INVARIANT TORI FOR NON CONSERVATIVE PERTURBATIONS OF INTEGRABLE SYSTEMS

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Abstract We prove some results on the persistence of invariant tori for non hamiltonian perturbations of integrable systems. We also obtain persistence for the case where the unperturbed torus is resonant. In such a case the persistence of invariant tori is ensured for ε small, but outside of a neighbourhood of zero.

1. Introduction and statement of the results

Hamiltonian integrable systems are the subject of extended studies; this is due not only to their intrinsical interest, but also to the fact that they represent a starting point for the study of more general (non-integrable) systems by means of perturbative techniques. Among the most relevant – and celebrated – results in this context are those concerning the persistence (maybe up to some distortion) of invariant tori of the integrable systems under Hamiltonian perturbations; this fact is central to many theoretical sides and applications of Hamiltonian perturbation theory, for which we will just refer e.g. to [2].

It is thus natural to wonder if some invariant tori can survive also a sufficiently small non-hamiltonian perturbation; in this paper we will indeed investigate this question.

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More precisely, we address the problem of existence of (attractive) invariant tori for systems of the form

$$\begin{aligned} \dot I &= \varepsilon\ f_\varepsilon(I,\varphi) \\ \dot\varphi &= \omega(I) + \varepsilon\ g_\varepsilon(I,\varphi) \end{aligned} \tag{1}$$

where $I \in \mathcal{G} \subset \mathbb{R}^n$ (with $\mathcal{G}$ open) are the slow variables, $\varphi \in \mathbf{T}^m$ are the fast angular variables, and ε is a (small) real parameter. We assume that the functions ω, f, g are analytic in each of their arguments.

For $m = 1$ the problem is of a different nature, as on the one side we are dealing with periodic solutions, and on the other we have no problems due to resonances. Thus, in the following we will assume we have to deal with a proper torus $\mathbf{T}^m$, i.e. we will assume $m > 1$ (for the case $m = 1$ see e.g. [2]). We also stress that we focus on the case of attractive tori, which is at the same time the simplest and the most interesting in applications; we expect this approach would apply also for the case of normally hyperbolic tori, in a substantially equivalent way, although technically slightly more complicate.

We will proceed in three steps: (i) we perform a second order averaging: we use average to break the degeneracy, and we go to second order for reasons explained below; (ii) we look for attractive hyperbolic invariant tori of the truncation of this averaged system; (iii) we continue the invariant torus of the averaged system to an invariant torus of (1). To this end we apply a suitable version of the theorem on persistence of normally hyperbolic manifolds (it is in this step that the need for second order averaging arises).

It is simple (although non trivial) to make the above scheme rigorous when the frequency vector is diophantine and constant, i.e. independent of the slow variables I. Indeed, in such a case the averaging transformation which maps (1) into the averaged system is globally defined (on $\mathcal{G}$), and the problem of continuing a torus, namely to achieve step (iii) above, can be solved in different ways. Essentially one can use a KAM theory, or the theory of normally hyperbolic manifold.

Here we are interested in the general case where $\omega = \omega(I)$ is non constant (the case of constant but not diophantine frequencies will also be of interesy).

The main difficulties in step (i) are due to the fact that in the case of non constant frequencies the averaging transformation is defined only on the set of slow variables on which the frequency vector is sufficiently non resonant; therefore the system (1) is equivalent to the averaged system only on such a set. Moreover, it will turn out that the nonresonant set depends on ε, and when $\varepsilon = 0$ it has a dense complement; therefore the question of existence of invariant tori for (1) requires some care.

To state our main theorem we have to introduce some notations.

Given a function $h : \mathcal{G} \times \mathbf{T}^m \to \mathbb{R}^q$ we define its average by

$$\langle h\rangle(I) := \frac{1}{(2\pi)^m} \int_{\mathbf{T}^m} h(I,\varphi) d\varphi\ .$$

We assume that $\langle f_0\rangle := \langle f_\varepsilon\rangle\big|_{\varepsilon=0}$ has an attractive zero J_*, namely a zero such that all the eigenvalues of $d\langle f_0\rangle(J_*)$ are strictly negative.

Moreover, we assume that J_* is nonresonant at order N, namely that there exist positive γ, N and a real τ such that

$$|\omega(J_*) \cdot k| \geq \frac{\gamma}{|k|^\tau} \ , \quad \forall k \in \mathbf{Z}^m \backslash \{0\} \ , \quad |k| := \sum_{i=1}^{m} |k_i| \leq N \ .$$

Theorem 1.1. *There exist positive constants ε_*, $\sigma_\sharp$, C_1 (independent of ε, and N) such that, provided*

$$N \geq |\ln \varepsilon_*|/\sigma_\sharp$$

and, with $\varepsilon_\sharp = \exp(-\sigma_\sharp N)$,

$$\varepsilon_\sharp \leq \varepsilon \leq \varepsilon_*$$

system (1) admits an attractive invariant torus of the form $(I, \varphi) = (\mathcal{I}(\varphi), \varphi)$ where the function $\mathcal{I}(\varphi)$ satisfies

$$|\mathcal{I}(\varphi) - J_*| \leq C_1 \varepsilon |\ln \varepsilon|^{\tau+1} \ . \tag{2}$$

We emphasize that our theorem ensures the preservation of a torus on which the frequencies are close to diophantine, but can satisfy some resonance relations. In such a case the persistence of the torus is ensured only for ε larger than some nonvanishing value. As far as we know this phenomenon has never been pointed out before.

In connection with this point it should be mentioned that the persistence of tori close to approximately diophantine equilibrium points was studied, among other papers, in [5,8], where a phenomenon which could have some relation to the one mentioned here was also observed.

The condition that T_0 is attractive (obviously, the repulsive case would be equivalent) comes into play through the negativity of nonzero Lyapounov exponents; should we have a generic transversally hyperbolic T_0, we should consider negative and positive Lyapounov exponents, and correspondingly a stable and an unstable manifold; we should then consider their perturbations, and the intersection of the perturbed manifolds, in order to obtain the perturbed invariant torus. Thus, we expect that the present scheme could be extended to the general case along these lines.

In the last part of this Introduction we would like to mention some works which are closely related to the present one. First of all, we mention that a preliminary version of our result appeared (without proof) in [3].

After the appearance of this preprint, A. Neishtadt called our attention to a series of papers by Gurtovnik and Neimark [7], starting at the end of the sixties, dealing with the same problem of non-Hamiltonian perturbations of a Hamiltonian integrable system. In these papers the authors proved persistence of normally hyperbolic diophantine tori under sufficiently smooth (not necessarily analytic) perturbation. These authors did not consider the case of resonant frequencies so we think that in this case our result is new. For what concerns the nonresonant case our present paper obtains essentially the same results of Gurtovnik and Neimark making available a complete selfcontained proof, based on the well established ideas of normally hyperbolic manifolds [6,9]. We emphasize that the proof of the theorems by Gurtovnik and Neimark appeared only in russian and moreover in journals which are (according to our experience) difficult to access for western readers.

Also, KAM techniques have been used to study persistence of invariant tori in the context of eq. (1) (see [4]) obtaining that there exists a large set $\mathcal{C} \subset \mathbb{R}^m$ such that if the frequency vector belongs to such a set, and if a suitable function (on which is only known that it is close to $\langle f_0 \rangle$) has a zero in $\mathcal{C}$, then the original system has an invariant torus filled by quasiperiodic orbits with diophantine frequencies; however an explicit application of these techniques has been implemented only in the case of constant frequencies. We would expect KAM technique to be applicable also in the case of nonconstant frequencies.

In this context, we recall that KAM techniques in hamiltonian systems allow to ensure that there exists a large Cantor-like set $\mathcal{W}$ in the action space such that in correspondence to any $I \in \mathcal{W}$ there exists an invariant torus of the system; the set $\mathcal{W}$ does usually depend on ε. In the present nonconservative case, one can expect that KAM techniques would allow to prove (under nondegeneracy conditions) that, if the solution of an equation of the kind $F(J, \varepsilon) = 0$ is in a suitable Cantor-like set, then correspondingly there exists an invariant torus for (1). Then typically an invariant torus should exist for a Cantor set of values of ε.

Notice that here we obtain instead existence of the torus for ε in an open set, and also in the case of resonant frequencies. Moreover we do not need any nondegeneracy condition on the dependence of the frequencies on the actions. Thus it is to be expected that a KAM approach could provide results complementing those obtained here.

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2. The normal form lemma

To state (and proof) the normal form theorem we first introduce some notations. Fix $\tau \geq m-1$, $\gamma > 0$ and $K > 0$, and define the domain

$$\Gamma^K(\tau,\gamma) := \left\{ I \in \mathcal{G} \ : \ |\omega(I)\cdot k| \geq \frac{\gamma}{|k|^\tau} \ ; \ \forall k \in \mathbf{Z}^m \backslash \{0\} \ ; \ |k| := \sum_{i=1}^m |k_i| \leq K \right\} .$$

and the complex extended domains with parameters ρ, σ

$$\Gamma^K_\rho(\tau,\gamma) := \bigcup_{I\in\Gamma^K(\tau,\gamma)} B_\rho(I) \ ,$$
$$\Delta^K_{\rho,\sigma} := \Gamma^K_\rho(\tau,\gamma) \times \mathbf{T}^m_\sigma$$

where $B_\rho(I)$ is the *complex ball* of radius ρ and center I and $\mathbf{T}^m_\sigma$ is the set of the φ's in the complexified torus such that $|\mathrm{Im}\,\varphi_i| \leq \sigma$.

Theorem 2.1. (Normal form) *Fix τ and γ, then there exist positive constants ε_0, σ, $\sigma_\sharp$, C_2, C_3 independent of ε with the following properties: define $K := |\ln\varepsilon|/\sigma_\sharp$, $\rho := C_2|\ln\varepsilon|^{-(\tau+1)}$ then, provided $0 \leq |\varepsilon| \leq \varepsilon_0$ there exists an analytic coordinate transformation $(J,\psi) = \mathcal{T}_\varepsilon(I,\varphi)$ defined on $\Delta^K_{\rho/2,\sigma/2}$ which puts the system in the form*

$$\begin{aligned} \dot{J} &= \varepsilon\langle f_\varepsilon\rangle(J) + \varepsilon^2|\ln\varepsilon|^{\tau+1}N^\varepsilon_I(J) + \varepsilon^3|\ln\varepsilon|^{2\tau+4}\mathcal{R}^\varepsilon_I(J,\psi) \\ \dot{\psi} &= \omega(J) + \varepsilon N^\varepsilon_\varphi(J) + \varepsilon^3|\ln\varepsilon|^{2\tau+4}\mathcal{R}^\varepsilon_\varphi(J,\psi) \end{aligned} \tag{3}$$

Moreover $N^\varepsilon_I, \mathcal{R}^\varepsilon_I, N^\varepsilon_\varphi$ and $\mathcal{R}^\varepsilon_\varphi$ are analytic and bounded on $\Delta^K_{\rho/2,\sigma/2}$, uniformly with respect to $\varepsilon \in [0,\varepsilon_0]$, and depend in a C^∞ way on $\varepsilon \in (0,\varepsilon_0]$. Moreover,

$$\sup_{\Delta^K_{\rho/2,\sigma/2}} \|\mathcal{T}(I,\varphi) - (I,\varphi)\| \leq C_3\varepsilon \quad , \quad \sup_{\Delta^K_{\rho/2,\sigma/2}} \left\|\mathcal{T}^{-1}(J,\psi) - (J,\psi)\right\| \leq C_3\varepsilon \ .$$

The coordinate transformation will be constructed as the composition of two transformations which increase by one the order of normalization. The procedure is essentially standard; the only novelty consists in the fact that we are in the case where the frequencies depend on the actions, and therefore the nonresonant domain has size which depends on ε and which shrinks to zero (logarithmically) with ε.

We come now to the proof of theorem 2.1. It will be a simple corollary of the forthcoming proposition 2.2. In the statement of proposition 2.2 the quantities $K, \rho, \sigma, \varepsilon$ will be considered as independent parameters.

First remark that, by hypothesis there exists a positive $\bar{\rho}$, such that

$$\sup_{ReI\in\mathcal{G}\ |\mathrm{Im}\,I_i|\leq\bar{\rho}} \|d\omega(I)\| \leq \Omega \tag{4}$$

Proposition 2.2. *On a domain $\Delta^K_{\rho,\sigma}$ consider an analytic system of the form*

$$\begin{aligned} \dot I &= z_I(I) + r_I(I,\varphi) \\ \dot\varphi &= \omega(I) + z_\varphi(I) + r_\varphi(I,\varphi) \ , \end{aligned} \tag{5}$$

assume that there exist constants Z and R such that

$$\sup_{\Delta^K_{\rho,\sigma}} \|z_I\| \le Z \qquad \sup_{\Delta^K_{\rho,\sigma}} \|z_\varphi\| \le Z$$
$$\sup_{\Delta^K_{\rho,\sigma}} \|r_I\| \le R \qquad \sup_{\Delta^K_{\rho,\sigma}} \|z_\varphi\| \le R$$

Fix $d < 1/2$ then there exist positive constants C_4, C_5, C_6 (independent of Z, R, K, ρ), such that, if $R/\rho < C_4$, and

$$\rho \le \frac{\gamma}{2\Omega K^{\tau+1}}$$

then there exists a coordinate transformation $(J,\psi) = \mathcal{T}(I,\varphi)$, analytic on $\widetilde\Delta$ and with inverse analytic on $\widehat\Delta$, these being defined by

$$\widetilde\Delta := \Delta^K_{\rho(1-d),\sigma(1-d)} \quad , \quad \widehat\Delta := \Delta^K_{\rho(1-2d),\sigma(1-2d)} \ ,$$

fulfilling the estimates

$$\sup_{\widetilde\Delta} \|\mathcal{T}(I,\varphi) - (I,\varphi)\| \le C_5 R \qquad \sup_{\widehat\Delta} \left\|\mathcal{T}^{-1}(J,\psi) - (J,\psi)\right\| \le C_5 R \ ,$$

wich transform (5) into a system of the same form, but defined on $\widehat\Delta$. More precisely the new "normal form terms" are given by

$$z'_I(J) = z_I(J) + \langle r_I\rangle(J) \ , \quad z'_\varphi(J) = z_\varphi(J) + \langle r_\varphi\rangle(J) \ .$$

Moreover the new normal forms z' and remainders r' fulfill the following estimates:

$$\sup_{\widehat\Delta} \|z'_I\| \le Z + R \quad , \quad \sup_{\widehat\Delta} \left\|z'_\varphi\right\| \le Z + R$$
$$\sup_{\widehat\Delta} \|r'_I\| \le C_6 R \left[\frac{Z}{\rho} + \frac{R}{\rho} + K^{m-1} e^{-d\sigma K}\right] \quad , \quad \sup_{\widehat\Delta} \left\|z'_\varphi\right\| \le C_6 R \left[\frac{Z}{\rho} + \frac{R}{\rho} + K^{m-1} e^{-d\sigma K}\right] .$$

The rest of the section is devoted to the proof of this theorem and to the easy conclusion of the proof of theorem 2.1.

First we split the perturbation into an essential part containing only harmonics of order less than K and a higher order part. To obtain a smooth dependence on ε this will be done in a smooth way. Let $\chi_K : \mathbb{R}^+ \to [0,1]$ be a C^∞ function with $\chi_K(x) \equiv 1$

for $x \leq K-1$ and $\chi_K(x) \equiv 0$ for $x \geq K$; assume that it depends smoothly also on K. Consider now the Fourier expansions

$$r_I(I,\varphi) = \sum_{k\in\mathbf{Z}^m} r_{I,k}(I)e^{ik\cdot\varphi} \ , \quad r_\varphi(I,\varphi) = \sum_{k\in\mathbf{Z}^m} r_{\varphi,k}(I)e^{ik\cdot\varphi}$$

and denote

$$r_I^-(I,\varphi) = \sum_{|k|\leq K} r_{I,k}(I)\chi_K(|k|)e^{ik\cdot\varphi} \ , \quad r_I^+ := r_I - r_I^- \ ,$$

and similarly for r_φ. We have the following

Lemma 2.3. *Fix a positive $d < \sigma$, then there exists a constant C independent of K such that one has*

$$\sup_{\Delta^K_{\rho,\sigma(1-d)}} \left\| r_I^+ \right\| \leq CRK^{m-1}e^{-d\sigma K} \ .$$

An equal estimate holds for r_φ.

Proof. By standard decay properties of analytic functions one has

$$\|r_{I,k}(I)\| \leq Re^{-\sigma|k|} \ .$$

So, for $(I,\varphi) \in \Delta^K_{\rho,\sigma(1-d)}$ one has

$$\begin{aligned}\left\| r_I^+(I,\varphi) \right\| &\leq \left\| \sum_{|k|>K-1} r_{I,k}(I)e^{ik\cdot\varphi} \right\| \leq \sum_{|k|>K-1} Re^{-d\sigma|k|} \\ &\leq 2^{2m-1}RK^{m-1}e^{-d\sigma K}\left(\frac{1}{1-e^{-d\sigma}}\right)^m\end{aligned}$$

$\square$

Consider now a coordinate transformation of the form

$$J = I + A(I,\varphi) \ , \quad \psi = \varphi + B(I,\varphi) \ ;$$

substituting in (5) one has, the following equations for J and ψ

$$\begin{aligned}\dot{J} &= z_I(I) + r_I^-(I,\varphi) + r_I^+(I,\varphi) \\ &\quad + d_I A(I,\varphi)(z_I(I,\varphi) + r_I(I,\varphi)) + d\varphi A(I,\varphi)\omega(I) \\ &\quad + d_\varphi A(I,\varphi)[z_\varphi(I) + r_\varphi(I,\varphi)] \ , \\ \dot{\psi} &= \omega(J) + z_\varphi(I) + r_\varphi^-(I,\varphi) + r_\varphi^+(I,\varphi) + d_I B(I,\varphi)z_I(I) + d_I B(I,\varphi)r_I(I,\varphi) \\ &\quad + d_\varphi B(I,\varphi)\omega(I) + d_\varphi B(I,\varphi)z_\varphi(I) + d_\varphi B(I,\varphi)r_\varphi(I,\varphi) \\ &\quad - d\omega(I)A(I,\varphi) + d\omega(I)A(I,\varphi) - d\omega(J)A(I,\varphi) + \omega(I) - \omega(J) + d\omega(J)A(I,\varphi)\end{aligned}$$

Suppose one can construct A and B in such a way that

$$r_I^- + d_\varphi A\omega = r_{I,0} \ , \quad r_\varphi^- - d\omega A + d_\varphi B\omega = r_{\varphi,0} \ , \tag{6}$$

where $r_{I,0}$ is the zero-th Fourier coefficient of r_I and similarly for $r_{\varphi,0}$, then the equation of motion assume again the form (5) provided one defines the new normal forms z'_I, z'_φ and the new remainders r'_I, r'_φ by

$$\begin{aligned}
z'_I(J,\psi) &:= z_I(J) + r_{I,0}(J)\ , \quad z'_\varphi(J,\psi) := z_\varphi(J) + r_{\varphi,0}(J)\ ,\\
r'_I(J,\psi) &:= z_I(I) - z_I(J) + r_{I,0}(I) - r_{I,0}(J) + d_I A(I,\varphi)(z_I(I,\varphi) + r_I(I,\varphi))\\
&\quad + d_\varphi A(I,\varphi)[z_\varphi(I) + r_\varphi(I,\varphi)] + r_I^+(I,\varphi)\ ,\\
r'_\varphi(J,\psi) &:= r_\varphi^+(I,\varphi) + d_I B(I,\varphi) z_I(I) + d_I B(I,\varphi) r_I(I,\varphi)\\
&\quad + d_\varphi B(I,\varphi) z_\varphi(I) + d_\varphi B(I,\varphi) r_\varphi(I,\varphi) + z_\varphi(I) - z_\varphi(J) + r_{\varphi,0}(I) - r_{\varphi,0}(J)\\
&\quad + [d\omega(I) - d\omega(J)]A(I,\varphi) + \omega(I) - \omega(J) + d\omega(J)A(I,\varphi)\ .
\end{aligned} \tag{7}$$

Now we construct and estimate the solutions of the equations (6). As usual the solution of the Homological equation is constructed by passing to Fourier transform and is given by

$$A(I,\varphi) := \sum_{0\neq|k|\leq K} \frac{r^-_{I,k}(I)\chi_K(|k|)}{i\omega(I)\cdot k} e^{ik\cdot\varphi}\ , \tag{8}$$

and

$$B(I,\varphi) := \sum_{0\neq|k|\leq K} \frac{\chi_K(|k|)}{i\omega(I)\cdot k}\left(r^-_{\varphi,k}(I) - d\omega(I)\frac{r^-_{I,k}(I)}{i\omega(I)\cdot k}\right) e^{ik\cdot\varphi}\ , \tag{9}$$

The following lemma ensures that the denominators in such an expression do not vanish.

Lemma 2.4. (lemma 4.1 of [1]) *Assume*

$$\rho \leq \frac{\gamma}{2\Omega K^{\tau+1}}$$

then, for all $I \in \Gamma^K_\rho(\gamma,\tau)$ one has

$$|\omega(I)\cdot k| \geq \frac{\gamma}{2|k|^\tau} \qquad 0 \neq k \leq K\ .$$

Proof. For any $I \in \Gamma^K_\rho$ there exists, at a distance smaller than ρ an $I_0 \in \Gamma^K$. So one has

$$|\omega(I)\cdot k| \geq |\omega(I_0)\cdot k| - |[\omega(I) - \omega(I_0)]\cdot k| \geq \frac{\gamma}{|k|^\tau} - \Omega\,\|I - I_0\|\,K\ ,$$

from which the thesis follows immediately. □

Lemma 2.5. *There exist solutions A and B of the homological equation (6) which, for any $d \leq 1/2$, are analytic in $\Delta^K_{\rho,\sigma(1-d)}$ and $\Delta^K_{\rho,\sigma(1-d)}$ respectively, and satisfy the estimates*

$$\sup_{\Delta^K_{\rho,\sigma(1-d)}} \|A\| \leq C_7 R\ , \qquad \sup_{\Delta^K_{\rho,\sigma(1-2d)}} \|B\| \leq C_7 R$$

with a suitable C_7.

Proof. We consider only A. The esitimate of B is identical. From the exponential decay of the Fourier coefficients of r_I and the estimate of the small denominators provided by lemma 2.4 one has ,for $(I,\varphi)\in\Delta^K_{\rho,\sigma(1-d)}$,

$$\|A(I,\varphi)\|\leq\sum_{0\neq|k|\leq K}\frac{R|k|^\tau}{2\gamma}e^{-d\sigma|k|}<\sum_{0\neq|k|}\frac{R|k|^\tau}{2\gamma}e^{-d\sigma|k|}\leq\frac{2^mR}{\gamma}\left(\frac{1}{1-e^{-d\sigma}}\right)^{m+\tau-1}\ .$$

$\square$

Proof of proposition 2.2. The proof of proposition 2.2 can be obtained by applying Cauchy estimates of derivatives and Lagrange mean value theorem. As an example we give explicitely the estimate of one of the terms of (7). All the other terms can be estimated in the same way. One has

$$\sup_{(J,\psi)\in\widehat{\Delta}}\|z_I(I(J,\psi))-z_I(J)\|\leq\sup_{(J,\psi)\in\widehat{\Delta}}\|d_Jz_I(J)\|\sup_{(J,\psi)\in\widehat{\Delta}}\|I(J,\psi)-J\|\leq\frac{CZ}{\rho}R\ .$$

$\square$

Proof of theorem 2.1. Assume first that there exist γ,τ such that $\Gamma^\infty(\gamma,\tau)\neq\emptyset$, then there exist $\bar\rho$, σ such that f and g extend to a bounded complex analytic functions on $\bigcup_{I\in\mathcal{G}}B_{\bar\rho}(I)\times\mathbf{T}^m_\sigma\supset\Delta^K_{\rho,\sigma}$ ($\rho\leq\bar\rho, K\leq K_*$). So system (1) fulfills the assumptions of proposition 2.2 (with $Z=0$). Fix $K:=(8/\sigma)|\ln\varepsilon|$ as well as $\rho:=(\gamma/2\Omega K^{\tau+1})$ and $d=1/8$; then, provided $\varepsilon|\ln\varepsilon|^{\tau+1}$ is small enough, the smallness condition of proposition 2.2 is satisfied and one finds coordinates in which the system has the form (5), with the quantities Z and R given by $Z=C\varepsilon$, $R=C\varepsilon^2|\ln\varepsilon|^{\tau+1}$ (C independent of ε). Then one applies proposition 2.2 one more time (with $d=1/6$) and the thesis follows. To obtain the statement one has to consider the case where $\Gamma^\infty(\gamma,\tau)=\emptyset$. But this is easily done since as far as $\Gamma^{(8/\sigma)|\ln\varepsilon|}(\gamma,\tau)=\emptyset$ the statement of the theorem becomes empty. $\square$

3. The proof of persistence of stable hyperbolic tori.

We consider here a system of the form (3). We assume that $\langle f_0\rangle:=\langle f_\varepsilon\rangle\big|_{\varepsilon=0}$ has an attractive zero J_*.

We first continue it to a zero of $(\langle f_\varepsilon\rangle+\varepsilon\left|\ln\varepsilon\right|^{\tau+1}N^\varepsilon_I)$. This requires some care since N^ε_I is defined only on the nonresonant set, which can be empty for small ε.

Lemma 3.1. *Assume that there exist $\tau,\gamma,K<\infty$ such that the zero J_* of $\langle f_0\rangle$ belongs to $\Gamma^K(\tau,\gamma)$; then there exist $\varepsilon_*>0$ and C_8 such that the following holds true: if*

$$K\geq|\ln\varepsilon_*|/\sigma_\sharp \tag{10}$$

then we define $\varepsilon_\sharp = e^{-K\sigma_\sharp} \le \varepsilon_*$, and for any ε satisfying

$$\varepsilon_\sharp \le \varepsilon \le \varepsilon_* \tag{11}$$

the function $\langle f_\varepsilon\rangle + \varepsilon\left|\ln\varepsilon\right|^{\tau+1} N_I^\varepsilon$ has an attractive zero $J_0(\varepsilon)$ which depends in a C^∞ way on $\varepsilon \in [\varepsilon_\sharp, \varepsilon_*]$ and satisfies

$$\|J_0(\varepsilon) - J_*\| \le C_8 \varepsilon |\ln\varepsilon|^{\tau+1} \; ; \tag{12}$$

moreover there exists $\lambda > 0$ such that all the eigenvalues $\lambda_i = \lambda_i(\varepsilon)$ of

$$d\langle f_\varepsilon\rangle(J_0) + \varepsilon\left|\ln\varepsilon\right|^{\tau+1} dN_I^\varepsilon(J_0)$$

satisfy

$$\inf_{\varepsilon_\sharp \le \varepsilon \le \varepsilon_*} |\lambda_1(\varepsilon)| \ge \lambda \; . \tag{13}$$

Remark 3.2. *If J_* is such that $\omega(J_*)$ is diophantine, then $K = \infty$ and, by (12) the function $J_0(\varepsilon)$ is continuous for $\varepsilon \in [0, \varepsilon_*]$.*

Proof. Consider the equation

$$\langle f_\varepsilon\rangle + \varepsilon\left|\ln\varepsilon\right|^{\tau+1} N_I^\varepsilon \equiv \langle f_0\rangle + \varepsilon\left|\ln\varepsilon\right|^{\tau+1} \tilde N_I^\varepsilon = 0 \tag{14}$$

where $\tilde N_I^\varepsilon$ contains also the difference between $\langle f_0\rangle$ and $\langle f_\varepsilon\rangle$. Remark that for fixed ε the l.h.s. of (14) is analytic on $\Gamma_{\rho/2}^{|\ln\varepsilon|/\sigma_\sharp}(\gamma,\tau)$, and that it has a C^∞ dependence on ε.

Fix $\varepsilon > e^{-K\sigma/8}$ and make a translation so that $J_* = 0$. We write equation (14) as $J = G_\varepsilon(J)$ with

$$G_\varepsilon(J) := -[d\langle f_0\rangle(0)]^{-1}[\langle f_0\rangle(J) - d\langle f_0\rangle(0)J + \varepsilon\left|\ln\varepsilon\right|^{\tau+1} \tilde N_I^\varepsilon(J)] \; .$$

It is straightforward to see, following the standard proof of the implicit function theorem, that there exists ε_* such that, if $\varepsilon < \varepsilon_*$ (and also $\varepsilon > e^{-K\sigma/8}$) then G is a contraction of a ball of radius of order $\varepsilon\left|\ln\varepsilon\right|^{\tau+1}$, and therefore it has a unique fixed point in this ball. By following the proof of smoothness of the implict function theorem one also gets smooth dependence of the fixed point on ε in the considered domain and also property (13). $\square$

In the rest of the paper we will use the quantity $\rho := C_2 |\ln\varepsilon|^{-(\tau+1)}$ which was defined in theorem 2.1.

Notice that under the hypotheses of lemma 3.1 the system

$$\begin{aligned} \dot J &= \varepsilon\langle f_\varepsilon\rangle + \varepsilon\left|\ln\varepsilon\right|^{\tau+1} N_I^\varepsilon \\ \dot\psi &= \omega + \varepsilon N_\varphi^\varepsilon \end{aligned} \tag{15}$$

has an attractive torus $J_0(\varepsilon) \times \mathbf{T}^m$ close to the torus $J_* \times \mathbf{T}^m$. We are going to continue such a normally hyperbolic torus to a torus of system (3). We will follow almost step by step the proof of persistence of normally hyperbolic manifolds given by Fenichel [6][9], just introducing the necessary quantitative estimates on the threshold. We stress that such estimates, not necessary in the "non-degenerate" case originally considered by Fenichel, are needed in our case: this is due to the fact that the Lyapunov exponents of the invariant manifold and the size of the perturbation are related.

In all the rest of the paper we will *assume* that ε is small enough and that J_* is sufficiently nonresonant. By this we mean that $J_* \in \Gamma^K(\tau,\gamma)$ with $K > |\ln\varepsilon|/\sigma_\sharp$.

We give now a suitable form to system (3). First we make an ε dependent translation in J space so that the attractive zero $J_0(\varepsilon)$ coincides with 0, namely so that $\langle f_\varepsilon\rangle(0) + \varepsilon\,|\ln\varepsilon|^{\tau+1} N_I^\varepsilon(0) = 0$. Then we write our system as

$$\begin{aligned} \dot J &= A_\varepsilon J + \mathcal{N}(J,\varepsilon) + \varepsilon^{2+p}\mathcal{S}_I(J,\psi,\varepsilon)\\ \dot\psi &= w(J,\varepsilon) + \varepsilon^{1+p}\mathcal{S}_\varphi(J,\psi,\varepsilon)\ , \end{aligned} \tag{16}$$

where p is a real number (to be fixed later) satisfying $0 < p < 1$, and

$$A_\varepsilon := d\langle f_\varepsilon\rangle(0) + \varepsilon\,|\ln\varepsilon|^{\tau+1}\, dN_I^\varepsilon(0)\quad,\quad \mathcal{N}(J,\varepsilon) := \langle f_\varepsilon\rangle(J) + \varepsilon\,|\ln\varepsilon|^{\tau+1} N_I^\varepsilon(J) - A_\varepsilon J$$

$$w(J,\varepsilon) := \omega(J) + \varepsilon N_\varphi^\varepsilon(J)\quad,\quad \mathcal{S}_I = \frac{\varepsilon|\ln\varepsilon|^{2\tau+4}\mathcal{R}_I^\varepsilon}{\varepsilon^p}\quad,\quad \mathcal{S}_\varphi = \frac{\varepsilon|\ln\varepsilon|^{2\tau+4}\mathcal{R}_\varphi^\varepsilon}{\varepsilon^p}\ .$$

In what follows we will systematically omit the ε dependence of the various functions. This simplified notation is not dangerous by virtue of the following

Lemma 3.3. *For any integer q the C^q norms of $\mathcal{N},\mathcal{S}_I,\mathcal{S}_\varphi,w$ are bounded on $B_{\rho/4}(0)\times \mathbf{T}^m$, uniformly in ε; moreover there exists C_N such that*

$$\|\mathcal{N}(J)\| \le C_N\,\|J\|^2\quad,\quad \|d\mathcal{N}(J)\| \le 2C_N\,\|J\|\ . \tag{17}$$

Proof. . Consider first $\mathcal{S}_I$. By Cauchy inequality its C^q norm is bounded on $B_{\rho/4}(0)\times\mathbf{T}^m_{\sigma/4}$ by a constant multiplied by

$$\frac{\varepsilon^3\ |\ln\varepsilon|^{4\tau+2}\ |\ln\varepsilon|^{q(\tau+1)}}{\varepsilon^{2+p}}\ ,$$

which is bounded provided $p < 1$. Similar considerations hold for $\mathcal{S}_\varphi$ and $\mathcal{N}$. The estimate of w is easily obtained by remarking that ω is analytic and bounded on $B_{\bar\rho}$ [see (4)]. The estimate (17) is a simple consequence of the fact that $\langle f_\varepsilon\rangle + \varepsilon\,|\ln\varepsilon|^{\tau+1} N_I^\varepsilon$ is twice differentiable with respect to J. □

We now fix p and q, with $0 < p < 1$ and $q \ge 1$, and introduce some notations: we denote by C_J, C_φ the constants bounding the C^q norm of $\mathcal{S}_I$ and $\mathcal{S}_\varphi$ respectively; the C^{q-1} norm of dw is bounded by 2Ω.

We denote by $F^t(J,\psi)$ the solution of (16) starting at the point (J,ψ). With an obvious notation we write $(J(t),\psi(t)) = F^t(J,\psi)$. We will also use the notations

$$H_t(J,\psi) := J(t)\ ,\quad \Phi_t(J,\psi) = \psi(t)\ ,\quad w_* := w(J_0) \tag{18}$$

and, for $\rho_1 < \rho/4$,

$$\mathcal{U}_{\rho_1} := (\mathcal{G} \cap B_{\rho_1}(0)) \times \mathbf{T}^m \ , \ .$$

We will often use the formula of variation of constants, which gives

$$H_t(J,\psi) = e^{\varepsilon A t} J + \varepsilon \int_0^t e^{\varepsilon A(t-s)} \left(\mathcal{N}(H_s(J,\psi)) + \varepsilon^{1+p} \mathcal{S}_I(H_s(J,\psi), \Phi_s(J,\psi)) \right) ds \ , \tag{19}$$

and the formula

$$\Phi_t(J,\psi) = \psi + \int_0^t \left[w(H_s(J,\psi)) + \varepsilon^{1+p} \mathcal{S}_\varphi(H_s(J,\psi), \Phi_s(J,\psi)) \right] ds \ . \tag{20}$$

Fenichel's method consists in looking for an invariant manifold of the map F^T, where T is a fixed positive time. We now proceed to implement this.

We fix

$$T := \varepsilon^{-7p/8} \ . \tag{21}$$

Notice that this establishes a relation between the time T and the smallest Lyapunov exponent $\varepsilon\lambda$ of the unperturbed invariant torus $J = 0$; in particular one has $\varepsilon\lambda T \ll 1$, but $\varepsilon \ll \varepsilon T$ (this is the key trick for the proof).

We have the following

Lemma 3.4. *For any ρ_1 satisfying $\rho/4 > \rho_1$ and $\lambda\rho_1 \geq 2\varepsilon^{1+p} C_J$, and $t \leq T$ one has*

$$F^t(\mathcal{U}_{\rho_1}) \subset \mathcal{U}_{\rho_1} \ .$$

Proof. Using (19) and (17) we have

$$\begin{aligned}
\|J(t)\| &\leq e^{-\varepsilon\lambda t} \rho_1 + \varepsilon \int_0^t e^{\lambda\varepsilon(t-s)} [C_N \rho_1^2 + \varepsilon^{1+p} C_J] ds \\
&\leq e^{-\varepsilon\lambda t} \rho_1 + \varepsilon t (C_N \rho_1^2 + \varepsilon^{1+p} C_J) \\
&= (1 - \lambda\varepsilon t)\rho_1 + \varepsilon t (C_N \rho_1^2 + \varepsilon^{1+p} C_J) + o(\varepsilon t)\rho_1 \ ,
\end{aligned}$$

and this is smaller than ρ_1 provided ρ_1 is small, εt is small, and $\varepsilon^{1+p} C_J + o(\varepsilon t)\rho_1 \leq \lambda\rho_1$. □

We fix now $\rho_1 := (2C_J/\lambda)\varepsilon^{1+p}$.

We recall that, following Fenichel [6], the invariant manifold is found as the graph of a section $\sigma : \mathbf{T}^m \to \mathbb{R}^n$ [where $J = \sigma(\psi)$]. Moreover, σ is characterized as the unique fixed point of the "graph transform" G. In order to define G in a precise way we need some estimates. These are given in the following lemmas.

Lemma 3.5. *For $(J,\psi) \in \mathcal{U}_{\rho_1}$, the following relations hold:*

$$\begin{aligned}
&\sup_{J,\psi} \|d_J H_T(J,\psi)\| \leq 1 - \varepsilon T \lambda + \text{h.o.t.} \ , \\
&\sup_{J,\psi,s\leq t} \|d_J \Phi_s(J,\psi)\| \leq t 2\Omega + \text{h.o.t.} \ , \\
&\sup_{J,\psi,s\leq t} \|d_\psi H_s(J,\psi)\| \leq \varepsilon^{2+p} t C_J + \text{h.o.t.} \\
&\sup_{J,\psi,s\leq t} \|d_\psi \Phi_s(J,\psi)\| \leq 1 + 2\Omega t^2 \varepsilon^{2+p} C_J + \text{h.o.t.} \ ,
\end{aligned} \tag{22}$$

where h.o.t. means higher order terms.

Proof. Let us write

$$\alpha_1 := \sup_{t\in[0,T]}\sup_{J,\psi\in\mathcal{U}_{\rho_1}}\|d_J H_t(J,\psi)\| \ , \quad \alpha_2(s) := \sup_{t\in[0,s]}\sup_{J,\psi\in\mathcal{U}_{\rho_1}}\|d_\psi H_t(J,\psi)\| \ ,$$
$$\alpha_3(s) := \sup_{t\in[0,s]}\sup_{J,\psi\in\mathcal{U}_{\rho_1}}\|d_J \Phi_t(J,\psi)\| \ , \quad \alpha_4(s) := \sup_{t\in[0,s]}\sup_{J,\psi\in\mathcal{U}_{\rho_1}}\|d_\psi \Phi_t(J,\psi)\| \ .$$

Differentiating eqs. (19) and (20) with respect to J, one has

$$d_J H_t = e^{\varepsilon A t} + \varepsilon\int_0^t e^{\varepsilon A(t-s)}\left[d\mathcal{N} d_J H_s + \mu\left(d_J\mathcal{S}_\varphi d_J H_s + d_\psi\mathcal{S}_\varphi d_J\Phi_s\right)\right] ds \ ,$$

and a similar equation holds for $d_j\Phi_t$. From these equations one obtains that

$$\begin{aligned} \|d_J H_t\| \leq & e^{-\lambda t\varepsilon} + \varepsilon t\left[2C_{\mathcal{N}}\rho_1\alpha_1 + \varepsilon^{1+p}\left(\alpha_1+\alpha_3\right)C_J\right] \ , \\ \|d_J\Phi_t\| \leq & t\left[2\Omega\alpha_1 + \varepsilon^{1+p}\left(\alpha_1+\alpha_3\right)C_\varphi\right] \ . \end{aligned} \tag{23}$$

Now, it is clear that $\alpha_1 = 1$, while the r.h.s. of the second of (23) is a function increasing with time; so we have that there exists $\tilde{\Phi} \geq \alpha_3$ such that $\tilde{\Phi} \leq t\Omega + \varepsilon^{1+p}t\left(C_\varphi + \tilde{\Phi}C_\varphi\right)$; neglecting higher order terms this gives $\|d_J\Phi_t\| \leq \tilde{\Phi} \leq t\Omega$. Substituting in the first of (23) we get the first two of (22) (recall that $\varepsilon T \ll 1$). Differentiating (19) and (20) with respect to ψ, and making some standard estimates, we get a system of inequalities for α_2, α_4 which gives the remaining of (22). □

Define now the space S_d of the Lipschitz functions $h : \mathbf{T}^m \to \mathcal{U}_{\rho_1}$ with Lipschitz constants $\mathrm{Lip}(h) < d$. We fix $d = \varepsilon^{1+p/2}$.

Fix a function $\sigma \in S_d$, and consider the map defined by

$$\psi \mapsto \xi = \xi(\psi) := \Phi_T(\sigma(\psi - w_* T), \psi - w_* T) \ , \tag{24}$$

where we used the notation of (18); it will be needed to invert such a map. To this end it is useful to introduce the function $\mathcal{S}(\psi) := \xi(\psi) - \psi$, so that $\xi(\psi) = \psi + \mathcal{S}(\psi)$.

Lemma 3.6. *Let $\sigma \in S_d$; then $\|\mathcal{S}(\psi)\| \leq (4\lambda C_J + C_\varphi)\varepsilon^{1+p}T \ll 1$ and Lip($\mathcal{S}$) $\leq$ $2\Omega T d +$ h.o.t..*

Proof. By (20) one has

$$\begin{aligned} \mathcal{S}(\psi) &= \int_0^T\left[w(H_s(J,\psi)) + \varepsilon^{1+p}\mathcal{S}_\varphi(H_s(J,\psi),\Phi_s(J,\psi))\right] ds - w_* T \\ &= \int_0^T\left[w(H_s(J,\psi)) - w_*\right] ds + \varepsilon^{1+p}\int_0^T \mathcal{S}_\varphi(H_s(J,\psi),\Phi_s(J,\psi)) ds \ , \end{aligned}$$

with $J = \sigma(\psi)$. But, taking into account lemma 3.4, one has $\|H_s\| \leq \rho_1$, and therefore the argument of the first integral is the difference between the value of the function w at points ρ_1 close, and moreover $\|\mathcal{S}_\varphi\|$ is bounded. So we have $\|\mathcal{S}(\psi)\| \leq 2\Omega\rho_1 T + \varepsilon^{1+p}C_\varphi T$, which – taking into account the definition of ρ_1 – gives the estimate of $\mathcal{S}$. We come to the Lipschitz constant of $\mathcal{S}$. We split the estimate in two parts, corresponding to the two parts of the argument of the integral in (20). We begin by

$$\int_0^T \| w\left(H_s\left(\sigma(\psi_1 - w_*T), \psi_1 - w_*T\right)\right) - w\left(H_s\left(\sigma(\psi_2 - w_*T), \psi_2 - w_*T\right)\right) \| \, ds$$
$$\leq T 2\Omega \left[d \|\psi_1 - \psi_2\| + \varepsilon^{2+p} T C_J \|\psi_1 - \psi_2\| \right] \simeq T 2\Omega d \|\psi_1 - \psi_2\| \ ,$$

where we neglected higher order terms, and used lemma 3.5. This gives the estimate of the Lipschitz constant of such a term. Proceeding analogously for the other term one obtains that its Lipschitz constant is smaller than

$$\varepsilon^{1+p} T C_\varphi (d + \varepsilon^{2+p} T C_J + 2\Omega T d + 1 + 2\Omega T^2 \varepsilon^{2+p} C_J) \simeq \varepsilon^{1+p} T C_\varphi \ ,$$

so that $\mathrm{Lip}(\mathcal{S}) \leq 2\Omega T d$ (up to h.o.t.). □

Corollary 3.7. *The function $\xi(\psi)$ is invertible.*

Proof. This follows from this lemma 3.6 and the implicit function theorem. □

We define the *graph transform* by writing $G\sigma(\xi) := H_T(\sigma(\psi - wT), \psi - wT)$, where ψ, and ξ are related by (24).

Lemma 3.8. *For any $\sigma \in S_d$, one has* $\mathrm{Lip}(G\sigma) \leq [1 + (2\Omega T d - \varepsilon\lambda T)]\, d + \text{h.o.t.}$*; hence* $\mathrm{Lip}(G\sigma) < d$*, and G maps S_d into itself.*

Proof. . Let ξ_1, ξ_2 be related to ψ_1, ψ_2 as in (24). We have to estimate

$$\|G\sigma(\xi_1) - G\sigma(\xi_2)\| = \|H_T\left(\sigma(\psi_1 - w_*T), \psi_1 - w_*T\right) - H_T\left(\sigma(\psi_2 - w_*T), \psi_2 - w_*T\right)\| \ ,$$

which is less than

$$\mathrm{Lip}\left[H_T(\sigma(. - w_*T), . - w_*)\right] \|\psi_1 - \psi_2\| \ .$$

Such a Lipschitz constant is estimated using lemma 3.5 by

$$\|d_J H_T\|\, d + \|d_\psi H_T\| \leq \left[(1 - \varepsilon\lambda T) d + \varepsilon^{2+p} T C_J\right] \ ,$$

where we neglected higher order terms. So, we need an estimate of $\|\psi_1 - \psi_2\|$ in terms of $\|\xi_1 - \xi_2\|$. One has

$$\|\psi_1 - \psi_2\| = \|\xi_1 - \xi_2 - \mathcal{S}(\psi_1) + \mathcal{S}(\psi_2)\|$$
$$\leq \|\xi_1 - \xi_2\| + \|\mathcal{S}(\psi_1) - \mathcal{S}(\psi_2)\| \leq \|\xi_1 - \xi_2\| + [\mathrm{Lip}\mathcal{S}]\, \|\psi_1 - \psi_2\| \ ,$$

so that (at first order)

$$\|\psi_1 - \psi_2\| \leq \frac{\|\xi_1 - \xi_2\|}{1 - 2\Omega T d} \simeq (1 + 2\Omega T d)\, \|\xi_1 - \xi_2\| \ . \tag{25}$$

This completes the proof. □

Lemma 3.9. *The graph transform $G\big|_{S_d}$ is Lipschitz and its Lipschitz constant is bounded by $1 + d4\Omega T - \varepsilon T\lambda$.*

Proof. . We have to estimate $\|G\sigma(\xi) - G\sigma'(\xi)\|$; this is equal to

$$\|H_T\left(\sigma(\psi - w_*T), \psi - w_*T\right) - H_T\left(\sigma'(\psi' - w_*T), \psi' - w_*T\right)\| \tag{26}$$

[where ψ' is related to ξ by (24), via the section σ']. This is estimated, using Lagrange theorem and lemma 3.5, by

$$\begin{aligned}(1 - \varepsilon\lambda T)\,\|\sigma(\psi - w_*T) - \sigma'(\psi' - w_*T)\| + \varepsilon^{2+p}TC_J\,\|\psi - \psi'\| \leq \\ \leq (1 - \varepsilon\lambda T)\left[\|\sigma(\psi - w_*T) - \sigma(\psi' - w_*T)\| + \right. \\ \left. + \|\sigma(\psi' - w_*T) - \sigma'(\psi' - w_*T)\|\right] + \varepsilon^{2+p}TC_J\,\|\psi - \psi'\|\ .\end{aligned} \tag{27}$$

To estimate this quantity we need an estimate of $\psi - \psi'$. Defining J and J' by

$$\begin{aligned}(J, \xi) &= F^T(\sigma(\psi - wT), \psi - wT) \\ (J', \xi) &= F^T(\sigma'(\psi' - w_*T), \psi' - w_*T)\end{aligned} \tag{28}$$

one has that $F^{-T}(J, \xi)$ and $F^{-T}(J', \xi')$ are both in $\mathcal{U}_{\rho_1}$, and moreover

$$\psi = \Phi_{-T}(\xi, J) + w_*T\ , \quad \psi' = \Phi_{-T}(\xi, J') + w_*T\ ;$$

from this, using (20) one has $\|\psi - \psi'\| \leq T[4\Omega + 2\varepsilon^{1+p}C_\varphi]$. Inserting this in (27) we complete the proof. $\square$

With this we have completed the proof of our Theorem:

Proof of Theorem 2.1 (end). Indeed, we have proved with lemma 3.9 that G is a contraction and therefore has a unique fixed point which is the invariant manifold. Moreover, by construction the fixed point is a section $\sigma : \mathbf{T}^m \to \mathcal{U}_{\rho_1}$, and therefore the perturbed invariant torus is $\mathcal{O}(\rho_1)$ close to the the torus $J = 0$, which in turn is $\mathcal{O}(\varepsilon\,|\ln\varepsilon|^{\tau+1})$-close to the invariant torus $J_* \times \mathbf{T}^m$ of the averaged system. Finally the change of variables of proposition 2.1 is ε close to the identity, and therefore the invariant torus satisfies (2). $\square$

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