Interpolation of an exact symplectomorphism by a Hamiltonian flow.

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Abstract. Let $\mathcal{O}$ be the zero-section of the cotangent bundle $T^*\mathcal{M}$ of a real analytic manifold $\mathcal{M}$. Let $F : (T^*\mathcal{M}, \mathcal{O}) \to (T^*\mathcal{M}, \mathcal{O})$ be a real analytic local diffeomorphism preserving the canonical symplectic form $\omega = \mathrm{d}\alpha$ of T^*M (α denotes the Liouville form). Suppose that $F^*\alpha - a$ is an exact form $\mathrm{d}S$. Then:

- We can reconstruct F from S and $f = F_{|\mathcal{O}}$.
- F can be included into a Hamiltonian flow, provided f is included into a flow.

The proofs are constructive. They are related with a derivation on the Lie algebra of functions (endowed with the Poisson bracket).

1 Introduction

As the time-1 flow of a Hamiltonian vector field on an exact symplectic manifold is exact symplectic, a natural question arises:

> Given an exact symplectomorphism, is it the time-1 flow of a time-dependent Hamiltonian vector field?

In such a case, we shall say that our symplectomorphism is *homologous to the identity.* Once we have interpolated our exact symplectomorphism by a time-dependent Hamiltonian flow, next question is:

> Can we get our Hamiltonian be 1-periodic in time?

This subject has been studied for many authors, and it has many variants. It is a particular case of the more general problem of *inclusion of a map into a flow.* Moser [10] already dealt with this problem when he proved the analyticity of the Birkhoff normal form around a hyperbolic fixed point of an area preserving map. Douady [4] solved the problem in the smooth symplectic case provided our map is given by a generating function and Conley and Zehnder [3] solved it for smooth diffeomorphism of a torus which leaves the center of mass fixed. On the other side, Douady [4], Kuksin [6] and Kuksin and Pöschel [7] solved the problem in analytic set up for maps which are close to integrable ones.

Our exact symplectic manifold is $T^*\mathcal{M}$, endowed with the canonical symplectic form given by the differential of the Liouville form $\alpha = y\ \mathrm{d}x$. $z : \mathcal{M} \to T^*\mathcal{M}$ is the canonical inclusion and $\mathcal{O} = z(M)$ is the zero-section of $T^*\mathcal{M}$. $q : T^*\mathcal{M} \to \mathcal{M}$ is the standard projection. In fact, we shall work in a tubular neighbourhood of $\mathcal{O}$, $\mathcal{N}$. We want to include a symplectomorphism $F : \mathcal{N} \to T^*M$ into a Hamiltonian flow, and, hence, F has to be exact. That is, there exists a function $S : \mathcal{N} \to \mathbb{R}$ such that $F^*\alpha - \alpha = \mathrm{d}S$. This function is a *primimitive function* of F. Moreover, we shall suppose that the zero section is invariant.

This reduction is not so restrictive, due to a Weinstein's theorem [13]. It states that we can send via a symplectomorphism a certain neighbourhood of any Lagrangian manifold onto a neighborhood of the zero-section of its cotangent bundle. Moreover, using a generalized Poincaré's lemma, he also proved that if our Lagrangian manifold is exact then the symplectomorphism is also exact (between two different manifolds, of course). See also [11] for analytic versions of this theorem. Lagrangian manifolds are important in Hamiltonian dynamics. For instance, KAM tori are Lagrangian, and also the stable and unstable manifolds of a hyperbolic fixed point.

The theorem that we shall prove along this paper is

Theorem *Let $\mathcal{M}$ be a real analytic manifold, and $\mathcal{N} = T^*\mathcal{M}$ a tubular neighbourhood of its zero-section. Let $F : \mathcal{N} \to T^*M$ be a real analytic exact symplectomorphism, such that the zero-section is invariant, and $f = q \circ F \circ z : \mathcal{M} \to \mathcal{M}$ is its dynamics. Suppose that f is interpolated by the flow $f_t = f_{t,0}$ of a real analytic time-dependent vector field $X_t \in \mathcal{X}(\mathcal{M})$: $f = f_1$. Then,*

> *F is (analytically) homologous to the identity (at least in a tubular neighbourhood of the zero-section).*

So then, we deal with the first problem. In order to get periodicity in time we can apply, in some cases, a theorem by Pronin and Treschev [12]. We remark that they studied the analytic case and applied a constructive method to obtain the Hamiltonian, a kind of averaging method, and they began from a non-periodic Hamiltonian. That is to say, the began from a homologous to the identity symplectomorphism, which is that we shall obtain. Moreover, they worked on a compact symplectic manifold, and we can apply their theorem in a relatively compact tubular neighbourhood of the zero-section, provided it is compact (say, a torus).

We also work in analytic set up, and our proof is also constructive. We remark that the only information that we need to obtain the Hamiltonian is the primitive function of our symplectomorphism and the vector field that interpolates the dynamics on the zero-section. If this dynamics is not included into a flow, then we conclude that our symplectomorphism is homologous to the corresponding lift.

2 Definitions

Though we shall work in the analytic category, the definitions can be done in the differentiable category. Let $\mathcal{M}$ be a real analytic manifold.

2.1 Exact symplectomorphisms

The *Liouville form* of the cotangent bundle $T^*\mathcal{M}$ of a manifold $\mathcal{M}$ is the Pfaffian form defined on each 'point' $\rho_x \in T^*\mathcal{M}$ and for any $X_{\rho_x} \in T_{\rho_x}T^*\mathcal{M}$ by

$$\alpha_{\rho_x}(X_{\rho_x}) = \rho_x(q_*(\rho_x)X_{\rho_x}).$$

Moreover, it is the unique Pfaffian form on $T^*\mathcal{M}$ which satisfies $\rho^*\alpha = \rho$ for any 1-form ρ on $\mathcal{M}$. Then, $\omega = \mathrm{d}\alpha$ is the *canonical symplectic structure* on $T^*\mathcal{M}$ (and α is an action form for ω). In cotangent coordinates $T^*\mathcal{M}$, $(x,y) = (x_1,\dots,x_d,y_1,\dots,y_d) \in \mathcal{U} \times \mathbb{R}^d$, these forms are $\alpha = y\ dx = \sum_{i=1}^d y_i\ dx_i,\ \omega = dy \wedge dx = \sum_{i=1}^d dy_i \wedge dx_i$.

Let $F : \mathcal{N} \to T^*\mathcal{M}$ be a diffeomorphism from a tubular neighbourhood of the zero-section, $\mathcal{N}$, onto its image. We shall refer to the first-order partial differential equation on $\mathcal{N}$

$$F^*\alpha - \alpha \quad = \quad \mathrm{d}S$$

as the *exactness equation* of F. If it is solvable, then we say that S is a *primitive function* of F, and hence F preserves the symplectic form. We say that F is an *exact symplectomorphism.*

Remark: If $\phi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism on the base $\mathcal{M}$ then its *lift* $\hat{\phi} \stackrel{\text{def}}{=} (\phi^{-1})^* : T^*\mathcal{M} \to T^*\mathcal{M}$ preserves not only the symplectic form $\omega = \mathrm{d}\alpha$ but also the Liouville form. All the *actionmorphisms* on the whole cotangent bundle can be obtained in this way (see, for instance, [8]). In fact, it is enough to be defined in a tubular neighbourhood of the zero-section.

In the literature, the primitive function is often called generating function. As we see, this function does not generate the symplectomorphism. This is the reason we have followed the nomenclature used in [2]. Anyway, the primitive function and the different types of generating functions are very close. $\triangleleft$

2.2 Hamiltonian flow.

It is well known that to a function $H : \mathcal{N} \to \mathbb{R}$ we associate a *Hamiltonian vector field* X_H, which is uniquely determined by $\mathbf{i}_{X_H}\omega = -\mathrm{d}H$. The *Poisson bracket* between two functions K,H is defined by $\{K, H\} = \omega(X_K, X_H) = -\mathrm{d}K(X_H) = \mathrm{d}H(X_K)$. These operations are natural with respect to pull back by symplectomorphisms, that is, for any symplectomorphism F: $F^*X_H \stackrel{\text{def}}{=} (F^{-1})_*X_H{\circ}F = X_{H\circ F}$, $\{K, H\}{\circ}F = \{K{\circ}F, H{\circ}F\}$.

In symplectic coordinates (x, y), we write

$$X_H = \begin{pmatrix} \dfrac{\partial H}{\partial y} & -\dfrac{\partial H}{\partial x} \end{pmatrix}^\top, \quad \{K, H\} = \frac{\partial K}{\partial y} \cdot \frac{\partial H}{\partial x} - \frac{\partial K}{\partial x} \cdot \frac{\partial H}{\partial y},$$

where $\cdot$ is the inner product.

Let H_t be a time-dependent Hamiltonian function (where the subscript t means the dependence on time), and X_{H_t} be the corresponding time-dependent Hamiltonian vector field. It is well known that the time-t flow from t_0, φ_{t,t_0}, is an exact symplectomorphism (for the sake of simplicity, we shall suppose completeness,that is, the flow is defined for all the values of t_0 and t). In fact, $\varphi^*_{t,t_0}\alpha - \alpha = \mathrm{d}S_{t,t_0}$, where

$$S_{t,t_0} = \int_{t_0}^{t} \mathbf{\Lambda}(H_s){\circ}\varphi_{s,t_0}\, ds$$

and $\mathbf{\Lambda}(H) = \alpha(X_H) - H$.

2.3 The Liouville derivative

On one hand, recall that the space of functions $\mathcal{F}(\mathcal{N})$ endowed with the Poisson bracket is a Lie algebra, and the relation between the Lie bracket and the Poisson bracket is given by $X_{\{K,H\}} = [X_K, X_H]$. On the other hand, we have defined an operator in this space, given by $\mathbf{\Lambda}(H) = \alpha(X_H) - H$.

Proposition 1 *The linear operator* $\mathbf{\Lambda}$ *associated to* α *is a derivation in the Lie algebra* $\mathcal{F}(\mathcal{N})$*. Moreover, it is natural with respect to pull back by actionmorphisms.*

Proof: The proof of the product rule is:

$$\begin{aligned} \mathbf{\Lambda}(\{H_1, H_2\}) &= \alpha([X_{H_1}, X_{H_2}]) - \{H_1, H_2\} = \mathrm{d}(\alpha(X_{H_2}))X_{H_1} - \mathbf{L}_{X_{H_1}}\alpha\ X_{H_2} - \{H_1, H_2\} \\ &= \{H_1, \alpha(X_{H_2})\} - \mathrm{d}(\mathbf{\Lambda}(H_1))X_{H_2} - \{H_1, H_2\} = \{H_1, \mathbf{\Lambda}(H_2)\} + \{\mathbf{\Lambda}(H_1), H_2\}. \end{aligned}$$

If $L^*\alpha = \alpha$ and $H \in \mathcal{F}(\mathcal{N})$: $\alpha(X_H){\circ}L = L^*(\mathbf{i}_{X_H}\alpha) = \mathbf{i}_{L^*X_H}L^*\alpha = \mathbf{i}_{X_{H\circ L}}\alpha = \alpha(X_{H\circ L})$. $\square$

In fact, to any action form of any exact symplectic form we can associate a derivation in the Lie algebra of functions. In our case, the derivation associated to the Liouville form is named the *Liouville derivative.* Moreover, $\mathbf{\Lambda}(H)$ is also known as the *elementary action* associated to the Hamiltonian H, because it is used in order to define a variational principle for its orbits (see, for instance, [1, 8]). It is very close to the *Legendre transformation.*

In cotangent coordinates $(x,y) \in \mathcal{U} \times \mathbb{R}^d$ the Liouville derivative is written

$$\mathbf{\Lambda}(H)(x,y) \quad = \quad y \cdot \nabla_y H(x,y) - H(x,y).$$

$\mathbf{\Lambda}$ is a vertical operator, because the value of $\mathbf{\Lambda}(H)$ on a fiber only depends on the value of H on such fiber.

3 Determination of an exact symplectomorphism.

As we have seen, an exact symplectomorphism $F : \mathcal{N} \to T^*\mathcal{M}$ is not determined by its primitive function S. In general, we can obtain all the exact symplectomorphisms with such primitive function composing F on the left with actionmorphisms L:

$$(L{\circ}F)^*\alpha - \alpha = F^*L^*\alpha - \alpha = F^*\alpha - \alpha = \mathrm{d}S.$$

So then, in order to answer the question

> What additional information do we need in order to determine an exact symplectomorphism from its primitive function?

we have to look for the actionmorphisms of our exact symplectic manifold, because

> An exact symplectomorphism is determined by its primitive function save an actionmorphism.

An actionmorphism defined in a tubular neighbourhood of the zero-section does not move the zero-section (see 4.3), and it is the lift of the diffeomorphism induced in the zero-section. In order to see this, it is enough to proof that the only actionmorphism $I : \mathcal{N} \to T^*\mathcal{M}$ that is the identity on the zero-section is the identity. We write the proof for completeness.

Proposition 2 *Let $I : \mathcal{N} \to T^*\mathcal{M}$ be an actionmorphism that fixes all the points of the zero-section. Then, I is the identity map.*

Proof: Take cotangent coordinates $(x,y) \in \mathcal{U} \times \mathbb{R}^d$ around each point of the zero-section $\{y = 0\}$. Hence, I is given by

$$\bar{x} = f(x,y) \ , \ \bar{y} = g(x,y),$$

with $f(x,0) = x, g(x,0) = 0$. The condition $I^*\alpha = \alpha$ is

$$0 = g(x,y)^\top A(x,y) - y^\top \ , \ 0 = g(x,y)^\top B(x,y),$$

where the matrices

$$A(x,y) = \frac{\partial f}{\partial x}(x,y) \ , \ B(x,y) = \frac{\partial f}{\partial y}(x,y),$$

satisfy $AB^\top = BA^\top$. Note that $g(x,0) = 0$, because the matrix $(A\ B)$ has maximum rank. Hence, we have

$$0 \quad = \quad (g(x,y)^\top A(x,y) - y^\top)B(x,y)^\top = -y^\top B(x,y)^\top,$$

and then each x-component f_i is a homogeneous function of degree 0 in the y-variables. By regularity on the zero-section, these functions f_i are constant with respect to the y-variables: $f(x,y) = f(x,0) = x$. Finally, $g(x,y) = y$, because A is the identity matrix. □

4 Proof of the theorem.

4.1 The homotopy method.

Let $H : \mathcal{N} \times \mathbb{R} \to \mathbb{R}$ be a time-dependent Hamiltonian function, X_{H_t} be the corresponding vector field and $\varphi_t = \varphi_{t,0}$ be the corresponding flow from $t_0 = 0$. We would like

$$\varphi_1^* \alpha - \alpha = \mathrm{d}S.$$

In fact, we impose $\forall t\ \varphi_t^* \alpha - \alpha = t\ \mathrm{d}S$ (this is the idea of a *homotopy method*). That is to say, we want that $S_{t,0} = t\ S$ (with the notation of 2.2). Hence, if we derive with respect to the time the previous homotopy formula we shall see that it is enough to impose $S = \mathbf{\Lambda}(H_t) \circ \varphi_t$. Therefore, if H_0 satisfies $S = \mathbf{\Lambda}(H_0)$ and

$$0 = \frac{d}{dt}(\mathbf{\Lambda}(H_t) \circ \varphi_t) = \mathrm{d}(\mathbf{\Lambda}(H_t))(\varphi_t) \frac{\partial \varphi_t}{\partial t} + \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t)) \circ \varphi_t = \{H_t, \mathbf{\Lambda}(H_t)\} \circ \varphi_t + \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t)) \circ \varphi_t,$$

then H_t is a time-dependent Hamiltonian whose time-1 flow is an exact symplectomorphism whose primitive function is S. We obtain the next *homotopy problem.*

Proposition 3 *Let $S : \mathcal{N} \to \mathbb{R}$ be a function. Then, the time-t flow $\varphi_t = \varphi_{t,0}$ of a time-dependent Hamiltonian $H_t : \mathcal{N} \to \mathbb{R}$ that satisfies $S = \mathbf{\Lambda}(H_0)$ and*

$$\frac{d}{dt}(\mathbf{\Lambda}(H_t)) \quad = \quad -\{H_t, \mathbf{\Lambda}(H_t)\},$$

is exact symplectic and its primitive function is $t\ S$.

4.2 Generating solutions from a particular one.

if we find a solution of the homotopy problem, we only can assure that the primitive function of its time-1 flow is S, but this does not determine F. This is related to the existence of functions whose $\mathbf{\Lambda}$-derivative vanish (the 'constants'). In fact, we can obtain many families of exact symplectomorphisms satisfying the same homotopy problem.

Proposition 4 *Let $S : \mathcal{N} \to \mathbb{R}$ be a function, $\bar{H}_t$ be a solution of the corresponding homotopy problem, and $\varphi_t = \varphi_{t,0}$ be the flow of the corresponding Hamiltonian vector field. Let $L_t = \hat{f}_t$ be a family of lifts generated (on the zero-section) by the flow $f_t = f_{t,0}$ of a vector field $X_t \in \mathcal{X}(\mathcal{M})$ on the zero-section. Consider the new family of exact symplectomorphisms $\psi_t = L_t \circ \varphi_t$, which is also connected to the identity. Then:*

- *$\psi_t = \psi_{t,0}$ is the flow of the Hamiltonian $H_t = h_t + \bar{H}_t \circ L_t^{-1}$, where h_t is the Hamiltonian lift of X_t;*
- *H_t is also a solution of the homotopy problem.*

Proof: The Hamiltonian lift of X_t is defined by $h_t(\rho_x) = \rho_x(X_t(x))$ (note that $h_t \in \ker(\mathbf{\Lambda})$). It is well known that its flow is just the lift of the flow on the zero section: $L_t = \hat{f}_t$. Then

$$\begin{aligned} \frac{d\psi_t}{dt} &= \frac{d}{dt}(L_t \circ \varphi_t) = \frac{\partial L_t}{\partial t} \circ \varphi_t + (L_t)_*(\varphi_t)\ \frac{d\varphi_t}{dt} = X_{h_t} \circ L_t \circ \varphi_t + ((L_t)_*\ X_{\bar{H}_t}) \circ \varphi_t \\ &= X_{h_t} \circ \psi_t + (L_t)_*\ X_{\bar{H}_t} \circ L_t^{-1} \circ \psi_t = X_{h_t} \circ \psi_t + X_{\bar{H}_t \circ L_t^{-1}} \circ \psi_t = X_{h_t + H_t \circ L_t^{-1}} \circ \psi_t. \end{aligned}$$

and the first point is proved (cf. [9]). The second one follows from

$$\begin{aligned}
\frac{d}{dt}(\mathbf{\Lambda}(H_t)) &= \frac{\partial}{\partial t}(\mathbf{\Lambda}(\bar{H}_t))\circ L_t^{-1} + \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t))(L_t^{-1})\ \frac{\partial L_t^{-1}}{\partial t} \\
&= -\{\bar{H}_t, \mathbf{\Lambda}(\bar{H}_t)\}\circ L_t^{-1} - \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t))(L_t^{-1})\ (L_t^{-1})_*\ X_{h_t} \\
&= -\{\bar{H}_t\circ L_t^{-1}, \mathbf{\Lambda}(\bar{H}_t\circ L_t^{-1})\} - \{h_t, \mathbf{\Lambda}(\bar{H}_t\circ L_t^{-1})\} \\
&= -\{H_t, \mathbf{\Lambda}(H_t)\}
\end{aligned}$$

and $\mathbf{\Lambda}(H_0) = \mathbf{\Lambda}(h_0 + \bar{H}_0\circ L_0^{-1}) = \mathbf{\Lambda}(\bar{H}_0) = S$. □

4.3 A condition on the primitive function.

Since the zero-section is fixed, we have some restrictions on the primitive function.

Proposition 5 *Let $F : \mathcal{N} \to T^*\mathcal{M}$ be an exact symplectomorphism and $S : \mathcal{N} \to \mathbb{R}$ be its primitive function. Then,*

$$F \textit{ fixes the zero-section } \Leftrightarrow \forall x \in \mathcal{M}\ \mathrm{d}S(0_x) = 0.$$

Proof: Let $x \in \mathcal{M}$ be any point on the zero-section. Since $\alpha(0_x) = 0$ and

$$(F^*\alpha)(0_x) = \alpha(F(0_x))\circ F_*(0_x) = F(0_x)\circ q_*(F(0_x))\circ F_*(0_x) = F(0_x)\circ(q\circ F)_*(0_x),$$

then

$$\mathrm{d}S(0_x) = F(0_x)\circ(q\circ F)_*(0_x).$$

Finally, as $(q\circ F)_*$ is an epimorphism in all points, we reach to the result. □

4.4 A splitting lemma

Lemma 1 *The space of functions $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{N})$ splits as*

$$\mathcal{F} = \ker\mathbf{\Lambda} \oplus \mathbf{\Lambda}(\mathcal{F}).$$

Moreover, $\mathbf{\Lambda}(\mathcal{F})$ is the subspace of functions whose vertical derivatives vanish on the zero-section and $\ker\mathbf{\Lambda}$ is the subspace of fiberwise homogeneous functions of degree 1.

Proof: We have to solve the equation $\mathbf{\Lambda}(H) = S$. Since $\mathbf{\Lambda}$ is a vertical operator, we can restrict our attention to each fiber, where is easy to work. On each fiber (fixed $x \in \mathcal{M}$) we have a linear operator transforming y-valued functions. Its properties are inherited by $\mathbf{\Lambda}$.

Hence, let $\mathcal{U} \subset \mathbb{R}^d$ be an open star-shaped neighbourhood of the origin in $\mathbb{R}^d$, with coordinates $y = (y_1, \ldots, y_d)$, and $S : \mathcal{U} \to \mathbb{R}$ be a function. We have to solve the p.d.e.

$$y \cdot \nabla_y H(y) - H(y) = S(y).$$

If $\nabla_y S(0) = 0$, its solutions are

$$H(y) = a \cdot y + \int_0^1 \frac{1}{t^2}(S(ty) - S(0))\ dt,$$

where $a \in \mathbb{R}^d$. □

4.5 An iterative method.

We can consider the homotopy problem as a family of evolution problems. We can specify a particular one, thanks to the previous lemma. It says that $\mathbf{\Lambda}_{|} \overset{\text{def}}{=} \mathbf{\Lambda}_{|\mathbf{\Lambda}(\mathcal{F})}$ is an isomorphism in $\mathbf{\Lambda}(\mathcal{F})$. If we look for a solution $\bar{H}_t \in \mathbf{\Lambda}(\mathcal{F})$, we define $S_t = \mathbf{\Lambda}(H_t) \in \mathbf{\Lambda}(\mathcal{F})$, and then we look for S_t as a solution of the *evolution problem*

$$\begin{cases} \dfrac{dS_t}{dt} = -\{\mathbf{\Lambda}_{|}^{-1}(S_t), S_t\}, \\ \\ S_0 = S. \end{cases}$$

If we solve this problem, then $\bar{H}_t = \mathbf{\Lambda}_{|}^{-1}(S_t)$ is a solution of the original one. Of course, it is necessary that $\mathbf{\Lambda}(\mathcal{F})$ be invariant under these operations to work inside this subspace. In fact, not only the vertical derivatives of S_0 vanish on the zero-section, but also *all* of them (recall that $\mathrm{d}S{\circ}z = 0$). Next lemma can be easily proved using cotangent coordinates.

Lemma 2 *Let $S, T \in \mathcal{F}(\mathcal{N})$ be two functions such that $\mathrm{d}S{\circ}z = 0$ and $\mathrm{d}T{\circ}z = 0$. Then,*

$$\mathrm{d}(\mathbf{\Lambda}_{|}^{-1}(S)){\circ}z = 0 \ , \ \mathrm{d}\{S, T\}{\circ}z = 0.$$

To obtain the solution we use expansions in powers of t. If $S_t = \sum_{k\geq 0} S_k t^k$ is the expansion of S_t (where $S_0 = S$), then we can compute all the terms by the recurrence

$$S_{k+1} = \frac{-1}{k+1} \sum_{u+v=k} \{\mathbf{\Lambda}_{|}^{-1}(S_u), S_v\}.$$

Hence, all the terms of the expansion verify $\mathrm{d}S_k{\circ}z = 0$. In fact, as the function $S_0 = S$ has y-order 2, the y-orders of the S_k increase: the y-order of S_k is $k+2$. This is the key point in order to prove the convergence of the expansions.

4.6 Proof of the convergence of the expansions.

We want the series be analytic in the 'spatial' variables, at least until a time $t > 1$ in a neighbourhood of the zero-section. Now, the analysis is local, and we shall prove it using majorant estimates.

Recall that for any two functions $f(z)$, $g(z)$ $(z = (z_1, \dots, z_m))$ analytic at $z = 0$:

$$f(z) = \sum_n f_n z^n, \ g(z) = \sum_n g_n z^n$$

(using multi-index notation), we say that g is a majorant for f ($f \ll g$) iff $\forall n \ |f_n| \leq g_n$.

Lemma 3 *The relation $\ll$ satisfies the following properties:*

1. $f_1 \ll g_1, \ f_2 \ll g_2 \ \Rightarrow \ f_1 + f_2 \ll g_1 + g_2, \ f_1 g_1 \ll g_1 g_2$;
2. $f \ll g \ \Rightarrow \ \dfrac{\partial f}{\partial z_i} \ll \dfrac{\partial g}{\partial z_i} \ (i = 1 \div m)$;
3. $\forall t \in [a, b] \ f_t \ll g_t \ \Rightarrow \ \displaystyle\int_a^b f_t(z)dt \ll \int_a^b g_t(z)dt.$

Let w_b be the product $w_b(z) = \prod_{i=1}^m (b - z_i)$, where $b > 0$. Hence:

4. $\forall i,k = 1 \div m$

$$1 \ll \frac{b}{b-z_i}\ ,\ \frac{z_i}{w_b} \ll \frac{b}{w_b}\ ,\ \frac{1}{(b-z_1)\dots(b-z_k)} \ll \frac{b^{m-k}}{w_b};$$

5. $\forall z \mid \|z\|_\infty < b,\ |f(z)| < c,\ \Rightarrow\ f \ll \dfrac{cb^m}{w_b}.$

We complete the previous rather standard lemma with a more specific one.

Lemma 4 *Take $m = 2d, z = (x_1,\dots,x_d,y_1,\dots,y_d), w(z) = \prod_{i=1}^d (b-x_i)(b-y_i)$. Then:*

6. $f \ll \prod_{t=1}^u y_{i_t} w^{-k}, g \ll \prod_{t=1}^v y_{j_t} w^{-l}\ (u>0, v>0) \Rightarrow$

$$\{f,g\} \ll b^{2d-1} \left(\begin{array}{c} c_{uvkl} \prod_{t=1}^u y_{i_t} \cdot \sum_{s=1}^v \prod_{t=1,t\neq s}^v y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^v y_{j_t} \cdot \sum_{s=1}^u \prod_{t=1,t\neq s}^u y_{i_t} \end{array} \right) w^{-(1+k+l)},$$

where $c_{uvkl} = k + \frac{2dkl}{u+v}$;

7. $f \ll \prod_{t=1}^u y_{i_t} w^{-k}\ (u>1) \Rightarrow \mathbf{\Lambda}_{|}^{-1}(f) \ll \dfrac{1}{u-1} \prod_{t=1}^u y_{i_t} w^{-k}.$

Proof:

6. Since

$$\sum_{j=1}^d \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} \ll \sum_{j=1}^d \left(\prod_{t=1}^u y_{i_t} \frac{kw^{-k}}{b-x_j} \left(\frac{\partial}{\partial y_j} \left(\prod_{t=1}^v y_{j_t} \right) w^{-l} + \prod_{t=1}^v y_{j_t} \frac{lw^{-l}}{b-y_j} \right) \right)$$

and

$$\sum_{j=1}^d \frac{\partial}{\partial y_j} \left(\prod_{t=1}^v y_{j_t} \right) = \sum_{s=1}^v \prod_{t=1,t\neq s}^v y_{j_t},$$

then

$$\{f,g\} \ll b^{2d-1} \left(\left(\begin{array}{c} l \prod_{t=1}^v y_{j_t} \cdot \sum_{s=1}^u \prod_{t=1,t\neq s}^u y_{i_t} \\ + \\ k \prod_{t=1}^u y_{i_t} \cdot \sum_{s=1}^v \prod_{t=1,t\neq s}^v y_{j_t} \end{array} \right) + \frac{2dkl}{b} \prod_{t=1}^u y_{i_t} \prod_{t=1}^v y_{j_t} \right) w^{-(1+k+l)}$$

$$\ll b^{2d-1} \left(\begin{array}{c} c_{uvkl} \prod_{t=1}^u y_{i_t} \cdot \sum_{s=1}^v \prod_{t=1,t\neq s}^v y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^v y_{j_t} \cdot \sum_{s=1}^u \prod_{t=1,t\neq s}^u y_{i_t} \end{array} \right) w^{-(1+k+l)}.$$

7. Finally,

$$\mathbf{\Lambda}_{|}^{-1}(f)=\int_0^1 t^{-2}f(x,ty)dt \ll \int_0^1 t^{-2}\prod_{s=1}^{u} y_{i_s}t^u w^{-k}(x,y)dt=\frac{1}{u-1}\prod_{s=1}^{u} y_{i_s}w^{-k}.$$

□

Finally, we are going to prove the convergence of the expansions until a large enough time.

Proposition 6 *Let $S:\mathcal{N}\to\mathbb{R}$ be a real analytic function, with $\mathrm{d}S\circ z=0$. Then, there exists a tubular neighbourhood of the zero-section where the solution of the evolution problem S_t is defined until a time $t>1$.*

Proof: We can use cotangent coordinates (x,y) in a neighbourhood of each point of the zero-section. It is sufficient to prove that we can get a small neighbourhood of zero where the series $\sum_{k\geq 0} S_k(x,y)t^k$ is defined for $t<T$ and $T\geq 1$.

$S(x,0)$ is constant, and we suppose that this constant is 0. Moreover, since $\forall x\ \frac{\partial S}{\partial y}(x,0)=0$ we can write

$$S(x,y)=\sum_{i,j} y_i y_j s_{ij}(x,y),$$

where de functions s_{ij} are real analytic (and $s_{ij}=s_{ji}$). Given a small enough radius $b>0$, let c be the maximum of the sup-norms of the functions s_{ij} on $\{\|(x,y)\|_\infty<b\}$. So then, $\forall i,j=1\div d\ s_{ij}\ll cb^{2d}w^{-1}$, where $w(x,y)=\prod_{i=1}^{d}(b-x_i)(b-y_i)$, and

$$S_0 \ll cb^{2d}\sum_{i_1,i_2} y_{i_1}y_{i_2}\ w^{-1},$$

Suppose that $\forall u\leq n$

$$S_u \ll \gamma_u \sum_{i_1,\ldots,i_{u+2}} \prod_{t=1}^{u+2} y_{i_t}\ w^{-(2u+1)},$$

where $\gamma_0=cb^{2d},\gamma_1,\ldots,\gamma_n$ are constants. We want to estimate S_{n+1}. So then, applying the majorant estimates of the previous lemma, we obtain

$$\begin{aligned} S_{n+1} &= \frac{-1}{n+1}\sum_{u+v=n}\{\mathbf{\Lambda}_{|}^{-1}(S_u),S_v\} \\ &\ll \frac{b^{2d-1}}{n+1}\sum_{\substack{u+v=n\\ i_1,\ldots,i_{u+2}\\ j_1,\ldots,j_{v+2}}}\frac{\gamma_u\gamma_v}{u+1}\begin{pmatrix} \hat{c}_{uv}\prod_{t=1}^{u+2} y_{i_t}\cdot\sum_{s=1}^{v+2}\prod_{t=1,t\neq s}^{v+2} y_{j_t} \\ + \\ \hat{c}_{vu}\prod_{t=1}^{v+2} y_{j_t}\cdot\sum_{s=1}^{u+2}\prod_{t=1,t\neq s}^{u+2} y_{i_t}\end{pmatrix} w^{-(2n+3)}, \end{aligned}$$

where $\hat{c}_{uv}=c_{(u+2),(v+2),(2u+1),(2v+1)}=(2u+1)+\frac{2d(2u+1)(2v+1)}{u+v+4}$. Since

$$\sum_{\substack{i_1,\ldots,i_{u+2}\\ j_1,\ldots,j_{v+2}}}\left(\prod_{t=1}^{u+2} y_{i_t}\cdot\sum_{s=1}^{v+2}\prod_{t=1,t\neq s}^{v+2} y_{j_t}\right) = d(v+2)\sum_{k_1,\ldots,k_{n+3}}\prod_{t=1}^{n+3} y_{k_t},$$

we reach to

$$S_{n+1} \ll \gamma_{n+1} \sum_{k_1,\ldots,k_{n+3}} \prod_{t=1}^{n+3} y_{k_t} \; w^{-(2n+3)},$$

where

$$\frac{db^{2d-1}}{n+1} \sum_{u+v=n} \frac{\gamma_u \gamma_v}{u+1} (\hat{c}_{uv}(v+2) + \hat{c}_{vu}(u+2)) \leq 4d(1+2d)b^{2d-1} \sum_{u+v=n} \gamma_u \gamma_v \stackrel{\text{def}}{=} \gamma_{n+1}.$$

Hence, we have majored the sequence of S_n by

$$S_n \ll \gamma_n \sum_{i_1,\ldots,i_{n+2}} \prod_{t=1}^{n+2} y_{i_t} \; w^{-(2n+1)},$$

where the sequence $\{\gamma_n\}$ is defined by

$$\begin{cases} \gamma_0 = cb^{2d}, \\ \gamma_{n+1} = K \displaystyle\sum_{u+v=n} \gamma_u \gamma_v, \end{cases}$$

where $K = 4d(1+2d)b^{2d-1}$. They are the coefficients of the Taylor series of the function

$$f(t) = \frac{1 - \sqrt{1 - 4K\gamma_0 t}}{2Kt},$$

and then $\lim_n \frac{\gamma_n}{\gamma_{n+1}} = \frac{1}{4K\gamma_0}$.

Finally, let $\rho \in [0, 1[$ be a ratio we shall choose later. If $\|x\|_\infty \leq \rho b$ and $\|y\|_\infty \leq \rho b$, then

$$|S_n(x,y)| \leq \gamma_n (d\rho b)^{n+2} (b(1-\rho))^{-2d(2n+1)} \stackrel{\text{def}}{=} \beta_n.$$

Therefore, we have bounded all the terms of the expansion in a domain of x, y:

$$\sum_{n \geq 0} |S_n(x,y)| t^n \leq \sum_{n \geq 0} \beta_n t^n.$$

Then, as

$$\lim_n \frac{\beta_n}{\beta_{n+1}} = \frac{b^{4d-1}(1-\rho)^{4d}}{4K\gamma_0 d\rho},$$

the convergence radius of $\sum_{n \geq 0} \beta_n t^n$ is greater than 1 provided that ρ is small enough. □

4.7 End of proof of the Theorem.

Let S_t be the solution of the evolution problem, that belongs to $\mathbf{\Lambda}(\mathcal{F})$. Then $\bar{H}_t = \mathbf{\Lambda}_{|}^{-1}(S_t)$ is a Hamiltonian whose flow $\varphi_t = \varphi_{t,0}$ has primitive function $t\, S$. Since $\mathrm{d}S_t \circ z = 0$ then $\mathrm{d}\bar{H}_t \circ z = 0$, and all the points of the zero-section are fixed. Hence, φ_1 is *the* exact symplectomorphism whose primitive function is S and fixes all the points of the zero-section.

Finally, if the dynamics *on* the zero-section can be interpolated by a flow, then the dynamics *around* the zero-section can be interpolated by a Hamiltonian flow. It is enough to lift the dynamics on the zero-section and apply 4.2, 4.5.

Conclusion

As we have seen, the primitive function of an exact symplectomorphism gives us some information about it, but not all the information. In order to obtain all the information we need, from a geometrical point of view, where and how our symplectomorphism sends an exact Lagrangian manifold.

The primitive function can be useful from a methodological point of view, because it let us obtain *all* the symplectic dynamics around exact Lagrangian manifolds. Dynamics which are not necessarily generated by generating functions. See the appendix to make effective computations of these dynamics.

As a corollary of our results, the dynamics of a symplectomorphism around an invariant torus whose dynamics is conjugated to an ergodic translation is homologous to the identity, and the Hamiltonian can be chosen periodic in time [12].

Appendix
Effective computation.

Assume that our manifold $\mathcal{M}$ is $\mathbb{R}^d$ and S is a function on $T^*\mathbb{R} \simeq \mathbb{R}^d \times \mathbb{R}^d$ that expands

$$S(x,y) = \sum_n s_n(x) y^n,$$

where the s_n are x-functions (we use multi-indices $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$). We look for the only symplectomorphism $F = (f, g)$ that fixes all the points of the zero-section $\{y = 0\}$ and whose primitive function is just S. We expand f and g by

$$f(x,y) = \sum_n f_n(x) y^n \ , \ g(x,y) = \sum_n g_n(x) y^n,$$

where $f_n = (f_n^1, \ldots, f_n^d)^\top$, $g_n = (g_n^1, \ldots, g_n^d)^\top$ are vector functions, being $f_0 = x$ and $g_0 = 0$.

We equate the terms of the same y-order in the exactness equation [5] for details). Order zero gives that s_0 is constant and each s_{e_i} vanishes: $\mathrm{D}S(x,0) = 0$ (we already knew this). Then, we obtain the next recurrence for the x-functions f_n and g_n (where $\sum_i$ means $\sum_{i=1}^d$, $u, v \in \mathbb{N}^d$ are multi-indices, $|n| = n_1 + \ldots + n_d$ and terms with 'wrong' multi-indices are taken zero):

- (Step 1) $\forall i, j = 1 \div d \ \ g_{e_i}^j = \delta_{ij} \ , \ f_{e_i}^j = (1 + \delta_{ij}) \ s_{e_i + e_j}$.
- (Step k) $\forall |n| = k, \ \forall j = 1 \div d \ \ g_n^j = G_n^j \ , \ f_n^j = \frac{n_j + 1}{k} \sum_i F_{n+e_j-e_i}^i - F_n^j$, where

$$\begin{aligned} G_n^j &= \frac{\partial s_n}{\partial x_j}(x) - \sum_i \sum_{\substack{u+v=n \\ u \neq 0, n}} \frac{\partial f_u^i}{\partial x_j}(x) g_v^i(x), \\ F_n^j &= (n_j + 1) s_{n+e_j}(x) - \sum_i \sum_{\substack{u+v=n \\ |v| > 1}} (u_j + 1) f_{u+e_j}^i(x) g_v^i(x). \end{aligned}$$

Remarks:

i) If $\mathcal{M} = \mathbb{T}^d$ the functions s_n, f_n and g_n 1-periodic in all their variables (save $f_0(x) = x$) and, hence, can be expanded in Fourier series. An example is that the dynamics on the zero-section $\mathbb{T}^d \times \{0\}$ is an ergodic shift $\phi(x) = x + \omega$, where $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

ii) This method does not uses the implicit function theorem, as when one tries to obtain a symplectomorphism from a generating function. Moreover, if the primitive function is real analytic, the expansions of f and g also converge. $\triangleleft$

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