcal{D}_e \cong \mathcal{O}/\mathcal{D} = \mathcal{R}$. Moreover, since $\mathbb{1} \in \mathcal{O} \subset \mathcal{O}_e$, and the $\mathcal{D}$–equivalence class of $\mathbb{1}$ is contained in the $\mathcal{D}_e$–equivalence class of $\mathbb{1}$, it follows that the identity maps to the identity. ∎

2.23 Remark (i) By a simple finite dimensional example one can verify that the condition $\mathcal{D} = \mathcal{D}_e \cap \mathcal{O}$ does not imply $\mathcal{O} \subset \mathcal{O}_e$.

(ii) Instead of the conditions in Lemma 2.22 another natural set of restriction conditions one can also choose is $\mathcal{D} = \mathcal{D}_e \cap \mathcal{F}$ and $\mathcal{O} = \mathcal{O}_e \cap \mathcal{F}$, but below we will see that these imply those in Lemma 2.22.

The next result gives sufficient conditions for the equation $\mathcal{D} = \mathcal{D}_e \cap \mathcal{O}$ to hold.

2.24 Lemma *$\mathcal{D} = \mathcal{D}_e \cap \mathcal{F}$ if either*

(i) $\mathfrak{S}_D = \mathfrak{S}_{De}\restriction\mathcal{F}$ *or*

(ii) $[\mathcal{F}\, \mathrm{C}^*(\mathcal{U} - \mathbb{1})] = [\mathcal{F}_e\, \mathrm{C}^*(\mathcal{U}_e - \mathbb{1})] \,\cap\, \mathcal{F}$.

Proof: Take $\omega \in \mathfrak{S}_{De}$ and recall the definition for the left kernel N_ω given in the preceding subsection. Then $N_\omega \cap \mathcal{F} = N_{\omega\restriction\mathcal{F}}$ and from (i) we get

$$\begin{aligned}
\mathcal{N} = \cap\,\{N_\omega \mid \omega \in \mathfrak{S}_D\} &= \cap\,\{N_\omega \mid \omega \in \mathfrak{S}_{De}\restriction\mathcal{F}\} \\
&= \cap\,\{N_\omega \cap \mathcal{F} \mid \omega \in \mathfrak{S}_{De}\} \\
&= \cap\,\{N_\omega \mid \omega \in \mathfrak{S}_{De}\} \cap \mathcal{F} = \mathcal{N}_e \cap \mathcal{F}\,.
\end{aligned}$$

This produces

$$\mathcal{N} = [\mathcal{F}\, \mathrm{C}^*(\mathcal{U} - \mathbb{1})] = [\mathcal{F}_e\, \mathrm{C}^*(\mathcal{U}_e - \mathbb{1})] \,\cap\, \mathcal{F}$$

as well as

$$\begin{aligned}
\mathcal{D} &= [\mathcal{F}\, \mathrm{C}^*(\mathcal{U} - \mathbb{1})] \,\cap\, [\mathrm{C}^*(\mathcal{U} - \mathbb{1})\, \mathcal{F}] \\
&= [\mathcal{F}_e\, \mathrm{C}^*(\mathcal{U}_e - \mathbb{1})] \,\cap\, [\mathrm{C}^*(\mathcal{U}_e - \mathbb{1})\, \mathcal{F}_e] \,\cap\, \mathcal{F} \\
&= \mathcal{D}_e \cap \mathcal{F}
\end{aligned}$$

and the proof is concluded. ■

2.25 Lemma *We have*

$$\mathcal{O} \subset \mathcal{O}_e \quad \textit{iff} \quad \mathcal{O} \subset \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\}\,.$$

Proof: By Theorem 2.3 (v) we have

$$\begin{aligned}
\mathcal{O}_e \cap \mathcal{F} &= \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in \mathcal{U}_e\} \\
&= \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \cap \mathcal{F} \quad \forall U \in \mathcal{U}\} \\
&\quad \cap \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\} \\
&= \Big(\mathcal{O} \,\cup\, \{F \in \mathcal{F} \mid UFU^{-1} - F \in (\mathcal{D}_e \cap \mathcal{F} \setminus \mathcal{D}) \quad \forall U \in \mathcal{U}\}\Big) \\
&\quad \cap \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\} \\
&= \Big(\mathcal{O} \,\cap\, \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\}\Big) \\
&\quad \cup \Big\{F \in \mathcal{F} \mid UFU^{-1} - F \in (\mathcal{D}_e \cap \mathcal{F} \setminus \mathcal{D}) \quad \forall U \in \mathcal{U} \\
&\qquad \text{and } UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\Big\}\,.
\end{aligned} \tag{4}$$

Now if $\mathcal{O} \subset \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\}$, then it is clear from the above equations that $\mathcal{O} \subset \mathcal{O}_e \cap \mathcal{F} \subset \mathcal{O}_e$.

To prove the other implication note that the second set in the union of Eqn. (4) is contained in $\mathcal{F} \setminus \mathcal{O}$ and therefore from $\mathcal{O} \subset \mathcal{O}_e$ we obtain

$$\mathcal{O} = \mathcal{O} \cap \mathcal{O}_e \cap \mathcal{F} = \mathcal{O} \cap \{F \in \mathcal{F} \mid UFU^{-1} - F \in \mathcal{D}_e \quad \forall U \in (\mathcal{U}_e \setminus \mathcal{U})\}\,,$$

which implies the desired inclusion. ■

2.26 Theorem *Given an included pair of first class constrained systems $(\mathcal{F}, \mathcal{U}) \subseteq (\mathcal{F}_e, \mathcal{U}_e)$ and notation as above, the following statements are in the relation $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ where:*

(i) $\mathfrak{S}_D \restriction \mathcal{O} = \mathfrak{S}_{De} \restriction \mathcal{O}$.

(ii) $\mathcal{D} = \mathcal{D}_e \cap \mathcal{O}$ *and* $\mathcal{O} \subseteq \mathcal{O}_e$.

(iii) $\mathcal{D} = \mathcal{D}_e \cap \mathcal{F}$ *and* $\mathcal{O} = \mathcal{O}_e \cap \mathcal{F}$.

(iv) $\mathfrak{S}_D = \mathfrak{S}_{De} \restriction \mathcal{F}$ *and* $\mathcal{O} \subseteq \mathcal{O}_e$.

Proof: We first prove the implication $(ii) \Rightarrow (i)$, so assume (ii). It suffices to show that all Dirac states on $\mathcal{O}$ extend to Dirac states on $\mathcal{O}_e$. Denote by $\mathfrak{S}(\mathcal{R})$ and $\mathfrak{S}_D(\mathcal{O})$ respectively the set of states on $\mathcal{R}$ and the set of Dirac states on $\mathcal{O}$ (assume also corresponding notation for $\mathcal{R}_e$ and $\mathcal{O}_e$). Then from Theorem 2.4 there exist w^*–continuous, isometric bijections θ and θ_e:

$$\theta \colon\ \mathfrak{S}_D(\mathcal{O}) \to \mathfrak{S}(\mathcal{R}) \quad \text{and} \quad \theta_e \colon\ \mathfrak{S}_{De}(\mathcal{O}_e) \to \mathfrak{S}(\mathcal{R}_e)\,.$$

Now take $\omega \in \mathfrak{S}_D(\mathcal{O})$, so that $\theta(\omega) \in \mathfrak{S}(\mathcal{R})$. From Lemma 2.22, $\mathcal{R} \subset \mathcal{R}_e$ so we can extend $\theta(\omega)$ to $\widetilde{\theta(\omega)} \in \mathfrak{S}(\mathcal{R}_e)$. Finally, $\theta_e^{-1}\Big(\widetilde{\theta(\omega)}\Big) \in \mathfrak{S}_{De}(\mathcal{O}_e)$ is an extension of ω, since for any $A \in \mathcal{O}$ we have

$$\theta_e^{-1}\Big(\widetilde{\theta(\omega)}\Big)(A) = \widetilde{\theta(\omega)}(\xi_e(A)) = \theta(\omega)(\xi(A)) = \omega(A)\,,$$

where $\xi \colon \mathcal{O} \to \mathcal{R}$ and $\xi_e \colon\ \mathcal{O}_e \to \mathcal{R}_e$ are the canonical factorization maps. This proves (i).

Next we prove $(iii) \Rightarrow (ii)$. Obviously $\mathcal{O} = \mathcal{O}_e \cap \mathcal{F}$ implies that $\mathcal{O} \subseteq \mathcal{O}_e$. Since $\mathcal{D} \subset \mathcal{O} \subset \mathcal{F}$ we have $\mathcal{D} = \mathcal{D}_e \cap \mathcal{F} \subset \mathcal{O}$ and so $\mathcal{D} = (\mathcal{D}_e \cap \mathcal{F}) \cap \mathcal{O} = \mathcal{D}_e \cap \mathcal{O}$.

Finally we prove $(iv) \Rightarrow (iii)$. By Lemma 2.24(i), $\mathfrak{S}_D = \mathfrak{S}_{De} \restriction \mathcal{F}$ implies that $\mathcal{D} = \mathcal{D}_e \cap \mathcal{F}$. By Theorem 2.3(iii) and the fact that $\mathcal{O}$ is a C*–algebra (hence the span of its selfadjoint elements), we have

$$\begin{aligned} \mathcal{O} &= [\{F \in \mathcal{F}_{sa} \mid [F, \mathcal{U}] \subset \mathcal{D}\}] \\ &= [\{F \in \mathcal{F}_{sa} \mid \omega([F, U^*] \cdot [F, U]) = 0 \quad \forall\, \omega \in \mathfrak{S}_D,\ U \in \mathcal{U}\}] \end{aligned}$$

where the last equality follows from $\mathcal{D} = \mathcal{N} \cap \mathcal{N}^*$, $[F, U]^* = -[F, U^*]$ for $F = F^*$, and $\mathcal{U} = \mathcal{U}^*$. Since $\mathfrak{S}_D = \mathfrak{S}_{De} \restriction \mathcal{F}$ we have

$$\begin{aligned} \mathcal{O} &= [\{F \in \mathcal{F}_{sa} \mid \omega([F, U^*] \cdot [F, U]) = 0 \quad \forall\, \omega \in \mathfrak{S}_{De},\ U \in \mathcal{U}\}] \\ &\supseteq [\{F \in \mathcal{F}_{sa} \mid \omega([F, U^*] \cdot [F, U]) = 0 \quad \forall\, \omega \in \mathfrak{S}_{De},\ U \in \mathcal{U}_e\}] \\ &= \mathcal{O}_e \cap \mathcal{F}\,. \end{aligned}$$

Thus since $\mathcal{O} \subseteq \mathcal{O}_e$ we conclude that $\mathcal{O} = \mathcal{O}_e \cap \mathcal{F}$. ■

2.27 Remark $\mathfrak{S}_D \restriction \mathcal{O} = \mathfrak{S}_{De} \restriction \mathcal{O}$ does not seem to imply that $\mathfrak{S}_D = \mathfrak{S}_{De} \restriction \mathcal{F}$ since for a state in $\mathfrak{S}_D$, if one restricts it to $\mathcal{O}$ and then extends to a state in $\mathfrak{S}_{De}$, it seems nontrivial whether this extension can coincide with the original state on $\mathcal{F}$.

3 Local Quantum Constraints

In practice, a large class of constraint systems occur in quantum field theory (henceforth denoted by QFT), for instance gauge theories. A prominent property of a QFT, is space–time locality, and usually when constraints occur, they also have this property. Heuristically such constraints are written as

$$\chi(x)\,\psi = 0 \qquad \text{for} \quad \psi \in \mathcal{H}^{(p)}\,,\ x \in M^4\,,$$

where Minkowski space is $M^4 = (\mathbb{R}^4,\,\eta)$ with metric $\eta := \mathrm{diag}(+,-,-,-)$, and this makes the locality explicit. Since χ is actually an operator–valued distribution, the correct expression should be of the form

$$\chi(f)\,\psi = 0 \qquad \text{for} \quad \psi \in \mathcal{H}^{(p)}\,,\ f \in C_c^\infty(\mathbb{R}^4)\;.$$

In this section we now want to analyze how locality intertwines with the T–procedure of constraint reduction. To make this precise, recall that the Haag–Kastler axioms [14, 13] express locality for a QFT as follows:

3.1 Definition *A* **Haag–Kastler QFT** *(or HK–QFT for short) consists of the following.*

- *A directed set $\Gamma \subseteq \{\Theta \subset M^4 \mid \Theta \ \ \text{open and bounded}\}$ partially ordered by set inclusion, such that $\mathbb{R}^4 = \cup\{\Theta \mid \Theta \in \Gamma\}$ and under the action of the orthochronous Poincaré group $\mathcal{P}^\uparrow_+$ on M^4 we have $g\Theta \in \Gamma$ for all $\Theta \in \Gamma$, $g \in \mathcal{P}^\uparrow_+$.*
- *A directed set $\widetilde{\Gamma}$ of C*–algebras with a common identity $\mathbb{1}$, ordered by inclusion, with an inductive limit C*–algebra $\mathcal{F}_0$ (over $\widetilde{\Gamma}$). We will call the elements of $\widetilde{\Gamma}$ the* **local field algebras** *and $\mathcal{F}_0$ the* **quasi–local algebra**.
- *A surjection $\mathcal{F}\colon\ \Gamma \to \widetilde{\Gamma}$, satisfying:*
  1. *(Isotony) $\mathcal{F}$ is order preserving, i.e. $\mathcal{F}(\Theta_1) \subseteq \mathcal{F}(\Theta_2)$ if $\Theta_1 \subseteq \Theta_2$.*
  2. *(Causality) If $\Theta_1,\ \Theta_2 \in \Gamma$ are spacelike separated, (henceforth denoted $\Theta_1 \perp \Theta_2$), then $[\mathcal{F}(\Theta_1),\,\mathcal{F}(\Theta_2)] = 0$ in $\mathcal{F}_0$.*
  3. *(Covariance) There is an action $\alpha\colon\ \mathcal{P}^\uparrow_+ \to \mathrm{Aut}\,\mathcal{F}_0$ such that $\alpha_g(\mathcal{F}(\Theta)) = \mathcal{F}(g\Theta)$, $g \in \mathcal{P}^\uparrow_+$, $\Theta \in \Gamma$.*

3.2 Remark In the usual algebraic approach to QFT (cf. [13]), there are additional axioms, e.g. that $\mathcal{F}_0$ must be primitive, that there is a vacuum state with GNS–representation in which the generators of the translations in the covariant representation of $\mathcal{P}^\uparrow_+$ have spectra in the forward light cone, that there is a compact gauge group, local definiteness, local normality etc. We will return to some of these axioms later, but for now, we restrict our analysis to those listed in Definition 3.1. There is some redundancy in this definition e.g. for the results in this Section, we will not need the assumption that $\Theta \in \Gamma$ is bounded.

In the context of the Haag–Kastler axioms, we would like to define a system of "local quantum constraints" in such a way that it includes the major examples from QFT.

3.3 Definition *A system of* **local quantum constraints** *consists of a surjection $\mathcal{F}\colon\ \Gamma \to \widetilde{\Gamma}$ as in Definition* 3.1 *satisfying isotony as well as*

(4) *(Local Constraints) There is a map $\mathcal{U}$ from Γ to the set of first class subsets of the unitaries in the local field algebras such that*

$\mathcal{U}(\Theta) \subset \mathcal{F}(\Theta)_u$ *for all* $\Theta \in \Gamma$, *and* *if $\Theta_1 \subseteq \Theta_2$, then $(\mathcal{F}(\Theta_1), \mathcal{U}(\Theta_1)) \subseteq (\mathcal{F}(\Theta_2), \mathcal{U}(\Theta_2))$ (i.e. $\mathcal{U}(\Theta_1) = \mathcal{U}(\Theta_2) \cap \mathcal{F}(\Theta_1)$, cf. Definition 2.21).*

3.4 Remark In this definition we have made the minimum assumptions to start the analysis. We have omitted causality and covariance because these are physical requirements which one should demand for the final physical theory, not the initial (unconstrained) theory which contains nonphysical objects. There are examples of constrained QFTs satisfying these conditions, e.g. [17, Remark 4.3], [18, Chapter 4] and see also below in the next section our example of Gupta–Bleuler electromagnetism.

Given a system of local quantum constraints, $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$, we can apply the T–procedure to each local system $(\mathcal{F}(\Theta), \mathcal{U}(\Theta))$, to obtain the "local" objects:

$$\begin{aligned}
\mathfrak{S}_D^\Theta &:= \Big\{\omega \in \mathfrak{S}(\mathcal{F}(\Theta)) \mid \omega(U) = 1 \quad \forall\, U \in \mathcal{U}(\Theta)\Big\} \;=\; \mathfrak{S}_D(\mathcal{F}(\Theta))\,, \\
\mathcal{D}(\Theta) &:= [\mathcal{F}(\Theta)\, \mathcal{C}(\Theta)] \cap [\mathcal{C}(\Theta)\, \mathcal{F}(\Theta)]\,, \\
\mathcal{O}(\Theta) &:= \{F \in \mathcal{F}(\Theta) \mid FD - DF \in \mathcal{D}(\Theta) \quad \forall\, D \in \mathcal{D}(\Theta)\} \;=\; M_{\mathcal{F}(\Theta)}(\mathcal{D}(\Theta))\,, \\
\mathcal{R}(\Theta) &:= \mathcal{O}(\Theta)/\mathcal{D}(\Theta)\,.
\end{aligned}$$

We can be now more precise about what our task is in this section:

> **Problem:** For a system of local quantum constraints $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$, find minimal conditions such that the net of local physical observables $\Theta \to \mathcal{R}(\Theta)$ becomes a HK–QFT.

We propose:

3.5 Definition *Fix a system of local quantum constraints $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$, then we say that it satisfies:*

(5) **reduction isotony** *if $\Theta_1 \subseteq \Theta_2$ implies $\mathcal{O}(\Theta_1) \subseteq \mathcal{O}(\Theta_2)$ and $\mathcal{D}(\Theta_1) = \mathcal{D}(\Theta_2) \cap \mathcal{O}(\Theta_1)$ (cf. Lemma 2.22 for motivation).*

(6) **weak causality** *if for $\Theta_1 \perp \Theta_2$ there is some $\Theta_0 \supset \Theta_1 \cup \Theta_2$, $\Theta_0 \in \Gamma$ such that*

$$\Big[\mathcal{O}(\Theta_1)\,,\, \mathcal{O}(\Theta_2)\Big] \subset \mathcal{D}(\Theta_0)\,.$$

(7) **weak covariance** *if there is an action $\alpha\colon \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{F}_0$ such that $\alpha_g(\mathcal{O}(\Theta)) = \mathcal{O}(g\Theta)$ and $\alpha_g(\mathcal{U}(\Theta)) \sim \mathcal{U}(g\Theta)$, for all $g \in \mathcal{P}_+^\uparrow$, $\Theta \in \Gamma$ (cf. Definition 2.9).*

3.6 Remark (i) Given a system with reduction isotony, we have by Lemma 2.22 that when $\Theta_1 \subseteq \Theta_2$, then $\mathcal{R}(\Theta_1)$ is isomorphic to a C*–subalgebra of $\mathcal{R}(\Theta_2)$, which we will denote as $\iota_{12}\colon \mathcal{R}(\Theta_1) \to \mathcal{R}(\Theta_2)$.

(ii) The weak causality condition is considerably weaker than requiring causality (cf. (2) in Definition 3.1) for the field algebra, and this will be crucial below for Gupta–Bleuler electromagnetism.

(iii) For the weak covariance condition, we do not need to state in which algebra the equivalence of constraints holds, since this only depends on the unital C*–algebra generated by the two constraint sets involved (cf. Remark 2.10 (ii)). Note that weak covariance follows from the covariance condition $\alpha_g(\mathcal{F}(\Theta)) = \mathcal{F}(g\Theta)$ and $\alpha_g(\mathcal{U}(\Theta)) \sim \mathcal{U}(g\Theta)$, for all $g \in \mathcal{P}_+^\uparrow$, using the fact that equivalent constraint sets produce the same observable algebra.

(iv) It is instructive to compare the conditions in Definition 3.5 with those of the Doplicher-Haag–Roberts analysis (DHR for short [5]), given that both are intended for application to gauge QFTs. First, in DHR analysis one assumes that the actions of the gauge group and the Poincaré group commute, which limits the analysis to gauge transformations of the first kind (and hence excludes quantum electromagnetism). In contrast, we assume weak covariance, hence include gauge transformations of the second kind (and also QEM). The DHR analysis also assumes field algebra covariance, which we omit. Second, the DHR analysis is done concretely in a positive energy representation, whereas we assume an abstract C*–system, hence we can avoid the usual clash between regularity and constraints appearing as continuous spectrum problems for the constraints (cf. Subsection 2.5) and which generally leads to indefinite metric representations. At the concrete level this problem manifests itself in the inability of constructing the vector potential satisfying Maxwell's equations as a covariant or causal quantum field on a space with an invariant vacuum, cf. [25, 26, 1] and [31, Eq. 8.1.2].

(v) In the next section we will construct an example (Gupta–Bleuler electromagnetism) which satisfies the conditions 3.5.

Now we show that the conditions in Definition 3.5 are sufficient to guarantee that the net of local physical observables $\Theta \to \mathcal{R}(\Theta)$ is a HK–QFT.

3.7 Theorem *Let $\Gamma \ni \Theta \to (\mathcal{F}(\Theta),\mathcal{U}(\Theta))$ be a system of local quantum constraints.*

(i) *If it satisfies reduction isotony, then $\Theta \to \mathcal{R}(\Theta)$ has isotony, i.e. $\Theta_1 \subset \Theta_2$, implies $\mathcal{R}(\Theta_1) \subset \mathcal{R}(\Theta_2)$. In this case, the net $\Theta \to \mathcal{R}(\Theta)$ has an inductive limit, which we denote by $\mathcal{R}_0 := \varinjlim \mathcal{R}(\Theta)$, and call it the* **quasi–local physical algebra**.

(ii) *If it satisfies weak causality and reduction isotony, then $\Theta \to \mathcal{R}(\Theta)$ has causality, i.e. $\Theta_1 \perp \Theta_2$, implies $[\mathcal{R}(\Theta_1), \mathcal{R}(\Theta_2)] = 0$.*

(iii) *If it satisfies weak covariance, then for each Θ we have $\widetilde{\alpha}_g(\mathcal{R}(\Theta)) = \mathcal{R}(g\Theta)$, $g \in \mathcal{P}_+^\uparrow$, where $\widetilde{\alpha}_g$ is the lifting of α_g (cf. (7) in Definition 3.5) to the local factor algebra $\mathcal{R}(\Theta)$. If the system of constraints in addition satisfies reduction isotony, then the isomorphisms $\widetilde{\alpha}_g\colon \mathcal{R}(\Theta) \to \mathcal{R}(g\Theta)$, $\Theta \in \Gamma$, are the restrictions of an automorphism $\widetilde{\alpha}_g \in \operatorname{Aut}\mathcal{R}_0$, and moreover, $\widetilde{\alpha}\colon \mathcal{P}_+^\uparrow \to \operatorname{Aut}\mathcal{R}_0$ is an action, i.e. the net $\Theta \to \mathcal{R}(\Theta)$ satisfies covariance.*

Thus if the system of local constraints satisfies all three conditions, then $\Theta \to \mathcal{R}(\Theta)$ is a HK–QFT.

Proof: (i) By reduction isotony we obtain from Lemma 2.22 for $\Theta_1 \subset \Theta_2$ a unital monomorphism $\iota_{12}\colon \mathcal{R}(\Theta_1) \to \mathcal{R}(\Theta_2)$. To get isotony from these monomorphisms, we have to verify that they satisfy Takeda's criterion: $\iota_{13} = \iota_{23} \circ \iota_{12}$ (cf. [29]), which will ensure the existence of the inductive limit $\mathcal{R}_0$, and in which case we can write simply inclusion $\mathcal{R}(\Theta_1) \subset \mathcal{R}(\Theta_2)$ for ι_{12}. Recall that $\iota_{12}(A + \mathcal{D}(\Theta_1)) = A + \mathcal{D}(\Theta_2)$ for $A \in \mathcal{O}(\Theta_1)$. Let $\Theta_1 \subset \Theta_2 \subset \Theta_3$, then by reduction isotony, $\mathcal{O}(\Theta_1) \subset \mathcal{O}(\Theta_2) \subset \mathcal{O}(\Theta_3)$, and so for $A \in \mathcal{O}(\Theta_1)$, $\iota_{23}(\iota_{12}(A + \mathcal{D}(\Theta_1))) = \iota_{23}(A + \mathcal{D}(\Theta_2)) = A + \mathcal{D}(\Theta_3) = \iota_{13}(A + \mathcal{D}(\Theta_1))$. This establishes Takeda's criterion.

(ii) Let $\Theta_1 \perp \Theta_2$ with $\Theta_0 \supset \Theta_1 \cup \Theta_2$ such that $[\mathcal{O}(\Theta_1), \mathcal{O}(\Theta_2)] \subset \mathcal{D}(\Theta_0)$ as in Definition 3.5 (6), then since $\mathcal{O}(\Theta_1) \cup \mathcal{O}(\Theta_2) \subset \mathcal{O}(\Theta_0)$ (by reduction isotony), the commutation relation is in $\mathcal{O}(\Theta_0)$, so when we factor out by $\mathcal{D}(\Theta_0)$, the right hand side vanishes and since factoring is a homomorphism, we get $[\mathcal{R}(\Theta_1), \mathcal{R}(\Theta_2)] = 0$ in $\mathcal{R}(\Theta_0)$ and therefore in $\mathcal{R}_0$.

(iii) Let $\alpha\colon \mathcal{P}_+^\uparrow \to \operatorname{Aut}\mathcal{F}_0$ be the action introduced by the weak covariance assumption in Definition 3.5 (7). Now $\alpha_g(\mathcal{O}(\Theta))$ are the observables of the constraint system $(\alpha_g(\mathcal{F}(\Theta)),\, \alpha_g(\mathcal{U}(\Theta)))$ with maximal constraint algebra $\alpha_g(\mathcal{D}(\Theta)) \subset \alpha_g(\mathcal{O}(\Theta)) = \mathcal{O}(g\Theta) \supset \mathcal{D}(g\Theta)$. Since $\alpha_g(\mathcal{U}(\Theta)) \sim$

$\mathcal{U}(g\Theta)$, they have the same Dirac states and so on $\mathcal{O}(g\Theta)$ the same maximal C*–algebra contained in the kernels of all Dirac states. Thus $\alpha_g(\mathcal{D}(\Theta)) = \mathcal{D}(g\Theta)$. Denote the factoring map $\xi_\Theta\colon \mathcal{O}(\Theta) \to \mathcal{R}(\Theta)$, i.e. $\xi_\Theta(A) = A + \mathcal{D}(\Theta)$ for all $A \in \mathcal{O}(\Theta)$. Then we lift $\alpha_g\colon \mathcal{O}(\Theta) \to \mathcal{O}(g\Theta)$ to a map $\widetilde{\alpha}_g\colon \mathcal{R}(\Theta) \to \mathcal{R}(g\Theta)$ by

$$\widetilde{\alpha}_g(\xi_\Theta(A)) := \alpha_g(A) + \alpha_g(\mathcal{D}(\Theta)) = \alpha_g(A) + \mathcal{D}(g\Theta) = \xi_{g\Theta}(\alpha_g(A)),$$

and this is obviously an isomorphism.

Next assume in addition reduction isotony, then we show that the isomorphisms $\widetilde{\alpha}_g$ defined on the net $\Theta \to \mathcal{R}(\Theta)$ are the restrictions of an automorphism $\widetilde{\alpha}_g \in \mathrm{Aut}\,\mathcal{R}_0$. Indeed, for $\Theta_1 \subset \Theta_2$ and any $A \in \mathcal{O}(\Theta_1)$ we have using equation $\mathcal{D}(g\Theta_1) = \mathcal{D}(g\Theta_2) \cap \mathcal{O}(g\Theta_1)$ and the monomorphisms $\iota_{12}\colon \mathcal{R}(\Theta_1) \to \mathcal{R}(\Theta_2)$, $\iota_{12}^g\colon \mathcal{R}(g\Theta_1) \to \mathcal{R}(g\Theta_2)$ that

$$\begin{aligned} \iota_{12}^g\Big(\widetilde{\alpha}_g(\xi_{\Theta_1}(A))\Big) &= \iota_{12}^g\Big(\alpha_g(A) + \mathcal{D}(g\Theta_1)\Big) = \alpha_g(A) + \mathcal{D}(g\Theta_2) = \widetilde{\alpha}_g\Big(\xi_{\Theta_2}(A)\Big) \\ &= \widetilde{\alpha}_g(\iota_{12}(\xi_{\Theta_1}(A)))\,. \end{aligned}$$

This shows that the diagram

$$\begin{array}{ccc} \mathcal{R}(\Theta_1) & \xrightarrow{\iota_{12}} & \mathcal{R}(\Theta_2) \\ \big\downarrow \widetilde{\alpha}_g & & \big\downarrow \widetilde{\alpha}_g \\ \mathcal{R}(g\Theta_1) & \xrightarrow{\iota_{12}^g} & \mathcal{R}(g\Theta_2) \end{array}$$

commutes. Therefore by the uniqueness property of the inductive limit [16, Section 11.4] the isomorphisms $\widetilde{\alpha}_g$ of the local observable algebras characterize an automorphism of $\mathcal{R}_0$ which we also denote by $\widetilde{\alpha}_g$. Since $\alpha\colon \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{F}$ is a group homomorphism, we see for the local isomorphisms that the composition of $\widetilde{\alpha}_g\colon \mathcal{R}(\Theta) \to \mathcal{R}(g\Theta)$ with $\widetilde{\alpha}_h\colon \mathcal{R}(g\Theta) \to \mathcal{R}(hg\Theta)$ is $\widetilde{\alpha}_h \circ \widetilde{\alpha}_g = \widetilde{\alpha}_{hg}\colon \mathcal{R}(\Theta) \to \mathcal{R}(hg\Theta)$. From this it follows that $\widetilde{\alpha}\colon \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{R}_0$ is a group homomorphism. ■

In the following sections we will construct field theory examples of local systems of quantum constraints which satisfy the weak conditions of Definition 3.5, hence define HK–QFTs for their net of physical algebras.

3.8 Proposition *Given a system of local quantum constraints, $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$, which satisfies reduction isotony and weak covariance, then the net $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}_m(\Theta))$ (where $\mathcal{U}_m(\Theta)$ is the maximal constraint group of $\mathcal{U}(\Theta)$ in $\mathcal{F}(\Theta)$, cf. Theorem 2.11 (ii)) is a system of local quantum constraints satisfying reduction isotony and covariance, i.e.*

$$\alpha_g(\mathcal{O}(\Theta)) = \mathcal{O}(g\Theta) \qquad \textit{and} \qquad \alpha_g(\mathcal{U}_m(\Theta)) = \mathcal{U}_m(g\Theta)\,, \quad g \in \mathcal{P}_+^\uparrow\,,\ \Theta \in \Gamma\,.$$

The system $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}_m(\Theta))$ is clearly locally equivalent to $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$, in the sense that $\mathcal{U}_m(\Theta) \sim \mathcal{U}(\Theta)$ for all $\Theta \in \Gamma$, from which it follows that if one of these two systems has weak causality, so has the other one.

Proof: Let $\Theta_1 \subseteq \Theta_2$, then by $(\mathcal{F}(\Theta_1), \mathcal{U}(\Theta_1)) \subseteq (\mathcal{F}(\Theta_2), \mathcal{U}(\Theta_2))$ and reduction isotony, we conclude from Theorem 2.26 that all Dirac states on $\mathcal{O}(\Theta_1) \subset \mathcal{O}(\Theta_2)$ extend to Dirac states on $\mathcal{O}(\Theta_2)$. Thus by Remark 2.12,

$$\begin{aligned} \mathcal{U}_m(\Theta_2) \cap \mathcal{F}(\Theta_1) &= \{U \in \mathcal{O}_u(\Theta_2) \mid \omega(U) = 1 \quad \forall\, \omega \in \mathfrak{S}_D^{\Theta_2}\} \cap \mathcal{F}(\Theta_1) \\ &= \{U \in \mathcal{F}_u(\Theta_1) \cap \mathcal{O}(\Theta_2) \mid \omega(U) = 1 \quad \forall\, \omega \in \mathfrak{S}_D^{\Theta_1}\} \\ &= \{U \in \mathcal{O}_u(\Theta_1) \mid \omega(U) = 1 \quad \forall\, \omega \in \mathfrak{S}_D^{\Theta_1}\} = \mathcal{U}_m(\Theta_1)\,, \end{aligned}$$

where we used the fact that if for a unitary U we have $\omega(U) = 1$ for all Dirac states ω, then $U \in \mathcal{O}$. Thus $(\mathcal{F}(\Theta_1), \mathcal{U}_m(\Theta_1)) \subseteq (\mathcal{F}(\Theta_2), \mathcal{U}_m(\Theta_2))$ and so the system $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}_m(\Theta))$ is a system of local quantum constraints. Reduction isotony follows from that of the original system and the equivalences $\mathcal{U}_m(\Theta) \sim \mathcal{U}(\Theta)$ for all $\Theta \in \Gamma$.

To prove the covariance property of $\mathcal{U}_m(\Theta)$ recall that from weak covariance we have $\alpha_g(\mathcal{U}(\Theta)) \sim \mathcal{U}(g\Theta)$ for all Θ. We show first that if $\mathcal{U}_1(\Theta) \sim \mathcal{U}(\Theta)$, then $\alpha_g(\mathcal{U}_1(\Theta)) \subset \mathcal{U}_m(g\Theta)$. We have

$$\begin{aligned} \alpha_g\Big(\mathcal{U}_1(\Theta)\Big) - \mathcal{U}(g\Theta) &= \alpha_g\Big(\mathcal{U}_1(\Theta) - \mathbb{1}\Big) + \mathbb{1} - \mathcal{U}(g\Theta) \\ &\subset \alpha_g\Big(\mathcal{D}(\Theta)\Big) + \mathcal{D}(g\Theta) \\ &= \mathcal{D}(g\Theta) \;=\; \alpha_g(\mathcal{D}(\Theta)), \end{aligned}$$

where the last equality follows from the proof of the previous theorem, and we used also that $\mathcal{D}_1(\Theta) = \mathcal{D}(\Theta)$. Since $\alpha_g(\mathcal{D}(\Theta))$ is the $\mathcal{D}$–algebra of $\alpha_g(\mathcal{U}_1(\Theta))$ in $\alpha_g(\mathcal{O}(\Theta)) = \mathcal{O}(g\Theta)$, this implies by Theorem 2.11 (i) that $\alpha_g(\mathcal{U}_1(\Theta)) \sim \mathcal{U}(g\Theta)$ and therefore it must be contained in $\mathcal{U}_m(g\Theta)$. Thus $\alpha_g(\mathcal{U}_m(\Theta)) \subset \mathcal{U}_m(g\Theta)$, $g \in \mathcal{P}$, and finally the inclusion $\alpha_{g^{-1}}(\mathcal{U}_m(g\Theta)) \subset \mathcal{U}_m(\Theta)$ proves covariance for $\Theta \to \mathcal{U}_m(\Theta)$. ■

4 Example: Gupta–Bleuler electromagnetism

Quantum electromagnetism, in the heuristic Gupta–Bleuler formulation, has a number of special features, cf. [12, 2]. First, it is represented on an indefinite inner product space, second, gauge invariance is imposed by the noncausal constraint

$$\chi(x) := \Big(\partial^\mu A_\mu\Big)^{(+)}(x),$$

and third, Maxwell's equations (in terms of the vector potential) are imposed as state conditions instead of as operator identities. This is necessary, because from the work of Strocchi e.a. [27, 28], we know that Maxwell's equations are incompatible with the Lorentz covariance of the vector potential. Gupta–Bleuler electromagnetism has been rigorously reconstructed in a C*–algebra context [6], in a way which allows one to avoid indefinite inner product representations (using instead representations which are nonregular on nonphysical objects). Here we will refine that approach in order to include the local constraint structure and to make contact with Haag–Kastler QFT.

4.1 Gupta–Bleuler electromagnetism as a local constraint system.

Our aim here is to define Gupta–Bleuler electromagnetism as a local system of constraints as in Definition 3.3 and subsequently to show that it has reduction isotony, weak causality and covariance. Our starting point for defining this system, is [6, Sections 4 and 5] where motivation and further results can be found.

For the field algebra we take the CCR algebra $\mathcal{A} := \overline{\Delta(\mathfrak{X},\, B)}$ where the symplectic space $(\mathfrak{X}, B)$ is constructed as follows. Consider the real linear space $\widehat{\mathcal{S}} := \{\widehat{f} \mid f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}^4)\} = \{f \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \mid \overline{f(p)} = f(-p)\}$, where $\widehat{f}(p) := \int_{\mathbb{R}^4} e^{-ipx} f(x) d^4x$ is the Fourier transform of f. Equip $\widehat{\mathcal{S}}$ with the presymplectic form

$$D(f,h) := \int_{C_+} \Big(f_\mu(p)\, \overline{h^\mu}(p) - \overline{f_\mu}(p)\, h^\mu(p)\Big) \frac{d^3p}{p_0} \tag{5}$$

where $C_+ := \{p \in \mathbb{R}^4 \mid p_\mu p^\mu = 0,\ p_0 \geq 0\}$ is the mantle of the positive light cone: $V_+ := \{p \in \mathbb{R}^4 \mid p_\mu p^\mu \geq 0,\ p_0 \geq 0\}$. This is motivated by the heuristic commutation relations $[A(f),\, A(h)] = iD(\widehat{f}, \widehat{h})$ using the smearing formula $A(f) := \int A_\mu(x) f^\mu(x)\, d^4x,\ f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}^4)$. Now define $\mathfrak{X} := \widehat{\mathcal{S}}/\mathrm{Ker}(D)$ which is a symplectic space with symplectic form B defined as the lift of D to the factor space $\mathfrak{X}$. Since this is an outer constraint situation, $\mathcal{A}$ does not contain the constraints, so we need to extend $\mathcal{A}$ to obtain the field algebra $\mathcal{F}_e$. For each $h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$ we have a one–parameter group of gauge transformations $T_h^t : \mathfrak{X} \to \mathfrak{X}$ (cf. [6]) given by

$$(T_h^t f)_\mu(p) = f_\mu(p) - it\pi\, p_\mu \widehat{h}(p) \int_{C_+} f^\nu(p')\, p'_\nu \overline{\widehat{h}(p')}\, \frac{d^3p'}{p'_0} \tag{6}$$

which corresponds in the heuristic picture to transformations

$$\exp(i\,A(f)) \mapsto \mathrm{Ad}\Big(\exp(it\,\chi(h)^*\chi(h))\Big)\exp(i\,A(f))\,,$$

where formally $\chi(h) := \int \chi(x)h(x)\, d^4x,\ h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$. Then $\{T_h^t \mid t \in \mathbb{R},\, h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})\}$ is a commutative set of symplectic transformations, hence preserve $\mathrm{Ker}(D)$ and so lift to the factor space $\mathfrak{X}$. Each T_h^t is a one–parameter group in t, but due to the nonlinearity in h, the map $h \to T_h^1 =: T_h$ is not a group homomorphism of $\mathcal{S}(\mathbb{R}^4, \mathbb{R})$. Define the automorphisms $\theta_h \in \mathrm{Aut}\Big(\overline{\Delta(\mathfrak{X},\, B)}\Big)$ by $\theta_h(\delta_f) := \delta_{T_h f},\ f \in \mathfrak{X}$, and let $\mathcal{G}$ be the discrete group generated in $\mathrm{Aut}\Big(\overline{\Delta(\mathfrak{X},\, B)}\Big)$ by all θ_h. Our field algebra will be the discrete crossed product $\mathcal{F}_e := \mathcal{G} \underset{\beta}{\times} \mathcal{A}$, where the action $\beta : \mathcal{G} \to \mathrm{Aut}(\mathcal{A})$ means just the inclusion. As a C*–algebra $\mathcal{F}_e$ is generated by $\mathcal{A}$ and a set of commuting unitaries $U_\mathcal{G} := \{U_\gamma \mid \gamma \in \mathcal{G}\}$ such that $\gamma(F) = U_\gamma\, F\, U_\gamma^*,\ F \in \mathcal{A}$, $U_{\gamma^{-1}} = U_\gamma^*$ and $U_{\gamma\gamma'} = U_\gamma U_{\gamma'},\ \gamma, \gamma' \in \mathcal{G}$.

4.1 Remark Sometimes we need a more concrete characterization of the space $\mathfrak{X}$. Now $\mathfrak{X} = \widehat{\mathcal{S}}/\mathrm{Ker}(D)$ and $\widehat{\mathcal{S}} = \{f \in \mathcal{S}(\mathbb{R}^4,\, \mathbb{C}^4) \mid \overline{f(p)} = f(-p)\} = \mathcal{S}_+ + i\mathcal{S}_-$ where $\mathcal{S}_\pm := \{u \in \mathcal{S}(\mathbb{R}^4,\, \mathbb{R}^4) \mid u(p) = \pm u(-p)\}$. From Eqn. (5) we see that $\mathrm{Ker}(D) = \{f \in \widehat{\mathcal{S}} \mid f\restriction C_+ = 0\}$, and hence factoring by $\mathrm{Ker}(D)$ is the same as restriction to C_+, i.e. $\mathfrak{X} = \widehat{\mathcal{S}}\restriction C_+$, and since $f(p)\restriction C_+ = f(\|\mathbf{p}\|,\, \mathbf{p})$ we can identify these functions with a subspace of $C(\mathbb{R}^3,\, \mathbb{C}^4)$. Since we are restricting Schwartz functions, we note that these functions on $\mathbb{R}^3$ are smooth except at the origin, and Schwartz on the complement of any open neighbourhood of the origin. The conditions $u(p) = \pm u(-p)$ for $u \in \mathcal{S}_\pm$ involve points outside C_+, so through smoothness they will influence the behaviour of $u\restriction C_+$ near the origin. Specifically if $u \in \mathcal{S}_+$ (resp. $u \in \mathcal{S}_-$), then on each line through the origin in C_+, $\{ta \mid t \in \mathbb{R}\}$, $a \in C_+\backslash 0$, the function $u_a(t) := u(ta)$ is smooth and even (resp. odd), hence all its derivatives of odd degree must be odd (resp. even) and its derivatives of even degree must be even (resp. odd). Thus the derivatives of u_a of odd (resp. even) degree are zero at the origin. This is a property which does restrict to C_+, and distinguishes between $\mathcal{S}_+\restriction C_+$ and $\mathcal{S}_-\restriction C_+$. Note from the above discussion, that $\mathfrak{X} = \widehat{\mathcal{S}}\restriction C_+$ contains all smooth functions with compact support away from zero.

To construct the net of local field algebras $\mathcal{F} : \Gamma \to \widetilde{\Gamma}$ as in Definition 3.1, let Θ be any open set in $\mathbb{R}^4$ and define

$$\begin{aligned}
\mathcal{S}(\Theta) &:= \{f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}^4) \mid \mathrm{supp}(f) \subset \Theta\} \\
\mathfrak{X}(\Theta) &:= \widehat{\mathcal{S}(\Theta)}/\mathrm{Ker}(D) \\
\mathcal{U}(\Theta) &:= \{U_{\theta_h} \mid h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}),\ \mathrm{supp}(h) \subset \Theta\} \\
\mathcal{F}(\Theta) &:= \mathrm{C}^*\Big(\delta_{\mathfrak{X}(\Theta)} \cup \mathcal{U}(\Theta)\Big) \subset \mathcal{F}_e.
\end{aligned}$$

Note that if Θ is bounded, then $\mathcal{S}(\Theta) = C_c^\infty(\Theta,\, \mathbb{R}^4)$. Moreover $T_h\mathfrak{X}(\Theta) \subset \mathfrak{X}(\Theta)$ when $\mathrm{supp}(h) \subset \Theta$. Thus if we let $\mathcal{G}(\Theta)$ be the discrete group generated in $\mathrm{Aut}(\mathcal{F})$ by $\{\theta_h \mid \mathrm{supp}(h) \subset \Theta\}$, then it preserves $\mathrm{C}^*(\delta_{\mathfrak{X}(\Theta)}) = \overline{\Delta(\mathfrak{X}(\Theta),\, B)}$, so that it makes sense to define $\mathcal{G}(\Theta) \underset{\beta}{\times} \mathrm{C}^*(\delta_{\mathfrak{X}(\Theta)})$.

4.2 Lemma *We have:* $\mathcal{F}(\Theta) = \mathcal{G}(\Theta) \underset{\beta}{\times} \mathrm{C}^*(\delta_{\mathfrak{X}(\Theta)}) = [U_{\mathcal{G}(\Theta)}\, \delta_{\mathfrak{X}(\Theta)}] = [\delta_{\mathfrak{X}(\Theta)}\, U_{\mathcal{G}(\Theta)}]$.

Proof: For the first equality recall the fact that the crossed product $\mathcal{G}(\Theta) \underset{\beta}{\times} \mathrm{C}^*(\delta_{\mathfrak{X}(\Theta)})$ is constructed from the twisted convolution algebra of functions (twisted by β), $f : \mathcal{G}(\Theta) \to \mathrm{C}^*(\delta_{\mathfrak{X}(\Theta)})$ of finite support, and these form a subalgebra of $\mathcal{F}_e$. The enveloping C*–norm on this convolution algebra coincides with the C*–norm of $\mathcal{F}_e$, and now the equality follows from the fact that the *–algebra $\mathcal{A}(\Theta)$ generated by $\{\delta_{\mathfrak{X}(\Theta)} \cup \mathcal{U}(\Theta)\}$ is dense in this convolution algebra. For the next two equalities note that $\mathcal{A}(\Theta)$ consists of linear combinations of products of unitaries in $\delta_{\mathfrak{X}(\Theta)}$ and unitaries in $\mathcal{U}(\Theta)$. Each such a product of unitaries can be written as a constant times a product of the form $U_\gamma \cdot \delta_f$, $\gamma \in \mathcal{G}$, $f \in \mathfrak{X}(\Theta)$ as well as a product of the form $\delta_{f'} \cdot U_{\gamma'}$, using the Weyl relation together with the implementing relation $U_{\theta_h}\delta_f = \delta_{T_h(f)}U_{\theta_h}$ to rearrange the order of the products. Clearly now the last two relations follow from this. ∎

By setting $\Theta = \mathbb{R}^4$, the global objects are included in this lemma. Also observe that whilst $U_{\mathcal{G}(\Theta)}$ is clearly an equivalent set of constraints to $\mathcal{U}(\Theta)$, in general it is strictly larger. Now to define a system of local quantum constraints (cf. Def. 3.3) let Γ be any directed set of open bounded sets of $\mathbb{R}^4$ which covers $\mathbb{R}^4$, and such that orthochronous Poincaré transformations map elements of Γ to elements of Γ. Then the map $\mathcal{F}$ from Γ to subalgebras of $\mathcal{F}_e$ by $\Theta \to \mathcal{F}(\Theta)$ satisfies isotony.

4.3 Theorem *The map* $\Gamma \ni \Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ *defines a system of local quantum constraints.*

Proof: The net $\Theta \to \mathcal{F}(\Theta)$ is isotone and by construction of the cross product we also have $\mathcal{U}(\Theta_1) = \mathcal{U}(\Theta_2) \cap \mathcal{F}(\Theta_1)$ if $\Theta_1 \subseteq \Theta_2$. It remains to show that $\mathcal{U}(\Theta)$ is first class in $\mathcal{F}(\Theta)$, $\Theta \in \Gamma$. Consider the central state ω_0 on $\mathcal{A}$ (cf. Eqn. (3)). This is $\mathcal{G}$–invariant, and its restriction to $\mathrm{C}^*(\delta_{\mathfrak{X}(\Theta)})$ is clearly $\mathcal{G}(\Theta)$–invariant. By Theorem 2.16 it extends to a nontrivial Dirac state on $\mathcal{F}(\Theta)$, hence $\mathcal{U}(\Theta)$ is first–class. ∎

4.4 Remark Observe that as Γ is preserved by translations (cf. Definition 3.1), we can cover each compact set in $\mathbb{R}^4$ by a finite number of elements in Γ. Hence, since Γ is a directed set, each compact set in $\mathbb{R}^4$ is contained in an element of Γ. Thus $\cup\{\mathcal{S}(\Theta) \mid \Theta \in \Gamma\} = C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and so

$$\mathcal{F}_0 = \varinjlim \mathcal{F}(\Theta) = \mathrm{C}^*\left(\delta_{\mathfrak{X}_{(0)}} \cup \mathcal{U}_{(0)}\right) \subset \mathcal{F}_e$$

where $\mathfrak{X}_{(0)} := C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)\widehat{\ }/\mathrm{Ker}\, D$ and $\mathcal{U}_{(0)} := \{U_{\theta_h} \mid h \in C_c^\infty(\mathbb{R}^4, \mathbb{R})\}$.

4.2 Reduction isotony and weak causality.

In order to prove reduction isotony, we first need to enforce the T–procedure locally as in Section 3, to obtain the objects:

$$\begin{aligned}
\mathfrak{S}_D^\Theta &:= \{\omega \in \mathfrak{S}(\mathcal{F}(\Theta)) \mid \omega(U_{\theta_h}) = 1 \quad \forall\, h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}), \quad \mathrm{supp}(h) \subset \Theta\} \\
\mathcal{D}(\Theta) &:= [\mathcal{F}(\Theta)(\mathcal{U}(\Theta) - \mathbb{1})] \cap [(\mathcal{U}(\Theta) - \mathbb{1})\mathcal{F}(\Theta)], \\
\mathcal{O}(\Theta) &:= \{F \in \mathcal{F}(\Theta) \mid FD - DF \in \mathcal{D}(\Theta) \quad \forall\, D \in \mathcal{D}(\Theta)\} \;=\; M_{\mathcal{F}(\Theta)}(\mathcal{D}(\Theta)), \\
\mathcal{R}(\Theta) &:= \mathcal{O}(\Theta)/\mathcal{D}(\Theta) \quad \text{where } \Theta \text{ is any open set in } \mathbb{R}^4.
\end{aligned}$$

We need to prove that if $\Theta_1 \subseteq \Theta_2$ then $\mathcal{D}(\Theta_1) = \mathcal{D}(\Theta_2) \cap \mathcal{O}(\Theta_1)$ and $\mathcal{O}(\Theta_1) \subseteq \mathcal{O}(\Theta_2)$, and this requires more explicit characterization of the local algebras involved.

4.5 Theorem *We have:*

$$\begin{aligned}
\mathcal{O}(\Theta) &= \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)} \cup \mathcal{D}(\Theta)) = [\delta_{\mathfrak{p}(\Theta)} \cup \mathcal{D}(\Theta)] = \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) + \mathcal{D}(\Theta) \qquad \textit{where} \\
\mathfrak{p}(\Theta) &:= \{ f \in \mathfrak{X}(\Theta) \mid T_h(f) = f \; \forall \, h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}), \;\; \mathrm{supp}(h) \subset \Theta \} \\
&= \{ f \in \mathfrak{X}(\Theta) \mid B(f, G_h(f)) = 0 \; \forall \, h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}), \;\; \mathrm{supp}(h) \subset \Theta \} \\
&= \{ f \in \mathfrak{X}(\Theta) \mid p_\mu f^\mu \restriction C_+ = 0 \}
\end{aligned}$$

with $G_h(f) := f - T_h(f)$*. Moreover* $\mathcal{R}(\Theta) \cong \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)})$*.*

Proof: For any $f \in \mathfrak{p}(\Theta)$ one has $\delta_f = \theta_h(\delta_f) = U_{\theta_h} \, \delta_f \, U^*_{\theta_h}$, $\mathrm{supp}(h) \subset \Theta$, so that $\delta_{\mathfrak{p}(\Theta)} \subset \mathcal{U}(\Theta)'$ and Theorem 2.3 (v) implies $\delta_{\mathfrak{p}(\Theta)} \subset \mathcal{O}(\Theta)$. Further $\mathcal{D}(\Theta) \subset \mathcal{O}(\Theta)$ proves the inclusion $\mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)} \cup \mathcal{D}(\Theta)) \subseteq \mathcal{O}(\Theta)$. To show the reverse inclusion take $A \in \mathcal{O}(\Theta) \subset \mathcal{F}(\Theta)$ and from Lemma 4.2 there is a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathrm{span}\{\delta_{\mathfrak{X}(\Theta)} U_{\mathcal{G}(\Theta)}\}$ converging in the C*–norm to A. Put $A_n := \sum_{i=1}^{k_n} \lambda_i^n \, \delta_{f_i^n} \, U_{\gamma_i^n}$, $\lambda_i^n \in \mathbb{C}$, $f_i^n \in \mathfrak{X}(\Theta)$, $\gamma_i^n \in \mathcal{G}(\Theta)$, and since $\{f_i^n \mid i = 1, \ldots, k_n \, , \; n \in \mathbb{N}\}$ is a denumerable set we can rearrange it into a single sequence $\{f_i\}_{i \in \mathbb{N}}$, where $f_i \neq f_j$ if $i \neq j$. Thus we can rewrite

$$A_n = \sum_{i=1}^{N_n} \delta_{f_i} \sum_{j=1}^{L_n} \lambda_{ij}^{(n)} \, U_{\gamma_{ij}^{(n)}} \, .$$

Observe that for $\omega \in \mathfrak{S}_D^\Theta$ we have $\pi_\omega(A_n)\Omega_\omega = \sum_i \zeta_i^{(n)} \pi_\omega(\delta_{f_i})\Omega_\omega$ where $\zeta_i^{(n)} := \sum_j \lambda_{ij}^{(n)}$. Now $A \in \mathcal{O}(\Theta)$ and therefore we have for any $\omega \in \mathfrak{S}_D^\Theta$,

$$\begin{aligned}
0 &= \pi_\omega((U_{\theta_h} - \mathbb{1}) \cdot A)\Omega_\omega = \lim_{n \to \infty} \pi_\omega((U_{\theta_h} - \mathbb{1}) \cdot A_n)\Omega_\omega \\
&= \lim_{n \to \infty} \pi_\omega(\theta_h(A_n) - A_n)\Omega_\omega = \lim_{n \to \infty} \sum_{i=1}^{N_n} \zeta_i^{(n)} \pi_\omega(\delta_{T_h(f_i)} - \delta_{f_i})\Omega_\omega
\end{aligned}$$

for $h \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$, $\mathrm{supp}(h) \subset \Theta$. In particular let ω be an extension of the central state ω_0 defined in Eq. (3) (which has Dirac state extensions by Theorem 2.16). Then for all h:

$$\begin{aligned}
0 &= \omega(A^*(U_{\theta_h} - \mathbb{1})^*(U_{\theta_h} - \mathbb{1})A) \\
&= \lim_{n \to \infty} \Big\{ 2\omega(A_n^* A_n) - 2\,\mathrm{Re}\Big(\sum_{i,j}^{N_n} \overline{\zeta}_i^{(n)} \zeta_j^{(n)} \exp[i \, B(f_j, \, T_h(f_i))/2] \; \omega(\delta_{f_j - T_h(f_i)}) \Big) \Big\} \\
&= 2 \lim_{n \to \infty} \Big\{ \sum_i^{N_n} |\zeta_i^{(n)}|^2 - \mathrm{Re}\Big(\sum_{(i,j) \in P_h(n)} \overline{\zeta}_i^{(n)} \zeta_j^{(n)} \Big) \Big\} \qquad (7)
\end{aligned}$$

where $P_h(n) := \{(i, \, j) \in \{1, \ldots, \, N_n\}^2 \mid f_j = T_h(f_i)\} \subset P_h(n+m)$. Observe that if $f_i \in \mathfrak{p}(\Theta)$, then $(i, \, i) \in P_h(n)$ for all h, and that these terms cancel in Eqn. (7), i.e. we may assume that $f_i \notin \mathfrak{p}(\Theta)$ in (7). Furthermore by the Cauchy–Schwarz inequality

$$\Big| \sum_{(i,j) \in P_h(n)} \overline{\zeta}_i^{(n)} \zeta_j^{(n)} \Big| \leq \Big(\sum_{i \in D_h(n)} |\zeta_i^{(n)}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{j \in R_h(n)} |\zeta_j^{(n)}|^2 \Big)^{\frac{1}{2}} \leq \sum_{i=1}^{N_n} |\zeta_i^{(n)}|^2 \qquad (8)$$

where $D_h(n) := \{i \mid (i,j) \in P_h(n)\}$ and $R_h(n) := \{j \mid (i,j) \in P_h(n)\}$ i.e. the domain and range of the relation defined by $P_h(n)$. If $D_h(n)$ or $R_h(n)$ is not $\{1, \, 2, \ldots, \, N_n\}$, then the last inequality is strict and Eqn. (7) cannot hold unless $\lim_{n \to \infty} \zeta_i^{(n)} = 0$ for all $i \leq N_n$ not in $D_h(n)$ or

$R_h(n)$. Given that $f_i \notin \mathfrak{p}(\Theta)$ in the surviving terms of the sum, for each i, choose an h such that $f_i \neq T_h(f_i)$, then by Eqn. (6) $T_{th}(f_i) = f_i + t^2(T_h(f_i) - f_i)$ for $t \in \mathbb{R}_+$, and so $\{T_{th}(f_i) \mid t \in \mathbb{R}_+\}$ is a continuous family of distinct elements of $\mathfrak{X}(\Theta)$. Since $\{f_j \mid j \in \mathbb{N}\}$ is denumerable, there exists a t_0 such that $T_{t_0h}(f_i) \neq f_j$ for all $j \in \mathbb{N}$, i.e. $i \notin D_{t_0h}(n)$ for all n, and so Eqn. (7) can only hold if $\lim\limits_{n\to\infty} \zeta_i^{(n)} = 0$. We conclude that in the original expression for A_n, if $f_i \notin \mathfrak{p}(\Theta)$, then $\lim\limits_{n\to\infty} \zeta_i^{(n)} = 0$.

Recall by Subsection 2.2 that the factorization map $\mathcal{O}(\Theta) \to \mathcal{R}(\Theta)$ is precisely the restriction of $\mathcal{O}(\Theta)$ in the universal representation π_u to the subspace $\mathcal{H}_u^{(p)} := \{\psi \in \mathcal{H}_u \mid \pi_u(\mathcal{U}(\Theta))\psi = \psi\}$, so if we can show that $\pi_u(\mathcal{O}(\Theta))\restriction\mathcal{H}_u^{(p)} = \pi_u(\mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}))\restriction\mathcal{H}_u^{(p)}$, that suffices to prove that $\mathcal{O}(\Theta) = \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) + \mathcal{D}(\Theta)$. Let $\psi \in \mathcal{H}_u^{(p)}$, and $f_j \notin \mathfrak{p}(\Theta)$:

$$\begin{aligned}
\pi_u(A)\psi &= \lim_{n\to\infty} \pi_u(A_n)\psi = \lim_{n\to\infty} \sum_{i=1}^{N_n} \zeta_i^{(n)} \pi_u(\delta_{f_i})\psi \\
&= \lim_{n\to\infty} \Big(\sum_{i\neq j}^{N_n} \zeta_i^{(n)} \pi_u(\delta_{f_i}) + \zeta_j^{(n)} \pi_u(\delta_{f_j}) \Big)\psi \\
&= \lim_{n\to\infty} \sum_{i\neq j}^{N_n} \zeta_i^{(n)} \pi_u(\delta_{f_i})\psi \qquad (9)
\end{aligned}$$

so we can omit all contributions where $f_j \notin \mathfrak{p}(\Theta)$ from the sum:

$$\pi_u(A)\psi = \lim_{n\to\infty} \sum_{f_i\in\mathfrak{p}(\Theta)}^{N_n} \zeta_i^{(n)} \pi_u(\delta_{f_i})\psi \in \pi_u\left(\mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)})\right)\psi \qquad \forall\, \psi \in \mathcal{H}_u^{(p)},$$

hence $\mathcal{O}(\Theta) = \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) + \mathcal{D}(\Theta)$, using $\delta_{\mathfrak{p}(\Theta)} \subset \mathcal{O}(\Theta)$, (cf. [6]).

For the last two equivalent characterisations of $\mathfrak{p}(\Theta)$, observe first that if $T_h(f) = f$, then $B(f,\, G_h(f)) = B(f,\, f - T_h(f)) = 0$ and conversely

$$0 = B(f,\, G_h(f)) = 2\pi\, i \Big| \int_{C_+} \frac{d^3p}{p_0} p_\mu\, f^\mu(p)\, \overline{\widehat{h}(p)} \Big|^2$$

which by Eqn. (6) implies that $G_h(f) = 0$. Choose $\widehat{h} = p_\mu f^\mu$ (which is in the allowed class of functions) to see the equivalence with $p_\mu f^\mu \restriction C_+ = 0$.

Finally, to prove that $\mathcal{R}(\Theta) = (\mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) + \mathcal{D}(\Theta))/\mathcal{D}(\Theta) \cong \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)})$, it suffices to show that the ideal $\mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) \cap \mathcal{D}(\Theta) = \{0\}$. Consider a sequence

$$A_n = \sum_{i=1}^{N_n} \lambda_i^{(n)} \delta_{f_i} \in \Delta(\mathfrak{p}(\Theta),\, B), \;\; \lambda_i^{(n)} \in \mathbb{C}, \qquad \text{converging to} \quad A \in \mathcal{D}(\Theta)$$

then we show that it converges to zero. Now $\delta_{f_j} \in \mathcal{O}(\Theta)$ for $f_j \in \mathfrak{p}(\Theta)$, so for $N_n > j$ we have

$$\delta_{-f_j} \cdot A_n = \sum_{i\neq j}^{N_n} \lambda_i^{(n)} \delta_{f_i - f_j} \exp\left(iB(f_i,\, f_j)/2\right) + \lambda_j^{(n)} \mathbb{1} \longrightarrow \delta_{-f_j} \cdot A \in \mathcal{D}(\Theta)\,.$$

Since the central state ω_0 (cf. Eqn. (3)) extends to a Dirac state we have

$$0 = \omega_0(\delta_{-f_j} \cdot A) = \lim_{n\to\infty} \omega_0(\delta_{-f_j} A_n) = \lim_{n\to\infty} \lambda_j^{(n)} \quad \forall\, j\,.$$

This implies that $A=0$, because we can realize A as an ℓ^2–sequence over $\mathfrak{p}(\Theta)$, and the evaluation map at a point in $\mathfrak{p}(\Theta)$ is ℓ^2–continuous, hence C*–continuous so by the previous equation evaluation of A at each point is zero. ■

One can now set $\Theta=\mathbb{R}^4$ to get the global version of this theorem.

4.6 Theorem *The system of local constraints defined here satisfies reduction isotony.*

Proof: Let $\Theta_1\subseteq\Theta_2$, then we start by showing that $\mathcal{O}(\Theta_1)\subseteq\mathcal{O}(\Theta_2)$, i.e. by Theorem 4.5 we show that $C^*(\delta_{\mathfrak{p}(\Theta_1)})+\mathcal{D}(\Theta_1)\subseteq C^*(\delta_{\mathfrak{p}(\Theta_2)})+\mathcal{D}(\Theta_2)$. This follows directly from $\mathcal{D}(\Theta_1)\subseteq\mathcal{D}(\Theta_2)$ and $\mathfrak{p}(\Theta_1)\subseteq\mathfrak{p}(\Theta_2)$ where the last inclusion comes from the last characterisation of $\mathfrak{p}(\Theta)$ in Theorem 4.5.

It only remains to show that $\mathcal{D}(\Theta_1)=\mathcal{D}(\Theta_2)\cap\mathcal{O}(\Theta_1)$. Recall that $\mathcal{D}(\Theta)\subset\mathcal{O}(\Theta)$, and that $\mathcal{D}(\Theta)$ is the largest C*–algebra in $\mathcal{F}(\Theta)$ (hence in $\mathcal{O}(\Theta)$) which is annihilated by all $\omega\in\mathfrak{S}_D^{\Theta}$. Since $\mathcal{O}(\Theta_1)\subseteq\mathcal{O}(\Theta_2)$, it suffices to show that every Dirac state on $\mathcal{O}(\Theta_1)$ extends to a Dirac state on $\mathcal{O}(\Theta_2)$.

Recall that $\mathcal{O}(\Theta_1)=C^*(\delta_{\mathfrak{p}(\Theta_1)})+\mathcal{D}(\Theta_1)$, so a Dirac state on $\mathcal{O}(\Theta_1)$ is uniquely determined by its values on $\delta_{\mathfrak{p}(\Theta_1)}$. Moreover, from the fact that $f\in\mathfrak{p}(\Theta_1)$ implies $p_\mu f^\mu\restriction C_+=0$ and Eqn. (6), we see that $\mathcal{U}(\Theta_2)$ commutes with $C^*(\delta_{\mathfrak{p}(\Theta_1)})$:

$$U_{\theta_h}\delta_f U_{\theta_h}^{-1}=\delta_{T_hf}=\delta_f\quad\text{since}\quad (T_hf)_\mu(p)=f_\mu(p)-i\pi\,p_\mu\widehat{h}(p)\int_{C_+}f^\nu(p')\,p'_\nu\overline{\widehat{h}(p')}\,\frac{d^3p'}{p'_0}=f_\mu(p)$$

for $f\in\mathfrak{p}(\Theta_1)$. Next define $\widetilde{\mathcal{O}}:=C^*(\delta_{\mathfrak{p}(\Theta_1)}\cup\mathcal{U}(\Theta_2))\subset\mathcal{O}(\Theta_2)$. Now $\widetilde{\mathcal{O}}$ is generated by the two mutually commuting C*–algebras $C^*(\delta_{\mathfrak{p}(\Theta_1)})$ and $C^*(\mathcal{U}(\Theta_2))\cong C^*(\mathcal{G}(\Theta_2))$ where the latter is Abelian. If $AB=0$ for $A\in C^*(\delta_{\mathfrak{p}(\Theta_1)})$ and $B\in C^*(\mathcal{U}(\Theta_2))$, then either $A=0$ or $B=0$. This we can see from the realisation of $C^*(\mathcal{U}(\Theta_2))$ as scalar valued functions of denumerable support in $\mathcal{G}(\Theta_2)$, so (pointwise) multiplication by a nonzero $A\in C^*(\delta_{\mathfrak{p}(\Theta_1)})$ cannot change support. Then by an application of the result in [30, Exercise 2, p. 220], we conclude that the map $\varphi(A\otimes B):=AB$, $A\in C^*(\delta_{\mathfrak{p}(\Theta_1)})$, $B\in C^*(\mathcal{U}(\Theta_2))$ extends to an isomorphism $\varphi: C^*(\delta_{\mathfrak{p}(\Theta_1)})\otimes C^*(\mathcal{U}(\Theta_2))\to\widetilde{\mathcal{O}}$. (Note that since $C^*(\mathcal{U}(\Theta_2))$ is commutative it is nuclear, hence the tensor norm is unique).

Let $\omega\in\mathfrak{S}_D^{\Theta_1}\restriction\mathcal{O}(\Theta_1)$ and define a product state $\widetilde{\omega}$ on $\widetilde{\mathcal{O}}$ by $\widetilde{\omega}:=\omega\otimes\hat{\omega}$, where $\hat{\omega}$ is the state $\hat{\omega}(U_\theta)=1$ for all $\theta\in\mathcal{G}(\Theta_2)$. Now extend $\widetilde{\omega}$ arbitrarily to $\mathcal{O}(\Theta_2)$, then since it coincides with ω on $\delta_{\mathfrak{p}(\Theta_1)}$ and $\widetilde{\omega}(\mathcal{U}(\Theta_2))=1$, it is a Dirac state on $\mathcal{O}(\Theta_2)$ which extends $\omega\restriction\mathcal{O}(\Theta_1)$. ■

4.7 Theorem *The system of local quantum constraints $(\mathcal{F}(\Theta),\mathcal{U}(\Theta))$ satisfies weak causality, i.e. if $\Theta_1\perp\Theta_2$ then $[\mathcal{O}(\Theta_1)\,,\,\mathcal{O}(\Theta_2)]\subset\mathcal{D}(\Theta_0)$ for some $\Theta_0\supset\Theta_1\cup\Theta_2$, $\Theta_i\in\Gamma$.*

Proof: Since $\mathcal{O}(\Theta)=C^*(\delta_{\mathfrak{p}(\Theta)})+\mathcal{D}(\Theta)$ it is sufficient to consider commutants of $A_i\in\mathcal{O}(\Theta_i)$ being generating elements: $A_i=\delta_{f_i}+D_i$ for $f_i\in\mathfrak{p}(\Theta_i)$, $D_i\in\mathcal{D}(\Theta_i)$, $i=1,2$. Now

$$\begin{aligned}[A_1,A_2] &= [\delta_{f_1},\delta_{f_2}]+[\delta_{f_1},D_2]+[D_1,\delta_{f_2}]+[D_1,D_2]\\ &= \Big(e^{\frac{i}{2}B(f_1,\,f_2)}-e^{-\frac{i}{2}B(f_1,\,f_2)}\Big)\delta_{f_1+f_2}+[\delta_{f_1},\,D_2]+[D_1,\,\delta_{f_2}]+[D_1,\,D_2]\,.\end{aligned}$$

The first term vanishes because $\Theta_1\perp\Theta_2$ implies the supports of f_1 and f_2 are spacelike separated, so

$$B(f_1,\,f_2)=\int dx\,dx'\,\Delta(x-x')\,f_1^\mu(x)\,f_{2\mu}(x')=0$$

because the Pauli–Jordan distribution Δ has support disjoint from the spacelike part of $\mathbb{R}^4$ [23, p. 107]. Further for any $\Theta_0\supset\Theta_1\cup\Theta_2$ reduction isotony implies $\mathcal{D}(\Theta_1)\subset\mathcal{D}(\Theta_0)\supset\mathcal{D}(\Theta_2)$ and $\mathcal{O}(\Theta_1)\subset\mathcal{O}(\Theta_0)\supset\mathcal{O}(\Theta_2)$. But $\mathcal{D}(\Theta_0)$ is a closed 2–sided ideal in $\mathcal{O}(\Theta_0)$ and therefore the last 3 terms of the sum above are contained in $\mathcal{D}(\Theta_0)$ and the proof is concluded. ■

4.8 Remark Note that the net $\Theta \to \mathcal{F}(\Theta)$ does not satisfy the causality property, as we expect from the choice of noncausal constraints $(\partial^\mu A_\mu)^{(+)}(x)$. To see this, let $\Theta_1 \perp \Theta_2$, and let $\delta_f \in \mathcal{F}(\Theta_1)$ and $U_{\theta_h} \in \mathcal{F}(\Theta_2)$, then the commutator $[\delta_f,\, U_{\theta_h}]$ need not vanish because in Eqn. (6) we can have that

$$\int_{C_+} \widehat{f}^\nu(p)\, p_\nu\, \overline{\widehat{h}(p)} \frac{d^3p}{p_0} \neq 0$$

for $\mathrm{supp}(f) \subset \Theta_1$ and $\mathrm{supp}(h) \subset \Theta_2$.

4.3 Covariance.

In order to examine weak covariance for this system of local constraints, we first need to define the action of $\mathcal{P}_+^\uparrow$ on $\mathcal{F}_e$. We start with the usual action of $\mathcal{P}_+^\uparrow$ on $\widehat{\mathcal{S}}$. Define

$$(V_g f)(p) := e^{-ipa}\, \Lambda\, f(\Lambda^{-1}p) \qquad \forall\; f \in \widehat{\mathcal{S}},\; g = (\Lambda, a) \in \mathcal{P}_+^\uparrow. \tag{10}$$

Then V_g is symplectic, hence lifts to a symplectic transformation of $\mathfrak{X}$, and this defines an action $\alpha : \mathcal{P}_+^\uparrow \to \mathrm{Aut}(\overline{\Delta(\mathfrak{X},\, B)})$ by $\alpha_g(\delta_f) := \delta_{V_g f}$, $f \in \mathfrak{X}$.

4.9 Lemma *The definition $\alpha_g(U_{\theta_h}) := U_{\theta_{W_g h}}$, where $(W_g h)(x) := h(\Lambda^{-1}x)$, extends α_g from $\overline{\Delta(\mathfrak{X},\, B)}$ to $\mathcal{F}_e$, producing a consistent action $\alpha : \mathcal{P}_+^\uparrow \to \mathrm{Aut}(\mathcal{F}_e)$.*

Proof: We need to show that if we extend α_g from the set $\overline{\Delta(\mathfrak{X},\, B)} \cup \{U_{\theta_h} \mid h\}$ to the *-algebra generated by it using the homomorphism property of α_g, then this is consistent with all relations of the U_{θ_h} amongst themselves, and between them and $\overline{\Delta(\mathfrak{X},\, B)}$. First we need to establish how V_g and T_h intertwines.

$$\begin{aligned}
\Big(V_g(T_h f)\Big)_\mu(p) &= e^{-ia\cdot p}\Big(\Lambda(T_h f)\Big)_\mu(\Lambda^{-1}p) \\
&= (V_g f)_\mu(p) - i\pi\, p_\mu \widehat{h}(\Lambda^{-1}p)\, e^{-ia\cdot p} \int_{C_+} \frac{d^3p'}{p'_0} f^\nu(\Lambda^{-1}p')(\Lambda^{-1}p')_\nu \overline{\widehat{h}(\Lambda^{-1}p')} \\
&= \Big(T_{W_g h} V_g h\Big)_\mu(p).
\end{aligned}$$

Thus $V_g T_h = T_{W_g h} V_g$. Now the basic relation between $\overline{\Delta(\mathfrak{X},\, B)}$ and $\{U_{\theta_h} \mid h\}$ is the implementing relation, so

$$\alpha_g(U_{\theta_h} \delta_f U_{\theta_h}^*) = \alpha_g(\delta_{T_h f}) = \delta_{V_g T_h f} = \delta_{T_{W_g h} V_g f} = U_{\theta_{W_g h}} \delta_{V_g f} U_{\theta_{W_g h}}^* = \alpha_g(U_{\theta_h})\alpha_g(\delta_f)\alpha_g(U_{\theta_h})^*$$

thus α_g is consistent with this. Finally we need to show that α_g respects any group identities in $\mathcal{G} \subset \mathrm{Aut}\overline{\Delta(\mathfrak{X},\, B)}$. Recalling that $\mathcal{G}$ consists of finite products of θ_h, let $\gamma = \theta_{h_1} \ldots \theta_{h_n} \in \mathcal{G}$, then $\gamma \to \theta_{W_g h_1} \ldots \theta_{W_g h_n}$ defines a consistent group homomorphism because

$$\begin{aligned}
\alpha_g(\gamma(\delta_f)) &= \alpha_g(\delta_{T_{h_1} \cdots T_{h_n} f}) = \delta_{V_g T_{h_1} \ldots T_{h_n} f} \\
&= \delta_{T_{W_g h_1} \ldots T_{W_g h_n} V_g f} = (\theta_{W_g h_1} \cdots \theta_{W_g h_n})\alpha_g(\delta_f)
\end{aligned}$$

i.e. $\theta_{W_g h_1} \cdots \theta_{W_g h_n} = \alpha_g \circ \gamma \circ \alpha_g^{-1}$. Thus $\alpha_g(U_{\theta_h}) = U_{\theta_{W_g h}}$ extends consistently to $U_{\mathcal{G}}$. ■

4.10 Theorem *Consider the action α: $\mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{F}_e$ defined above. Then the system of local quantum constraints $\Gamma \ni \Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ satisfies $\alpha_g(\mathcal{U}(\Theta)) = \mathcal{U}(g\Theta)$ and the net $\Gamma \ni \Theta \to \mathcal{F}(\Theta)$ transforms covariantly, i.e. $\alpha_g(\mathcal{F}(\Theta)) = \mathcal{F}(g\Theta)$, $\Theta \in \Gamma$. Therefore the local observables define a covariant net, i.e. $\alpha_g(\mathcal{O}(\Theta)) = \mathcal{O}(g\Theta)$.*

Proof: We have:

$$\alpha_g(\mathcal{U}(\Theta)) = \alpha_g(\{U_{\theta_h} \mid \mathrm{supp}(h) \subset \Theta\}) = \{U_{\theta_{V_g h}} \mid \mathrm{supp}(h) \subset \Theta\} \subseteq \{U_{\theta_h} \mid \mathrm{supp}(h) \subset g\Theta\} = \mathcal{U}(g\Theta)$$

and replacing g by g^{-1} gives the converse inclusion.

For covariance of the net $\mathcal{F}(\Theta)$, recall that each $\mathcal{F}(\Theta)$ is generated by $\mathcal{U}(\Theta)$ and $\delta_{\mathfrak{X}(\Theta)}$, so since

$$\alpha_g(\delta_{\mathfrak{X}(\Theta)}) = \delta_{V_g \mathfrak{X}(\Theta)} = \delta_{\mathfrak{X}(g\Theta)}\,,$$

it follows that $\alpha_g(\mathcal{F}(\Theta)) = \mathcal{F}(g\Theta)$. The covariance property for the net of local observables follows from Remark 3.6 (iii) ■

Finally putting together Theorems 4.6, 4.7, 4.10 and 3.7 we have proved for the Gupta–Bleuler model

4.11 Theorem *The system of local quantum constraints $\Gamma \ni \Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ satisfies reduction isotony, weak causality and covariance and therefore the corresponding net of local physical observables $\Gamma \ni \Theta \to \mathcal{R}(\Theta)$ is a HK–QFT.*

4.4 A simple physical observable algebra

The net $\Theta \to \mathcal{R}(\Theta) = \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)})$ produces a quasi–local physical algebra

$$\mathcal{R}_0 = \varinjlim \mathcal{R}(\Theta) = \varinjlim \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) = \mathrm{C}^*(\delta_{\mathfrak{p}}) = \overline{\Delta(\mathfrak{p},\, B)}\,,$$

where $\mathfrak{p} := \mathrm{Span}\{\mathfrak{p}(\Theta) \mid \Theta \in \Gamma\} = \bigcup_{\Theta\in\Gamma} \mathfrak{p}(\Theta)$ since Γ is a directed set. Since B is degenerate on $\mathfrak{p}$ (see below), $\mathcal{R}_0$ is not simple and thus cannot be the final physical algebra. Moreover since we have not enforced Maxwell's equations, from a physical point of view $\mathcal{R}_0$ cannot be considered as representing the observables of an electromagnetic field as yet. To solve these problems, we now do a second stage of constraining where we choose for our constraint system $(\mathcal{R}_0, \widetilde{\mathcal{U}})$ where $\widetilde{\mathcal{U}} := \delta_{\mathfrak{p}_0}$ and $\mathfrak{p}_0$ is the kernel of $B\restriction\mathfrak{p}$. The T–procedure applied to this pair will result in a simple algebra via Corollary 5.4 in [6]. For the connection with the Maxwell equations, we need the following theorem:

4.12 Theorem *We have:*

$$\begin{aligned} \mathfrak{p}_0 \;&:=\; \{f \in \mathfrak{p} \mid B(f,\, k) = 0 \quad \forall\, k \in \mathfrak{p}\} \\ &=\; \{f \in \mathfrak{p} \mid f_\mu(p) = p_\mu\, h(p) \;\; \textit{for}\;\; p \in C_+, \;\; \textit{where}\;\; h : C_+ \to \mathbb{C} \\ &\qquad\qquad \textit{is any function such that} \quad p \to p_\mu h(p) \;\; \textit{is in}\;\; \mathfrak{X}_{(0)}\,\} \end{aligned}$$

Proof: Now $\mathfrak{p} \subset \mathfrak{X}_{(0)} \subset \mathfrak{X}$ (cf. Remark 4.4). Define the map γ: $\mathcal{S}(\mathbb{R}^4, \mathbb{R}^4) \to \mathfrak{X}$ by $\gamma(f) := \widehat{f}\restriction C_+$, then by definition $D(f,\, k) =: B(\gamma(f), \gamma(k))$ for $f,\; k \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}^4)$ and $\gamma(C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)) = \mathfrak{X}_{(0)}$ (again by Remark 4.4). Furthermore, by Theorem 4.5, $\mathfrak{p}$ is the image under γ of $\mathfrak{P} := \{f \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4) \mid \partial^\mu f_\mu = 0\}$. Since the smooth functions of compact support are dense with respect to the Schwartz topology in the Schwartz space, and the divergence operator is continuous for the Schwartz topology, the closure of $\mathfrak{P}$ in the Schwartz topology is $\widetilde{\mathfrak{P}} := \{f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}^4) \mid \partial^\mu f_\mu = 0\}$. It is well–known that D is a tempered distribution (it is the two–point function for the free electromagnetic field), hence it is continuous with respect to the Schwartz topology on $\mathcal{S}(\mathbb{R}^4, \mathbb{R}^4)$ in each entry. Thus $D(f,\, k) = 0$ for all $k \in \mathfrak{P}$ iff $D(f,\, k) = 0$ for all $k \in \widetilde{\mathfrak{P}}$ and hence $\mathfrak{p}_0 = \{f \in \mathfrak{p} \mid\; B(f,\, k) = 0 \;\; \forall\, k \in \widetilde{\mathfrak{p}}\}$ where $\widetilde{\mathfrak{p}} := \gamma(\widetilde{\mathfrak{P}})$.

Let $f \in \mathfrak{p}$, so $p_\mu f^\mu(p) = 0$ for $p \in C_+$, i.e. $\mathbf{p} \cdot \mathbf{f}(p) = \|\mathbf{p}\|\, f_0(p)$, so for $p \in C_+\backslash 0$, we have $f_0(p) = \mathbf{p} \cdot \mathbf{f}(p)/\|\mathbf{p}\| = \mathbf{e}(p) \cdot \mathbf{f}(p)$ where $\mathbf{e}(p) := \mathbf{p}/\|\mathbf{p}\|$. Now in terms of real and imaginary parts $f = u + iv \in \mathfrak{p}_0$ iff for all $k = w + ir \in \widetilde{\mathfrak{p}}$ we have that

$$\begin{aligned} 0 &= D(f,\,k) = 2i \int_{\mathbb{R}^3\backslash 0} (v_\mu w^\mu - u_\mu r^\mu) \frac{d^3p}{\|\mathbf{p}\|} \qquad \text{(using eqn. 5)} \\ &= 2i \int_{\mathbb{R}^3\backslash 0} \Big(\mathbf{u}\cdot\mathbf{r} - \mathbf{v}\cdot\mathbf{w} + (\mathbf{e}(p)\cdot\mathbf{v})(\mathbf{e}(p)\cdot\mathbf{w}) - (\mathbf{e}(p)\cdot\mathbf{u})(\mathbf{e}(p)\cdot\mathbf{r}) \Big) \frac{d^3p}{\|\mathbf{p}\|}\,. \end{aligned}$$

Choose $w = 0$ (which is possible in $\widetilde{\mathfrak{p}}$) to get that for all $r \in \widetilde{\mathfrak{p}} \cap (\mathcal{S}_- \restriction C_+)$ (recall Remark 4.1):

$$\begin{aligned} 0 &= D(f,\,k) = 2i \int_{\mathbb{R}^3\backslash 0} \Big(\mathbf{u}\cdot\mathbf{r} - (\mathbf{e}(p)\cdot\mathbf{u})(\mathbf{e}(p)\cdot\mathbf{r}) \Big) \frac{d^3p}{\|\mathbf{p}\|} \\ &= 2i \int_{\mathbb{R}^3\backslash 0} \mathbf{r} \cdot \Big(\mathbf{u} - \mathbf{e}(p)(\mathbf{e}(p)\cdot\mathbf{u}) \Big) \frac{d^3p}{\|\mathbf{p}\|}\,. \end{aligned} \tag{11}$$

Now let $m : C_+ \to \mathbb{R}_+$ be a smooth bump function with compact support away from zero, then we know that that the function s given by

$$\mathbf{s}(p) := (\mathbf{u}(p) - \mathbf{e}(p)(\mathbf{e}(p)\cdot\mathbf{u}(p)))\, m(p) \qquad \text{and} \qquad s_0(p) := \mathbf{e}(p)\cdot\mathbf{s}(p)$$

is in $\mathfrak{X}$ by the characterisation of $\mathfrak{X}$ given in Remark 4.1, that it contains all smooth functions with compact support away from zero. Moreover, since $p_\mu s^\mu = 0$, we conclude $s \in \widetilde{\mathfrak{p}}$. (Note that $s \notin \mathfrak{p}$, hence the extension to $\widetilde{\mathfrak{p}}$ in the first part of the proof). So we can choose $r = s$ above in Eqn. (11), then by continuity, positivity and by ranging over all m, we conclude that

$$\mathbf{u}(p) - \mathbf{e}(p)(\mathbf{e}(p)\cdot\mathbf{u}(p)) = 0 \quad \forall\, p \in C_+$$

and as the second term is just the projection of $\mathbf{u}(p)$ in the direction of $\mathbf{p}$, this means $\mathbf{u}(p)$ must be proportional to $\mathbf{p}$ for all $p \in C_+\backslash 0$, i.e. $\mathbf{u}(p) = \mathbf{p}\, q(p)$, for some suitable scalar function q. Since $u_0(p) = \mathbf{e}(p)\cdot\mathbf{u}(p) = \|\mathbf{p}\|\, q(p) = p_0\, q(p)$, $p \in C_+$, this means $u_\mu(p) = p_\mu\, q(p)$, $p \in C_+$. By setting $r = 0$, we obtain a similar result for v, and hence $f_\mu(p) = p_\mu\, h(p)$, $p \in C_+$. The only restriction on h is that $f \in \mathfrak{X}_{(0)}$, since f is automatically in $\mathfrak{p}$ by its form. ■

4.13 Remark (i) In the above, the fact that $f \in \mathfrak{X}$ means that h must be smooth away from the origin. Since h is undefined at the origin in the proof, consider the behaviour of $f \in \mathfrak{p}_0$ at zero. Let $a \in C_+\backslash 0$, then by continuity of f:

$$\mathbf{f}(0) = \lim_{t\to 0^+} \mathbf{f}(ta) = \mathbf{a} \lim_{t\to 0^+} t\, h(ta) = \mathbf{a} \lim_{t\to 0^+} \frac{\mathbf{a}\cdot\mathbf{f}(ta)}{\|\mathbf{a}\|^2} = \mathbf{e}(a)(\mathbf{e}(a)\cdot\mathbf{f}(0))$$

which can only be true for all a if $f(0) = 0$.

(ii) Now recall Maxwell's equations $F_{\mu\nu}{}^{,\nu}(x) = 0$. In the heuristic version of Gupta–Bleuler QEM, these need to be imposed as state conditions to define the physical field. Using the smearing formula for $A(f)$, $F_{\mu\nu}{}^{,\nu}(x)$ corresponds to the space

$$\mathfrak{f} := \Big\{ f \in \mathfrak{X} \mid f_\mu(p) = p_\mu p^\nu k_\nu(p), \quad p \in C_+,\ k \in \mathfrak{X} \Big\}.$$

By Theorem 4.12 we observe that $\mathfrak{f} \subset \mathfrak{p}_0$, and thus enforcing the second stage of constraints $\widetilde{\mathcal{U}} = \delta_{\mathfrak{p}_0}$ will also impose the Maxwell equations. Note however that the inclusion $\mathfrak{f} \subset \mathfrak{p}_0$ is proper, which we see as follows. Consider a line $t \to ta$, $a \in C_+\backslash 0$ and let $f \in \mathfrak{f}$ and $g \in \mathfrak{p}_0$, then $\lim\limits_{t\to 0^+}\frac{d}{dt}f_\mu(ta)=0$, but $\lim\limits_{t\to 0^+}\frac{d}{dt}g_\mu(ta)=a_\mu\lim\limits_{t\to 0^+}(h(ta)+t\frac{d}{dt}h(ta))$ if $g_\mu(p)=p_\mu\,h(p)$, and we can easily choose an $h\in\mathcal{S}(\mathbb{R}^4,\,\mathbb{C})$ which makes the latter nonzero. Thus merely imposing the Maxwell equations does not appear to be sufficient to make the physical algebra simple (contrary to a claim in [6]).

For completeness we would also like to consider the local structure of the constraint system $(\mathcal{R}_0,\widetilde{\mathcal{U}})$. Define $\widetilde{\mathcal{U}}(\Theta):=\widetilde{\mathcal{U}}\cap\mathcal{R}(\Theta)=\delta_{\mathfrak{s}(\Theta)}$, $\Theta\in\Gamma$, where $\mathfrak{s}(\Theta):=\mathfrak{p}_0\cap\mathfrak{p}(\Theta)$, then it is clear that $\Theta\to(\mathcal{R}(\Theta),\widetilde{\mathcal{U}}(\Theta))$ is a system of local quantum constraints. Since $\widetilde{\mathcal{U}}\subset Z(\mathcal{R}_0)$, a local T–procedure produces:

$$\begin{aligned}\widetilde{\mathcal{D}}(\Theta)&=[\mathcal{R}(\Theta)(\mathbb{1}-\widetilde{\mathcal{U}}(\Theta))]\\ \widetilde{\mathcal{O}}(\Theta)&=\mathcal{R}(\Theta)\\ \widetilde{\mathcal{R}}(\Theta)&=\mathcal{R}(\Theta)/[\mathcal{R}(\Theta)(\mathbb{1}-\widetilde{\mathcal{U}}(\Theta))]\,.\end{aligned}$$

4.14 Theorem *The system of local constraints $\Theta\to(\mathcal{R}(\Theta),\widetilde{\mathcal{U}}(\Theta))$ satisfies reduction isotony, causality and weak covariance, hence $\Theta\to\widetilde{\mathcal{R}}(\Theta)$ is a HK–QFT. Moreover*

$$\widetilde{\mathcal{R}}(\Theta)\cong\overline{\Delta(\mathfrak{p}(\Theta)/\mathfrak{s}(\Theta),\,\widetilde{B})}\cong\overline{\Delta(\mathfrak{c}(\Theta),\,B)}$$

where $\mathfrak{c}(\Theta):=\{f\in\mathfrak{p}(\Theta)\mid\mathbf{p}\cdot\mathbf{f}(p)=0,\;\;p\in C_+\}$ is the "Coulomb space", and

$$\widetilde{\mathcal{R}}_0\cong\overline{\Delta(\mathfrak{p}/\mathfrak{p}_0,\,\widetilde{B})}\cong\overline{\Delta(\mathfrak{c},\,B)}$$

where $\widetilde{B}$ is B lifted to $\mathfrak{p}/\mathfrak{p}_0$, and $\mathfrak{c}:=\{f\in\mathfrak{p}\mid\mathbf{p}\cdot\mathbf{f}(p)=0,\;\;p\in C_+\}$.

Proof: For reduction isotony, since it is obvious that if $\Theta_1\subseteq\Theta_2$ then $\widetilde{\mathcal{O}}(\Theta_1)=\mathcal{R}(\Theta_1)\subseteq\mathcal{R}(\Theta_2)=\widetilde{\mathcal{O}}(\Theta_2)$, we only need to show that $\widetilde{\mathcal{D}}(\Theta_1)=\widetilde{\mathcal{D}}(\Theta_2)\cap\mathcal{R}(\Theta_1)$, which by Lemma 2.24 will be the case if every Dirac state on $\mathcal{R}(\Theta_1)$ extends to a Dirac state on $\mathcal{R}(\Theta_2)$. We first prove that $\mathfrak{p}(\Theta)=\mathfrak{c}(\Theta)\oplus\mathfrak{s}(\Theta)$ where $\mathfrak{c}(\Theta)$ is the "Coulomb space" above, and $\mathfrak{s}(\Theta):=\mathfrak{p}_0\cap\mathfrak{p}(\Theta)$. Let $f\in\mathfrak{p}(\Theta)$, so $0=p_\mu f^\mu(p)$, $p\in C_+$ and write

$$f_\mu=g_\mu+p_\mu h\qquad\text{where}\qquad h(p):=f_0(p)/\|\mathbf{p}\|=\mathbf{p}\cdot\mathbf{f}/\|\mathbf{p}\|^2\qquad\text{and}\qquad g_\mu(p):=f_\mu(p)-p_\mu h(p)\,.$$

Then obviously $p_\mu h$ is in $\mathfrak{s}(\Theta)$ and $\mathbf{p}\cdot\mathbf{g}(p)=\mathbf{p}\cdot\mathbf{f}(p)-\|\mathbf{p}\|^2h(p)=0$, so $g\in\mathfrak{c}(\Theta)$. To see that the decomposition is unique, let $g,\;k\in\mathfrak{c}(\Theta)$ such that $g_\mu-k_\mu=p_\mu h$. Then $0=\mathbf{p}\cdot(\mathbf{g}-\mathbf{k})=\|\mathbf{p}\|^2h$, i.e. $h=0$.

Since for $\Theta_1\subseteq\Theta_2$ we have $\mathfrak{s}(\Theta_1)=\mathfrak{s}(\Theta_2)\cap\mathfrak{p}(\Theta_1)$ and thus $\mathrm{Span}(\mathfrak{p}(\Theta_1)\cup\mathfrak{s}(\Theta_2))=\mathfrak{c}(\Theta_1)\oplus\mathfrak{s}(\Theta_2)$, so $\mathcal{A}:=\mathrm{C}^*(\delta_{\mathrm{Span}(\mathfrak{p}(\Theta_1)\cup\mathfrak{s}(\Theta_2))})$ is generated by two mutually commuting C*–algebras $\mathrm{C}^*(\delta_{\mathfrak{c}(\Theta_1)})$ and $\mathrm{C}^*(\delta_{\mathfrak{s}(\Theta_2)})$ where the last one is commutative. Now let $A\in\mathrm{C}^*(\delta_{\mathfrak{c}(\Theta_1)})$ and $B\in\mathrm{C}^*(\delta_{\mathfrak{s}(\Theta_2)})$ such that $AB=0$. Then we want to show that $A=0$ or $B=0$. Let $A_n:=\sum\limits_{i=1}^{N_n}\alpha_i^{(n)}\delta_{f_i}\longrightarrow A=\sum\limits_{i=1}^{\infty}\alpha_i\delta_{f_i}$ where $f_i\in\mathfrak{c}(\Theta_1)$, and $B_n:=\sum\limits_{j=1}^{M_n}\beta_j^{(n)}\delta_{k_j}\longrightarrow B=\sum\limits_{j=1}^{\infty}\beta_j\delta_{k_j}$ with $k_j\in\mathfrak{s}(\Theta_2)$. Then $0=AB=\lim\limits_{n\to\infty}\sum\limits_{i,j}^{N_n,M_n}\alpha_i^{(n)}\beta_j^{(n)}\delta_{f_i+k_j}$. However $f_i+k_j\neq f_{i'}+k_{j'}$ for $i\neq i'$ and $j\neq j'$ since $\mathfrak{c}(\Theta_1)$ and $\mathfrak{s}(\Theta_2)$ are linear independent spaces intersecting only in $\{0\}$. Thus the set $\{\delta_{f_i+k_j}\mid i\in\mathbb{N},\;j\in\mathbb{N}\}$ is linearly independent and so $0=\lim\limits_{n\to\infty}\alpha_i^{(n)}\beta_j^{(n)}=\alpha_i\beta_j$. Since this holds for all possible pairs $i,\;j$, there is no pair $\alpha_i,\;\beta_j$ such that $\alpha_i\beta_j\neq0$ and so either all $\alpha_i=0$ or all $\beta_j=0$, i.e. $A=0$ or $B=0$. Thus from Takesaki [30, Exercise 2, p. 220] we conclude that $\mathcal{A}$ is isomorphic to $\mathrm{C}^*(\delta_{\mathfrak{c}(\Theta_1)})\otimes\mathrm{C}^*(\delta_{\mathfrak{s}(\Theta_2)})$ by the map $\varphi(A\otimes B):=AB$.

Let ω be a Dirac state on $\mathcal{R}(\Theta_1)$, i.e. $\omega(\delta_{\mathfrak{s}(\Theta_1)})=1$, and then define a state $\widetilde{\omega}$ on $\mathcal{A}$ by $\widetilde{\omega}:=(\omega\otimes\widehat{\omega})\circ\varphi^{-1}$ where $\widehat{\omega}$ is the state on $\mathrm{C}^*(\delta_{\mathfrak{s}(\Theta_2)})$ satisfying $\widehat{\omega}(\delta_{\mathfrak{s}(\Theta_2)})=1$. Now extend $\widetilde{\omega}$

arbitrarily to $\mathcal{R}(\Theta_2) \supset \mathcal{A}$, then it coincides with ω on $\mathcal{R}(\Theta_1)$ and satisfies $\widehat{\omega}(\delta_{\mathfrak{s}(\Theta_2)}) = 1$ hence is a Dirac state on $\mathcal{R}(\Theta_2)$. This establishes reduction isotony.

For causality, the fact that $\Theta \to \mathcal{R}(\Theta)$ is a HK–QFT already implies that $[\mathcal{R}(\Theta_1),\, \mathcal{R}(\Theta_2)] = 0$ when $\Theta_1 \perp \Theta_2$, so $[\widetilde{\mathcal{O}}(\Theta_1),\, \widetilde{\mathcal{O}}(\Theta_2)] = 0$.

For covariance, we already have that $\alpha_g(\widetilde{\mathcal{O}}(\Theta)) = \alpha_g(\mathcal{R}(\Theta)) = \mathcal{R}(g\Theta) = \widetilde{\mathcal{O}}(g\Theta)$ for $g \in \mathcal{P}^\uparrow_+$, $\Theta \in \Gamma$. Now

$$\alpha_g(\widetilde{\mathcal{U}}(\Theta)) = \alpha_g(\delta_{\mathfrak{s}(\Theta)}) = \delta_{V_g \mathfrak{s}(\Theta)} \quad \text{and} \quad \widetilde{\mathcal{U}}(g\Theta) = \delta_{\mathfrak{s}(g\Theta)} \,.$$

To see that these are equal, note that $V_g\mathfrak{s}(\Theta) = V_g(\mathfrak{p}_0 \cap \mathfrak{p}(\Theta))$, V_g is symplectic, and $V_g\mathfrak{p}(\Theta) \subseteq \mathfrak{p}(g\Theta)$ by

$$p^\mu (V_g f)_\mu(p) = p^\mu (\Lambda f(\Lambda^{-1}p))_\mu e^{-ip\cdot a} = (\Lambda^{-1}p)^\mu f_\mu(\Lambda^{-1}p) e^{-ip\cdot a} = 0$$

for $p \in C_+$ and $f \in \mathfrak{p}(\Theta)$. Thus $V_g(\mathfrak{p}_0 \cap \mathfrak{p}(\Theta)) \subseteq \mathfrak{p}_0 \cap \mathfrak{p}(g\Theta)$. For the reverse inclusion: $V_{g^{-1}}(\mathfrak{p}_0 \cap \mathfrak{p}(g\Theta)) \subseteq \mathfrak{p}_0 \cap \mathfrak{p}(\Theta)$ implies that $\mathfrak{p}_0 \cap \mathfrak{p}(g\Theta) \subseteq V_g(\mathfrak{p}_0 \cap \mathfrak{p}(\Theta))$. Thus $\alpha_g(\widetilde{\mathcal{U}}(\Theta)) = \widetilde{\mathcal{U}}(g\Theta)$.

Finally, for the last two isomorphism claims, recall that $\mathcal{R}(\Theta) = \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)}) = \overline{\Delta(\mathfrak{p}(\Theta),\, B)}$ and so since $\widetilde{\mathcal{R}}(\Theta) = \mathcal{R}(\Theta)/[\mathcal{R}(\Theta)(\mathbb{1} - \widetilde{\mathcal{U}}(\Theta))]$ and $\widetilde{\mathcal{U}}(\Theta) = \delta_{\mathfrak{s}(\Theta)}$ where $\mathfrak{s}(\Theta)$ is the degenerate part of $\mathfrak{p}(\Theta)$, we conclude from Theorem 2.20 that $\widetilde{\mathcal{R}}(\Theta) \cong \overline{\Delta(\mathfrak{p}(\Theta)/\mathfrak{s}(\Theta),\, \widetilde{B})}$ (providing the symplectic commutant of $\mathfrak{s}(\Theta)$ in $\mathfrak{p}(\Theta)$, $\mathfrak{s}(\Theta)' = \mathfrak{p}(\Theta)$, and this is obvious since $\mathfrak{s}(\Theta) = \mathfrak{p}_0 \cap \mathfrak{p}(\Theta)$ and $\mathfrak{p}(\Theta) \subset \mathfrak{p}$). Since $\mathfrak{p}(\Theta) = \mathfrak{c}(\Theta) \oplus \mathfrak{s}(\Theta)$, this is isomorphic to $\overline{\Delta(\mathfrak{c}(\Theta),\, B)}$. Since for $\Theta_1 \subseteq \Theta_2$ the inclusion $\widetilde{\mathcal{R}}(\Theta_1) \subseteq \widetilde{\mathcal{R}}(\Theta_2)$ comes from $\mathfrak{p}(\Theta_1) \subseteq \mathfrak{p}(\Theta_2)$, and this inclusion factors through to produce $\mathfrak{p}(\Theta_1) \subseteq \mathfrak{p}(\Theta_2)$, hence $\mathfrak{c}(\Theta_1) \subseteq \mathfrak{c}(\Theta_2)$, so the last isomorphism is clear. ∎

Thus the quasi–local algebra $\widetilde{\mathcal{R}}_0$ is simple. (Unless we can show that B is nondegenerate on each $\mathfrak{c}(\Theta)$, it is not clear that the local algebras are simple).

An apparent puzzle raised by the isomorphisms $\widetilde{\mathcal{R}}(\Theta) \cong \overline{\Delta(\mathfrak{c}(\Theta),\, B)}$ here, is the noncovariance of the spaces $\mathfrak{c}(\Theta)$ under V_g, $g \in \mathcal{P}^\uparrow_+$, given that the net $\widetilde{\mathcal{R}}(\Theta)$ is covariant under the isomorphisms derived from V_g. The resolution is that V_g maps an equivalence class $f + \mathfrak{s}(\Theta)$ in $\mathfrak{p}(\Theta)$ to the equivalence class $V_g f + \mathfrak{s}(g\Theta)$ in $\mathfrak{p}(g\Theta)$, and these equivalence classes correspond to elements $h \in \mathfrak{c}(\Theta)$ and $k \in \mathfrak{c}(g\Theta)$ repectively, but it is not true that $k = V_g h$.

In this example we have done our constraint reduction in two stages, and the question arises as to whether we would have obtained the same physical algebra from a single reduction by the full set of constraints. This will be examined in the next section.

5 Further topics

5.1 Global vs local constraining

For a system of local constraints $\Theta \to (\mathcal{F}(\Theta),\, \mathcal{U}(\Theta))$ as in Definition 3.3, a natural question to ask is the following. What is the relation between the limit algebra $\mathcal{R}_0 := \varinjlim \mathcal{R}(\Theta)$ and the algebra $\mathcal{R}_e$ obtained from enforcing the full constraint set $\bigcup\limits_{\Theta\in\Gamma} \mathcal{U}(\Theta) =: \mathcal{U}_e$ in the quasi–local algebra $\mathcal{F}_0$? In particular, when will $\mathcal{R}_0 = \mathcal{R}_e$? In other words, we compare the local constrainings of the net to a single global constraining. (This has bearing on the BRST–constraining algorithm).

Now $\mathcal{R}_e = \mathcal{O}_e/\mathcal{D}_e$ where as usual we have $\mathcal{D}_e = [\mathcal{F}_0(\mathcal{U}_e - \mathbb{1})] \cap [(\mathcal{U}_e - \mathbb{1})\mathcal{F}_0]$ and $\mathcal{O}_e = \{F \in \mathcal{F}_0 \mid [F,\, D] \in \mathcal{D}_e \;\forall\, D \in \mathcal{D}_e\}$.

5.1 Theorem *Let the system of local constraints* $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ *have reduction isotony, then* $\mathcal{R}_0 := \varinjlim \mathcal{R}(\Theta) = \mathcal{O}_0/(\mathcal{D}_e \cap \mathcal{O}_0)$ *where* $\mathcal{O}_0 := \varinjlim \mathcal{O}(\Theta)$. *Moreover, there is an injective homomorphism of* $\mathcal{R}_0$ *into* $\mathcal{R}_e$.

Proof: First observe that $\mathcal{D}_e \cap \mathcal{O}_0 = [\mathcal{O}_0(\mathcal{U}_e - \mathbb{1})] \cap [(\mathcal{U}_e - \mathbb{1})\mathcal{O}_0]$ because $\mathcal{U}_e \subset \mathcal{O}_0$ hence every Dirac state on $\mathcal{O}_0$ extends to one on $\mathcal{F}_0$, and the $\mathcal{D}$–algebra is characterised as the maximal C^*–algebra in the kernels of all the Dirac states.

Now denote by $\xi_\Theta : \mathcal{O}(\Theta) \to \mathcal{R}(\Theta)$ the factoring map by $\mathcal{D}(\Theta)$. Let $\Theta_1 \subseteq \Theta_2$, then the diagram

$$\begin{array}{ccc} \mathcal{O}(\Theta_1) & \xrightarrow{\text{inclusion}} & \mathcal{O}(\Theta_2) \\ \Big\downarrow \xi_{\Theta_1} & & \Big\downarrow \xi_{\Theta_2} \\ \mathcal{R}(\Theta_1) & \xrightarrow{\iota_{12}} & \mathcal{R}(\Theta_2) \end{array}$$

commutes by reduction isotony and the proof of Lemma 2.22. Thus there exists a surjective homomorphism for the inductive limit algebras: $\xi_0 : \mathcal{O}_0 \to \mathcal{R}_0$, such that $\xi_0 \upharpoonright \mathcal{O}(\Theta) = \xi_\Theta$, $\Theta \in \Gamma$. Clearly $\mathcal{D}(\Theta) \subset \operatorname{Ker} \xi_0$ for all $\Theta \in \Gamma$, hence $\mathcal{U}_e - \mathbb{1} \subset \operatorname{Ker} \xi_0$, and so by the previous paragraph $\mathcal{D}_e \cap \mathcal{O}_0 \subseteq \operatorname{Ker} \xi_0$. Thus $\mathcal{D}_e \cap \mathcal{O}(\Theta) \subseteq \operatorname{Ker} \xi_0 \cap \mathcal{O}(\Theta) = \mathcal{D}(\Theta)$. Since $\mathcal{D}(\Theta) \subset \mathcal{D}_e$ we conclude $\mathcal{D}_e \cap \mathcal{O}(\Theta) = \mathcal{D}(\Theta)$, and so the global factorisation map $\mathcal{O}_0 \to \mathcal{O}_0/(\mathcal{O}_0 \cap \mathcal{D}_e)$ coincides on each $\mathcal{O}(\Theta)$ with ξ_Θ. Thus it is ξ_0, i.e. $\operatorname{Ker} \xi_0 = \mathcal{D}_e \cap \mathcal{O}_0$, so $\mathcal{R}_0 = \xi_0(\mathcal{O}_0) = \mathcal{O}_0/(\mathcal{O}_0 \cap \mathcal{D}_e)$.

To prove the last claim, we just apply Lemma 2.22 to the pair $(\mathcal{O}_0, \mathcal{U}_e) \subset (\mathcal{F}_0, \mathcal{U}_e)$. To verify its two conditions, note that we already know by the first part of the proof that $\mathcal{D}_e \cap \mathcal{O}_0 = [\mathcal{O}_0(\mathcal{U}_e - \mathbb{1})] \cap [(\mathcal{U}_e - \mathbb{1})\mathcal{O}_0]$ so we only need to check the second condition. We also saw above that $\mathcal{D}_e \cap \mathcal{O}_0$ is an ideal in $\mathcal{O}_0$ (the kernel of a homomorphism), hence the algebra of observables in $\mathcal{O}_0$ of the constraints $\mathcal{U}_e$ is all of $\mathcal{O}_0$ (using Theorem 2.3(ii)). Thus we only need to show that $\mathcal{O}_0 \subset \mathcal{O}_e$, i.e. that $\mathcal{O}(\Theta) \subset \mathcal{O}_e$ for all $\Theta \in \Gamma$. By Theorem 2.3(iii) we only need to show that $[F, \mathcal{U}_e] \subset \mathcal{D}_e \ \forall F \in \mathcal{O}(\Theta)$, but this follows immediately from the last paragraph since for an observable $F \in \mathcal{O}_0$ we have: $[F, \mathcal{U}_e] \subset \mathcal{D}_e \cap \mathcal{O}_0 \subset \mathcal{D}_e$. ■

We do not as yet have useful general criteria to ensure that $\mathcal{R}_0 = \mathcal{R}_e$, though we now verify that it holds for both stages of constraining in the Gupta–Bleuler example.

Example. Recall the first stage of constraining in the previous example. We had a system of local constraints $\Gamma \ni \Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ where $\mathcal{F}(\Theta) = \mathrm{C}^*\left(\delta_{\mathfrak{X}(\Theta)} \cup \mathcal{U}(\Theta)\right)$ and $\mathcal{U}(\Theta) = \{U_{\theta_h} \mid h \in C_c^\infty(\Theta, \mathbb{R}^4)\}$. By Theorem 4.5 we have $\mathcal{R}(\Theta) \cong \mathrm{C}^*(\delta_{\mathfrak{p}(\Theta)})$ and so $\mathcal{R}_0 = \varinjlim \mathcal{R}(\Theta) = \overline{\Delta(\mathfrak{p}, B)}$ where $\mathfrak{p} = \bigcup_{\Theta \in \Gamma} \mathfrak{p}(\Theta)$. We need to compare this to $\mathcal{R}_e$ which we obtain from the system $(\mathcal{F}_0, \mathcal{U}_e)$ where $\mathcal{F}_0 = \varinjlim \mathcal{F}(\Theta) = \mathrm{C}^*(\mathcal{U}_e \cup \delta_{\mathfrak{X}_{(0)}})$, (cf. Remark 4.4) and $\mathcal{U}_e = \bigcup_{\Theta \in \Gamma} \mathcal{U}(\Theta) = \mathcal{U}_{(0)}$. Now the method in the proof of Theorem 4.5 did not use the assumption $\Theta \in \Gamma$, hence it can be transcribed to prove that $\mathcal{O}_e = \mathrm{C}^*(\delta_{\mathfrak{p}_e}) + \mathcal{D}_e$ where $\mathfrak{p}_e := \{f \in \mathfrak{X}_{(0)} \mid T_h(f) = f \ \forall h \in C_c^\infty(\Theta, \mathbb{R}),\ \Theta \in \Gamma\}$. Each $f \in \mathfrak{p}_e \subset \mathfrak{X}_{(0)}$ is in some $\mathfrak{X}(\Theta)$, so is in $\mathfrak{p}(\Theta)$ by the defining condition. Thus $\mathfrak{p}_e \cap \mathfrak{X}(\Theta) \subseteq \mathfrak{p}(\Theta)$. However, by Theorem 4.5, these are characterised by $p_\mu f^\mu \upharpoonright C_+ = 0$, and by Eqn. (6) this implies $T_h(f) = f$ for all $h \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$. Thus $\mathfrak{p}_e = \mathfrak{p}$, so $\mathcal{O}_e = \mathrm{C}^*(\delta_\mathfrak{p}) + \mathcal{D}_e$ and hence by the argument in the last part of the proof of Theorem 4.5 we have $\mathcal{R}_e \cong \mathrm{C}^*(\delta_\mathfrak{p}) \cong \mathcal{R}_0$.

Next we verify for the second stage of constraining that the local and global constrainings ultimately coincide. Here we have the system of local constraints: $\Gamma \ni \Theta \to (\mathcal{R}(\Theta), \widetilde{\mathcal{U}}(\Theta))$ where $\mathcal{R}(\Theta) = \overline{\Delta(\mathfrak{p}(\Theta), B)}$ and $\widetilde{\mathcal{U}}(\Theta) = \delta_{\mathfrak{s}(\Theta)}$, $\mathfrak{s}(\Theta) = \mathfrak{p}_0 \cap \mathfrak{p}(\Theta)$. By Theorem 4.14 we have $\widetilde{\mathcal{R}}_0 \cong \overline{\Delta(\mathfrak{p}/\mathfrak{p}_0, \widetilde{B})}$. We need to compare this to the physical algebra $\widetilde{\mathcal{R}}_e$ obtained from the system $(\mathcal{R}_0, \widetilde{\mathcal{U}}_e)$ where $\mathcal{R}_0 = \varinjlim \mathcal{R}(\Theta) = \overline{\Delta(\mathfrak{p}, B)}$ and $\widetilde{\mathcal{U}}_e = \bigcup_{\Theta \in \Gamma} \widetilde{\mathcal{U}}(\Theta) = \delta_{\mathfrak{p}_0}$. By Theorem 5.2 in [6] we have $\widetilde{\mathcal{R}}_e \cong \overline{\Delta(\mathfrak{p}/\mathfrak{p}_0, \widetilde{B})} \cong \widetilde{\mathcal{R}}_0$, and this proves the claim.

5.2 Reduction by stages

In this subsection we address the problem reduction by stages, i.e. subdivide the initial constraint set, then impose these constraint sets along an increasing chain (terminating with the full set of constraints), and analyse when the final physical algebra of the chain is the same as that obtained from a single constraining by the full set. This problem occurred in the Gupta–Bleuler example, and is related also to the one in the previous subsection.

5.2 Definition *An* **n–chain of constraints** *consists of a first–class constraint system* $(\mathcal{F},\,\mathcal{C})$ *and a chain of subsets*

$$\{0\} \neq \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_n = \mathcal{C}$$

such that $\mathcal{C} \subset \mathcal{O}_i\ \forall\, i = 1, 2, \ldots, n$, *where we henceforth denote by* $(\mathfrak{S}_{D_i},\, \mathcal{D}_i,\, \mathcal{O}_i,\, \mathcal{R}_i,\, \xi_i)$ *the data resulting from application of a T–procedure to* $(\mathcal{F},\, \mathcal{C}_i)$. *(Recall that* $\xi_i : \mathcal{O}_i \to \mathcal{R}_i$ *denotes the canonical factorization map). By convention we will omit the subscript* i *when* $i = n$.

Note that $\mathfrak{S}_D = \mathfrak{S}_{D_n} \subset \mathfrak{S}_{D_{n-1}} \subset \cdots \subset \mathfrak{S}_{D_2} \subset \mathfrak{S}_{D_1}$ and $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$. The condition $\mathcal{C} \subset \mathcal{O}_i$ is nontrivial, but necessary for the procedure in the next theorem. Below we will use subscript notation A_i, $A_{(i)}$ and $A_{\{i\}}$ to distinguish between similar objects in different contexts.

5.3 Theorem *Given an n–chain of constraints as above, we define inductively the following cascade of first–class constraint systems* $(\mathcal{R}_{(k-1)},\, \xi_{\{k-1\}}(\mathcal{C}_k))$, $k = 1, \ldots, n$ *with T–procedure data* $(\mathfrak{S}_{D(k)},\, \mathcal{D}_{(k)},\, \mathcal{O}_{(k)},\, \mathcal{R}_{(k)},\, \xi_{(k)})$ *and notation* $\xi_{\{k\}} := \xi_{(k)} \circ \xi_{(k-1)} \circ \cdots \circ \xi_{(1)}$ *and conventions* $\xi_{\{0\}} := \mathrm{id}$, $\mathcal{R}_{(0)} = \mathcal{F}$. *Then*

- *(i)* $\mathrm{Dom}\,\xi_{\{k\}} = \mathcal{O}_1 \cap \mathcal{O}_2 \cap \cdots \cap \mathcal{O}_k =: \mathcal{O}_{\{k\}}$, $\mathrm{Ker}\,\xi_{\{k\}} = \mathcal{O}_{\{k\}} \cap \mathcal{D}_k$, *and* $\mathrm{Ran}\,\xi_{\{k\}} = \mathcal{R}_{(k)}$ *where we use the conventions* $\mathcal{O}_{\{0\}} := \mathcal{F}$ *and* $\mathcal{D}_0 = \{0\}$.
- *(ii)* $\mathcal{D}_{(k)} = \xi_{\{k-1\}}(\mathcal{O}_{\{k-1\}} \cap \mathcal{D}_k)$,
- *(iii)* $\mathcal{O}_{(k)} = \xi_{\{k-1\}}(\mathcal{O}_{\{k\}})$,
- *(iv)* $\mathcal{R}_{(k)} \cong \mathcal{O}_{\{k\}}/(\mathcal{O}_{\{k-1\}} \cap \mathcal{D}_k) \subset \mathcal{R}_k$,
- *(v) the lifting map* $\varphi_k : \mathfrak{S}_{D_k}\!\restriction\!\mathcal{O}_{\{k\}} \to \mathfrak{S}(\mathcal{R}_{(k)})$ *defined by* $\varphi_k(\omega)(\xi_{\{k\}}(A)) := \omega(A)$, $A \in \mathcal{O}_{\{k\}}$, *gives a bijection* $\varphi_k : \mathfrak{S}_{D_{k+1}}\!\restriction\!\mathcal{O}_{\{k\}} \to \mathfrak{S}_{D(k+1)}$.

We will call the application of a T–procedure to $(\mathcal{R}_{(k-1)},\, \xi_{\{k-1\}}(\mathcal{C}_k))$, $k = 1, \ldots, n$ *to produce the data* $(\mathfrak{S}_{D(k)},\, \mathcal{D}_{(k)},\, \mathcal{O}_{(k)},\, \mathcal{R}_{(k)},\, \xi_{(k)})$ *the* k^{th}**stage reduction** *of the given n-chain.*

Proof: We apply the second principle of induction, and also remind the reader that $\mathcal{C} \subset \mathcal{O}_i$ $\forall i$. For $k = 1$ we have by convention that $(\mathcal{R}_{(0)},\, \xi_{\{0\}}(\mathcal{C}_1)) = (\mathcal{F},\, \mathcal{C}_1)$ which is first–class, and $(\mathfrak{S}_{D(1)},\, \mathcal{D}_{(1)},\, \mathcal{O}_{(1)},\, \mathcal{R}_{(1)},\, \xi_{(1)}) = (\mathfrak{S}_{D_1},\, \mathcal{D}_1,\, \mathcal{O}_1,\, \mathcal{R}_1,\, \xi_1)$. Thus $\mathrm{Dom}\,\xi_{\{1\}} = \mathrm{Dom}\,\xi_1 = \mathcal{O}_1 = \mathcal{O}_{\{1\}}$, $\mathrm{Ker}\,\xi_{\{1\}} = \mathrm{Ker}\,\xi_1 = \mathcal{D}_1 = \mathcal{O}_{\{1\}} \cap \mathcal{D}_1$, $\mathrm{Ran}\,\xi_{\{1\}} = \mathrm{Ran}\,\xi_1 = \mathcal{R}_1 = \mathcal{R}_{(1)}$. Moreover $\mathcal{D}_{(1)} = \mathcal{D}_1 = \xi_{\{0\}}(\mathcal{O}_{\{0\}} \cap \mathcal{D}_1)$, $\mathcal{O}_{(1)} = \xi_{\{0\}}(\mathcal{O}_{\{1\}}) = \mathcal{O}_1$ and $\mathcal{R}_{(1)} = \mathcal{R}_1 = \mathcal{O}_{\{1\}}/(\mathcal{O}_{\{0\}} \cap \mathcal{D}_1)$. Now using $\mathcal{C}_2 \subset \mathcal{C} \subset \mathcal{O}_1$, the bijection $\varphi_1 : \mathfrak{S}_{D_1}\!\restriction\!\mathcal{O}_1 \to \mathfrak{S}(\mathcal{R}_1)$ given by Theorem 2.4, produces for $\omega \in \mathfrak{S}_{D_2}$:

$$\varphi_1(\omega)(\xi_1(C^*C)) = \omega(C^*C) = 0 = \omega(CC^*) = \varphi_1(\omega)(\xi_1(CC^*))$$

for all $C \in \mathcal{C}_2$, i.e. $\varphi_1(\omega) \in \mathfrak{S}_{D(2)}$. Conversely, if $\varphi_1(\omega) \in \mathfrak{S}_{D(2)}$ then $\omega \in \mathfrak{S}_{D_2}$. Thus the theorem holds for $k = 1$.

For the induction step, fix an integer $m \geq 1$ and assume the theorem is true for all $k \leq m$. We prove that it holds for $m + 1$. Now $(\mathcal{R}_{(m)},\, \xi_{\{m\}}(\mathcal{C}_{m+1}))$ is first–class, because by (v),

$\varphi_m(\mathfrak{S}_{D_{m+1}} \restriction \mathcal{O}_{\{m\}}) = \mathfrak{S}_{D(m+1)} \neq \emptyset$ since by $\mathcal{C}_{m+1} \subset \mathcal{C} \subset \mathcal{O}_i$ we have $\emptyset \neq \mathfrak{S}_D \restriction \mathcal{O}_{\{m\}}$ and φ_m is a bijection. We first prove (ii).

$$\begin{aligned}
\mathcal{D}_{(m+1)} &= \{\xi_{\{m\}}(F) \mid F \in \mathcal{O}_{\{m\}} \quad \text{and} \quad \omega(\xi_{\{m\}}(F^*F)) = 0 = \omega(\xi_{\{m\}}(FF^*)) \quad \forall\, \omega \in \mathfrak{S}_{D(m+1)}\} \\
&= \{\xi_{\{m\}}(F) \mid F \in \mathcal{O}_{\{m\}} \quad \text{and} \quad \widehat{\omega}(F^*F) = 0 = \widehat{\omega}(FF^*) \quad \forall\, \widehat{\omega} \in \mathfrak{S}_{D_{m+1}}\} \\
&\qquad\qquad \text{(using (v) of induction assumption)} \\
&= \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})\,.
\end{aligned}$$

For (iii) we see:

$$\begin{aligned}
\mathcal{O}_{(m+1)} &= \{\xi_{\{m\}}(F) \mid F \in \mathcal{O}_{\{m\}} \quad \text{and} \quad [\xi_{\{m\}}(F),\, \xi_{\{m\}}(\mathcal{C}_{m+1})] \subset \mathcal{D}_{(m+1)}\} \\
&\qquad\qquad \text{(by Theorem 2.3(iii))} \\
&= \{\xi_{\{m\}}(F) \mid F \in \mathcal{O}_{\{m\}} \quad \text{and} \quad \xi_{\{m\}}([F,\mathcal{C}_{m+1}]) \subset \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})\} \\
&= \{\xi_{\{m\}}(F) \mid F \in \mathcal{O}_{\{m\}} \quad \text{and} \quad [F,\mathcal{C}_{m+1}] \subset \mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}} + \mathcal{O}_{\{m\}} \cap \mathcal{D}_m\}\,.
\end{aligned}$$

Now since $\mathcal{D}_m \subset \mathcal{D}_{m+1}$, we have $\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}} + \mathcal{O}_{\{m\}} \cap \mathcal{D}_m = \mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}}$ and $\mathcal{C}_{m+1} \subset \mathcal{O}_{\{m\}}$ and so $[F,\mathcal{C}_{m+1}] \subset \mathcal{O}_{\{m\}}$ for all $F \in \mathcal{O}_{\{m\}}$. Thus

$$\begin{aligned}
\mathcal{O}_{(m+1)} &= \{\xi_{\{m\}}(F) \mid F \in \mathcal{O}_{\{m\}} \quad \text{and} \quad [F,\mathcal{C}_{m+1}] \subset \mathcal{D}_{m+1}\} \\
&= \xi_{\{m\}}(\mathcal{O}_{\{m\}} \cap \mathcal{O}_{m+1}) = \xi_{\{m\}}(\mathcal{O}_{\{m+1\}})\,.
\end{aligned}$$

For (iv), note that

$$\mathcal{R}_{(m+1)} = \mathcal{O}_{(m+1)}/\mathcal{D}_{(m+1)} = \xi_{\{m\}}(\mathcal{O}_{\{m+1\}})\Big/\xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})\,.$$

Define a map $\psi : \mathcal{R}_{(m+1)} \to \mathcal{R}_{m+1}$ by

$$\psi\left(\xi_{\{m\}}(A) + \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})\right) := A + \mathcal{D}_{m+1} \qquad A \in \mathcal{O}_{\{m+1\}}\,.$$

To see that it is well–defined, let $B \in \mathcal{O}_{\{m+1\}}$ be such that $\xi_{\{m\}}(A) - \xi_{\{m\}}(B) \in \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})$, i.e. $A - B \in \mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}} + \mathcal{O}_{\{m\}} \cap \mathcal{D}_m = \mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}}$, and so

$$\psi\left(\xi_{\{m\}}(B) + \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})\right) = B + \mathcal{D}_{m+1} = A + \mathcal{D}_{m+1}\,.$$

It is easy to see that ψ is a *–homomorphism onto the subalgebra $\mathcal{O}_{\{m+1\}}/\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}} \subset \mathcal{R}_{m+1}$ and since $\operatorname{Ker}\psi = \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}}) = \mathcal{D}_{(m+1)}$ which is the zero of $\mathcal{R}_{(m+1)}$, ψ is a monomorphism.

To prove (i), recall that $\xi_{\{m+1\}} = \xi_{(m+1)} \circ \xi_{\{m\}}$, so

$$\begin{aligned}
\operatorname{Dom}\xi_{\{m+1\}} &= \{F \in \operatorname{Dom}\xi_{\{m\}} \mid \xi_{\{m\}}(F) \in \operatorname{Dom}\xi_{(m+1)} = \mathcal{O}_{(m+1)}\} \\
&= \{F \in \mathcal{O}_{\{m\}} \mid \xi_{\{m\}}(F) \in \xi_{\{m\}}(\mathcal{O}_{\{m+1\}})\} \\
&= \{F \in \mathcal{O}_{\{m\}} \mid F \in \mathcal{O}_{\{m+1\}} + \mathcal{O}_{\{m\}} \cap \mathcal{D}_m = \mathcal{O}_{\{m+1\}}\}
\end{aligned}$$

because $\mathcal{D}_m \subset \mathcal{D}_{m+1} \subset \mathcal{O}_{m+1}$. Thus $\operatorname{Dom}\xi_{\{m+1\}} = \mathcal{O}_{\{m+1\}}$. Now

$$\begin{aligned}
\operatorname{Ker}\xi_{\{m+1\}} &= \{F \in \operatorname{Dom}\xi_{\{m+1\}} \mid \xi_{\{m\}}(F) \in \operatorname{Ker}\xi_{(m+1)} = \mathcal{D}_{(m+1)}\} \\
&= \{F \in \mathcal{O}_{\{m+1\}} \mid \xi_{\{m\}}(F) \in \xi_{\{m\}}(\mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}})\} \\
&= \{F \in \mathcal{O}_{\{m+1\}} \mid F \in \mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}} + \mathcal{O}_{\{m\}} \cap \mathcal{D}_m = \mathcal{D}_{m+1} \cap \mathcal{O}_{\{m\}}\} \\
&= \mathcal{O}_{\{m+1\}} \cap \mathcal{D}_{m+1}\,. \\
\operatorname{Ran}\xi_{\{m+1\}} &= \xi_{(m+1)}\left(\xi_{\{m\}}(\operatorname{Dom}\xi_{\{m+1\}})\right) = \xi_{(m+1)}\left(\xi_{\{m\}}(\mathcal{O}_{\{m+1\}})\right) \\
&= \xi_{(m+1)}(\mathcal{O}_{\{m+1\}}) = \mathcal{R}_{(m+1)}\,.
\end{aligned}$$

Finally, to prove (v), since φ_{m+1} is a surjection, each $\omega \in \mathfrak{S}_{D(m+2)}$ is of the form $\omega = \varphi_{m+1}(\widehat{\omega})$ for some $\widehat{\omega} \in \mathfrak{S}_{D_{m+1}} \upharpoonright \mathcal{O}_{\{m+1\}}$. Now $\omega \in \mathfrak{S}_{D(m+2)}$ iff $\omega(\xi_{\{m+1\}}(C^*C)) = 0 = \omega(\xi_{\{m+1\}}(CC^*))$ for all $C \in \mathcal{C}_{m+2}$ iff $\widehat{\omega}(C^*C) = 0 = \widehat{\omega}(CC^*)$ for all $C \in \mathcal{C}_{m+2}$ iff $\widehat{\omega} \in \mathfrak{S}_{D_{m+2}} \upharpoonright \mathcal{O}_{\{m+1\}}$. ■

5.4 Corollary *Given an n–chain of constraints as above, a sufficient condition to obtain $\mathcal{R}_n \cong \mathcal{R}_{(n)}$ is that $\mathcal{O}_n \backslash \mathcal{O}_{\{n\}} \subset \mathcal{D}_n$.*

Proof:

$$\begin{aligned} \mathcal{R}_n &= \mathcal{O}_n/\mathcal{D}_n = \mathcal{O}_{\{n\}}/(\mathcal{D}_n \cap \mathcal{O}_{\{n\}}) \qquad (\text{since } \mathcal{O}_n \backslash \mathcal{O}_{\{n\}} \subset \mathcal{D}_n) \\ &= \mathcal{O}_{\{n\}}/(\mathcal{D}_n \cap \mathcal{O}_{\{n-1\}}) \cong \mathcal{R}_{(n)} \end{aligned}$$

where we used Theorem 5.3(iv) for the last isomorphism. ■

Example. The Gupta–Bleuler model of the previous section provide examples of 2–chains of constraints both at the local and the global levels. We will only consider the global level, and refer freely to the example of the last section where both global constrainings were done. Let $\mathcal{C}_1 := \{U_{\theta_{\underset{\sim}{h}}} - \mathbb{1} \mid h \in C_c^\infty(\mathbb{R}^4, \mathbb{R})\}$ and let the total constraint set in $\mathcal{F}_0$ be $\mathcal{C} = \mathcal{C}_2 := \mathcal{C}_1 \cup \widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}} := \mathbb{1} - \widetilde{\mathcal{U}}_e = \{\mathbb{1} - \delta_f \mid f \in \mathfrak{p}_0\}$.

5.5 Claim *$\mathcal{C}_1 \subset \mathcal{C}_2 = \mathcal{C}$ is a 2–chain of constraints in $\mathcal{F}_0 = \mathrm{C}^*(\delta_{\mathfrak{X}_{(0)}} \cup \mathcal{U}_{(0)})$. (Notation as in Remark 4.4).*

Proof: To see that $\mathcal{C}$ is first–class, define a state $\widehat{\omega}$ on $\overline{\Delta(\mathfrak{X}_{(0)}, B)} \subset \mathcal{F}_0$ by $\widehat{\omega}(\delta_f) = 1$ if $f \in \mathfrak{p}_0$, and otherwise $\widehat{\omega}(\delta_f) = 0$ (that this defines a state is easy to check). Since by Theorem 4.5 the space $\mathfrak{p}$ is pointwise invariant under T_h for $h \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$, (also using Eqn. (6)) so is $\mathfrak{p}_0$, hence $\widehat{\omega}$ is invariant under $\mathcal{G}_{(0)} =$ the group generated by $\mathcal{G}(\Theta)$, $\Theta \in \Gamma$. Thus $\widehat{\omega}$ extends (uniquely) to a Dirac state on $\mathcal{F}_0$ (by a trivial application of Theorem 2.16). Thus $\mathcal{C}$ is first class.

It is obvious that $\mathcal{C} \subset \mathcal{D}_2 \subset \mathcal{O}_2$, so we only need to show that $\mathcal{C} \subset \mathcal{O}_1$. By Theorem 4.5 and the last subsection we have $\mathcal{O}_1 = \mathrm{C}^*(\delta_{\mathfrak{p}}) + \mathcal{D}_1$ and as $\mathcal{C}_1 \subset \mathcal{D}_1$ and $\delta_{\mathfrak{p}_0} \subset \mathrm{C}^*(\delta_{\mathfrak{p}})$ it follows that $\mathcal{C} \subset \mathcal{O}_1$. ■

We now show that $\mathcal{O}_2 \backslash \mathcal{O}_{\{2\}} \subset \mathcal{D}_2$, i.e. since $\mathcal{O}_{\{2\}} = \mathcal{O}_1 \cap \mathcal{O}_2$, that $\mathcal{O}_2 \backslash \mathcal{O}_1 \subset \mathcal{D}_2$. Then by Corollary 5.4 we have $\mathcal{R}_2 = \mathcal{R}_{(2)}$, i.e. the two–step reduction by stages produces the same physical algebra as a single reduction by the full constraint set $\mathcal{C}$. This, we will do as follows. We know that from the last subsection that $\mathcal{O}_1 = \mathrm{C}^*(\delta_{\mathfrak{p}}) + \mathcal{D}_1$ and in the next two claims we prove that $\mathcal{O}_2 = \mathrm{C}^*(\delta_{\mathfrak{p}}) + \mathcal{D}_2$. Then since $\mathcal{D}_1 \subset \mathcal{D}_2$ it follows that $\mathcal{O}_2 \backslash \mathcal{O}_1 \subset \mathcal{D}_2$. We first prove:

5.6 Claim *$\mathcal{O}_2 = \mathrm{C}^*(\delta_{\mathfrak{p}_0'}) + \mathcal{D}_2$ where $\mathfrak{p}_0' = \{f \in \mathfrak{X}_{(0)} \mid B(f, s) = 0 \quad \forall\, s \in \mathfrak{p}_0\}$.*

Proof: We adapt the proof of Theorem 4.5. Since $\mathcal{O}_2 \subset \mathcal{F}_0 = \mathrm{C}^*(\delta_{\mathfrak{X}_{(0)}} \cup \mathcal{U}_{(0)})$, for a general $A \in \mathcal{O}_2$ we can write

$$A = \lim_{n\to\infty} A_n \qquad \text{where} \qquad A_n = \sum_{i=1}^{N_n} \delta_{f_i} \sum_{j=1}^{L_n} \lambda_{ij}^{(n)} U_{\gamma_{ij}^{(n)}} \tag{12}$$

where $f_i \neq f_j$ if $i \neq j$, $f_i \in \mathfrak{X}_{(0)}$, $\gamma_{ij}^{(n)} \in \mathcal{G}_{(0)}$ and $\lambda_{ij}^{(n)} \in \mathbb{C}$. Consider the equivalence classes of $\mathfrak{X}_{(0)}/\mathfrak{p}_0$. If $f_i - f_j =: s \in \mathfrak{p}_0$, then $\delta_{f_i} = \delta_{f_j} \cdot \delta_s \exp(iB(s, f_j)/2)$ and $\delta_s \in \widetilde{\mathcal{U}}_e$. Thus we can write Eqn. (12) in the form

$$A_n = \sum_{i=1}^{N_n} \delta_{f_i} \sum_{j=1}^{L_n} \sum_{k=1}^{K_n} \lambda_{ijk}^{(n)} \delta_{s_{ij}} U_{\gamma_{ik}^{(n)}} \tag{13}$$

where $f_i - f_j \notin \mathfrak{p}_0$ if $i \neq j$ and $s_{ij} \in \mathfrak{p}_0$. Let $\omega \in \mathfrak{S}_{D_2}$, then

$$\begin{aligned} \pi_\omega(A_n)\Omega_\omega &= \sum_{i=1}^{N_n} \zeta_i^{(n)}\, \pi_\omega(\delta_{f_i})\, \Omega_\omega \\ \text{where} \qquad \zeta_i^{(n)} &= \sum_{j=1}^{L_n}\sum_{k=1}^{K_n} \lambda_{ijk}^{(n)} \in \mathbb{C}\,. \end{aligned} \tag{14}$$

Let $h \in \mathfrak{p}_0$, then from $\delta_h \in \widetilde{\mathcal{U}}_e$ and $A \in \mathcal{O}_2$, we get, using Eqn. (14):

$$\begin{aligned} 0 &= \omega(A^*(\delta_h - \mathbb{1})^*(\delta_h - \mathbb{1})A) = \lim_{n\to\infty} \omega(A_n^*(2\mathbb{1} - \delta_h - \delta_{-h})A_n) \\ &= \lim_{n\to\infty} \Big\{ 2\omega(A_n^* A_n) - \sum_{i,j=1}^{N_n} \overline{\zeta}_i^{(n)} \zeta_j^{(n)}\, \omega(\delta_{-f_i}(\delta_h + \delta_{-h})\delta_{f_j}) \Big\} \end{aligned} \tag{15}$$

The state $\widehat{\omega}$ on $\mathcal{F}_0$ defined by $\widehat{\omega}(\delta_f) := \chi_{\mathfrak{p}_0}(f)$ and $\widehat{\omega}(U_g) = 1\ \forall\, g \in \mathcal{G}_{(0)}$ (encountered in the proof of Claim 5.5) is in $\mathfrak{S}_{D_2}$, so Eqn. (15) becomes for it:

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big\{ 2\sum_{j=1}^{N_n} |\zeta_j^{(n)}|^2 - \sum_{j=1}^{N_n} |\zeta_j^{(n)}|^2 (e^{iB(h,\,f_j)} + e^{-iB(h,\,f_j)}) \Big\} \\ &= 2\lim_{n\to\infty} \sum_{j=1}^{N_n} |\zeta_j^{(n)}|^2 (1 - \cos\, B(h,\, f_j)) \end{aligned} \tag{16}$$

where we made use of $f_i - f_j \notin \mathfrak{p}_0$ if $i \neq j$ and the equation

$$\pi_\omega(\delta_{-f_i}\delta_h\delta_{f_j})\,\Omega_\omega = \pi_\omega(\delta_{-f_i} e^{iB(h,\,f_j)}\delta_{f_j}\delta_h)\,\Omega_\omega = e^{iB(h,\,f_j)} e^{-iB(f_i,\,f_j)/2} \pi_\omega(\delta_{f_j - f_i})\,\Omega_\omega\,.$$

Now the terms in the sum of Eqn. (16) are all positive so in the limit these must individually vanish, i.e.

$$\lim_{n\to\infty} |\zeta_j^{(n)}|^2 (1 - \cos\, B(h,\, f_j)) = 0 \qquad \forall\, h \in \mathfrak{p}_0\,.$$

Thus since the first factor is independent of h and the second is independent of n, either $\lim_{n\to\infty} |\zeta_j^{(n)}|^2 = 0$ or $B(h,\, f_j) = 0\ \ \forall\, h \in \mathfrak{p}_0$ (i.e. $f_j \in \mathfrak{p}_0'$). Now we can rewrite the argument around Eqn. (9) almost verbatim to obtain $\mathcal{O}_2 = \mathrm{C}^*(\delta_{\mathfrak{p}_0'}) + \mathcal{D}_2$. ■

5.7 Claim $\mathfrak{p}_0' = \mathfrak{p}$.

Proof: Clearly $\mathfrak{p}_0' \supseteq \mathfrak{p}$ by definition. For the converse inclusion, let $f \in C_c^\infty(\mathbb{R}^4,\, \mathbb{R}^4)$ such that $\gamma(f) := \widehat{f} \restriction C_+ \in \mathfrak{p}_0'$, i.e. we have $B(\gamma(f),\, \gamma(k)) = 0\ \forall\, \gamma(k) \in \mathfrak{p}_0$. Now by Theorem 4.12, if $\gamma(k)$ is in $\mathfrak{p}_0$, then $\gamma(k)_\mu(p) = ip_\mu h(p)$ for some h such that $\gamma(k) \in \mathfrak{X}_{(0)}$. In particular, we can take $h = \gamma(r)$ for $r \in C_c^\infty(\mathbb{R}^4,\, \mathbb{R})$, in which case $\gamma(k)_\mu = \gamma(\partial_\mu r)$, and so

$$\begin{aligned} 0 &= B(\gamma(f),\, \gamma(k)) = D(f,\, k) \\ &= \int\!\!\int f_\mu(x)\, k^\mu(y)\, \Delta(x-y)\, d^4x\, d^4y = \int\!\!\int f_\mu(z+y)\, k^\mu(y)\, \Delta(z)\, d^4z\, d^4y \\ &= \int \Big(\int f_\mu(z+y)\, \frac{\partial r(y)}{\partial y_\mu}\, d^4y \Big)\, \Delta(z)\, d^4z \\ &= -\int \Big(\int (\partial^\mu f_\mu)(z+y)\, r(y)\, d^4y \Big)\, \Delta(z)\, d^4z \\ &= -\int\!\!\int (\partial^\mu f_\mu)(x)\, r(y)\, \Delta(x-y)\, d^4y\, d^4x \\ &= -\widetilde{D}(\gamma(\partial^\mu f_\mu),\, \gamma(r)) \end{aligned}$$

for all $r \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$ and where Δ is the Pauli–Jordan distribution and $\widetilde{D}$ is the symplectic form for the free neutral scalar bosonic field. It is well–known that $\widetilde{D}$ is nondegenerate on $\gamma(C_c^\infty(\mathbb{R}^4, \mathbb{R}))$, (to see this, use the Schwartz density argument in the proof of Theorem 4.12) hence since $\gamma(\partial^\mu f_\mu)$ is also in this space, we conclude $\gamma(\partial^\mu f_\mu) = 0$, i.e. $p^\mu \gamma(f_\mu) = 0$, i.e. by Eqn. (6) $\gamma(f) \in \mathfrak{p}$. Hence we have the reverse inclusion, so $\mathfrak{p}_0' = \mathfrak{p}$. ■

Thus $\mathcal{O}_2 = \mathrm{C}^*(\delta_\mathfrak{p}) + \mathcal{D}_2$ and so $\mathcal{O}_2 \backslash \mathcal{O}_1 \subset \mathcal{D}_2$.

5.3 The weak spectral condition

A very important additional property which is used in the analysis of algebraic QFT, is that of the spectral condition.

5.8 Definition *An action $\alpha : \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{F}_0$ on a C^*–algebra $\mathcal{F}_0$ satisfies the* **spectral condition** *if there is a state $\omega \in \mathfrak{S}(\mathcal{F}_0)$ such that*

(i) ω is translation–invariant, i.e. $\omega \circ \alpha_g = \omega \ \forall\, g \in \mathbb{R}^4 \subset \mathcal{P}_+^\uparrow$,

(ii) the spectrum of the generators of translations in π_ω is in the forward light cone V_+.

Let now $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ be a system of local constraints with reduction isotony and weak covariance. We want to find the weakest requirement on $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ to ensure that $\widetilde{\alpha} : \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{R}_0$ (cf. Theorem 3.7(iii)) satisfies the spectral condition. We propose:

5.9 Definition *The given action $\alpha : \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{F}_0$ on $\mathcal{F}_0 = \varinjlim \mathcal{F}(\Theta)$ satisfies the* **weak spectral condition** *iff the set*

$$\mathcal{C} := (\mathcal{U}_e - \mathbb{1}) \cup \{\alpha_f(A) \mid A \in \mathcal{O}_0,\ f \in F(V_+)\} \subset \mathcal{O}_0 := \varinjlim \mathcal{O}(\Theta)$$

is a first–class constraint set, where we used the notation

$$\begin{aligned} \mathcal{U}_e &:= \bigcup_{\Theta\in\Gamma} \mathcal{U}(\Theta), \qquad\qquad \alpha_f(A) := \int_{\mathbb{R}^4} \alpha_t(A)\, f(t)\, d^4t\,, \\ F(V_+) &:= \{f \in L^1(\mathbb{R}^4) \mid \operatorname{supp}\widehat{f} \subset \mathbb{R}^4 \backslash V_+ \quad \textit{and}\ \operatorname{supp}\widehat{f}\ \textit{is compact}\} \end{aligned}$$

where $\widehat{f}$ denotes the Fourier transform of f.

5.10 Theorem *Let $\Theta \to (\mathcal{F}(\Theta), \mathcal{U}(\Theta))$ be a local system of contraints with reduction isotony and weak covariance. Then $\widetilde{\alpha} : \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{R}_0$ satisfies the spectral condition iff $\alpha : \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\mathcal{F}_0$ satisfies the weak spectral condition.*

Proof: We will use the notation in Subsection 5.1. By the last definition, α satisfies the weak spectral condition iff $\mathcal{C}$ is first–class iff the left ideal $[\mathcal{O}_0\mathcal{C}]$ in $\mathcal{O}_0$ is proper iff the left ideal $\xi_0([\mathcal{O}_0\mathcal{C}]) = [\mathcal{R}_0\xi_0(\mathcal{C})]$ is proper in $\mathcal{R}_0$ where $\xi_0 : \mathcal{O}_0 \to \mathcal{O}_0/(\mathcal{D}_e \cap \mathcal{O}_0) = \mathcal{R}_0$ is the canonical factoring map, and we used $\mathrm{Ker}\,\xi_0 = \mathcal{D}_e \cap \mathcal{O}_0 \subset [\mathcal{O}_0(\mathcal{U}_e - \mathbb{1})] \subset [\mathcal{O}_0\mathcal{C}]$. Now $\xi_0(\mathcal{C}) = \{\xi_0(\alpha_f(A)) \mid A \in \mathcal{O}_0,\ f \in F(V_+)\}$ because $\xi_0(\mathcal{U}_e - \mathbb{1}) = 0$. Let $A \in \mathcal{O}_0$ and $f \in F(V_+)$, then

$$\begin{aligned} \xi_0(\alpha_f(A)) &= \xi_0\Big(\int_{\mathbb{R}^4} \alpha_t(A)\, f(t)\, d^4t\Big) = \int_{\mathbb{R}^4} \xi_0(\alpha_t(A))\, f(t)\, d^4t \\ &= \int_{\mathbb{R}^4} \widetilde{\alpha}_t(\xi_0(A))\, f(t)\, d^4t = \widetilde{\alpha}_f(\xi_0(A))\,. \end{aligned}$$

Thus $[\mathcal{R}_0\xi_0(\mathcal{C})]$ is the left ideal generated in $\mathcal{R}_0$ by $\{\widetilde{\alpha}_f(B)) \mid B \in \mathcal{R}_0,\ f \in F(V_+)\}$ and this is precisely Doplicher's left ideal (cf. [4, 24]), which is proper iff $\widetilde{\alpha}$ satisfies the spectral condition by 2.7.2 in Sakai [24]. ■

In general the weak spectral condition seems very difficult to verify directly. However, for the Gupta–Bleuler example in this paper, it is easily verified via the last theorem:

Example. We show that the final HK–QFT of the Gupta–Bleuler example (as expressed in Theorem 4.14) satisfies the spectral condition, and hence the initial system must satisfy the weak spectral condition. Thus we need to show the existence of a state on $\widetilde{\mathcal{R}}_0 := \varinjlim \widetilde{\mathcal{R}}(\Theta)$ which satisfies the two conditions in Definition 5.8 for the action $\widetilde{\alpha} : \mathcal{P}_+^\uparrow \to \mathrm{Aut}\,\widetilde{\mathcal{R}}_0$. Now in the usual Gupta–Bleuler theory, the indefinite inner product

$$K(f,\, h) := \int_{C_+} \overline{\widehat{f}^\mu(p)}\, \widehat{h}_\mu(p)\, \frac{d^3p}{p_0}\,, \qquad \forall\, f,\, h \in \mathfrak{X}_0$$

and the Fock–type representation constructed from it features prominently (cf. discussion in [6]). We now show that on $\widetilde{\mathcal{R}}_0 = \overline{\Delta(\mathfrak{p}/\mathfrak{p}_0,\, \widetilde{B})}$ the formula

$$\omega(\delta_{\zeta(f)}) := \exp\big(-k(f,\, f)/4\big)\,, \qquad f \in \mathfrak{p} \quad \text{where} \qquad k := 2\,\mathrm{Re}\, K$$

and $\zeta : \mathfrak{p} \to \mathfrak{p}/\mathfrak{p}_0$ is the usual factorisation map, defines a state satisfying the spectral condition.

5.11 Claim *We have* $K(f,\, f) \geq 0\ \forall\, f \in \mathfrak{p}$ *and* $\mathrm{Ker}\,(K\restriction\mathfrak{p}) = \mathfrak{p}_0$.

Proof: Let $f \in \mathfrak{p}$, then by Theorem 4.5 we see $p_\mu f^\mu(p) = 0$ for all $p \in C_+$, and thus by the proof of Theorem 4.12, $f_0(p) = \mathbf{p} \cdot \mathbf{f}(p)/\|\mathbf{p}\|$ for $p \neq 0$. Thus

$$K(f,\, f) \;:=\; \int_{C_+} \overline{f^\mu(p)}\, f_\mu(p)\, \frac{d^3p}{p_0} = \int_{C_+} \big(\overline{\mathbf{f}} \cdot \mathbf{f}(p) - |f_0(p)|^2\big) \frac{d^3p}{p_0}\,. \tag{17}$$

Now $$\overline{\mathbf{f}} \cdot \mathbf{f}(p) \;=\; \|\mathbf{f}(p)\|^2 = \left\|\frac{\mathbf{p}}{\|\mathbf{p}\|}\right\|^2 \|\mathbf{f}(p)\|^2 \geq \left|\frac{\mathbf{p}}{\|\mathbf{p}\|} \cdot \mathbf{f}(p)\right|^2 = |f_0(p)|^2$$

and so $K(f,\, f) \geq 0$ for all $f \in \mathfrak{p}$.

Let $K(f,\, f) = 0$ for $f \in \mathfrak{p}$, then since by the preceding the integrand in Eqn. (17) is positive, we conclude that $\|\mathbf{f}(p)\| = |\mathbf{p}\cdot\mathbf{f}(p)|/\|\mathbf{p}\|$ on C_+. Thus $\mathbf{f}(p)$ must be parallel to $\mathbf{p}$, i.e. $\mathbf{f}(p) = \mathbf{p}\, h(p)$ for some h. Since $f_0(p) = \mathbf{p} \cdot \mathbf{f}(p)/\|\mathbf{p}\| = \|\mathbf{p}\|\, h(p) = p_0\, h(p)$ we conclude $f_\mu(p) = p_\mu\, h(p)$, i.e. $f \in \mathfrak{p}_0$ by Theorem 4.12. Thus $\mathrm{Ker}\,(K\restriction\mathfrak{p}) \subseteq \mathfrak{p}_0$. The reverse inclusion is obvious. ∎

Thus K lifts to a well–defined positive nondegenerate inner product on $\mathfrak{p}/\mathfrak{p}_0$, so ω is well–defined and because $B = 2\,\mathrm{Im}\, K$, i.e. $2K = k + iB$, we conclude that ω is a Fock–state. It is $\widetilde{\alpha}$–invariant since via Eqn. (10) we have

$$\begin{aligned} K(V_g f,\, V_g h) &= \int_{C_+} (\overline{V_g f})^\mu(p)\, (V_g h)_\mu(p)\, \frac{d^3p}{p_0} \\ &= \int_{C_+} (\Lambda\overline{f})^\mu(\Lambda^{-1}p)\, (\Lambda h)_\mu(\Lambda^{-1}p)\, \frac{d^3p}{p_0} \\ &= \int_{C_+} \overline{f}^\mu(\Lambda^{-1}p)\, h_\mu(\Lambda^{-1}p)\, \frac{d^3p}{p_0} = k(f,\, h) \end{aligned}$$

since the measure d^3p/p_0 is Lorenz invariant on the light cone.

The implementer of a Poincaré transformation g is just the second quantization of the lift of V_g to $\mathfrak{p}/\mathfrak{p}_0$, i.e. $U_g := \Gamma(\widetilde{V}_g)$ where $\widetilde{V}_g$ is the lift of V_g to $\mathfrak{p}/\mathfrak{p}_0$. We need to verify the spectral condition for the generators of the translations. Recall for $g = (\Lambda,\, a)$ we have $(V_g f)(p) = e^{-ipa}\, \Lambda f(\Lambda^{-1}p)$. Translation by a therefore acts by multiplication operators $(V_a f)(p) = e^{-ipa} f(p)$

with infinitesimal generators $\widetilde{P}_\mu$ of $\widetilde{V}_a$ being the lift to $\mathfrak{p}/\mathfrak{p}_0$ of the multiplication operators $f(p) \to p_\mu f(p)$. Now for $f \in \mathfrak{p}$:

$$K(\zeta(f),\, \widetilde{P}_0\,\zeta(f)) = \int_{C_+} \overline{f^\mu(p)}\, p_0\, f_\mu(p)\, \frac{d^3p}{p_0} = \int_{C_+} \overline{f^\mu(p)}\, f_\mu(p)\, d^3p \geq 0$$

since we have shown in the proof of the last claim that $\overline{f^\mu(p)}\, f_\mu(p) \geq 0 \quad \forall f \in \mathfrak{p},\ p \in C_+$. So $\widetilde{P}_0 \geq 0$. Since $U_a = \Gamma(\widetilde{V}_a) = \exp\big(-ia_\mu\, d\Gamma(\widetilde{P}^\mu)\big)$, the generators for translation for ω are $d\Gamma(\widetilde{P}^\mu)$, and so $\widetilde{P}_0 \geq 0$ implies $d\Gamma(\widetilde{P}_0) \geq 0$.

To conclude, notice from the fact that $\widetilde{P}_\mu$ acts on $\mathfrak{p}/\mathfrak{p}_0$ and in $\mathfrak{p}$ we have restriction to C_+, that the spectrum of $\widetilde{P}_\mu$ must be in C_+. Now in second quantization on an n–particle space:

$$d\Gamma(\widetilde{P}_\mu) = \widetilde{P}_\mu \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes \widetilde{P}_\mu \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \widetilde{P}_\mu$$

and the spectrum for this will be all possible sums of n vectors in C_+, and this will always be in V_+. Since the spectrum for the full $d\Gamma(\widetilde{P}_\mu)$ is the sum over all those on the n–particle spaces, this will still be in V_+ since V_+ is a cone. Thus we have verified the spectral condition as claimed.

6 Conclusions.

In this paper we introduced the concept of a system of local quantum constraints and we obtained "weak" versions of the Haag–Kastler axioms of isotony, causality, covariance and spectrality in such a way that after a local constraining procedure the resulting system of physical algebras satisfies the usual versions of these axioms. We analyzed Gupta–Bleuler electromagnetism in detail and showed that it satisfies these weak axioms, but that it violates the causality axiom. This example was particularly satisfying, in that we obtained by pure C*–algebra techniques the correct physical algebra and positive energy Fock–representation without having to pass through an indefinite metric representation.

There are many future directions of development for this project, and a few of the more evident ones are:

- Find an example of a realistic constrained local field theory which satisfies the weak Haag–Kastler axioms, but violates the usual covariance axiom (a variant of the Coulomb gauge may work).
- Continue the analysis here for the rest of the Haag–Kastler axioms, i.e. find the appropriate weak versions of e.g. the axioms of additivity, local normality, local definiteness etc., as well as examples which satisfy the weak axioms but not the usual ones.
- In the present paper we assumed a system of local constraints which are first–class. Now a reduction procedure at the C*–level exists also for second class constraints (cf. [11]) and so one can therefore ask what the appropriate weakened form of the Haag–Kastler axioms should be for such a system. A possible example for such an analysis is electromagnetism in the Coulomb gauge.

7 Appendix

We will give in this appendix the proof of Theorem 2.19. Recall the notation and results of Subsection 2.5.

7.1 Theorem *Given nondegenerate $(\mathfrak{X},\, B)$ and $\mathfrak{s} \subset \mathfrak{X}$ as in Subsection 2.5, where $\mathfrak{s} \subset \mathfrak{s}'$ and $\mathfrak{s} = \mathfrak{s}''$, then*

$$\mathcal{O} = \mathrm{C}^*(\delta_{\mathfrak{s}'} \cup \mathcal{D}) = [\mathrm{C}^*(\delta_{\mathfrak{s}'}) \cup \mathcal{D}]\,.$$

Proof: From Theorem 2.3 (ii) and (v) it is clear that $\mathcal{O} \supset \overline{C^*(\delta_{\mathfrak{s}'} \cup \mathcal{D})}$ and we only have to prove the converse inclusion. We first show that for any Dirac state ω of $\overline{\Delta(\mathfrak{X}, B)}$ we have that $\pi_\omega(\delta_f)\Omega_\omega \perp \mathcal{H}_\omega^{(p)} := \{\psi \in \mathcal{H}_\omega \mid \pi_\omega(\delta_\mathfrak{s})\psi = \psi\}$ for all $f \notin \mathfrak{s}'$. For any $f \notin \mathfrak{s}'$, choose a $k \in \mathfrak{s}$ such that $B(f, k) \notin 2\pi\mathbb{Z}$. Then

$$\pi_\omega(\delta_k)(\pi_\omega(\delta_f)\Omega_\omega) = e^{iB(k,f)}\pi_\omega(\delta_f\delta_k)\Omega_\omega = e^{iB(k,f)}\pi_\omega(\delta_f)\Omega_\omega$$

and so $\pi_\omega(\delta_f)\Omega_\omega$ is in a different eigenspace of $\pi_\omega(\delta_k)$ than $\mathcal{H}_\omega^{(p)}$, hence must be perpendicular to it.

Now recall (cf. Remark 2.8 (ii)) that if $A \in \mathcal{O}$, then $\pi_\omega(A)$ preserves $\mathcal{H}_\omega^{(p)}$ (here ω is a Dirac state). Thus $\pi_\omega(A)\Omega_\omega \perp \pi_\omega(\delta_f)\Omega_\omega$ for $f \notin \mathfrak{s}'$, and so $\omega(\delta_f A) = 0$ for $f \notin \mathfrak{s}'$. Now let $A_n = \sum_{i=1}^{N_n} \lambda_i^{(n)}\delta_{h_i} \subset \Delta(\mathfrak{X}, B)$ be a sequence converging to A. We furthermore partition the set $\{h_i \mid i \in \mathbb{N}\}$ into $\mathfrak{s}$–equivalence classes and choose one representative in each, so that we can write

$$A_n = \sum_j \delta_{h_j} \sum_i \beta_{ij}^{(n)}\delta_{c_{ij}} \qquad \text{where} \quad c_{ij} \in \mathfrak{s} \quad \text{and} \quad h_j - h_k \notin \mathfrak{s} \quad \text{if} \quad j \neq k$$

i.e. the first sum is over representatives in different equivalence classes. Thus we get for $A \in \mathcal{O}$ that for all $f \notin \mathfrak{s}'$:

$$0 = \lim_{n\to\infty} \omega(\delta_f A_n) = \lim_{n\to\infty} \sum_{j,i} \beta_{ij}^{(n)}\omega(\delta_f\delta_{h_j}\delta_{c_{ij}}) \; . \tag{18}$$

Now we make a particular choice for ω, by setting $\omega(\delta_f) = 1$ if $f \in \mathfrak{s}$ and zero otherwise. (To see that this indeed defines a state, note that if we lift the central state on $\overline{\Delta(\mathfrak{s}'/\mathfrak{s}'', \widetilde{B})}$ to $C^*(\delta_{\mathfrak{s}'})$, then by [8, p. 387] it extends uniquely to $\overline{\Delta(\mathfrak{X}, B)}$ and will coincide with ω). Then

$$\begin{aligned} \omega(\delta_f\delta_{h_j}\delta_{c_{ij}}) &= \exp\left[i(B(f, h_j) + B(f + h_j, c_{ij}))/2\right] \omega(\delta_{f+h_j+c_{ij}}) \\ &= e^{\frac{i}{2}B(f,h_j)}\chi_\mathfrak{s}(f + h_j) \end{aligned}$$

where $\chi_\mathfrak{s}$ denotes the characteristic function of the set $\mathfrak{s}$. Thus Eqn. (18) becomes:

$$\lim_{n\to\infty} \sum_{j,i} \beta_{ij}^{(n)} e^{\frac{i}{2}B(f,h_j)}\chi_\mathfrak{s}(f + h_j) = 0 \qquad \forall\, f \notin \mathfrak{s}' \; .$$

If there is a k such that $h_k \notin \mathfrak{s}'$, we can choose $f = -h_k$ in the previous equation, then since $h_j - h_k \notin \mathfrak{s}$ for $j \neq k$, we get $\chi_\mathfrak{s}(h_j - h_k) = \delta_{jk}$ and thus $\lim_{n\to\infty} \sum_i \beta_{ik}^{(n)} = 0$.

Consider now the universal representation $\pi_u : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u)$ and let $\psi \in \mathcal{H}_u^{(p)} := \{\psi \in \mathcal{H}_u^{(p)} \mid \pi_u(\mathcal{D})\psi = 0\}$. Then for each k such that $h_k \notin \mathfrak{s}'$ we have

$$\lim_n \pi_u\Big(\delta_{h_k} \sum_i \beta_{ik}^{(n)}\delta_{c_{ik}}\Big)\,\psi = \pi_u(\delta_{h_k}) \lim_n \sum_i \beta_{ik}^{(n)}\psi = 0 \; .$$

Hence in the sums involved in $\pi_u(A_n)\psi$ we can drop all terms where $h_j \notin \mathfrak{s}'$, i.e.

$$\begin{aligned} \pi_u(A)\psi &= \lim_n \pi_u(A_n)\psi = \lim_n \sum_{h_j\in\mathfrak{s}'} \pi_u(\delta_{h_j}) \sum_i \beta_{ik}^{(n)}\pi_u(\delta_{c_{ik}})\,\psi \\ &= \lim_n \sum_{h_j\in\mathfrak{s}'} \pi_u(\delta_{h_j}) \sum_i \beta_{ik}^{(n)}\psi \;\in\; \pi_u(C^*(\delta_{\mathfrak{s}'}))\,\psi \qquad \forall \psi \in \mathcal{H}_u^{(p)} \; . \end{aligned}$$

Thus the restriction of $\mathcal{O} = \pi_u(\mathcal{O})$ to $\mathcal{H}_u^{(p)}$ is the same as the restriction of $C^*(\delta_{\mathfrak{s}'})$ to $\mathcal{H}_u^{(p)}$ (given that $C^*(\delta_{\mathfrak{s}'}) \subset \mathcal{O}$). However, recall from Theorem 2.7 and the preceding remarks, that $\mathcal{O}/\mathcal{D} = \mathcal{R} \cong \mathcal{O}\restriction\mathcal{H}_u^{(p)} = C^*(\delta_{\mathfrak{s}'})\restriction\mathcal{H}_u^{(p)}$ (with respect to the open projection P, we have $\mathcal{H}_u^{(p)} = (\mathbb{1} - P)\mathcal{H}_u$). Thus $\mathcal{O} = C^*(\delta_{\mathfrak{s}'}) + \mathcal{D}$. ∎

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References

[1] A.O. Barut and R. Rączka, *Properties of non–unitary zero mass induced representations of the Poincaré group on the space of tensor–valued functions*, Ann. Inst. H. Poincaré **17** (1972), 111–118.

[2] K. Bleuler, *Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen*, Helv. Phys. Acta **23** (1950), 567.

[3] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, 1964.

[4] S. Doplicher, *An algebraic spectrum condition*, Commun. Math. Phys. **1** (1965), 1–5.

[5] S. Doplicher, R. Haag, and J.E. Roberts, *Fields, observables and gauge transformations I*, Commun. Math. Phys. **13** (1969), 1–23.

[6] H. Grundling, *Systems with outer constraints. Gupta–Bleuler electromagnetism as an algebraic field theory*, Commun. Math. Phys. **114** (1988), 69–91.

[7] ______, *A group algebra for inductive limit groups. Continuity problems of the canonical commutation relations*, Acta Applicandae Mathematicae **46** (1997), 107–145.

[8] H. Grundling and C.A. Hurst, *Algebraic quantization of systems with a gauge degeneracy*, Commun. Math. Phys. **98** (1985), 369–390.

[9] ______, *Algebraic structures of degenerate systems and the indefinite metric*, J. Math. Phys. **28** (1987), 559–572.

[10] ______, *A note on regular states and supplementary conditions*, Lett. Math. Phys. **15** (1988), 205–212, [Errata: *ibid.* **17**, 173–174 (1989)].

[11] ______, *The quantum theory of second class constraints: Kinematics*, Commun. Math. Phys. **119** (1988), 75–93, [Erratum: *ibid.* **122**, 527–529 (1989)].

[12] S.N. Gupta, *The theory of longitudinal photons in quantum electrodynamics*, Proc. Phys. Soc. **A63** (1950), 681.

[13] R. Haag, *Local Quantum Physics*, Springer Verlag, Berlin, 1992.

[14] R. Haag and D. Kastler, *An algebraic approach to quantum field theory*, J. Math. Phys. **5** (1964), 848–861.

[15] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras I*, Academic Press, New York, 1983.

[16] ———, *Fundamentals of the Theory of Operator Algebras II*, Academic Press, Orlando, 1986.

[17] F. Lledó, *A family of examples with quantum constraints*, Lett. Math. Phys. **40** (1997), 223–234.

[18] ———, *Algebraic properties of massless free nets*, Ph.D. thesis, University of Potsdam, 1998.

[19] J. Manuceau, C^**–algèbre de relations de commutation*, Ann. Inst. H. Poincaré **VIII** (1968), 139–161.

[20] J. Manuceau, M. Sirugue, D. Testard, and A. Verbeure, *The smallest* C^**–algebra for canonical commutations relations*, Commun. Math. Phys. **32** (1973), 231–243.

[21] G.J. Murphy, C^**–Algebras and Operator Theory*, Academic Press, Boston, 1990.

[22] G.K. Pedersen, C^**–Algebras and their Automorphism Groups*, Academic Press, London, 1989.

[23] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self–Adjointness*, Academic Press, New York, 1975.

[24] S. Sakai, *Operator Algebras in Dynamical Systems*, Encyclopedia of Mathematics and its Applications vol. 41, Cambridge University Press, Cambridge, 1991.

[25] F. Strocchi, *Gauge problem in quantum field theory*, Phys. Rev. **162** (1967), 1429–1438.

[26] ———, *Gauge problem in quantum field theory. III. Quantization of Maxwell equations and weak local commutativity*, Phys. Rev. D **2** (1970), 2334–2340.

[27] ———, *Locality and covariance in QED and gravitation. General prove of Gupta–Bleuler type formulations*, Mathematical Methods in Theoretical Physics (Colorado Ass. Univ. Press, Boulder) (W.E. Brittin, ed.), 1973.

[28] F. Strocchi and A.S. Wightman, *Proof of the charge superselection rule in local relativistic quantum field theory*, J. Math. Phys. **15** (1974), 2198–2224, [Erratum: *ibid.* **17**, 1930–1931 (1976)].

[29] Z. Takeda, *Inductive limit and infinite direct product of operator algebras*, Tohoku Math. J. **7** (1955), 67–86.

[30] M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, New York 1979.

[31] S. Weinberg, *The Quantum Theory of Fields. Vol. I*, Cambridge University Press, Cambridge, 1995.