ON ABSENCE OF EMBEDDED EIGENVALUES FOR SCHRÖDINGER OPERATORS WITH PERTURBED PERIODIC POTENTIALS

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Abstract

The problem of absence of eigenvalues imbedded into the continuous spectrum is considered for a Schrödinger operator with a periodic potential perturbed by a sufficiently fast decaying "impurity" potential. Results of this type have previously been known for the one-dimensional case only. Absence of embedded eigenvalues is shown in dimensions two and three if the corresponding Fermi surface is irreducible modulo natural symmetries. It is conjectured that all periodic potentials satisfy this condition. Separable periodic potentials satisfy it, and hence in dimensions two and three Schrödinger operator with a separable periodic potential perturbed by a sufficiently fast decaying "impurity" potential has no embedded eigenvalues.

1991 Subject Classification: 35P99, 47A55, 47F05

Keywords: Schrödinger operator, periodic potential, embedded eigenvalues

1 Introduction

Consider the stationary Schrödinger operator

$$H_0 = -\Delta + q(x) \tag{1}$$

in $L^2(\mathbf{R^n})$ ($n = 2$ or 3), where the real potential $q(x) \in L^\infty(\mathbf{R^n})$ is periodic with respect to the integer lattice $\mathbf{Z^n}$: $q(x+l) = q(x)$ for all $l \in \mathbf{Z^n}$, $x \in \mathbf{R^n}$. The spectrum of this operator is absolutely continuous and has the well known band-gap structure (see [10], [13], [20], [29], [32], [36], [38]):

$$\sigma(H_0) = \underset{i\geq 1}{\cup} [a_i, b_i],$$

where $a_i < b_i$, and $\lim_{i\to\infty} a_i = \infty$. Let us introduce also a perturbed operator with an impurity potential $v(x)$:

$$H = H_0 + v(x) = -\Delta + q(x) + v(x), \tag{2}$$

where $v(x)$ is compactly supported or sufficiently fast decaying at infinity (the exact conditions will be introduced later). It is known (see [5], Section 18 in [15], and references therein) that the continuous spectrum of H coincides with the spectrum of H_0, and only some additional "impurity" point spectrum $\{\lambda_j\}$ can arise. A general understanding is that eigenvalues λ_j should normally arise only in the gaps of the continuous spectrum (i.e. in the gaps of $\sigma(H_0)$). In other words, the case of embedded eigenvalues when one of the eigenvalues λ_j belongs to one of the open segments (a_i, b_i) is prohibited (at least for a sufficiently fast decaying impurity potential $v(x)$). Physically speaking, the existence of an embedded eigenvalue means a strange situation when an electron is confined, in spite of having enough energy for propagation. There are known examples of embedded eigenvalues, first of which was suggested by J. von Neumann and E. Wigner (see [11] and section XIII.13 in [32] for discussion of this topic and related references). The problem of the absence of embedded eigenvalues has been intensively studied. There are many known results on non-existence of embedded eigenvalues for the case of zero underlying potential $q(x)$ (see, for instance, books [11], [18] Section 14.7, and [32] Section XIII.13). The case of a periodic potential $q(x)$ is much less studied. In [11] a one-dimensional example is provided where a not very fast decaying perturbation of a periodic potential creates embedded eigenvalues. There are many papers devoted to studying the behavior of the point spectrum in the gaps of the continuous spectrum (see, for instance, [1] - [6], [9], [14], [17], [23]-[25], [31], and [33] - [35]). However, apparently the only known result on the absence of embedded eigenvalues relates to the one-dimensional case of the Hill's operator

$$H_0 = -\frac{d^2}{dx^2} + q(x) + \nu(x).$$

(see [33], [34]). The purpose of this paper is to address the multi-dimensional case. Section 2 contains some necessary notions, auxiliary information, and our main condition on the periodic potential. It is conjectured that the condition is always satisfied. According to [3], this condition is satisfied at least when the potential is separable in 2D or of the form $q_1(x_1)+q_2(x_2,x_3)$ in 3D. In Section 3 we prove a conditional statement (in dimension less than four) on the absence of embedded eigenvalues for a periodic potential satisfying this condition. Finally, the Section 4 contains the main unconditional result. In our considerations we follow an approach that was suggested long ago by the second author for the case of zero underlying potential $q(x)$ (see [37]). Some recent developments made it possible to adjust this method to the periodic case. In particular, results on allowed rate of decay of solutions obtained in [12] and [27] play a crucial role.

A more limited version of the main result was announced by the authors without proof in [21].

2 Bloch and Fermi varieties

Let us introduce some notions and notations. We assume that $q(x) \in L^\infty(\mathbf{R^n})$ is a periodic potential.

Definition 1 *The (complex)* ***Bloch variety*** *$B(q)$ of the potential q consists of all pairs $(k,\lambda) \in \mathbf{C^{n+1}}$ (where $k \in \mathbf{C^n}$ is a* ***quasimomentum*** *and $\lambda \in \mathbf{C}$ is an eigenvalue) for which there exists a non-zero solution of the equation*

$$H_0 u = \lambda u \tag{3}$$

satisfying the so called Floquet-Bloch condition:

$$u(x+l) = e^{ik\cdot l}u(x),\ l \in \mathbf{Z^n},\ \mathbf{x} \in \mathbf{R^n}. \tag{4}$$

Here $k \cdot l = \sum k_i l_i$ is the standard dot-product.

Definition 2 *The projection $F_c(q)$ onto $\mathbf{C^n}$ of the intersection of the Bloch variety $B(q)$ with the subspace $\lambda = c$ is called the (complex)* ***Fermi variety*** *of the potential q at the level of energy c:*

$$F_c(q) = \{k \in \mathbf{C^n} |\ (\mathbf{k},\mathbf{c}) \in \mathbf{B(q)}\}.$$

In other words, $F_c(q)$ consists of all quasimomenta $k \in \mathbf{C^n}$ for which there exists a non-zero solution of the equation

$$H_0 u = cu$$

satisfying (4).

When $c = 0$ we will use the notation $F(q)$ for $F_0(q)$. Let us notice the following obvious relation between the Bloch and Fermi varieties:

$$\{(k,c) \in B(q)\} \Longleftrightarrow \{k \in F_c(q)\} \Longleftrightarrow \{k \in F(q-c)\}. \tag{5}$$

We will use the following notations for the real parts of the above varieties:

$$B_{\mathbf{R}}(q) = B(q) \cap \mathbf{R^{n+1}}, \mathbf{F_{R,\lambda}(q)} = \mathbf{F}_{\lambda}(\mathbf{q}) \cap \mathbf{R^n}.$$

The varieties $B_{\mathbf{R}}(q)$ and $F_{\mathbf{R},\lambda}(q)$ are called the **real Bloch variety** and the **real Fermi variety** respectively. The following statement is contained in Theorems 3.17 and 4.4.2 of [20] (with remarks of Section 3.4.D in [20] about conditions on potentials taken into account).

Lemma 3 *The Bloch variety $B(q) \subset \mathbf{C^{n+1}}$ is the set of all zeros of a non-zero entire function $f(k,\lambda)$ on $\mathbf{C^{n+1}}$ of order n (see [22]), i.e.*

$$|f(k,\lambda)| \leq C_{\epsilon} \exp\left((|k|+|\lambda|)^{n+\epsilon}\right), \ \forall\, \epsilon > 0.$$

The Fermi variety $F_{\lambda}(q)$ is the set of all zeros of a non-zero entire function of order n on $\mathbf{C^n}$.

Lemma 3 implies in particular that both Bloch and Fermi varieties are examples of what is called in complex analysis **analytic sets** (see for instance [8], [16], and [28]). Moreover, these are **principal** analytic sets in the sense that they are sets of all zeros of single analytic functions, while general analytic sets might require several analytic equations for their (local) description. Analyticity of these varieties (without estimates on the grows of the defining function) was obtained in [38].

Definition 4 *An analytic set $A \subset \mathbf{C^m}$ is said to be **irreducible**, if it cannot be represented as the union of two proper analytic subsets.*

Irreducibility of the zero set of an analytic function can be understood as absence of non-trivial factorizations of this function (i.e., of a factorization into analytic factors that have smaller zero sets).

Definition 5 *A point of an analytic set $A \subset \mathbf{C^m}$ is said to be **regular**, if in a neighborhood of this point the set A can be represented as an analytic submanifold of $\mathbf{C^m}$. The set of all regular points of A is denoted by $regA$.*

We collect in the following lemma several basic facts about analytic sets that we will need later. The reader can find them in many books on several complex variables. In particular, all these statements are proven in sections 2.3, 5.3, 5.4, and 5.5 of Chapter 1 of [8].

Lemma 6 *Let A be an analytic set.*

a) The set $regA$ is dense in A. Its complement in A is closed and nowhere dense in A.

b) The set A can be represented as a (maybe infinite) locally finite union of irreducible subsets

$$A = \bigcup_i A_i$$

*called its **irreducible components**.*

c) Irreducible components are closures of connected components of $regA$. In particular, set A is irreducible if and only if $regA$ is connected.

d) Let A be irreducible and A_1 be another analytic set such that $A \cap A_1$ contains a non-empty open portion of A. Then $A \subset A_1$. In particular, if f is an analytic function that vanishes on an open portion of A, then f vanishes on A.

e) Any analytic set A has a stratification $A = \cup A_j$ into disjoint complex analytic manifolds (strata) A_j such that the union $\cup A_j$ is locally finite, the closure $\overline{A_j}$ of each A_j and its boundary $\overline{A_j} \backslash A_j$ are analytic subsets, and such that if the intersection $A_j \cap \overline{A_k}$ of two different strata is not empty, then $A_j \subset \overline{A_k}$ and $dim A_j < dim A_k$.

We will need the following simple corollary from this lemma.

Corollary 7 *Let A be an irreducible proper analytic subset of $\mathbf{C^n}$ such that the intersection $A \cap \mathbf{R^n}$ contains an open part of a smooth $(n-1)$-dimensional submanifold $M \subset \mathbf{R^n}$. If f is an analytic function in a complex neighborhood of M such that $f = 0$ on M, then $f = 0$ on an open subset of A. In particular, if B is another analytic subset of $\mathbf{C^n}$ such that $M \subset B$, then $A \subset B$.*

Proof. Considering stratification $A = \cup A_j$ (see the last statement in the preceding lemma), one can conclude that only strata of dimension $n-1$ can contain open pieces of M. In fact, if there is an A_j of complex dimension at most $n-2$ containing an open piece of M, we get the contradiction as follows. Consider a point of M and the tangent space to A_j at this point. This is a complex linear subspace of complex dimension at most $n-2$, which contains a real subspace (tangent space to M) of real dimension $n-1$. This is obviously impossible. So, now we can assume that instead of A we are dealing with one of its strata of dimension $n-1$, i.e. we may assume that A is smooth. In appropriate analytic coordinates in a complex neighborhood of a point of M one can represent A locally as a complex hyperplane with the real part M. Then standard uniqueness theorem for analytic functions implies that any function f analytic on A and vanishing on M is identically equal to zero on A. If now B is an analytic subset of $\mathbf{C^n}$ such that $M \subset B$, then each of the analytic functions locally defining B has the properties of the function f above. Hence, we conclude that B contains an open part of A. Then statement d) of the lemma implies that $A \subset B$. This finishes the proof of the corollary.

After this brief excursion into complex analysis we return now to Bloch and Fermi varieties. The next statement follows by inspection of (4):

Lemma 8 *The sets $B(q)$ and $F_\lambda(q)$ are periodic with respect to the quasimomentum k with the lattice of periods $2\pi\mathbf{Z^n} \subset \mathbf{C^n}$ (i.e., the dual lattice to $\mathbf{Z^n}$).*

We choose as a fundamental domain of the group $2\pi\mathbf{Z^n}$ acting on $\mathbf{R^n}$ the following set called the (first) **Brillouin zone**:

$$B = \{k = (k_1, ..., k_n) \in \mathbf{R^n} |\ \mathbf{0 \leq k_j \leq 2\pi,\ j = 1, ..., n}\}.$$

It has been known for a long time (though not always formulated in these terms) that one can tell where the spectrum $\sigma(H_0)$ lies by looking at the Fermi variety. We will now briefly remind the reader this classical result and, at the same time, the definition of spectral bands.

Consider the problem (3) - (4). It has a discrete real spectrum $\{\lambda_j(k)\}$, where $\lambda_j \to \infty$ as $j \to \infty$. We number the eigenvalues in the increasing order. This way we obtain a sequence of continuous functions $\lambda_j(k)$ on the Brillouin zone B. They are usually called **band functions**, or branches of the **dispersion relation**. The values of the function $\lambda_j(k)$ for a fixed j span the jth band $[a_j, b_j]$ of the spectrum $\sigma(H_0)$. A reformulation of this statement is the following theorem, which can be found in equivalent forms in [10] (Section 6.6), [13], [20] (Theorem 4.1.1), [29], and [32] (Theorem XII.98).

Theorem 9 *A point λ belongs to the spectrum $\sigma(H_0)$ of the operator $H_0 = -\Delta + q(x)$ if and only if the real Fermi variety $F_{\mathbf{R},\lambda}(q)$ is non-empty.*

In other words, by changing λ one observes the Fermi variety $F_\lambda(q)$ and notices the moments when it touches the real subspace. This set of values of λ is the spectrum. Now the question arises how can one distinguish the interiors of the spectral bands. The natural idea is that when λ is in the interior of a spectral band, then the real Fermi variety will be massive. "Massive" means here "of dimension $n-1$", i.e. of maximal possible dimension for a proper analytic subset in $\mathbf{R^n}$. This is confirmed by the following statement.

Lemma 10 *If λ belongs to the interior of a spectral band, then the real Fermi variety $F_{\mathbf{R},\lambda}(q)$ contains an open piece of a submanifold M of dimension $n-1$ in $\mathbf{R^n}$.*

Proof of the lemma follows from the stratification of the real Fermi variety $F_{\mathbf{R},\lambda}(q)$ into smooth manifolds (see for instance propositions 17 and 18 of the Chapter V in [28]) and from the obvious remark that when λ belongs to the interior of a spectral band, then $F_{\mathbf{R},\lambda}(q)$ must separate $\mathbf{R^n}$. These two observations imply existence of a smooth piece in $F_{\mathbf{R},\lambda}(q)$ of dimension at least $(n-1)$.

Now we introduce our basic condition:

Condition 11 *Assume that for any λ that belongs to the interior of a spectral band of the Schrödinger operator*

$$H_0 = -\Delta + q(x)$$

any irreducible component of the Fermi variety $F_\lambda(q)$ intersects the real space $\mathbf{R^n}$ by a subset of dimension $n-1$ (i.e. by a subset that contains a piece of a smooth hypersurface).

In fact, the Lemma 10 says that for λ in the interior of a spectral band the Fermi variety $F_\lambda(q)$ does intersect the real space $\mathbf{R^n}$ over a subset of dimension $n-1$. We, however, need more, that every irreducible component of $F_\lambda(q)$

does the same. In other words, there are no "hidden" components that do not show up on the real subspace in any significant way. Thus, Lemma 10 and irreducibility of $F_\lambda(q)/2\pi\mathbf{Z^n}$ would imply validity of the Condition 11. In fact, we believe that (modulo the action of the dual lattice) the Fermi surface is irreducible. The following conjecture formulated in [3] is probably correct:

Conjecture 12 *For any periodic potential (from an appropriate functional class, for instance continuous, or locally square integrable) the surface $F_\lambda(q)/2\pi\mathbf{Z^n}$ is irreducible.*

It looks like this conjecture is very hard to prove (see related discussion in [3] and [19]). As the rest of the paper shows, proving it would lead to a result on the absence of embedded eigenvalues. The following weaker conjecture must be easier to prove:

Conjecture 13 *A generic periodic potential (from an appropriate functional class) satisfies the Condition 11.*

The word "generic" could mean "from a residual set", or something of this sort.

One can easily prove using separation of variables and simple facts about the Hill's equation irreducibility of $F_\lambda(q)/2\pi\mathbf{Z^n}$ for separable periodic potentials $q(x) = \sum_i q_i(x_i)$ in any dimension [3]. The paper [3], however, also contains a stronger non-trivial result:

Lemma 14 *([3]) If $d = 3$ and $q(x) = q_1(x_1) + q_2(x_2, x_3)$, then $F_\lambda(q)/2\pi\mathbf{Z^n}$ is irreducible.*

The proof of this lemma relies on the deep study of the Bloch variety done in [19].

3 The conditional result

We are ready to prove our main conditional statement.

Theorem 15 *If a real periodic potential $q(x) \in L^\infty(\mathbf{R^n})$ $(n \le 3)$ satisfies the Condition 11 and an impurity potential $v(x)$ is measurable and satisfies the estimate*

$$|\nu(x)| \le Ce^{-|x|^r}, r > 4/3 \tag{6}$$

almost everywhere in $\mathbf{R^n}$, then the spectrum of H contains no embedded eigenvalues. In other words,

$$\{\lambda_j\} \cap \bigcup_{i\ge1} (a_i, b_i) = \emptyset,$$

where $\{\lambda_j\}$ is the impurity point spectrum of H, and

$$\bigcup_{i\ge1} [a_i, b_i] = \sigma(H_0)$$

is the band structure of the essential spectrum of H.

Proof. Let us assume that there exists a λ that belongs to some (a_i, b_i) and to the point spectrum of H simultaneously. Then there exists a non-zero function $u(x) \in L_2(\mathbf{R^n})$ (an eigenfunction) such that

$$-\Delta u + qu + vu = \lambda u,$$

or

$$(H_0 - \lambda)u = -vu.$$

Let us denote the function in the right hand side by $\psi(x)$:

$$\psi(x) = -v(x)u(x).$$

Consider the fundamental domain $\mathcal{K}$ of the group $\mathbf{Z^n}$ of periods:

$$\mathcal{K} = \{(x_1, ..., x_n) \in R^n | \ 0 \leq x_i \leq 1, \ i = 1, ..., n\}.$$

Then (6) implies that the function $\psi(x)$ satisfies the estimate

$$||\psi||_{L^2(\mathcal{K}+l)} \leq Ce^{-|l|^r} \tag{7}$$

for all $l \in \mathbf{Z^n}$, where $r > 4/3$. Consider the following Floquet transform of functions defined on $\mathbf{R^n}$ (see section 2.2 in [20] about properties of this transform):

$$\mathcal{F} : \ f(x) \longrightarrow \widehat{f}(k, x) = \sum_{l \in \mathbf{Z^n}} f(x - l)e^{-ik\cdot(x-l)}.$$

The result is a function, which is $\mathbf{Z^n}$-periodic with respect to x. We can consider the resulting function as a function of k with values in a space of functions on the n-dimensional torus $\mathbf{T^n} = \mathbf{R^n}/\mathbf{Z^n}$. If we apply this transform to the function $\psi(x)$, then we get a function of $k \in \mathbf{C^n}$

$$\phi(k) = \widehat{\psi}(k, x) \tag{8}$$

with values in the space $L^2(\mathbf{T^n})$.

Lemma 16 *Function $\phi(k)$ is an entire function on $\mathbf{C^n}$ of the order $s = r/(r-1) < 4$.*

Proof of the lemma. From (7) we get

$$\left|\left|\psi(x - l)e^{-ik\cdot(x-l)}\right|\right|_{L^2(\mathcal{K})} \leq Ce^{C|k| + Im(k\cdot l) - |l|^r}.$$

Then

$$||\phi(k)||_{L^2(\mathcal{K})} \leq Ce^{C|k|} \sum_{l \in \mathbf{Z^n}} e^{-0.5|l|^r} e^{Im(k\cdot l) - 0.5|l|^r}.$$

A simple Legendre transform type estimate (finding extremal values of the exponent) shows that

$$e^{Im(k\cdot l) - 0.5|l|^r} \leq Ce^{C|k|^{r/(r-1)}},$$

which proves that the function $\phi(k)$ is an entire function of the order $s = r/(r-1)$ in $\mathbf{C^n}$. The lemma is proven.

As it is well known (see e.g. Chapter 2 of [20] or XIII in [32]), the transform $\mathcal{F}$ leads to the following operator equation with a parameter k:

$$(H_0(k) - \lambda)\,\widehat{u}(k) = \phi(k), \tag{9}$$

where

$$\widehat{u}(k) = (\mathcal{F}u)\,(k,x)$$

is defined for $k \in \mathbf{R^n}$ and

$$H_0(k) = (i\nabla - k)^2 + q(x)$$

is considered as an operator on the torus $\mathbf{T^n}$. According to Theorem 2.2.5 in [20],

$$\widehat{u}(k) \in L^2_{loc}(\mathbf{R^n}, \mathbf{L_2}(\mathbf{T^n})).$$

In fact, standard interior elliptic estimates show that

$$\widehat{u}(k) \in L^2_{loc}(\mathbf{R^n}, \mathbf{H^2}(\mathbf{T^n})),$$

where $H^2(\mathbf{T^n})$ is the Sobolev space of order two on $\mathbf{T^n}$. Theorem 3.1.5 of [20] implies that the operator

$$(H_0(k) - \lambda) : H^2(\mathbf{T^n}) \to \mathbf{L_2}(\mathbf{T^n})$$

is invertible if and only if $k \notin F_{\mathbf{R},\lambda}(q)$. As it is shown in the proof of Theorem 3.3.1 and in Lemma 1.2.21 of [20] (see also comments in Section 3.4.D on reducing requirements on the potential, in particular Theorem 3.4.2), the inverse operator can be represented as follows:

$$(H_0(k) - \lambda)^{-1} = B(k)/\zeta(k), \tag{10}$$

where $B(k)$ is a bounded operator from $L_2(\mathbf{T^n})$ into $H^2(\mathbf{T^n})$ and $B(k)$ and $\zeta(k)$ are correspondingly an operator and a scalar entire functions of order n in $\mathbf{C^n}$. Besides, the zeros of $\zeta(k)$ constitute exactly the Fermi variety $F_\lambda(q)$. We conclude now that the following representation holds on the set $\mathbf{R^n} \backslash \mathbf{F_{R,\lambda}(q)}$:

$$\widehat{u}(k) = \frac{B(k)\phi(k)}{\zeta(k)} = \frac{g(k)}{\zeta(k)},$$

where $g(k)$ is a $H^2(\mathbf{T^n})$-valued entire function of order n.

Now we need the following auxiliary result:

Lemma 17 *Let Z be the set of all zeros of an entire function $\zeta(k)$ in $\mathbf{C^n}$ and Z_j be its irreducible components. Assume that the real part $Z_{j,\mathbf{R}} = Z_j \cap \mathbf{R^n}$ of each Z_j contains a submanifold of real dimension $n-1$. Let also $g(k)$ be an* *entire function in $\mathbf{C^n}$ with values in a Hilbert space H such that on the real subspace $\mathbf{R^n}$ the ratio*

$$\widehat{u}(k) = \frac{g(k)}{\zeta(k)}$$

belongs to $L_{2,loc}(\mathbf{R^n}, \mathbf{H})$. Then $\widehat{u}(k)$ extends to an entire function with values in H.

Proof of the lemma. Applying linear functionals, one can reduce the problem to the case of scalar functions g, so we will assume that $g(k) \in \mathbf{C}$.

According to Lemma 6, the sets of regular points of components Z_j are disjoint. Hence, the traces of these sets on $\mathbf{R^n}$ are also disjoint. Consider one component Z_j. The intersection of $regZ_j$ with the real space $\mathbf{R^n}$ contains a smooth manifold of dimension $(n-1)$. Namely, we know that $Z_{j,\mathbf{R}}$ contains such a manifold, which we will denote M_j. The only alternative to our conclusion would be that the whole M_j sits inside the singular set of Z_j. The proof of Corollary 7 shows that this is impossible, since lower dimensional strata cannot contain any open pieces of M_j.

Let us denote by m_j the minimal order of zero of function $\zeta(k)$ on Z_j. (We remind the reader that the order of zero of an analytic function at a point is determined by the order of the first non-zero term of the function's expansion at this point into homogeneous polynomials.) Since the condition that an analytic function has a zero of order higher than a given number can be written down as a finite number of analytic equations, one can conclude that the order of zero of $\zeta(k)$ equals m_j on a dense open subset of Z_j, whose complement is an analytic subset of lower dimension. As it was explained in the proof of Corollary 7, lower dimensional strata cannot contain $(n-1)$-dimensional submanifolds of $\mathbf{R^n}$. This means that one can find a point $k^j \in Z_{j,\mathbf{R}}$ such that $Z_{j,\mathbf{R}}$ is a smooth hypersurface in a neighborhood U of k^j in $\mathbf{R^n}$ and such that the order of zero of $\zeta(k)$ on $Z_{j,\mathbf{R}}$ equals m_j in this neighborhood. Let us now prove that $g(k)$ has zeros on $Z_{j,\mathbf{R}} \cap U$ of at least the same order as $\zeta(k)$. If this were not so, then in appropriate local coordinates $k = (k_1, ..., k_n)$ the ratio g/ζ would have a singularity of at least the order k_1^{-1}, which implies local square non-integrability of the function $\widehat{u}(k)$. This contradicts our assumption, so we conclude that $g(k)$ has zeros on $Z_{j,\mathbf{R}} \cap U$ of at least the same order as $\zeta(k)$. Consider the (analytic) set A of all points in $\mathbf{C^n}$ where $g(k)$ has zeros of order at least m_j. We have just proven that the intersection of A with Z_j contains an $(n-1)$-dimensional smooth submanifold of R^n. Then Corollary 7 implies that $Z_j \subset A$, or $g(k)$ has zeros on Z_j of at least the order m_j. Hence, according to Proposition 3 in section 1.5 of Chapter 1 in [8] the ratio g/ζ is analytic everywhere in $\mathbf{C^n}$ except maybe at the union of subsets of Z_j, where the function ζ has zeros of order higher than m_j. This subset, however, is of dimension not higher than $n-2$. Now a standard analytic continuation theorem (see for instance Proposition 3 in Section 1.3 of the Appendix in [8]) guarantees that g/ζ is an entire function. This concludes the proof of the lemma.

Returning to the proof of the theorem and using the result of the Lemma 10 and the assumption that the Condition 11 is satisfied for the potential $q(x)$, we conclude that $\widehat{u}(k)$ is an entire function and that it is the ratio of two entire functions of order at most $w = \max(n, s) < 4$, where s is defined in Lemma 16. Using (in the radial directions) the estimate of entire functions from below contained in section 8 of Chapter 1 in [26] (see also Theorem 1.5.6 and Corollary 1.5.7 in [20] or similar results in [7]), we conclude that $\widehat{u}(k)$ is itself an entire function of order w with values in $H^2(\mathbf{T^n})$. Now Theorem 2.2.2 of [20] claims that the solution $u(x)$ satisfies the decay estimate:

$$||u||_{H^2(K+a)} \leq C_p \exp(-c_p\, |a|^p)$$

for any $p < w/(w-1)$, where K is an arbitrary compact in $\mathbf{R^n}$, and $a \in \mathbf{R^n}$. Using standard embedding theorems, we conclude that

$$|u(x)| \leq C_p \exp(-c_p\, |x|^p),\ \forall\, p < w/(w-1).$$

Since $w < 4$, one can choose a value p such that

$$4/3 < p < w/(w-1).$$

However, Remark 2.6 in [12] and Theorem 1 in [27] state that there is no non-trivial solution of the equation

$$-\Delta u + qu + vu - \lambda u = 0$$

with the rate of decay

$$|u(x)| \leq C \exp(-c\, |x|^p),\ p > 4/3.$$

This contradiction concludes the proof of the theorem.

Remark 1 *The condition $p > 4/3$ (and hence the dimension restriction $n < 4$) is essential for the validity of the result of [12] and [27], so this is the place where our argument breaks down for dimensions four and higher even if we require compactness of support of the perturbation potential $v(x)$. The rest of the arguments stay intact (the embedding theorem argument, which also depends on dimension, is not really necessary).*

4 Separable potentials

The result of the previous section leads to the problem of finding classes of periodic potentials that satisfy the Condition 11. As we stated in conjectures 12 and 13, we believe that all (or almost all) of periodic potentials satisfy this condition. Although we were not able to prove these conjectures, as an immediate corollary of the Lemma 14 and of the Theorems 15 and **??** we get the following result:

Theorem 18 *If for $n < 4$ the background periodic potential $q(x) \in L^\infty(\mathbf{R^n})$ is separable for $n = 2$ or $q(x) = q_1(x_1) + q_2(x_2, x_3)$ for $n = 3$, and the perturbation potential $v(x)$ satisfies the estimate 6, then there are no eigenvalues of the operator H in the interior of the bands of the continuous spectrum.*

5 Comments

1. We have only proven the absence of eigenvalues embedded into the interior of a spectral band. It is likely that eigenvalues cannot occur at the ends of the bands either (maybe except the bottom of the spectrum), if the perturbation potential decays fast enough. This was shown in the one-dimensional case in [33] under the condition

$$\int (1 + |x|)\, |v(x)|\, dx < \infty$$

on the perturbation potential. On the other hand, if $v(x)$ only belongs to L^1, then the eigenvalues at the endpoints of spectral bands can occur [34].

2. Most of the proof of the conditional Theorem 15 does not require the unperturbed operator to be a Schrödinger operator. One can treat general selfadjoint periodic elliptic operators as well. The only obstacle occurs at the last step, when one needs to conclude the absence of fast decaying solutions to the equation. Here we applied the results of [12] and [27], which are applicable only to the operators of the Schrödinger type. Carrying over these results to more general operators would automatically generalize Theorem 15. The restriction that the dimension n is less than four also comes from the allowed rate of decay stated in the result of [12] and [27].

3. It might seem that the irreducibility condition 11 arises only due to the way the proof is done. We believe that this is not true, and that the validity of the condition is essentially equivalent to the absence of embedded eigenvalues. To be more precise, we conjecture that existence for some λ in the interior of a spectral band of an irreducible component A of the Fermi surface such that $A \cap \mathbf{R^n} = \emptyset$ implies existence of a localized perturbation of the operator that creates an eigenvalue at λ. As a supporting evidence of this one can consider fourth order periodic differential operators, where the Fermi surface contains four points. In this case one can have λ in the continuous spectrum, while some points of the Fermi surface being complex. Then one can use these components of the Fermi surface "hidden" in the complex domain to cook up a localized perturbation that does create an eigenvalue at λ [30].

ACKNOWLEDGMENTS

The authors express their gratitude to Professors A. Figotin, T. Hoffmann-Ostenhof, H. Knörrer, S. Molchanov, and V. Papanicolaou for helpful discussions and information. The work of P. Kuchment was partly supported by the NSF Grant DMS 9610444 and by a DEPSCoR Grant. P. Kuchment expresses his gratitude to NSF, ARO, and to the State of Kansas for this support. The work of B. Vainberg was partly supported by the NSF Grant DMS-9623727. B. Vainberg expresses his gratitude to NSF for this support. The content of this paper does not necessarily reflect the position or the policy of the federal government, and no official endorsement should be inferred.

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