Exponential decay of averaged Green functions for random Schrödinger operators, a direct approach .

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Résumé. Sous des hypothèses d'analyticité convenables sur la densité de probabilité, nous étudions l'espérence de la fonction de Green. Nous donnons des résultats précis sur des domaines d'extension holomorphe en énergie et sur la décroissance exponentielle. L'ingrédient principal (comme dans [SW]) est la construction d'une mésure de probabilité dans le domaine complexe après déformation de contour. Ceci nous permet d'éviter d'utiliser des séries de perturbation. Comparée a la méthode de [SW], la variante ici semble limitée à l'équation de Schrödinger aléatoire, dans quel cas elle permet cependant de traiter des distributions de probabilité plus générales.

Abstract. Under suitable analyticity conditions on the probability distribution, we study the expectation of the Green function. We give precise results about domains of holomorphic extensions in energy and exponential decay. The key ingredient (as in [SW]) is the construction of a probability measure in the complex domain after contour deformation. This permits us to avoid the use of perturbation series. Compared to the method in [SW], the variant here seems limited to the random Schrödinger equation, in which case however it permits to treat more general probability distributions

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0. Introduction.

The purpose of this work is to present a variant of the method in [SW], which gives a more direct approach and which permits to treat more general probability distributions; not only perturbations of the Cauchy distributions but also for instance Gaussian ones. The main idea of the proof is the same as in [SW], namely to replace a certain complex density by a probability measure, but here we exploit in a more essential way some special structure in the problem and avoid the use of Fourier transform. On the other hand, we feel that the method of [SW] is of a more general nature and is likely to have applications to analyticity problems in statistical mechanics. Though the results below are more general (in the random Schrödinger case) they permit to recover only a slightly weakened version of the main result in [SW].

Let Δ be the discrete Schrödinger operator on $\ell^2(\mathbf{Z}^d)$, defined by

$$\Delta u(\nu) = \sum_{|\mu-\nu|_1=1} u(\mu),\ \ |\cdot|_1 = |\cdot|_{\ell^1}. \tag{0.1}$$

When $\Lambda \subset \mathbf{Z}^d$ is a finite subset, we put

$$\Delta_\Lambda = r_\Lambda \Delta r_\Lambda^*, \tag{0.2}$$

where $r_\Lambda : \ell^2(\mathbf{Z}^d) \to \ell^2(\Lambda)$ is the restriction operator, so that the adjoint $r_\Lambda^* = \ell^2(\Lambda) \to \ell^2(\mathbf{Z}^d)$ is the operator of extension by 0: $(r_\Lambda^* u)(\nu) = u(\nu)$, when $\nu \in \Lambda$, $= 0$, when $\nu \in \mathbf{Z}^d \setminus \Lambda$.

If $g(v)dv$ is a probability measure on $\mathbf{R}$ and $t > 0$, we are interested in the expectation value of the Green function

$$G_\Lambda(E) = (t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)^{-1},\ \ E \notin \sigma(t\Delta_\Lambda + \mathrm{diag}\,(v_j)), \tag{0.3}$$

given by

$$\langle G_\Lambda(E)(\mu,\nu)\rangle_g = \int (t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)^{-1}(\mu,\nu) \prod_{j\in\Lambda} (g(v_j)dv_j), \tag{0.4}$$

first for $\mathrm{Im}\, E > 0$, and then wherever this expression can be extended holomorphically w.r.t. E.

Let $K \subset \mathbf{C}$ be compact, symmetric around $\mathbf{R}$ and assume:

(H1) *g extends to a holomorphic function on $\mathbf{C} \setminus K$ (that we also denote by g), which satisfies*

$$|g(z)| \le C(1 + |z|)^{-2},\ \ |z| \ge C,$$

for some $C > 0$ with $K \subset D(0, C)$.

Here $D(0, C)$ denotes the open disc of center 0 and radius C.

Let $K_- = \{z \in K; \mathrm{Im}\, z \le 0\}$, and let $\mathrm{ch}\,(K_-)$ denote the convex hull of K_-. Our second assumption will need some further discussion:

(H2) *For every simple closed smooth negatively oriented curve* γ *in* $\mathbf{C}_- = \{z \in \mathbf{C};\ \mathrm{Im} z \leq 0\}$ *(with non-vanishing derivative) which is real in a neighborhood of all real points of* $\mathrm{ch}\,(K_-)$ *and with* $\{z \in \mathrm{ch}\,(K_-); \mathrm{Im}\, z < 0\} \subset \mathrm{int}\,(\gamma)$, *there exists a probability measure* $\mu_\gamma(dz)$ *supported in* γ, *such that*

$$\int_\gamma f(z)g(z)dz = \int_\gamma f(z)\mu_\gamma(dz),$$

for all functions f *which are holomorphic in* $\mathrm{int}\,(\gamma)$ *and smooth in* $\overline{\mathrm{int}\,(\gamma)}$.
Here $\mathrm{int}(\gamma)$ denotes the bounded open set which has γ as its (smooth) boundary.

We notice that (H2) is equivalent to the apparently weaker assumption that for every neighborhood V of $\mathrm{ch}\,(K_-)$ there exists a closed contour $\gamma = \gamma_V$ contained in V and a probability measure μ_γ with support in γ, with the properties described in (H2). In fact, if γ is a simple smooth closed loop (with non-vanishing derivative), let $P_\gamma : C(\gamma) \to C(\overline{\mathrm{int}\,(\gamma)})$ be the Poisson operator determined by $P_\gamma u_{|_\gamma} = u$, $P_\gamma u$ harmonic in $\mathrm{int}\,(\gamma)$. Since P_γ is positivity preserving, the adjoint P_γ^* maps a positive measure μ supported in $\overline{\mathrm{int}\,(\gamma)}$ to a positive measure $P_\gamma^*\mu$, supported on γ, and if f is holomorphic in $\mathrm{int}\,(\gamma)$, or more generally harmonic, and in $C(\overline{\mathrm{int}\,(\gamma)})$, then

$$\int f(z)\mu(dz) = \int P_\gamma(f_{|_\gamma})\mu(dz) = \int f_{|_\gamma}(z)(P_\gamma^*\mu)(dz). \tag{0.5}$$

Let γ, $\widetilde{\gamma}$ be two curves with the geometric properties described in (H2) and such that γ is contained in $\mathrm{int}\,\widetilde{\gamma}$ and carries a probability measure μ_γ with the properties in (H2). Then $P_{\widetilde{\gamma}}^*(\mu_\gamma)$ is carried by $\widetilde{\gamma}$ and has the properties of (H2), and we have proven the claim.

For $\lambda > 2d$, let

$$W(\lambda) = \{\eta \in \mathbf{R}^d;\ 2\sum_1^d \cosh \eta_j < \lambda\}. \tag{0.6}$$

This is a strictly convex bounded open neighborhood of 0 with smooth boundary, which is symmetric around 0. Let

$$p_\lambda(x) = \sup_{\eta \in W(\lambda)} x \cdot \eta,\ x \in \mathbf{R}^d \tag{0.7}$$

be the support function of $W(\lambda)$. p_λ is smooth outside $x = 0$, convex, even, positively homogeneous of degree 1, and $p_\lambda(x) > 0$ for $x \neq 0$. Let $D(0, r)$ denote the open disc of center 0 and radius r. A main result of this work is:

Theorem 0.1. *The expectation value (0.4) extends from the open upper half plane to a holomorphic function on* $\mathbf{C} \setminus (\mathrm{ch}\,(K_-) + \overline{D(0, 2td)})$, *that we continue to denote by* $\langle G_\Lambda(E)(\mu, \nu)\rangle$. *If* $E \in \mathbf{C}\setminus(\mathrm{ch}\,(K_-) + \overline{D(0, 2td)})$, *let* $\mathrm{dist}\,(E, \mathrm{ch}\,(K_-)) = \inf_{F \in \mathrm{ch}\,(K_-)} |E - F|$ *be the distance from* E *to* $\mathrm{ch}\,(K_-)$, *so that* $\frac{1}{t}\mathrm{dist}\,(E, \mathrm{ch}\,(K_-)) > 2d$. *Then for every* $\lambda \in]2d, \frac{1}{t}\mathrm{dist}\,(E, \mathrm{ch}\,(K_-))[$, *we have*

$$\|e^{\eta\cdot(\cdot)}\langle G_\Lambda(E)\rangle e^{-\eta\cdot(\cdot)}\|_{\mathcal{L}(\ell^2,\ell^2)} \leq \frac{1}{\mathrm{dist}\,(E, \mathrm{ch}\,(K_-)) - t\lambda}, \tag{0.8}$$

for every $\eta \in W(\lambda)$, and in particular,

$$|\langle G_\Lambda(E)(\mu,\nu)\rangle| \leq \frac{1}{\operatorname{dist}(E, \operatorname{ch}(K_-)) - t\lambda}\, e^{-p_\lambda(\mu-\nu)}. \tag{0.9}$$

The proof will also show that the limit of $\langle G_\Lambda(E)(\mu,\nu)\rangle$ exists when $\Lambda \nearrow \mathbf{Z}^d$, for the complex values of E in the theorem, and we then get (0.8), (0.9) also for the limit.

The first step in the proof is to notice that if γ is a curve as in (H2):

$$\langle G_\Lambda(E)\rangle = \int_{\gamma^\Lambda} (t\Delta_\Lambda + \operatorname{diag}(v_j) - E)^{-1} \prod_{j\in\Lambda} g(v_j)dv_j, \tag{0.10}$$

for $\operatorname{Im} E > 0$. Then since $(t\Delta_\Lambda + \operatorname{diag}(v_j) - E)^{-1}$ depends holomorphically on $v = \{v_j\}_{j\in\Lambda}$ in ${\Omega_E}^\Lambda$, where Ω_E is an E-dependent neighborhood of the closed lower half plane, we can apply Fubini's theorem and (H2), to get:

$$\langle G_\Lambda(E)\rangle = \int_{\gamma^\Lambda} (t\Delta + \operatorname{diag}(v_j) - E)^{-1} \prod_{j\in\Lambda} \mu_\gamma(dv_j). \tag{0.11}$$

The remainder of the proof is given in section 2, and consists in showing that $(t\Delta_\Lambda + \operatorname{diag}(v_j) - E)^{-1}$ exists when $v_j \in \gamma$, $\forall j \in \Lambda$, when γ is as above and convex, and $E \in \mathbf{C} \setminus (\overline{\operatorname{int}(\gamma)} + \overline{D(0, 2td)})$, and satisfies (0.8), (0.9), with K_- replaced by $\overline{\operatorname{int}(\gamma)}$.

It is clear that the proof gives a more general result. For $N \in \{1, 2, ..\}$ and $E_k \in \mathbf{C}$ with $\operatorname{Im} E_k > 0$, consider

$$\langle \prod_{k=1}^N (G_\Lambda(E_k)(\mu,\nu))\rangle_g = \int \prod_{k=1}^N ((t\Delta_\Lambda + \operatorname{diag}(v_j) - E_k)^{-1}(\mu,\nu)) \prod_{j\in\Lambda} (g(v_j)dv_j). \tag{0.4'}$$

Then we have the following generalization of Theorem 0.1:

Theorem 0.1'. *The expectation value (0.4') extends to a holomorphic function on $(\mathbf{C} \setminus (\operatorname{ch}(K_-) + \overline{D(0, 2td)}))^N$, that we denote by the LHS of (0.4'). For $(E_1, .., E_N)$ in this product domain, let $\lambda_k \in]2d, \frac{1}{t}\operatorname{dist}(E_k, \operatorname{ch} K_-)[$. Then*

$$|\langle \prod_{k=1}^N (G_\Lambda(E_k)(\mu,\nu))\rangle_g| \leq \prod_{k=1}^N \frac{e^{-p_{\lambda_k}(\mu-\nu)}}{(\operatorname{dist}(E_k, \operatorname{ch}(K_-)) - t\lambda_k)}. \tag{0.9'}$$

Remark. Using the fact that g is holomorphic, to show that $\langle G_\Lambda(E)\rangle$ is holomorphic, has already been done before Constantinescu, Fröhlich, Spencer [CFS]. Their proof uses the Neumann series of $(t\Delta + V - E)^{-1}$ for small t and contour deformation in v_j to show that the expectation value of the resulting series converges. However they did not try to show that the resulting measure on the new contour can be made positive. The method in [CFS] is effective when g decays fast enough at infinity. The measures considered in the present paper do not necessarily have this property.

The assumption (H2) is rather implicit and in order to have more applications, we shall prove in section 1 the following stability result:

Proposition 0.2. *Let $g_0(v)dv$ be a probability measure on $\mathbf{R}$ which satisfies (H1), (H2), with $K = K_0$. With γ as in (H2), we denote by $\mu_{g_0,\gamma}$ the corresponding probability measure given in (H2). Assume that for every γ as in (H2), there is a constant $C_\gamma > 0$ such that $\mu_{g_0,\gamma} \geq \frac{1}{C_\gamma}dt$ on $\mathbf{R}\cap\gamma\cap V_\gamma$, where V_γ is some neighborhood of $\mathrm{ch}\,(K_-)$.*

Let $g_j(v)dv$, $j = 1, 2, ..$ be a sequence of probability measures on $\mathbf{R}$ such that $g_j - g_0$ is of class C^2 and tends to 0 in C^2 when $j \to \infty$. Further, we assume that each g_j satisfies (H1) with $K = K_j \to K_0$ and that $g_j \to g_0$ on every compact subset of $\mathbf{C} \setminus K_0$.

Let $\widetilde{K} \subset \mathbf{C}$ be compact, symmetric around $\mathbf{R}$ and containing K_0 in its interior, and such that $\widetilde{K}_- = \{z \in z \in \widetilde{K};\, \mathrm{Im}\, z \leq 0\}$ is convex. Then for j sufficiently large, g_j satisfies (H1), (H2) with $K = \widetilde{K}$.

An immediate consequence of this result and Theorem 0.1 is:

Corollary 0.3. *We make the assumptions of Proposition 0.2. For every $\epsilon > 0$, there is a $j(\epsilon) \in \mathbf{N}$ such that the following holds for $j \geq j(\epsilon)$: $\langle G_\Lambda(E)\rangle_{g_j}$ extends holomorphically to $\mathbf{C} \setminus (\widetilde{K}_- + \overline{D(0, 2td + \epsilon)})$ and so that for every E in the latter set and every $\lambda \in\,]2d, \frac{1}{t}(\mathrm{dist}\,(E, \widetilde{K}_-) - \epsilon)[$:*

$$\|e^{\eta\cdot(\cdot)}\langle G_\Lambda(E)\rangle_{g_j} e^{-\eta\cdot(\cdot)}\|_{\mathcal{L}(\ell^2,\ell^2)} \leq \frac{1}{\mathrm{dist}\,(E, \widetilde{K}_-) - \epsilon - t\lambda}, \tag{0.12}$$

for all $\eta \in W(\lambda)$, and in particular,

$$|\langle G_\Lambda(E)(\mu, \nu)\rangle_{g_j}| \leq \frac{1}{\mathrm{dist}\,(E, \widetilde{K}_-) - \epsilon - t\lambda}\, e^{-p_\lambda(\mu-\nu)}. \tag{0.13}$$

(0.13) can be generalized as in Theorem 0.1'.

The following special case was considered in [SW]. Let

$$g_0(v) = \frac{1}{\pi}\frac{1}{v^2 + 1}, \tag{0.14}$$

so that (following Economou [E],) by the residue method, using the pole at $v = -i$:

$$\langle G_\Lambda(E)(\mu, \nu)\rangle_{g_0} = (t\Delta_\Lambda - (E + i))^{-1}(\mu, \nu). \tag{0.15}$$

Let $\widetilde{K} \subset \mathbf{C}$ be compact, symmetric around $\mathbf{R}$, with $-i \notin \mathrm{ch}\,\widetilde{K}_-$ and let g_j have the properties in Corollary 0.3 with $K = K_j$ there equal to $\widetilde{K} \cup \{-i, i\}$. We then recover the main result of [SW] in a slightly weakened form, by letting $E \in \mathbf{R}$ have the property that $\mathrm{dist}\,(E, \mathrm{ch}\,(\widetilde{K}_- \cup \{-i\})) = |E + i| > |E - F|$, for all $F \in (\mathrm{ch}\,(\widetilde{K}_- \cup \{-i\})) \setminus \{-i\}$.

The preceding example may be generalized. Take for instance

$$g_0(v) = \frac{C_1}{\pi} \frac{\beta_1}{(v-\alpha_1)^2+\beta_1^2} + \frac{C_2}{\pi} \frac{\beta_2}{(v-\alpha_2)^2+\beta_2^2}, \tag{0.16}$$

where $\alpha_1, \alpha_2 \in \mathbf{R}$, $\beta_1, \beta_2 > 0$, $C_1, C_2 \geq 0$, $C_1 + C_2 = 1$. Then

$$\langle G_\Lambda(E)(\mu,\nu)\rangle_{g_0} = \langle G_\Lambda(E)(\mu,\nu)\rangle_{C_1\delta_{\alpha_1-i\beta_1}+C_2\delta_{\alpha_2-i\beta_2}}$$

becomes the expectation value for a complex Bernoulli distribution, and we may consider perturbations as in Theorem 0.3.

The plan of the paper is the following: In section 1 we discuss the assumption (H2), prove Proposition 0.2 and check that the g_0's in (0.14) and in (0.16) satisfy (H2). In section 2, we complete the proof of Theorem 0.1. In section 3, we genealize our results further, by showing how to avoid making the assumption (H2).

1. More about the assumption (H2).

We first check that g_0 in (0.14) satisfies the assumptions (H1), (H2) with $K = \{i, -i\}$: (H1) is obviously fulfilled and in order to check (H2), let γ be a simple closed negatively oriented loop in the closed lower half-plane with $-i \in \mathrm{int}\,(\gamma)$. Let $f \in \mathrm{Hol}\,(\mathrm{int}\,(\gamma)) \cap C^\infty(\overline{\mathrm{int}\,(\gamma)})$. Here we let $\mathrm{Hol}\,(\Omega)$ denote the space of holomorphic functions on Ω, if $\Omega \subset \mathbf{C}$ is open. By the method of residues

$$\int_\gamma f(z) g_0(z) dz = f(-i). \tag{1.1}$$

But f is also harmonic, so

$$f(-i) = \int_\gamma f(z) P_\gamma(-i, dz), \tag{1.2}$$

where $P_\gamma(-i, dz) =: \mu_\gamma(dz)$ is the harmonic measure (i.e. the Poisson kernel). This is a probability measure on γ, given by a strictly positive density, so (H2) is fulfilled.

When g_0 is given by (0.16), we also have (H1), (H2) with $K = \{\alpha_1 + i\beta_1,\ \alpha_1 - i\beta_1,\ \alpha_2 + i\beta_2,\ \alpha_2 - i\beta_2\}$, and $\mathrm{ch}\,(K_-)$ is the closed segment joining $\alpha_1 - i\beta_1$ and $\alpha_2 - i\beta_2$.

We next prove Proposition 0.2. Let g_0, $\widetilde{K}$ be as in that proposition and choose γ as in (H2), with K there equal to $\widetilde{K}$. The discussion following (H2) shows that it suffices (for every such γ) to show that for j large enough depending on γ:

$$\int_\gamma f(z) g_j(z) dz = \int f(z) \mu_{\gamma,j}(dz),\ \ f \in \mathrm{Hol}\,(\mathrm{int}\,(\gamma)) \cap C^\infty(\overline{\mathrm{int}\,(\gamma)}),$$

for some probability measure $\mu_{\gamma,j}$ on γ.

We already know that

$$\int_\gamma f(z) g_0(z) dz = \int_\gamma f(z) \mu_{\gamma,0}(dz),$$

where $\mu_{\gamma,0}$ is a probability measure and from the discussion after (H2) and the positivity assumption on $\mu_{\gamma,0}$ in $\mathbf{R}\cap \operatorname{neigh}(K)$, it follows that $\mu_{\gamma,0}\geq \frac{1}{C}|dz|$ everywhere on γ. It is then clear that Proposition 0.2 will follow from:

Proposition 1.1. *Let $\Omega\subset\mathbf{C}$ be open bounded, simply connected with positively oriented C^∞ boundary $\partial\Omega=\gamma$. Let $g\in C^2(\gamma)$ be a complex-valued function such that*

$$\int_\gamma g(z)dz\in\mathbf{R}. \tag{1.3}$$

Then there exists a real-valued function $k\in C(\gamma)$, such that

$$\int_\gamma f(z)g(z)dz=\int_\gamma f(z)k(z)|dz|, \tag{1.4}$$

for all $f\in \operatorname{Hol}(\Omega)\cap C^\infty(\overline{\Omega})$.

Proof. The Riemann mapping theorem (see [B]) gives us a diffeomorphism $\kappa:\overline{\Omega}\to\overline{D(0,1)}$, holomorphic in the interior. By composing with κ^{-1}, we can therefore reduce ourselves to the case when $\Omega=D(0,1)$.

Expand in a Fourier series with $z=e^{it}$:

$$g(z)dz=(\sum_{-\infty}^{+\infty}\widehat{g}(j)e^{i(j+1)t})idt=(\sum_{-\infty}^{+\infty}\widetilde{g}(j)e^{ijt})dt, \tag{1.5}$$

where $\widetilde{g}(j)=i\widehat{g}(j-1)$ and where the C^2 assumption assures normal convergence of the series. The assumption (1.3) tells us that

$$\widetilde{g}(0)=i\widehat{g}(-1)\in\mathbf{R}. \tag{1.6}$$

For $f\in\operatorname{Hol}(D(0,1)\cap C^\infty(\overline{D(0,1)})$, we have

$$f(e^{it})=\sum_{\ell=0}^{\infty}\widehat{f}(\ell)e^{i\ell t}$$

so

$$\int_{\partial D(0,1)}f(z)g(z)dz=2\pi\sum_{\ell=0}^{\infty}\widehat{f}(\ell)\widetilde{g}(-\ell). \tag{1.7}$$

This expression does not change if we modify $\widetilde{g}(j)$ for $j\geq 1$ and we take

$$k(t)dt=(\widetilde{g}(0)+\sum_{-\infty}^{-1}\widetilde{g}(j)e^{ijt}+\sum_{1}^{\infty}\overline{\widetilde{g}(-j)}e^{ijt})dt, \tag{1.8}$$

which is real thanks to (1.6). #

We also give a proof which avoids the use of the mapping theorem. For simplicity we assume that all objects are smooth. Let γ be a positively oriented boundary of an open bounded simply connected set $\Omega \subset \mathbf{C}$ with smooth boundary. Let $f \in C^\infty(\gamma)$.

Proposition 1.2. *We have* $\int_\gamma \phi(z) f(z) dz = 0$ *for all* $\phi \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$ *iff* f *extends to an element in* $\mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$.

Proof. If f extends to $\mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$, then ϕf extends to an element in the same space and $\int \phi f dz = 0$.

Before proving the converse statement, let $f \in C^\infty(\gamma)$ and consider the two Cauchy integrals,

$$C_{\mathrm{int}} f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega,$$

$$C_{\mathrm{ext}} f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbf{C} \setminus \overline{\Omega}.$$

Then $C_{\mathrm{int}} f \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$, $C_{\mathrm{ext}} f \in \mathrm{Hol}\,(\mathbf{C} \setminus \overline{\Omega}) \cap C^\infty(\mathbf{C} \setminus \Omega)$, and

$$f = C_{\mathrm{int}} f_{|_\gamma} - C_{\mathrm{ext}} f_{|_\gamma}.$$

If we now assume that $\int_\gamma \phi(z) f(z) dz = 0$ for all $\phi \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$, then $C_{\mathrm{ext}} f = 0$, so $f = C_{\mathrm{int}} f_{|_\gamma}$ and f has the extension $C_{\mathrm{int}} f$ in $\mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$. #

Proposition 1.3. *For every* $g \in C^\infty(\gamma)$ *with* $\int_\gamma g(z) dz \in \mathbf{R}$*, there is a unique* $f \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$ *such that* $\mathrm{Im}\,(f dz_{|_\gamma}) = \mathrm{Im}\,(g dz_{|_\gamma})$.

Proof. We first discuss uniqueness. It suffices to show that if $f \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$, $\mathrm{Im}\,(f dz_{|_\gamma}) = 0$, then $f = 0$. Let $F \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$ be a primitive of f: $\frac{\partial F}{\partial z} = f$, or equivalently: $dF = f dz$. Hence $d\mathrm{Im}\, F = \mathrm{Im}\,(f dz)$, so $\mathrm{Im}\, F$ is a harmonic function on Ω with

$$d(\mathrm{Im}\, F_{|_\gamma}) = \mathrm{Im}\, f dz_{|_\gamma} = 0.$$

In other words, $\mathrm{Im}\, F_{|_\gamma}$ is constant, so the uniqueness in the standard Dirichlet problem implies that $\mathrm{Im}\, F = \mathrm{Const.}$ on Ω. Then F is constant, and $f = \frac{\partial F}{\partial z}$ vanishes, as claimed.

The proof of existence uses the same idea. Since $\int_\gamma \mathrm{Im}\,(g dz) = 0$, there exists $G \in C^\infty(\gamma; \mathbf{R})$ with $dG = \mathrm{Im}\,(g dz_{|_\gamma})$. Indeed the vanishing of the integral assures us that the primitive is single valued. Let $\mathcal{G} \in C^\infty(\overline{\Omega})$ be the solution of the Dirichlet problem:

$$\Delta \mathcal{G} = 0 \text{ on } \Omega, \ \mathcal{G}_{|_\gamma} = G.$$

Since $\mathcal{G}$ is harmonic and Ω simply connected, it is equal to the imaginary part of a holomorphic function F which is easily seen to belong to $\mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$. Let $f = \frac{\partial F}{\partial z}$. Then,

$$\mathrm{Im}\, f dz_{|_\gamma} = \mathrm{Im}\, dF_{|_\gamma} = d(\mathrm{Im}\, F_{|_\gamma}) = dG = \mathrm{Im}\,(g dz_{|_\gamma}).$$

#

Combining this result and the easy part of Proposition 1.2, we get a new proof of Proposition 1.1. Indeed, if g is given as in Proposition 1.1, then let f be as in Proposition 1.3. It follows from the proof that f is of class C^1. According to Proposition 1.2, the real measure $(gdz - fdz)_{|_\gamma}$ has the required properties and can be written as $k|dz|$ with k of class C^1.

We end this section by linking the above discussion to the Neumann problem. Let $g \in C^\infty(\gamma)$ with

$$\int g(z)dz = 1. \tag{1.9}$$

As above, we look for $f \in \mathrm{Hol}\,(\Omega) \cap C^\infty(\overline{\Omega})$ with

$$gdz_{|_\gamma} = fdz_{|_\gamma} + k|dz|_{|_\gamma}, \tag{1.10}$$

with the last term real. The unique solution of this problem is given by the function f in Proposition 1.3, which is of the form $f(z) = \frac{\partial F}{\partial z}$, where F is holomorphic in Ω and solves the Dirichlet problem

$$\Delta \mathrm{Im}\, F = 0 \text{ in } \Omega,\ \mathrm{Im}\, F_{|_\gamma}(z) = \int_{\gamma_{z_0,z}} \mathrm{Im}\,(g(z)dz),\ z \in \gamma, \tag{1.11}$$

where $\gamma_{z_0,z}$ denotes the positively oriented segment of γ which starts at z_0 and ends at z.

We observe that the conjugated function $\mathrm{Re}\, F$ satisfies on γ:

$$\frac{\partial}{\partial \nu_{\mathrm{int}}} \mathrm{Re}\, F = -\frac{\partial}{\partial t} \mathrm{Im}\, F(\gamma(t)),\ \nu_{\mathrm{int}} = \text{ interior unit normal},$$

provided that we parametrize γ by arc length with positive orientation, so that $|\gamma'(t)| = 1$. But $\frac{\partial}{\partial t}\mathrm{Im}\, F(\gamma(t)) = \mathrm{Im}\,(g(\gamma(t))\gamma'(t))$ according to the boundary condition in (1.11), so $\mathrm{Re}\, F$ is the solution (unique up to a constant) of the Neumann problem:

$$\Delta \mathrm{Re}\, F = 0 \text{ in } \Omega,\ \frac{\partial}{\partial \nu_{\mathrm{int}}} \mathrm{Re}\, F = -\mathrm{Im}\,(g\gamma') \text{ on } \gamma. \tag{1.12}$$

Then we get $\mathrm{Re}\,(fdz_{|_\gamma}) = d_t \mathrm{Re}\, F(\gamma(t))$, so

$$k(t) = \mathrm{Re}\,(g(\gamma(t))\gamma'(t)) + \frac{\partial}{\partial t}[(r_\gamma K_N(\mathrm{Im}\, g\gamma'))(\gamma(t))]. \tag{1.13}$$

Here K_N denotes the Poisson-Neumann operator which solves up to a constant the Neumann problem:

$$\Delta K_N v = 0 \text{ on } \Omega,\ \frac{\partial}{\partial \nu_{\mathrm{int}}} K_N v = v \text{ on } \gamma,\ \text{when } \int v(\gamma(t))dt = 0,$$

and r_γ is the restriction operator: $C^\infty(\overline{\Omega}) \to C^\infty(\gamma)$.

2. Coercivity and exponential decay.

For $\lambda > 2d$, let

$$W(\lambda) = \{\eta \in \mathbf{R}^d;\ 2\sum_1^d \cosh \eta_j < \lambda\}. \tag{2.1}$$

(We refer to [SW] for a more complete discussion, using also the Fourier transform.) $W(\lambda)$ is a convex bounded open set symmetric around 0, and we let

$$p_\lambda(x) = \sup_{\eta \in W(\lambda)} x \cdot \eta, \tag{2.2}$$

be the corresponding support function. p_λ is convex, smooth outside 0, and positively homogeneous of degree 1. Moreover $p_\lambda(x) > 0$ for $x \neq 0$. In [SW], we observed that for $\eta \in W(\lambda)$:

$$\| e^{\eta\cdot(\cdot)} \Delta_\Lambda e^{-\eta\cdot(\cdot)} \|_{\mathcal{L}(\ell^2,\ell^2)} \leq \lambda, \tag{2.3}$$

and similarly for $\Delta = \Delta_{\mathbf{Z}^d}$.

Let $E \in \mathbf{C}$ with $|E| > 2d$ and choose $\lambda \in]2d, |E|[$. Then for $\eta \in W(\lambda)$, we have in the sense of self-adjoint operators:

$$\mathrm{Re}\, e^{-i \arg E}(e^{\eta\cdot(\cdot)}(E - \Delta_\Lambda)e^{-\eta\cdot(\cdot)}) \geq |E| - \lambda, \tag{2.4}$$

and similarly with Δ_Λ replaced by Δ. As noticed in [SW] with a slightly different method, it follows that

$$\| e^{\eta\cdot(\cdot)} (E - \Delta_\Lambda)^{-1} e^{-\eta\cdot(\cdot)} \|_{\mathcal{L}(\ell^2,\ell^2)} \leq \frac{1}{|E| - \lambda}, \tag{2.5}$$

and in particular for the matrix of the inverse:

$$|(E - \Delta_\Lambda)^{-1}(\mu, \nu)| \leq \frac{1}{|E| - \lambda}\, e^{-p_\lambda(\mu - \nu)}. \tag{2.6}$$

Using an argument of regularization of the weights near infinity, we obtain (2.5), (2.6) also for Δ. In the present paper, we shall not use (2.5), (2.6) but rather (2.4) and establish:

Proposition 2.1. *Let $E \in \mathbf{C}$ with $|E| > 2d$ and let v_j, $j \in \Lambda$ satisfy $\mathrm{Re}\, e^{-i \arg E} v_j \leq 0$. Then $E - (\Delta_\Lambda + \mathrm{diag}\,(v_j))$ has a bounded inverse such that for every $\lambda \in]2d, |E|[$ and every $\eta \in W(\lambda)$:*

$$\| e^{\eta\cdot(\cdot)} (E - (\Delta_\Lambda + \mathrm{diag}\,(v_j)))^{-1} e^{-\eta\cdot(\cdot)} \|_{\mathcal{L}(\ell^2,\ell^2)} \leq \frac{1}{|E| - \lambda}. \tag{2.7}$$

In particular,

$$|(E - (\Delta_\Lambda + \mathrm{diag}\,(v_j)))^{-1}(\mu, \nu)| \leq \frac{1}{|E| - \lambda}\, e^{-p_\lambda(\mu - \nu)}. \tag{2.8}$$

The result remains valid with Λ replaced by $\mathbf{Z}^d$ if we assume that $\{v_j\}_{j \in \mathbf{Z}^d}$ is bounded.

Proof. Using (2.4), we get

$$\mathrm{Re}\, e^{-i\arg E}(e^{\eta\cdot(\cdot)}(E-(\Delta_\Lambda+\mathrm{diag}\,(v_j)))e^{-\eta\cdot(\cdot)}) \geq |E|-\lambda \tag{2.9}$$

and (2.7) and (2.8) are obtained as (2.5) and (2.6). The extension to the case $\Lambda = \mathbf{Z}^d$ is treated using regularization of the weights at infinity. #

We now return to the situation in the sections 0, 1. Let γ be a curve as in (H2) so that

$$\langle G(E)(\mu,\nu)\rangle = \int (t\Delta_\Lambda + \mathrm{diag}\,(v_j)-E)^{-1}(\mu,\nu)\prod_{j\in\Lambda}\mu_\gamma(dv_j). \tag{2.10}$$

We may assume without loss of generality that γ is convex. Let $E\in\mathbf{C}$ belong to the exterior of γ, and let $\pi_\gamma(E)\in\gamma$ be the point on γ, closest to E: $|\pi_\gamma(E)-E| = \mathrm{dist}\,(E,\gamma)$. Then the line through $\pi_\gamma(E)$ which is perpendicular to $E-\pi_\gamma(E)$ separates γ and E, and we have:

$$\mathrm{Re}\,(e^{-i\arg(E-\pi_\gamma(E))}(\pi_\gamma(E)-v)) \geq 0,\ \ \forall v\in\gamma. \tag{2.11}$$

Writing

$$\begin{aligned}&t\Delta + \mathrm{diag}\,(v_j) - E = -(E-\pi_\gamma(E)) + (t\Delta + \mathrm{diag}\,(v_j-\pi_\gamma(E))) = \\ &-t[\frac{1}{t}(E-\pi_\gamma(E)) - (\Delta+\mathrm{diag}\,(\frac{1}{t}(v_j-\pi_\gamma(E)))],\end{aligned}$$

we apply Proposition 2.1 with E there replaced by $\frac{1}{t}(E-\pi_\gamma(E))$ and v_j there replaced by $\frac{1}{t}(v_j-\pi_\gamma(E))$, and get for $\eta\in W(\lambda)$:

$$\begin{aligned}&\|e^{\eta\cdot(\cdot)}(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}e^{-\eta\cdot(\cdot)}\|_{\mathcal{L}(\ell^2,\ell^2)} \leq \\ &\frac{1}{t}\frac{1}{(\frac{1}{t}|E-\pi_\gamma(E)|-\lambda)} = \frac{1}{|E-\pi_\gamma(E)|-t\lambda},\end{aligned} \tag{2.12}$$

if $\frac{1}{t}\,|E-\pi_\gamma(E)|>2d$ and $\lambda\in]2d,\frac{1}{t}|E-\pi_\gamma(E)|[$, and in particular,

$$|(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}(\mu,\nu)| \leq \frac{1}{|E-\pi_\gamma(E)|-t\lambda}\,e^{-p_\lambda(\mu-\nu)}. \tag{2.13}$$

Since μ_γ is a probability measure, we get from this and (2.10):

$$\|e^{\eta\cdot(\cdot)}\langle G(E)\rangle e^{-\eta\cdot(\cdot)}\|_{\mathcal{L}(\ell^2,\ell^2)} \leq \frac{1}{|E-\pi_\gamma(E)|-t\lambda},\ \ \eta\in W(\lambda), \tag{2.14}$$

$$|\langle G(E)(\mu,\nu)| \leq \frac{1}{|E-\pi_\gamma(E)|-t\lambda}\,e^{-p_\lambda(\mu-\nu)}, \tag{2.15}$$

for $\lambda\in]2d,\frac{1}{t}|E-\pi_\gamma(E)|[$.

Since we can choose γ in an arbitrarily small neighborhood of $\mathrm{ch}\,(K_-)$, we may arrange so that $|E-\pi_\gamma(E)|\to\mathrm{dist}\,(E,\mathrm{ch}\,(K_-))$ and Theorem 0.1 follows. #

3. A more general result.

In this section we shall relax the assumption (H1) about holomorphic extendability, and suppress the assumption (H2). The price we have to pay, is that the result will be valid only for E sufficiently far away from the region of non-analyticity in some precise sense that we will explain. We start by proving some auxiliary results which contain the essential ideas. Write $\langle z\rangle = \sqrt{(1+|z|^2)}$.

Lemma 3.1. *Let $\mathbf{R}\ni t\mapsto \gamma(t)$ be a smooth curve in the open upper half plane $\mathbf{C}^+$, with $\gamma'(t)\neq 0$ and without self-intersections. Assume that $\gamma(t)=C_\pm + e^{i\theta_\pm}t$, $\pm t >> 0$, where $\theta_- < 0 < \theta_+ < \pi+\theta_-$, and let $\Omega\subset\mathbf{C}^+$ be the open set with $\partial\Omega=\gamma$. Let $g(z)$ be a continuous function on γ with $g(z)=\mathcal{O}(\langle z\rangle^{-2-\epsilon})$, for some $\epsilon>0$, and with*

$$\int_\gamma g(z)dz\in\mathbf{R} \tag{3.1}$$

Then there exists a real measure μ on γ of the form $\mu(dz)=m(z)|dz|$ with m continuous and $\mathcal{O}(\langle z\rangle^{-2-\epsilon})$, for some, possibly new $\epsilon>0$, such that

$$\int_\gamma \phi(z)g(z)dz = \int_\gamma \phi(z)\mu(dz),\ \ \forall\phi\in \mathrm{Hol}\,(\Omega)\cap C(\overline{\Omega})\cap L^\infty(\Omega). \tag{3.2}$$

Proof. Let $G\in C^1(\gamma)$ be a primitive of $\mathrm{Im}\,(gdz)$ with $G=\mathcal{O}(\langle z\rangle^{-1-\epsilon})$. Considering rotations of the functions $\mathrm{Re}\, z^{-1-\epsilon}$, we see that we can find a positive harmonic function q, defined near $\overline{\Omega}$, of the order of magnitude $|z|^{-1-\epsilon}$, for sufficiently small $\epsilon>0$, depending on θ_+, θ_-. Consider the Dirichlet problem for $\mathcal{G}\in C^1(\overline{\Omega})$:

$$\Delta\mathcal{G}=0 \text{ in } \Omega,\ \ \mathcal{G}_{|_\gamma}=G. \tag{3.3}$$

Approaching this problem with suitable problems on $\Omega_R := \Omega\cap D(0,R)$, using the maximum principle with q as a comparison function, and letting $R\to\infty$, we see that (3.3) has a unique solution with $\mathcal{G}=\mathcal{O}(\langle z\rangle^{-1-\epsilon})$, for some $\epsilon>0$.

Using a scaling argument we also see that

$$\nabla\mathcal{G}=\mathcal{O}(\langle z\rangle^{-2-\epsilon}). \tag{3.4}$$

Since $\mathcal{G}$ is harmonic in Ω, we have

$$\mathcal{G}=\mathrm{Im}\,\mathcal{F},\ \ \mathcal{F}\in\mathrm{Hol}\,(\Omega). \tag{3.5}$$

¿From the Cauchy-Riemann equations, we see that $\mathcal{F}\in C^1(\overline{\Omega})$, and

$$\nabla\mathcal{F}=\mathcal{O}(\langle z\rangle^{-2-\epsilon}). \tag{3.6}$$

Put $\mu(dz)=gdz-d\mathcal{F}_{|_\gamma}=gdz-fdz_{|_\gamma}$, where $f=\frac{\partial\mathcal{F}}{\partial z}$. Then,

$$\mathrm{Im}\,(gdz-d\mathcal{F}_{|_\gamma})=\mathrm{Im}\,(gdz)-dG=0, \tag{3.7}$$

so μ is real. Moreover,

$$\mu(dz)=m(z)|dz|,\ m\in C(\gamma),\ m=\mathcal{O}(\langle z\rangle^{-2-\epsilon}). \tag{3.8}$$

Finally we have (3.2), since

$$\int_\gamma \phi(z)f(z)dz=0,\ \phi\in \mathrm{Hol}\,(\Omega)\cap C(\overline{\Omega})\cap L^\infty(\Omega),$$

by a standard argument for Cauchy integrals. #

Remark. Assume that g satisfies the regularity and growth assumptions of the Lemma outside some compact subset K of γ and that g is a distribution near K. Then we can find a real distribution μ on γ which is of the form $m(z)|dz|$ outside K with m as in the lemma, such that the identity in (3.2) holds for all $\phi\in\mathrm{Hol}\,(\Omega)\cap C^\infty(\overline{\Omega})\cap L^\infty(\Omega)$. To see this we repeat the proof. G will now be a distribution near K, and with the same properties as before outside K. Then we can solve (3.3), where $\mathcal{G}$ is of temperate growth near K and elsewhere C^1 up to the boundary, and $=\mathcal{O}(\langle z\rangle^{-1-\epsilon})$ far away. We still have (3.4) far away and can define $\mathcal{F}$ as before, holomorphic in Ω, of temperate growth near K and C^1 up to the boundary away from K. Moreover $f:=\frac{\partial\mathcal{F}}{\partial z}=\mathcal{O}(\langle z\rangle^{-2-\epsilon})$ far away. Define $\mu(dz)$ as before, now with $fdz_{|_\gamma}$ interpreted as a boundary value in the sense of distributions. We get a real distribution, and our claim follows from the fact that

$$\int_\gamma \phi(z)f(z)dz=0,\ \phi\in\mathrm{Hol}\,(\Omega)\cap C^\infty(\overline{\Omega})\cap L^\infty(\Omega).$$

Let γ be as in the lemma and let $\mu(dz)=m(z)|dz|$ with m real, continuous and $=\mathcal{O}(\langle z\rangle^{-2-\epsilon})$. Let $p(z)=\frac{1}{\pi}\frac{\mathrm{Im}\,z}{|z|^2}$ and let

$$Pu(z)=\int p(z-t)u(t)dt$$

be the Poisson operator for the upper half plane $\mathbf{C}^+$, mapping bounded continuous functions on $\mathbf{R}$ to bounded continuous functions on $\overline{\mathbf{C}^+}$. We recall that P is positivity preserving. The adjoint P^* maps bounded measures (i.e. with finite total mass) on $\mathbf{C}^+$ to bounded measures on $\mathbf{R}$, and in the case of positive measures, the total mass is conserved. We are interested in $P^*(\mu)$ which is of the form $k(t)dt$ with $k(t)=\int p(z-t)\mu(dz)$.

Lemma 3.2. *Let γ_j, $j=1,2$ be as in Lemma 3.1, and let $\mu_j(dz)$ be a real measure on γ_j of the form $m_j(z)|dz|$ with m_j continuous and $\mathcal{O}(\langle z\rangle^{-2})$. Assume that*

$$\int_{\gamma_1}\phi(z)\mu_1(dz)=\int_{\gamma_2}\phi(z)\mu_2(dz), \tag{3.9}$$

for all $\phi\in\mathrm{Hol}\,(\mathbf{C}^+)$, which are bounded on every half-plane: $\mathrm{Im}\,z>\epsilon$, $\epsilon>0$. Then

$$P^*(\mu_1)=P^*(\mu_2). \tag{3.10}$$

Proof. Since μ_j are real, it follows that

$$\int_{\gamma_1} \mathrm{Re}\,(\phi(z))\mu_1(dz) = \int_{\gamma_2} \mathrm{Re}\,(\phi(z))\mu_2(dz),$$

for all ϕ as in the lemma. It then suffices to notice that $p(z-t) = \mathrm{Re}\,\phi_t$, where ϕ_t is as in the lemma. #

As in the remark after the proof of Lemma 3.1, we can relax the regularity assumptions on μ_j on some bounded part of γ_j.

Lemma 3.3. *Let γ be as in Lemma 3.1 and let $\mu(dz) = m(z)|dz|$ be a real density on γ with $m(z)$ continuous, $= \mathcal{O}(\langle z\rangle^{-2-\epsilon})$ for some $\epsilon > 0$.*
a) Then $P^*(\mu) = k(t)dt$ with $k(t)$ continuous and $\mathcal{O} = \mathcal{O}(\langle t\rangle^{-2})$.
b) For $T \geq 0$, let $\gamma_T(t) = \gamma(t) + iT$, and define μ_T on γ_T as $m(z-iT)|dz|$. If

$$\int_\gamma \mu(dz) = 1, \tag{3.11}$$

then there exists $T_0 \geq 0$, such that $P^*(\mu_T) \geq 0$ on $\mathbf{R}$, precisely for $T \geq T_0$.

Proof. We have

$$\begin{aligned}
&k(t) = \int_\gamma p(z-t)\mu(dt) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\,\gamma(s)}{|\gamma(s)-t|^2} m(\gamma(s))\gamma'(s)ds = \\
&\mathcal{O}(1)\int_{-\infty}^{+\infty} \frac{\langle s\rangle}{(1+|t|+|s|)^2}\langle s\rangle^{-2}ds = \mathcal{O}(1)\int_0^\infty \frac{1}{(1+|t|+s)^2}\frac{1}{1+s}ds = \mathcal{O}(\frac{1}{(1+|t|)^2}),
\end{aligned}$$

and we have proved a), using only the weaker assumption that $m = \mathcal{O}(\langle z\rangle^{-2})$.

If $P^*(\mu_{T_0}) \geq 0$, for some T_0 and $T > T_0$, then we can identify this measure with the measure $P^*_{T-T_0}(\mu_T)$ on the line $\mathrm{Im}\, z = T - T_0$, where P_{T-T_0} is the Poisson operator for the half plane $\mathrm{Im}\, z > T - T_0$. It is easy to see that $P^*(\mu_T) = P^* P^*_{T-T_0}(\mu_T) \geq 0$, so to prove b), it suffices to find one $T > 0$ such that $P^*(\mu_T) \geq 0$. Write $P^*(\mu_T) = k_T dt$, where

$$\begin{aligned}
&k_T(t) = \int_{\gamma_T} p(z-t)\mu_T(dz) = \\
&p(\gamma_T(0)-t) + \int_{\gamma_T} (p(z-t) - p(\gamma_T(0)-t))\mu_T(dz).
\end{aligned} \tag{3.12}$$

Here $p(\gamma_T(0)-t) \sim T/(T+|t|)^2$.

When $z \in \gamma_T$ and $|z - \gamma_T(0)| \leq T$, we have

$$|p(z-t) - p(\gamma_T(0)-t)| \leq C\frac{|z-\gamma_T(0)|}{|\gamma_T(0)-t|^2},$$

and the corresponding contribution to the last integral in (3.12) is

$$\mathcal{O}(1)\int_{-T}^{T}\frac{1+|s|}{(T+|t|)^2(1+|s|)^2}ds=\mathcal{O}(1)\frac{\log T}{(T+|t|)^2}=o(1)p(\gamma_T(0)-t), \tag{3.13}$$

when $T\to\infty$, uniformly in t.

The integral over $|z-\gamma_T(0)|>T$ can be split in two terms:

$$\mathcal{O}(1)\int_{|s|\geq T}\frac{|s|}{s^2+t^2}\frac{1}{s^{2+\epsilon}}ds, \tag{3.14}$$

and

$$\mathcal{O}(1)\int_{|s|\geq T}\frac{T}{T^2+t^2}\frac{1}{s^{2+\epsilon}}ds. \tag{3.15}$$

the last expression is $\mathcal{O}(1)\frac{1}{T^2+t^2}=o(1)p(\gamma_T(0)-t)$. (3.14) is

$$\mathcal{O}(1)\int_{T}^{\infty}\frac{1}{s^2+t^2}\frac{1}{s^{1+\epsilon}}ds. \tag{3.16}$$

If $T\geq|t|$, we estimate this by $\mathcal{O}(1)T^{-2}=o(1)p(\gamma_T(0)-t)$. If $T\leq|t|$, then the change of variables $s=|t|\sigma$ gives

$$\frac{\mathcal{O}(1)}{|t|^{2+\epsilon}}\int_{T/|t|}^{\infty}\frac{1}{1+\sigma^2}\frac{1}{\sigma^\epsilon}\frac{d\sigma}{\sigma}=\mathcal{O}(1)\frac{1}{T^\epsilon|t|^2}=o(1)p(\gamma_T(0)-t).$$

Summing up, we get

$$k_T(t)=(1+o(1))p(\gamma_T(0)-t),\ T\to\infty, \tag{3.17}$$

uniformly in t, and the positivity follows, when T is large enough. #

Remark. a) We can relax the continuity assumption on m and allow $\mu(dz)$ to be a distribution on some bounded set and elsewhere as in the lemma. Then a), b) still hold. In estimating for instance the last integral in (3.12), we decompose μ_T into a continuous density and a distribution with compact support. For the contribution of the latter, we only need to notice that for $k\geq 1$:

$$\nabla_z^k(p(z-t)-p(\gamma_T(0)-t))=\mathcal{O}_k(1)\frac{1}{(T+|t|)^{1+k}}=o(1)p(\gamma_T(0)-t).$$

b) If we map the upper half-plane conformally onto the unit disc, then the curves in the preceding lemmas close (at the image point of infinity), and we get a conceptual link with some of the arguments in section 1.

We can now start to formulate the main result of this section. Let $g(x)dx$ be a probability measure on $\mathbf{R}$ and assume that for some $\epsilon>0$:

$$g \text{ is continuous and } \mathcal{O}(\langle x\rangle^{-2-\epsilon}) \text{ outside some bounded set.} \tag{3.18}$$

$$g \text{ has a holomorphic extension to some set of the form} \\ |\mathrm{Im}\, z| < \frac{1}{C}(\mathrm{Re}\, z - C), \text{ which satisfies } g = \mathcal{O}(\langle z\rangle^{-2-\epsilon}). \tag{3.19}$$

A straight line L is called *admissible* if L is non-parallel to $\mathbf{R}$ and if g has a holomorphic extension $g = \mathcal{O}(\langle z\rangle^{-2-\epsilon})$ to a set of the form $\{z \in \mathbf{C}; \mathrm{dist}\,(z, K_L) < C^{-1}\langle z\rangle\}$, where $K_L := \mathbf{C}_- \cap \Pi_L^+$ and $\mathbf{C}_-$ is the closed lower half-plane, and Π_L^+ is the closed half plane with boundary L containing $[a, +\infty[$ for some a.

If L is admissible, represent L as $a + e^{-i\theta}\mathbf{R}$ for $a \in \mathbf{R}$, $0 < \theta < \pi$, and let γ be a curve obtained from $]-\infty, a-\epsilon] \cup (a-\epsilon) + e^{-i(\theta+\epsilon)}\mathbf{R}_+$, by smoothing in a small neighborhood of $a - \epsilon$. Here ϵ is sufficiently small. The complex density $g(z)dz_{|_\gamma}$ is then well defined. Let μ_γ be the corresponding normalized real measure on γ obtained from Lemma 3.1 (after a rotation + translation which maps L to $\mathbf{R}$). Let P_L be the Poisson operator associated to the closed half-plane Π_L^- opposite to Π_L^+. We say that L is *admitted* if $P_L^*(\mu_\gamma) \geq 0$. From Lemma 3.3 we know that if L is admissible, then L becomes admitted after a sufficiently long parallel translation to the right. If L is admitted, let $h_L(z)$ be the real affine linear form which vanishes on L with normalized gradient pointing in the direction of Π_L^+. Put,

$$h(z) = \sup_{L \text{ admitted}} h_L(z), \tag{3.20}$$

so that $h(z)$ is a convex function which tends to $+\infty$ when z tends to infinity in some conic neighborhood of $[0, +\infty[$.

Theorem 3.4. *Assume (3.18), (3.19) and define $h(z)$ by (3.20). Then $\langle(t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)^{-1}\rangle_g$ has a holomorphic extension from the open upper half-plane to the union of this half-plane with $\{E \in \mathbf{C}; h(E) > 2dt\}$. If E belongs to the latter set and $\lambda \in [2d, h(E)/t[$, then for every $\eta \in W(\lambda)$:*

$$\|e^{\eta\cdot(\cdot)}\langle(t\Delta + \mathrm{diag}\,(v_j) - E)^{-1}\rangle_g e^{-\eta\cdot(\cdot)}\|_{\mathcal{L}(\ell^2,\ell^2)} \leq \frac{1}{h(E) - t\lambda},$$

and in particular,

$$|\langle(t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)^{-1}(\mu,\nu)\rangle_g| \leq \frac{1}{h(E) - t\lambda}\, e^{-p_\lambda(\mu-\nu)}.$$

Proof. Let L be an admitted line and choose γ as above. For $\mathrm{Im}\, E > 0$, we have

$$\langle(t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)^{-1}\rangle_g = \int_{\gamma^\Lambda} (t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)^{-1} \prod g(v_j)dv_j. \tag{3.21}$$

Let $E \in \Pi_L^+$ with $\mathrm{dist}\,(E, L) > 2dt$ and let $v_j \in \Pi_L^-$, $\eta \in W(\lambda)$, $2d \leq \lambda < \mathrm{dist}\,(E, L)/t$. Then

$$-\mathrm{Re}\,(e^{\eta\cdot(\cdot)}e^{-i\arg(E-\pi_L(E))}(t\Delta_\Lambda + \mathrm{diag}\,(v_j) - E)e^{-\eta\cdot(\cdot)}) \geq \mathrm{dist}\,(E, L) - t\lambda,$$

where $\pi_L(E)$ is the point in L which realizes the distance from E to L. It follows that

$$\|e^{\eta\cdot(\cdot)}(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}e^{-\eta\cdot(\cdot)}\|\leq\frac{1}{\mathrm{dist}\,(E,L)-t\lambda},$$

$$|(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}(\mu,\nu)|\leq\frac{1}{\mathrm{dist}\,(E,L)-t\lambda}\,e^{-p_\lambda(\mu-\nu)}.$$

If in addition $\mathrm{Im}\,E>0$, then we can use (3.21) to get

$$\begin{aligned}\langle(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}\rangle_g&=\int_{\gamma^\Lambda}(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}\prod\mu_\gamma(dv_j)\\&=\int_{L^\Lambda}(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}\prod P_L^*(\mu_\gamma)(dv_j).\end{aligned}$$

¿From this identity and the preceding estimates, we see that $\langle(t\Delta_\Lambda+\mathrm{diag}\,(v_j)-E)^{-1}\rangle_g$ extends holomorphically to the set of E in Π_L^+ with $\mathrm{dist}\,(E,L)>t\lambda$ and satisfies the same estimates. It then suffices to vary L among all admitted lines. #

We have not tried to formulate a result of maximal generality or sharpness. One obvious generalization would be to consider holomorphic extensions to some Riemann surface. Another would be to consider the situation in the introduction, and only make the hypothesis (H1). When γ is a closed bounded curve, Lemma 3.1 reduces to Proposition 1.1, and Lemma 3.2, 3.3 remain valid. We leave the formulation of the analogue of Theorem 3.4 to the interested reader.

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