Resolution of Semilinear Equations by Fixed Point Methods

P. Amster and M. C. Mariani

ABSTRACT: *We give conditions in order to obtain solutions of quasilinear systems with periodic type conditions. Our main tool will be the use of fixed point theorems.*

Introduction

In this work we will study some special cases of ordinary semilinear differential equations of the type $X' = F(t, X)$, with boundary conditions $X(0) = g(X(\alpha))$.

The periodic problem may be regarded as a particular case, when $g = I$. In this case under some conditions and F Lipschitzian it is possible to obtain solutions by finding a fixed point of the Poincaré operator (see e.g. [8]).

Existence and uniqueness results for second order differential equations and systems with periodic conditions are given in [1], [2], [3], [4], [5], [6], [8], [9].

Existence by Fixed Point methods

We'll study the system

$$(2) \begin{cases} X' = F(t, X) & \text{in} \quad (0, \alpha) \\ X(0) = g(X(\alpha)) \end{cases}$$

where $F : [0, \alpha] \times R^m \longrightarrow R^m$ and $g : R^m \longrightarrow R^m$ are continuous.

Let us define

$$F_M = \sup_{t\in[0,\alpha],|x|\leq M} |F(t, x)|$$

$$g_M = \sup_{|x|\leq M} |g(x)|$$

B_M will denote the closed ball of radius M centered in 0 in the space $C([0,\alpha], R^m)$.

THEOREM 1

If $\dfrac{g_M}{M} + \alpha\dfrac{F_M}{M} \leq 1$ *then (2) admits a solution in* B_M. *Furthermore, if* F *and* g *are Lipschitz with constants* K_F *and* K_g, $K_g + K_F\alpha < 1$, *(2) has a unique solution.*

<u>*Proof*</u>

We consider the continuous operator

$$TX(t) = g(X(\alpha)) + \int_0^t F(s, X(s))ds \tag{3}$$

If $\|X\|_\infty \leq M$ then

$$|g(X(\alpha)) + \int_0^t F(s, X(s))ds| \leq g_M + \alpha F_M$$

On the other hand

$$|TX(t_1) - TX(t_2)| \leq |t_2 - t_1| F_M$$

By Arzela-Ascoli, we conclude that T is compact, and being $\dfrac{g_M}{M} + \alpha\dfrac{F_M}{M} \leq 1$, $T(B_M) \subset (B_M)$. By Schauder Theorem (see e.g. [7]), we conclude that T has a fixed point in B_M.

When F and g are Lipschitz with constant K_F, K_g, T is a contraction for $K_g + K_F\alpha < 1$.

REMARKS

i) If g *has continuous inverse, we may consider the operator*

$$TX(t) = g^{-1}(X(0)) - \int_t^\alpha F(s, X(s))ds$$

and get solutions of (2) under the same conditions of Theorem 1 for g^{-1}. *In particular, if* g *is a linear isomorphism,* $\|g\| \neq 1$, *the system (2) admits a solution in* B_M *when* $\|g\| + \alpha\dfrac{F_M}{M} \leq 1$ *or* $\|g^{-1}\| + \alpha\dfrac{F_M}{M} \leq 1$.

ii) For constant g, *Theorem 1 gives a proof of the well known existence result for ordinary equations with Cauchy data.*

For linear g, existence may be obtained from a different operator if $I - g$ is invertible:

Theorem 2

Let g be linear such that $I - g$ is invertible. We consider $G = B + \varphi I$, where $B = (I-g)^{-1}g$ and φ is defined by

$$\varphi(t,s) = \begin{cases} 1 & \text{if } t \geq s \\ 0 & \text{if } t < s \end{cases}$$

and assume, for a certain M, that $\int_0^\alpha |G(t,s)|ds\frac{F_M}{M} \leq 1$ for all t. Then (2) admits a solution in B_M.

Furthermore, if F is Lipschitz with constant K, $\int_0^\alpha |G(t,s)|dsK < 1$, (2) has a unique solution.

<u>*Proof*</u>

For any $X \in C([0,\alpha], R^m)$, we define

$$X_0 = \int_0^\alpha BF(s,X)ds$$

and

$$TX(t) = X_0 + \int_0^t F(s,X)ds = \int_0^\alpha G(t,s)F(s,X)ds$$

As in the previous theorem, T is compact and, for $\|X\|_\infty \leq M$,

$$\|TX\|_\infty \leq \int_0^\alpha |G(t,s)|dsF_M \leq M$$

By Schauder Theorem, T has a fixed point in B_M.

Moreover, if F is Lipschitz, T is a contraction.

As simple consequence we obtain the following result for $g = kI$, improving Theorem 1 when $k \leq 0$:

Corollary 3

Let $k \neq 1$, $g = kI$, and $c = \inf_{M>0} \frac{F_M}{M}$. Then the problem (2) admits a solution in $C([0,\alpha], R^m)$ in the following cases:

i) $|k| \geq 1$, $c\alpha < \dfrac{k-1}{k}$

ii) $|k| < 1$, $c\alpha < 1 - k$

In particular, if $\dfrac{F_M}{M} \longrightarrow 0$, then for any $k \neq 1$ (2) admits a solution in $C([0,\alpha], R^m)$.

Proof

It is immediate in this case that

$$G(t,s)=\begin{cases}\frac{1}{1-k} & \text{if } t\geq s\\ \frac{k}{1-k} & \text{if } t<s\end{cases}$$

and a simple computation shows that

$$\int_0^\alpha |G(t,s)|ds \leq \frac{k}{k-1} \qquad \text{if } |k|\geq 1$$

y

$$\int_0^\alpha |G(t,s)|ds \leq \frac{1}{1-k} \qquad \text{if } |k|< 1$$

REMARK

For $n>1$ *assumption* $\frac{F_M}{M}\longrightarrow 0$ *is not appliable to the equation* $u^{(n)}=f(t,u,...,u^{(n-1)})$.

For the periodic problem, which is not contemplated in the results above, we have the following criteria:

THEOREM 4 *If* X_n *is a bounded sequence in* $C([0,\alpha],R^m)$ *such that*

$$\begin{cases}X_n'=F_n(t,X_n) & in \quad (0,\alpha)\\ X_n(0)=g_n(X_n(\alpha_n))\end{cases}$$

with linear g_n, *and continuous* F_n *such that* $g_n\longrightarrow I$, $F_n\longrightarrow F$, *and* $\alpha_n\longrightarrow\alpha$ ($\alpha_n\leq\alpha$). *Then the periodic problem admits a solution in* $C([0,\alpha],R^m)$.

Proof

We consider the same operator as in (3) for $g=I$, then

$$(TX_n)'=F(t,X_n)=F(t,X_n)-F_n(t,X_n)+X_n'$$

and

$$(TX_n-X_n)(t)=(TX_n-X_n)(0)+\int_0^t F(s,X_n)-F_n(s,X_n)$$

Being T compact we may suppose that $TX_n\longrightarrow X$. Moreover, taking K compact big enough, we obtain:

$$|\int_0^t F(s,X_n)-F_n(s,X_n)|\leq\alpha\|F-F_n\|_{\infty,K}\longrightarrow 0$$

and

$$(TX_n - X_n)(0) = X_n(\alpha) - g_n(X_n(\alpha_n)) = (I - g_n)(X_n(\alpha)) + g_n(X_n(\alpha) - X_n(\alpha_n)) \longrightarrow 0$$

since X_n is bounded, $I - g_n \longrightarrow 0$ and $X_n(\alpha) - X_n(\alpha_n) = \int_{\alpha_n}^{\alpha} F_n(s, X_n) \longrightarrow 0$.

Then $X_n \longrightarrow X$, and X is a fixed point of T.

Theorem 5

Let us assume that the system

$$(4_r)\begin{cases} X' = \dfrac{1}{r}F(t,X) & \text{in} \quad (0,\alpha) \\ X(0) = \dfrac{1}{r}X(\alpha). \end{cases}$$

has no solution in ∂B_M for any $r \in (1, 1+\alpha\frac{F_M}{M}]$. Then the periodic problem (r=1) admits a solution in B_M.

<u>*Proof*</u>

We consider the same compact operator as in (3) for $g = I$, and define

$$T^*X = \begin{cases} TX & \text{if } \|TX\|_\infty \le M \\ \dfrac{MTX}{\|TX\|_\infty} & \text{if } \|TX\|_\infty \ge M. \end{cases}$$

$T^* : B_M \longrightarrow B_M$ is compact and in consequence it has a fixed point X. If X is not a fixed point of T, then $\|X\|_\infty = M$ and $TX = rX$, with

$$r = \frac{\|TX\|_\infty}{M}$$

Then the system (4_r) has a solution in ∂B_M, and $1 < r \le 1 + \alpha\frac{F_M}{M}$.

Example

Let F be continuous with $F(t,x).x < 0$ for any $(t,x) \in [0,\alpha] \times R^m$ such that $|x| = M$. Then any solution of (4_r) verifies that $\frac{1}{2}(X.X)' = X'.X = \frac{1}{r}F(t,X).X < 0$ when $|X(t)|$ is close to M. We conclude that if $|X(0)| < M$ then $\|X\|_\infty < M$. On the other hand, if $|X(0)| = M$, $|X(\alpha)| = r|X(0)| > M$. By theorem 5, the periodic problem admits a solution in B_M.

Uniqueness for the problem (2)

In theorems 1 and 2 we obtained uniqueness for problem (2) when F and g are Lipschitz with small constants. Now we'll prove uniqueness under some other assumptions:

THEOREM 6

Let g be linear and F continuously differentiable with respect to X and for every $(t,x) \in (0,\alpha) \times R^m$ let $A(t,x)$ denote the matrix $D_x F(t,x)$. Then (2) has at most one solution in any of the following cases:

i) $\|g\| < 1$ y $A(t,x) \leq 0$ for any $(t,x) \in (0,\alpha) \times R^m$.

ii) g invertible, $\|g^{-1}\| < 1$ y $A(t,x) \geq 0$ for any $(t,x) \in (0,\alpha) \times R^m$.

iii) g isometric, $A(t,x) > 0$ (or $A(t,x) < 0$) for any $(t,x) \in\in (0,\alpha) \times R^m$, $x \neq 0$.

<u>*Proof*</u>

Let us suppose that X and Y are solutions of (2) and take $Z = Y - X$. Then, $Z'.Z = (F(t,Y) - F(t,X))Z$, and applying for fixed t mean value theorem to $\varphi(u) = F(t, uY + (1-u)X)Z$ we see that

$$(Z.Z)' = 2A(t,\psi)Z.Z$$

for a certain $\psi(t)$.

Assuming i) we obtain that $|Z| = (Z.Z)^{1/2}$ decreases in $(0,\alpha)$, and the result follows since $|Z(0)| = |g(Z(\alpha))| < |Z(\alpha)|$. Under condition ii) the proof is analogous, considering $Z(\alpha) = g^{-1}(Z(0))$. If we assume iii), we obtain that $|Z|$ is monotone, and if $Z(t_0) \neq 0$ then $|Z|$ is strictly monotone in a neighborhood of t_0, a contradiction.

ACKNOWLEDGEMENT

The authors thank specially Prof. J. Mawhin for his careful reading of the manuscript and his fruitful suggestions and remarks.

References

[1] S. Ahmad: An existence theorem for periodically pertubed conservative systems, Michigan Math.J. 20 (1974), 385-392.

[2] S.Ahmad, J.Salazar: On existence of periodic solutions for nonlinearly perturbed conservative systems. Differential Equations, pp. 103-114, Academic Press, Orlando, FL (1980)

[3] P.W.Bates: Solutions of nonlinear elliptic systems with meshed spectra, J.Nonlinear Anal. 4 (1980), 1023-1030

[4] K.J.Brown, S.S.Lin: Periodically perturbed conservative systems and a global inverse function theorem. J.Nonlinear Anal. 4 (1980), 193-201

[5] A.Fonda, J.Mawhin:Iterative and Variational Methods for the Solvability of Some Semilinear Equations in Hilbert Spaces. Journal of Differential Equations, Vol 98, No. 2 (1992).

[6] A. C. Lazer: Application of a lemma on bilinear forms to a problem in nonlinear oscillations, Proc. Amer. Math. Soc. 33 (1972) 89-94.

[7] N.G.Lloyd: Degree Theory, Cambridge University Press, 1978.

[8] J.Mawhin, Continuation theorems and periodic solutions of ordinary differential equations. Recherches de mathématique 44 (1994), Inst. de Math Pure et Apliquée, Univ.Cath.de Louvain. Prepublication

[9] S.Tersian: On a class of abstract systems without resonance in a Hilbert Space. J.Nonlinear Anal. 6 (1982), 703-710

P.Amster

Dpto. de Matemática Fac. de Cs. Exactas y Naturales, UBA Pab. I, Ciudad Universitaria (1428) Capital, Argentina

CONICET

M. C. Mariani

Dpto. de Matemática Fac. de Cs. Exactas y Naturales, UBA Pab. I, Ciudad Universitaria (1428) Capital, Argentina

CONICET

Address for correspondence: Prof. M. C. Mariani, Dpto. de Matemática Fac. de Cs. Exactas y Naturales, UBA Pab. I, Ciudad Universitaria (1428) Capital, Argentina

E-mail: mcmarian@dm.uba.ar