Rigidity of higher dimensional conformal Anosov systems

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Abstract. We show that Anosov systems in manifolds with trivial tangent bundles and with the property that the derivatives of the return maps at periodic orbits are multiples of the identity in the stable and unstable bundles are locally rigid. That is, any other smooth map, in a C^1 neighborhood such that it has the same Jordan normal form at corresponding periodic orbits is smoothly conjugate to it. This generalizes results of [CM]. We present several arguments for the main results. In particular, we use quasi-conformal regularity theory. We also extend the examples of [L2] to show that some of the hypothesis we make are indeed necessary.

1. Introduction, notation and statement of main results.

1.1. Introduction.

The main goal of this paper is to prove some rigidity results for some Anosov systems in higher dimensions.

In our context, rigidity means that, for some systems, data at periodic orbits determine the smooth conjugacy class. The data we will consider turn out to be invariants of C^1 — or even Lipschitz — conjugacy so that, as a corollary we obtain that systems in our class which are conjugated by a Lipschitz homeomorphism are smoothly conjugate. We will give precise statements later, but we point out that the main novelty of the results presented here is that they hold for Anosov systems in arbitrarily high dimensions without using families. Also, we introduce the use of quasi-conformal techniques. In spite of the importance of quasi-conformal techniques in the geometric rigidity program [Mo1],[Mo2] they seem not to have played much role in the study or rigidity of Anosov dynamical systems.

We recall that in [L1], [LM] and, by other methods in [Po], [L2], it was proved that two 2-dimensional Anosov diffeomorphisms (two 3-dimensional Anosov flows) that are topologically conjugate and such that the Lyapunov exponents at corresponding periodic orbits agree are smoothly conjugate. Since the Lyapunov exponents at periodic orbits are invariants of Lipschitz conjugacy, we obtain as a corollary that two Lipschitz conjugate Anosov diffeomorphisms of a 2-dimensional manifold (or Anosov flows of a 3-dimensional manifold) are smoothly conjugate.

The fact that low dimensionality plays a role was clarified in [L2] which constructed examples of analytic families of analytic diffeomorphisms in $\mathbb{T}^d$ $d \geq 4$ which are C^k conjugate but not C^{k+1}. (To the best of our knowledge, the situation in $\mathbb{T}^3$ remains unexplored.) The examples in [L2] have the property that the germs of return maps around periodic orbits are C^∞ equivalent locally, so that, without extra hypothesis, it is hopeless to try to obtain rigidity out of local data around periodic orbits for general Anosov systems. In this paper, we will also extend the examples of [L2] to show that even assuming normal forms which are just one non-trivial Jordan block, rigidity is false. At the end of the statement of Theorem 1.1, we have collected some remarks on what is the role of the different assumptions in this theory. Roughly, it seems to us that the reason why one needs (in the present state of the theory) assumptions such as one-dimensionality or the conformal assumptions used here is that the one needs to rearrange the equations obtained taking derivatives of the equations into coboundary equations that have much better regularity and existence theory.

For certain automorphisms of $\mathbb{T}^d$, it was shown in [CM] that analytic families of analytic diffeomorphisms that preserved the data at periodic orbits, had to be analytic changes of variables of the map. The technique used was a KAM iteration.

In the present paper we present results that generalize the results of [CM] in that we do not require families, just proximity assumptions. Also, the class of systems that we can consider are somewhat wider in principle (as it is sometimes the case in rigidity theory, one may end up proving that classes that are wider in principle, are indeed equivalent. (See e.g Lemma 1.3.) We also can deal with regularities other that analytic.

We will present two proofs of the main rigidity result, one following the strategy in [L1],[LM] (we will refer to it as the "scattering method") and another one following the strategy of [Po],[L2] (we will refer to is as "using SRB measures"). Both will require some technical improvements which may be of independent interest for the rigidity program.

In the the proof using SRB measures, we need to introduce quasi conformal estimates and study regularities in quasi-conformal classes. The quasi-conformal classes appear naturally in this higher dimensional problems to supplement the use of invariant measures. This is because the use of invariant measures gives control over the Jacobian of the transformation and eventually shows it is smooth. For one dimensional functions, the fact that the Jacobian of a transformation is smooth is clearly enough to show that the function is smooth. Of course, this is not true in higher dimensions and we need to supplement the estimates of the Jacobian with estimates that show that the derivative is "diagonal". Of course, in a manifold, the notion of a derivative being diagonal does not make sense, since it maps one tangent space into a completely different one. One possible way out would be to assume that there is a trivialization, but it is more geometrically natural to use the notion – in principle weaker – of 1-quasiconformality. That is, that there exists a metric such that the lengths of all vectors in the unstable manifold gets expanded by the same amount.

1.2. Notation.

We will use C^α $0 < \alpha \le \mathrm{Lip}$, C^{Lip} means Lipschitz functions and $\mathrm{Lip} < 1$, $\mathrm{Lip} > \alpha \quad \forall \alpha < 1$. We also denote by $\|f\|_{C^\alpha}$ the semi-norm $\sup_{x\neq y} d(f(x), f(y))/d(x,y)^\alpha$. When f ranges on a Lie group, we can consider $\|f\|'_{C^\alpha} = \sup_{x\neq y} d(f(x)f^{-1}(y), \mathrm{Id})/d(x,y)^\alpha$.

These two seminorms are equivalent when the range of f is restricted to lie in a bounded neighborhood of the identity. Note that the use of seminorms is rather natural for cohomology equations since the solutions are defined up to additive (or multiplicative constants).

We recall that, given an Anosov diffeomorphism of (resp. an Anosov flow X_t on) a manifold M we can find a splitting of the tangent bundle

$$T_xM = E^s_x \oplus E^u_x \quad (\text{resp. } T_xM = \{\dot X(x)\} \oplus E^s_x \oplus E^u_x\ )$$

corresponding to stable and unstable directions. For Anosov flows, we will also introduce the center-stable, center-unstable bundles. $E^{cs}_x = E^s_x \oplus \{\dot X\}$, $E^{cu}_x = E^u_x \oplus \{\dot X\}$.

We also recall that, associated to the stable, unstable bundles (and center-stable, center-unstable bundles in the case of flows), we can associate the stable, unstable, center-stable, center-unstable foliations. We will denote by W^s_x, W^u_x, W^{cs}_x, W^{cu}_x. the leaf of one of the foliations passing through the point x When we want to use just an small disk of the leaf around the point, we will use the notation $W^{s,\mathrm{loc}}_x$ and similarly for the others. Since these foliations depend on the map or flow that we are considering, if there is possibility of confusion, we will make the dependence on the map explicit using $W_x^{s,(f)}$ and the like. In the course of the proofs we will need properties of these foliations but we will refer to the literature for the most standard ones.

We recall that when the map is C^r , $r = 1, 2, \ldots, \infty, \omega$, these foliations have C^r leaves and their r-jets are Hölder on the manifold. (In the case that $r = \omega$, the topology of r-jets is induced by a norm that implies the analyticity.) It then, makes sense to consider the restriction of an smooth functions to these leaves and the derivatives along these leaves. Following [LMM], we will say that a map is in C_s^r if the restriction to leaves of the stable foliation is C^r and the r-jet is continuous on the manifold. Similarly, we will denote by D_s the differential of the restrictions. That is, if $\phi : M \mapsto N$, $D_s\phi(x)$ will be a linear map from E_x^s to $T_{\phi(x)}N$. In the important case that $M = N$ and that $\phi = f$, since E^s is invariant, we can consider $D_s f$ as a map from E_x^s to $E_{f(x)}^s$. Analogous definitions are made for the other invariant bundles. Of course, these concepts depend on the foliation and, hence on the map. When there is danger of confusion, we will denote them as $C_{s,(f)}^r$ and $D_{s,(f)}$.

Since we will also be using some parts of the theory of quasi-conformal maps, we recall that, given a metric, we can define the "distortion" of a derivative by:

$$K(Df(x)) = \frac{\max_{v \in T_x M, ||v||=1} ||Df(x)v||}{\min_{v \in T_x M, ||v||=1} ||Df(x)v||} \tag{1.1}$$

We emphasize that the distortion of a map depends on the metrics on the domain and the range that we consider. Notably, if we have two metrics on a manifold, the identity map, may have distortion bigger than one if we consider it as a map between the manifold endowed with a metric to the manifold endowed with the other. This dependence on the target and domain metrics will not be included in the notation, following customary practice, but we will try to make clear what are the metrics in the domain and the range since it will be necessary to take advantage of of the freedom afforded by the choice of metrics with special properties.

For Anosov systems that leave invariant subbundles, we can also consider distortions along the subbundles. That is:

$$K_s(Df(x)) = \frac{\max_{v \in E_x^s, ||v||=1} ||Df(x)v||}{\min_{v \in E_x^s, ||v||=1} ||Df(x)v||} \tag{1.2}$$

We will refer to K_s as the stable distortion. Analogous definitions will hold for all the other invariant subbundles. Notice that in that case, we only need to have the metric defined on the subbundle.

If a map is differentiable almost everywhere, we will call the distortion of the map, the supremum of the distortions of the derivatives and say that the map is K-quasi-conformal if it is differentiable almost everywhere and the distortion is bounded by K.

Similarly, given any metric on the stable bundle, which is smooth restricted to the stable leafs, (recall that the stable bundle is the tangent bundle of the stable leafs) we will speak about stable quasi-conformal maps when almost everywhere – with respect to this metric – on any leaf of the stable foliation, the map restricted to the stable leaf is differentiable, the derivative sends elements of the stable bundle into elements of the stable bundle and the stable distortion is bounded.

Analogous definitions hold for the other bundles.

1.3. Statement of results.

Our main rigidity is the following:

Theorem 1.1. *Let M be a compact manifold with a trivial tangent bundle. Let f be a C^r ($r = 2,3,\ldots,\infty,\omega$) transitive Anosov diffeomorphism of (resp. X_t a transitive Anosov flow on) M such that:*

i) *If $f^n(x) = x$, then $Df^n(x) : E^s_x \longrightarrow E^s_x$ and $Df^n(x) : E^u_x \longrightarrow E^u_x$ are multiples of the identity. (resp. If $X_T(x) = x$, then $DX_T(x) : E^s_x \longrightarrow E^s_x$ and $DX_T(x) : E^u_x \longrightarrow E^u_x$ are multiples of the identity.)*

ii) *Assume moreover that we can find continuous trivialization $\Psi^s_x : E^s_x \to \mathbb{R}^s$, $\Psi^s_x : E^s_x \to \mathbb{R}^s$ such that*

$$\begin{aligned} \Psi^s_{f(x)} D_s f(x) (\Psi^s_x)^{-1} &= \gamma_s(x)\,\mathrm{Id}_s \\ \Psi^u_{f(x)} D_u f(x) (\Psi^u_x)^{-1} &= \gamma_u(x)\,\mathrm{Id}_u \end{aligned} \tag{1.3}$$

where γ_s, γ_u are real valued functions.

Assume that g is a C^r diffeomorphism of (resp. Y_t an Anosov flow on) M and $h : M \to M$ is a homeomorphism in such a way that

a) *$h \circ g = f \circ h$ (resp. $h \circ X_t = Y_t \circ h$)*

b) *If $f^n(x) = x$, then $Df^n(x)$ and $(Dg^n) \circ h(x)$ have the same Jordan normal form (resp. if $X_T(x) = x$ then $DX_T(x)$, (DY_T) $(h(x))$ have the same Jordan normal form)*

c) *g is sufficiently close to f in C^1. (resp. X is sufficiently C^1 close to X.)*

Then, h is $C^{r-\epsilon}$ for all $\epsilon > 0$.

In other words, the diffeomorphisms satisfying condition *i)* and *ii)* in Theorem 1.1 are "locally rigid". (Locally refers to the fact that we need to assume that the diffeomorphism g is in a neighborhood.) This is, of course, more restrictive than proving the result for all diffeomorphisms but is less restrictive than assuming that there is a family which satisfies the same assumptions for all its elements.

Note that the condition *i)* can be verified by knowing the conjugacy class of the derivatives at periodic points and that condition *b)* is a necessary condition for Lipschitz conjugacy.

Examples of diffeomorphisms and flows satisfying condition *i)* and *ii)* of Theorem 1.1 can be readily constructed by taking linear isomorphisms of tori or of nilmanifolds with the appropriate spectrum.

There is an small redundancy between conditions i) and ii). We will show that if there is a diffeomorphism f satisfying conditions i) and ii), then, in a C^1 neighborhood, all the diffeomorphisms satisfying condition i) also satisfy condition ii). (See Proposition 2.6.) In particular, given a linear automorphism satisfying i), we can find a neighborhood so that all the diffeomorphisms satisfying i) also satisfy ii).

We also note that it follows from assumption ii) that the trivializations are somewhat smoother. Indeed, we will show that it is possible to choose $\gamma_s \Psi^s, \Psi^u \in C^{r-1}$. (See Proposition 2.5.)

Remark. We are not sure that assumption ii) is necessary. The reason why it enters in the proof is that the derivative of the conjugacy equation (a) in Theorem 1.1) is

$$Df \circ hDh = Dh \circ gDh \tag{1.4}$$

Given that the conjugacy itself can be constructed, we can consider (1.4) as an equation for Dh. Unfortunately the only equations for which have a good theory (the smooth Livisic theory) is for equations of the form

$$Dh \circ gDh^{-1} = \eta(x) \tag{1.5}$$

Note that (1.5) does not have a geometric meaning. The maps Dh and $Dh\circ$ are supposed to be maps from some tangent spaces so that the product in (1.5) does not make geometric sense. Nevertheless, with a trivialization, we can indeed make sense of equations such as coboundary and apply the rich smooth Livsic theory. Such rearrangements already were implicit in the one dimensional results. It seems possible to us that, using more systematically the existence of a conformal metric, one could perhaps eliminate this assumption.

Remark. We note, however that condition ii) of Theorem 1.1 is satisfied for nilmanifolds and infranilmanifolds. These are the only manifolds for which examples of Anosov systems are known and, indeed, it is an outstanding conjecture that these are the only manifolds for which Anosov systems exist and that they are all transitive. For flows, the situation is much less clear, since it is known that there are non-algebraic Anosov flows. (See [HT])

Remark. The reason why we need to assume c) is related to the fact that we only have existence results for Livsic equation (1.5) in non-commutative groups when the range is sufficiently close to the identity. The result without such assumptions was stated in [Li2]. It is nevertheless well known and accepted that the proof presented has serious gaps. On the other hand, the smooth Livsic theorem as we will state it does not require smallness assumptions.

Remark. Note that we do not need to assume that the Lyapunov exponents at periodic orbits are the same for all the periodic orbits as it is the case for the maps that are produced using automorphisms of tori or of nilmanifolds. We only need that the return maps at periodic orbits are diagonal on the stable and unstable manifolds. It could well happen that the spectrum at different periodic points was different. Nevertheless, we will show (See Lemma 1.3) that when the stable and unstable manifolds are two dimensional, the fact that the return maps are diagonal implies that the Lyapunov exponents are constant.

We also note that, as a part of the proof, we also obtain are result that may be of independent interest.

Theorem 1.2. *Let f be a C^r transitive Anosov diffeomorphism on a compact manifold M with a trivial tangent bundle. (resp. X_t a transitive Anosov flow). Assume that:*

$i)$ f (resp. X_t) satisfies assumption $i)$ $ii)$ of Theorem 1.1.

Then there exists a C_u^{r-1} metric g^u on the bundle E^u such that, in that metric, $K_u(Df(x)) = 1 \forall x \in M$ (resp. $K_u(DX_t(x) = 1 \forall x \in M, t \in \mathbb{R}$.) And analogous results for the stable bundle and, in the case of flows, for the center-unstable, center-stable.

Remark. In the above result if one makes assumption $i)$, $ii)$ for one of the invariant bundles, it is possible to recover the conclusions for that bundle.

A consequence of Theorem 1.2 is the following:

Lemma 1.3. *Assume that a transitive Anosov system satisfies the conclusions of Theorem 1.2 (In particular, i) and ii) of Theorem 1.1) and that the unstable manifold is two-dimensional. Then, the unstable Lyapunov exponents are constant.*

Proof. Consider the stable manifold of a fixed point. The metric g^u on which the map is confomal restricted to the unstable manifold is differentiable and bounded with respect to the natural Riemannian metric.

Recall that the stable manifold is topologically $\mathbb{R}^2$. By the Riemann measurable mapping theorem, we can find a bounded and differentiable change of variables that reduces the metric g^u to standard and f to conformal. Now, given the fact that the map is bounded, we discover that restricted to the unstable manifold in these coordinates $|f(x)| \leq K|x|$ and is conformal. By Liouville's theorem, we obtain that fi reduces to multiplying by a constant in these coordinates.

Now, we recall that, by the transitivity of the Anosov diffeomorphism, the unstable manifold is dense. It follows that we can approximate arbitrary well the exponents in a periodic orbit by the exponents of a finite orbit in the unstable manifold. But those are constant.

∎

Note that a corollary of Theorem 1.2 is that if f is an Anosov system satisfying assumption $i)$ of Theorem 1.1, then, for any metric $K_u(f^n) \leq C$, $K_s(f^n) \leq C$ with C independent of n.

We will also prove the following result:

Theorem 1.4. *Let f be a transitive Anosov diffeomorphism satisfiying i), ii) of Theorem 1.1. Then, given $\epsilon > 0$ we can find $\delta > 0$ such that if $d_{C^1}(f, g) \le \delta$, the homeomorphism h satisfying*

$$f \circ h = h \circ g; \qquad d_{C^0}(h, \mathrm{Id}) \le K\delta,$$

given by structural stability satisfies

$$h, h^{-1} \in C^{1-\epsilon}.$$

This result is very similar to the result of [PV], which used very heavily that the systems had one-dimensional stable and unstable leaves.

We postpone the proof of this theorem till a subsequent section.

Another fact that reinforces the idea that systems satisfying i) and ii) of Theorem 1.1 are in some respects one dimensional is the following observation.

Lemma 1.5. *Let f be a transitive Anosov flow satisfying i) ii). Then, the Anosov splitting is $C^{1+\alpha}$ for every $\alpha > 0$.*

Proof. The standard proof of the regularity of the splitting is to use the invariant section theorem.

We see that the action on the C^r sections of the Grasmannian of maps close to the stable bundle has the norm

$$\sup_x |Df(x)|^r |D_s f(x)||(D_u f)^{-1}(x)| \tag{1.6}$$

Hence, provided that r is such that (1.6) is smaller than 1, we obtain that the bundle is C^r. (In the original paper [HP], (1.6) is substituted by the expression that is obtained taking the sup in all the factors. In [HK], one can find the remark that it is (1.6) what is needed. Morever, using the fact that the invariant splitting for f is the same as that for f^n one can use f^n in place of f in (1.6).)

Of course, we can obtain the same result in any smooth metric.

We have shown in Theorem 1.2 that we can obtain a metric $g_u \in C_u^{r-1}$ in the stable bundle for which $|Df(x)||Df^{-1}(f(x))| = 1$. We also have a metric on the stable bundle so that $|D_s f(x)| \le \lambda < 1$.

We can consider a metric which is the direct sum of the two for which obviously, (1.6) is satisfied for $r > 1$. This metric is not smooth, but if we replace it by an smooth approximation, the bounds do not change much and the argument shows that indeed there is a $C^{1+\alpha}$ splitting.

$\blacksquare$

2. Preliminaries for the proofs of Theorem 1.1

2.1. Overview of the arguments

The proofs we will present have several recognizable milestones

1) Replacing h by a smoother function.

2) Proving that h is absolutely continuous

3) Bootstrapping the smoothness

For each of the steps 2) and 3) we will present two independent proofs.

For 2) we will show how to obtain the result using the scattering method of [L1], [LM] and how — when dimension of the stable or unstable manifolds is bigger than 1 — using quasi-conformal estimates.

For 3) we will present a bootstrap argument similar to that in [L1], [LM] and another one based on the use of SRB measures. The latter one has the advantage that when the stable or unstable manifolds are 1-dimensional, it makes unnecessary step 2. Also, it is the only bootstrap we are aware of that can establish analytic regularity.

We hope that presenting alternative arguments may be useful to the readers interested in extending the theory.

For the experts, we point out that the step of local smoothing has been significantly stremlined because we have removed the restriction that the smoothed map is a homeomorphism. (The need to get a local homeomorphism was the main reason why the result in [L1] was only local.) This is accopplished by observing that the construction of [Sh] produces a homeomorphism even if the starting point of the iterative process is not a homemorphism. (See Section 2.2)

As usual in many rigidity results for Anosov systems, an important role will be played by theorems that assert existence and smoothness of solutions of cohomology equations. Here, we will have to consider cohomology equations over non-commutative groups and we review briefly the theory we need.

To simplify the notation we will do the detailed work only with diffeomorphisms and deduce the results for flows by taking time one maps of the later. This will require that our regularity results are worked both for Anosov diffeomorphisms and for time maps of Anosov flows.

If we take $f = X_1$, $g = Y_1$, the conjugacy between X and Y is a conjugacy between f, g. In the proof we will show that provided that f, g have stable and unstable manifolds, the conjugacies are smooth when restricted to the stable and unstable manifolds.

If f and g are Anosov diffeomorphisms, by Lemma 2.3 of [LMM] or [Jo], this implies that they are smooth. If f and g are time one maps of Anosov flows, we only have to remark that smoothness of h

along the direction of the flow is obvious and that the regularity lemmas of [LMM], [Jo] can be made to work in this situation (Lemma 2.3 of [LMM] does not require any change. The result in [Jo] only works when we have two transverse foliations, but we can first work on the leaves of W^{cs}_x and apply [Jo] to the orbits and the stable foliation and then apply the result to the transverse foliations of W^{cs}_x and W^u_x.)

2.2. M. Shub's theory of conjugacy of expanding maps and local smoothing.

We recall that in [Sh] it was proved that if f, g are expanding maps of a manifold in the same homotopy class

$$h = \lim_{n \to \infty} f^n g^{-n}$$

exists and is a homeomorphism that satisfies

$$h \circ g = f \circ h$$

Here we want to state another theorem that is proved by the same methods as the result in [Sh].

We will be considering maps that have a contracting foliation. That is, we assume that and we can find $C > 0$, $0 < \lambda < 1$, $\lambda < \mu$

$$\begin{aligned} &T_x M = E^s_x \oplus E^{ns}_x \\ &v \in E^s_x \Leftrightarrow |Df^n(x)v| \le C\lambda^n |v| \qquad n \ge 0 \\ &v \in E^{ns}_x \Leftrightarrow |Df^n(x)v| \le C\mu^n |v| \qquad n \le 0 \end{aligned} \tag{2.1}$$

We recall (see [HP], [Fe1], [Fe2]) that we can associate a strong stable foliation whose leaves are characterized by

$$y \in W^s_x \Leftrightarrow d(f^n x, f^n y) \le C\lambda^n d(x, y) \qquad n \ge 0 \tag{2.2}$$

This foliation has C^r leaves, and the C^r jets of the leaves are Hölder functions on the manifold. Also, $T_x W^s_x = E^s_x$.

We also remark that we can introduce an adapted metric so that all the C's in the previous estimates (2.1) (2.2) become 1. We will do that since it simplifies the estimates without any loss of generality.

Note that because the leaves are smooth, it makes sense to restrict the ambient Riemannian metrics to them.

If x and y are in the same leaf, we define $d_s(x, y)$ to be the distance along W^s_x considered as a Riemannian manifold on its own. We emphasize that, even if $d_s(x, y) \le d(x, y)$, $d_s(x, y)$ can be much bigger than $d(x, y)$ since W^s_x can wrap around the manifold and come back close to itself.

For the applications we have in mind (Anosov systems and time-1 maps of Anosov flows) we can take μ to be equal to 1. In that case, we point out that the stable foliation is invariant under homeomorphisms.

That is, if f, g are as above, and $h \circ f = g \circ h$, with h homeomorphism then,

$$
\begin{aligned}
(2.3) \qquad & y \in W_x^{s,f} \Leftrightarrow d(f^n(x), f^n(y)) \to 0 \Leftrightarrow \\
& \Leftrightarrow d(h \circ f^n(x), h \circ f^n(y)) \to 0 \\
& \Leftrightarrow d(g^n \circ h(x), g^n \circ h(y)) \to 0 \Leftrightarrow h(y) \in W_{h(x)}^{s,g}
\end{aligned}
$$

Note, however, that we have formulated the following result in such a way that this is not needed.

Theorem 2.1. *Let f, g be diffeomorphic with stable foliations. Let h be a homeomorphism of the manifold such that:*

i) $h \circ f = g \circ h$ *(and, therefore, by (2.3)* $h(W_x^{s,f}) = h(W_x^{s,g})$ *)*

Let k be a map — not necessarily invertible — such that

ii) $k(W_x^{s,f}) \subset W_{h(x)}^{s,g}$

iii) $\sup_x d_s(k(x), h(x)) < \infty$. *Then*

$$
(2.4) \qquad h(x) = \lim g^n \circ h \circ f^{-n}(x)
$$

and the limit is reached uniformly in the distance d_s.

Proof. We just observe that if we consider the space of maps k satisfying ii) in Theorem 2.1 topologized with

$$
d(k_1, k_2) = \sup_x d_s(k_1(x), k_2(x))
$$

it is a complete space and that the map $k \to \mathcal{C}(k) = f \circ k \circ g^{-1}$ is a contraction because

$$
\begin{aligned}
d_s(f \circ k_1 \circ g^{-1}_{(x)}, f \circ k_2 \circ g^{-1}(x)) &\le \lambda d_s(k_1 \circ g^{-1}(x), k_2 \circ g^{-1}(x)) \\
&\le \lambda \sup_y d_s(k_1(y), k_2(y))
\end{aligned}
$$

Since that last term is independent of x, when we take $\sup_x$ in the left hand side, we prove that $\mathcal{C}$ is a contraction and hence, $\mathcal{C}^n(k)$ converges to the unique fixed point of $\mathcal{C}$.

■

The space of maps satisfying ii) is not empty because we assume that there exists an h that satisfies that.

Remark. Even if we will not need it, we point out the map k produced in Theorem 2.1 is a homeomorphism if it is homotopic to the identity. This is because, if we reverse the role of f and g we can obtain another map $\tilde{k}$ that satisfies $\tilde{k} \circ g = f \circ \tilde{k}$. Then we have, $k \circ \tilde{k} \circ g = g \circ k \circ \tilde{k}$, $\tilde{k} \circ k \circ f = f \circ \tilde{k} \circ k$. Since f and g are expansive, we obtain the $k \circ \tilde{k} = \mathrm{Id}$, $\tilde{k} \circ k = \mathrm{Id}$.

We will also need the following result that tells us that we can start the scattering process in Theorem 2.1 from a slightly smooth map.

Lemma 2.2. *Let f, g be either a C^r Anosov ($r = 1, \dots, \infty, \omega$) Anosov diffeomorphism or a time one map of an Anosov flow. Let h be a homeomorphism satisfying:*

i) $h \circ f = g \circ h$

(Hence $h(W_x^{s,f}) = W_{h(x)}^{s,g}$.) Then, we can find a map k that:

a) $k(W_x^{s,f}) = W_{h(x)}^{s,g}$

b) $\sup_x d_s(k(x), h(x)) < \infty$

c) $k \in C_s^{\infty,f}$

d) *The r-jets of k are C^α on the manifold.*

e) *The map k can be chosen to be as C^0 close to h as we desire.*

Proof. The idea of the proof is the observation that, given a map h that sends the stable foliation of f into that of g, we can produce — using local coordinates — another map which is smooth in all the places where h was and, moreover, in a neighborhood, is as smooth as in the claim of the result. By repeating the process a finite number of times, we end up with a map which is smooth everywhere.

Recall that, by the standard theory of invariant (un)stable foliations, (see e.g., [A, §16]), in any neighborhood of a point, we can find a coordinate system that sends the foliation into the standard foliation obtained taking Cartesian products of open sets in Euclidean spaces.

More precisely, denoting by B_ρ^s, B_ρ^u the unit balls in $\mathbb{R}^s, \mathbb{R}^u$, we can find maps

$$\Lambda_x^f : B_\rho^s \times B_\rho^u \to M$$

in such a way that

a) $\Lambda_x^f(0,0) = x$

b) $\Lambda_x^f(B_\rho^s \times \{y\}) = W_{\Lambda_x^f(0,y)}^{s,f,\mathrm{loc}}$

c) The mapping $\Lambda_x^f(\cdot, y)$ is a C^∞ local diffeomorphism and the C^∞ jets of these maps depend continuously on y.

d) A finite number of these coordinate patches covers the manifold.

Of course, analogous maps can be constructed for y.

If h is a map that sends the stable foliation of f into that of g, we have:

$$\Lambda^{g^{-1}}_{h(x)} \circ h \circ \Lambda^f_x(s,u) = (\alpha(s,u), \beta(u))$$

with α, β continuous mappings on an open neighborhood of the origin.

We can now approximate the function $\alpha(s,u)$ by another function $\tilde{\alpha}$ such that

a) $\tilde{\alpha}(s,u)$ agrees with $\alpha(s,u)$ if $|s| > \tilde{\rho}$, $|u| > \tilde{\rho}$

b) $\tilde{\alpha}(\cdot,u)$ is C^∞ and the C^∞ jet depends continuously on u.

c) The modulus of continuity of $\tilde{\alpha}(\cdot,u)$ is not bigger than that of $\alpha(\cdot,u)$.

(A convenient way of doing this is setting

$$\tilde{\alpha}(s,u) = \int K(s,s';u)\alpha(s',u)\,ds$$

where $K(s,s';u) = \delta(s-s')$ if u or s are large.)

The map $\tilde{h} = \Lambda^g_{h(x)}(\tilde{\alpha},\beta) \circ \Lambda^{f^{-1}}_x$ agrees with h outside of a neighborhood — so that it can be extended by h. Notice also that $\tilde{h}$ is certainly $C^{\infty,f}_s$ in the places where h is and also in a neighborhood of x.

By repeating the process a finite number of times we end up with a map that satisfies the conclusions of the theorem.

∎

Remark. We also note that the proofs we present could work with considerably less smoothness of the map k. For the first proof using the scattering method it would suffice that k is in $C^{1,f}_s$ and for the proof using quasi-conformal estimates it would suffice that it is quasi-conformal.

Smooth Livsic theory for non-commutative maps.

Existence and properties of trivializations.

2.3. Proof of Theorem 1.2

A very important role in the argument will be played by the Livsic theorems that discuss the existence and smoothness of solutions to cohomology equations over transitive Anosov systems.

In this sections, we will review them briefly and show how they can be used to prove existence and regularity of trivializations and also to prove Theorem 1.2.

We will find very important the following results.

Theorem 2.3. *Let f be a transitive Anosov diffeomorphism of (resp. X_t a transitive Anosov flow on) a compact manifold M. Let G be a Lie group, $\mathcal{G}$ its algebra. Let $\eta : M \to G$ (resp. $\eta : M \to \mathcal{G}$) be a C^α function $0 < \alpha \leq \mathrm{Lip}$.*

Assume that: i) *Whenever* $f^n(x) = x$*, then* $\eta(f^{n-1}(x)) \cdots \eta(x) = \mathrm{Id}$ *(resp. whenever* $X_T(x) = x$ *then* $\Gamma(t)$ *the solution of* $\frac{d}{dt}\Gamma(t) = \eta(X_t(x))\Gamma(t)$*;* $\Gamma(0) = \mathrm{Id}$ *verifies* $\Gamma(T) = \mathrm{Id}$*)*

ii) *If* G *is neither compact, abelian nor nilpotent assume* $\eta(M)$ *lies in a sufficiently small neighborhood of the identity in* G*.*

Then,

a) *There exist a* $\varphi : M \to G$ *such that*

$$\varphi(f(x))\,\eta(x) = \varphi(x) \tag{2.5}$$

(resp.

$$-\varphi^{-1}(x)\,\eta(x) = \dot{\varphi}(x) \tag{2.6}$$

where $\dot{\varphi}(x) = \frac{d}{dt}\varphi(X_t(x))|_{t=0}$*)*

b) φ *is* C^α *and* $\|\varphi\|_{C^\alpha} \le K\|\eta\|_{C^\alpha}$

Remark. The statement of Theorem 2.3 for diffeomorphisms can be found in [Li2]. In that paper, one can also find a statement of the theorem without assumption ii). Unfortunately, it seems well known among the experts that the argument presented there for this part is false and indeed, proving Theorem 2.3 without assumption *ii*) is one of the outstanding problems. The argument for the result as stated can be found in [Li2]. More modern expositions can be found in [NT1] and [L3].

Remark. Note that (2.6) is an infinitesimal version of (2.5). If we write, $f(x) = 1 + \varepsilon\dot{X}$ and $\eta(x) = \mathrm{Id} + \varepsilon\hat{\eta}(x)$ — with $\hat{\eta}$ an element of the Lie algebra! — then, substituting

$$\varphi(x + \varepsilon\dot{X})^{-1} = \varphi^{-1}(x) - \varepsilon\varphi^{-1}(x)\dot{\varphi}(x)\varphi^{-1}(x)$$

into (2.5) we derive (2.6). This heuristic way of thinking will show how we can go from one to the other by "integrating" or "taking derivatives."

Remark. Note that condition i) is clearly a necessary condition for the existence of solutions to (2.5). In effect, if such solution existed

$$\begin{aligned}
&\eta(f^{n-1}(x)) \cdots \eta(x) = \\
&(\varphi(f^n(x)))^{-1}\varphi(f^{n-1}(x))\ (\varphi(f^{n-1}(x)))^{-1}\varphi(f^{n-2}(x)) \cdots (\varphi(f(x)))^{-1}\varphi(x) = \\
&= (\varphi(f^n x))^{-1}\varphi(x) = \mathrm{Id}
\end{aligned}$$

A similar argument works for the case of flows.

The regularity of the solutions is (2.5), (2.6) is addressed by the following result.

Theorem 2.4. *Let f be a C^{r+1} $r = 1, 2, \ldots, \infty, \omega$ Anosov diffeomorphism of (resp. X_t a C^{r+1} Anosov flow on) a compact manifold M. Let $\eta(x)$ be a C^r_s function from M to a Lie group G (resp. to a Lie algebra $\mathcal{G}$). Let ϕ be a continous function from M to G such that it satisfies (2.5) (resp. (2.6))*

Then ϕ is C^r_s. Analogous ressults hold for regularity along the unstable foliation. In particular, if $\eta \in C^r \subset C^r_s \cap C^r_u$, then, $\phi \in C^r_s \cap C^r_u \subset C^{r-1+\mathrm{Lip}}$.

The inclusion $C^r_s \cap C^r_u \subset \Lambda^r \subset C^{r-\epsilon}$ was established in [LMM] for $r \le \infty$. Another proof can be found in [Jo]. The paper [HK] contains a proof for the $r = \infty$ case. Even if the method of proof there leads to a higher loss that ϵ (with this method one can only conclude that $C^r_s \cap C^r_u \subset C^{r-n-2}$), the method can be extended (See [L4]) to conclude the result for $r = \omega$.

The proof of Theorem 2.4 can be found in [NT1] and in [L3].

As a simple corollary of Theorem 2.4 is the following:

Proposition 2.5. *Let f be as in Theorem 1.1. Then, it is possible to choose $\gamma_s, \Psi^s \in C^{r-1}_s$. Analogous results hold for the trivialization of the unstable subbundle.*

Actually we will show that the only thing that we need to do to obtain an smooth Ψ^s is to multiply Ψ^s by a real valued function.

Proof. We can assume that there is a C^{r-1} metric on the manifold. For $r \le \infty$ this is an easy argument based on particions of unity. For $r = \omega$ it is a deep theorem in [Gr].

We note that we can multiply Ψ^s by the factor $(\det \Psi_s)^{1/d_s}$, where d_s is the dimension of the stable subbundle without altering that it is a trivialization – note that the determinant cannot vanish because of the assumption that it is a trivialization which is invertible– . Hence, we can assume without loss of generality that $\det \Psi^s = 1$. Of course, the γ_s gets modified accordingly.

When these normalizations are enforced, we obtain that

$$\gamma_s = \det D_s f$$

which, of course belongs to C^{r-1}_s.

We can then write (1.3) as

$$\Psi_{f(x)} \Psi_x^{-1} = (1/\gamma_s) D_s f(x)$$

This is an equation that falls under the scope of Theorem 2.4.

∎

Proposition 2.6. *Assume that f is an Anosov diffeomorphism in a manifold satisfying ii) of Theorem 1.1. Then, there exist a C^1 neighborhood V of f such that if $g \in V$ satisfies i), it also satisfies ii).*

Proof. We denote by $\Psi^{s,(g)}, \Psi^{s,(f)}$ the trivializations corresponding to the maps f, g and, similarly $\gamma_s^{(f)}, \gamma_s^{(g)}$.

As in Proposition 2.5, we can assume that there is an analytic metric in the mainfold and that $\gamma_s^{(f)} = (\det D_s f)^{1/d_s}$, $\gamma_s^{(g)} = (\det D_s g)^{1/d_s}$.

We note that because $\Psi^{s,(f)} \circ f D_s f (\Psi^{s,(f)})^{-1} = \gamma_s^{(f)} \operatorname{Id}$ and f, g are C^1 close, we have

$$\Psi^{s,(f)} \circ g D_s g (\Psi^{s,(f)})^{-1} = \gamma_s^{(f)} \operatorname{Id} + \Delta$$

where Δ is small.

Note also that

$$(1/\gamma_s^{(g)}) \Psi^{s,(f)} \circ g D_s g (\Psi^{s,(f)})^{-1} = \gamma_s^{(f)}/\gamma_s^{(g)} \operatorname{Id} + (1/\gamma_s^{(g)}) \Delta$$

satisfies the hypothesis of Theorem 2.3 when g satisfies hypothesis $i)$ of Theorem 1.1.

Then, we conclude that we can find Γ such that

$$\Gamma \circ g (\gamma_s^{(f)}/\gamma_s^{(g)} \operatorname{Id} + (1/\gamma_s^{(g)} \Delta) (\Gamma \circ g)^{-1} = \operatorname{Id}$$

Hence, we can take $\Psi_s^{(g)} = \Gamma \Psi_s^{(f)}$.

■

Once we have the trivilizations, it is elementary to give a proof of Theorem 1.2 in our case. The metric is just the pull back of the standard metric of $\mathbb{R}^n$ given the trivializations.

We point out that it is possible to give a much more geometric proof that does not require that the bundle of the manifold is trivial. The idea is that we fix the metric at one point with a dense orbit. We can propagate the metric along the orbit by just doing a push forward and multiplying by a conveniently chosen factor. The same arguments used in the proof of the non-commutative Livsic theorem yield the result. Full details of this construction appear in [L3]. Nevertheless, for our purposes, the proof using the trivialization is enough.

3. Almost everywhere differentiability of the conjugacy.

The main goal of this section is to present two proofs of the following result.

Theorem 3.1. *With the notations above, and in the conditions of Theorem 1.1 and Theorem 2.1, the map*

$$h = \lim_{n\to\infty} g^n \circ k \circ f^{-n}$$

is absolutely continuous when restricted to any $W_x^{s,f}$.

In a first subsection, using the scattering method, we will prove it is Lipschitz and in a second subsection, using quasi-conformal maps, we will show that the limit is quasiconformal.

3.1. Proof of Theorem 3.1 using the scattering method

We note that, by Theorem 2.1 the maps $h_n = g^n \circ k \circ f^n$ are converging uniformly so that it suffices to show that the stable derivatives of these maps are bounded independently of x and of n.

Since the trivialization maps Ψ and their inverse norms which are bounded independently of x and n, it suffices to bound

$$\|\Psi^{s,g}_{R_n(x)} D_s h_n(x) (\Psi^{s,f}_x)^{-1}\| \equiv E_n(x) \tag{3.1}$$

independently of n and x.

Taking D_s in the definition of h_n, applying the chain rule and introducing the notation

$$\begin{aligned} M(x) &= \Psi^{s,g}_{g(x)} Dg(x) (\Psi^{s,g}_x)^{-1} \\ N(x) &= \Psi^{s,f}_x Df^{-1}(x) (\Psi^{s,f}_{f(x)})^{-1} \end{aligned} \tag{3.2}$$

we see that we can rewrite (3.1) as:

$$\begin{aligned} E_n(x) = \|M(g^{n-1} \circ k \circ k \circ f^{-n}(x) \cdots M(k \circ f^{-n}(x)) \Psi^{s,g}_{kf^{-n}(x)} D_s k(f^{-n}(x)) \\ (\Psi^{s,f}_{f^{-n}(x)})^{-1} N(f^{-n+1}(x)) \cdots N(x)\| \end{aligned} \tag{3.3}$$

Note that the Ψ's that appear in the previous formula and the $D_s k$ are uniformly bounded, so that it suffices to show that the product of M's and N's is uniformly bounded.

Note that the M and N's are matrices that are multiples of the identity so that the proof can be carried out in a manner very similar to that in [L1].

We define $m(x)$, $\ell(x)$ by:

$$e^{m(x)}\,\mathrm{Id} = M(x) \qquad e^{\ell(x)}\,\mathrm{Id} = N(x)$$

Then, with this notation, to show that (3.3) is bounded it suffices to show that

$$\sup_x \left|[m \circ g^{n-1} \circ k + \cdots + m \circ k + \ell \circ f + \cdots + \ell \circ f^n] \circ f^{-n}(x)\right| \tag{3.4}$$

is bounded independently of n.

Note first that in taking suprema over x we can ignore the last f^{-n}. We also note that since $d(g^{n-1} \circ h(x), g^{n-1} \circ k(x)) \leq K\lambda^n$ for some $0 < \lambda < 1$ and since m is a C^α function

$$\begin{aligned}
&\sup_x |m \circ g^{n-1} \circ k(x) + \cdots + m \circ k(x) - m \circ g^{n-1} \circ h(x) + \cdots + m \circ h(x)| \\
&\quad \leq \sup_x |m \circ g^{n-1} \circ k(x) - m \circ g^{n-1} \circ h(x)| \\
&\qquad + \sup_x |m \circ g^{n-2} \circ k(x) - m \circ g^{n-2} \circ h(x)| \\
&\qquad + \cdots + \\
&\qquad + \sup_x |m \circ k(x) - m \circ h(x)| \\
&\quad \leq K\lambda^{\alpha n} + K\lambda^{\alpha(n-1)} + \cdots + K\lambda \leq K
\end{aligned}$$

so that to show that (3.4) is bounded it suffices to show that

$$\begin{aligned}
&\sup_x |m \circ g^{n-1} \circ h(x) + \cdots + m \circ h(x) + \ell \circ f^n + \cdots + \ell \circ f| = \\
&\quad = \sup_x |m \circ h \circ f^{n-1}(x) + \cdots + m \circ h(x) + \\
&\qquad \ell \circ f^{n-1}(x) + \cdots + \ell(x) + [\ell \circ f^n(x) - \ell(x)]|
\end{aligned}$$

where we have used that h satisfies the intertwining relation. Ignoring the terms in $[\cdots]$, which are obviously bounded, we have that the term to estimate becomes, calling $r(x) = m \circ h(x) + \ell(x)$

$$\sup_x |r \circ f^{n-1}(x) + \cdots + r(x)| \tag{3.5}$$

We now observe that, if $f^n(x) = x \sum_{i=0}^{n-1} n(f^i(x))$ is the Lyapunov exponent of $f^{-n}(x)$ and $\sum_{i=0}^{n-1} m \circ h \circ f^i(x)$ is the Lyapunov exponent of g^n at $h(x)$. By the hypothesis of equality of Lyapunov exponents at periodic orbits, they are equal. Hence r satisfies the compatability conditions of the commutative Livsic theorem. Hence:

$$r(x) = -s(f(x)) + s(x)$$

and substituting in (3.5) we obtain a telescoping sum that reduces the sum to

$$\sup_x |-s \circ f^n(x) + s(x)|$$

This is clearly bounded and the theorem is established.

■

3.2. Proof of Theorem 3.1 using quasi-conformal estimates.

We will show that

$$h = \lim_{n\to\infty} g^n \circ k \circ f^{-n}$$

is quasi-conformal when restricted to any $W_x^{s,f}$.

We estimate the distortions of the $h_n = g^n \circ k \circ f^{-n}$ when restricted to the stable manifold. We claim that they are uniformly bounded.

Recall that $K(h_n \mid W_x^f) = \sup_x K(D_s h_n)$. Since the distortion of $\Psi_x^{s,f}, \Psi_x^{s,g}$ and their inverses are uniformly bounded to show that $K(D_s h_n(x))$ is uniformly bounded it suffices to show that $K(\Psi^{s,g}_{h_n(x)} D_s h_n(x) (\Psi_x^{s,f})^{-1})$ is uniformly bounded.

Again, using the chain rule and introducing the notation (3.2) we see that

$$\begin{aligned}
&K(\Psi^{s,g}_{h_n(x)} D_s h_n(x)(\Psi_x^{s,f})^{-1}) = \\
&= K(M(g^{n-1} \circ k \circ f^n(x)) \cdots M(k \circ f^{-n}(x))(\Psi^{s,g}_{kf^{-n}(x)})^{-1} D_s k(f^{-n}x) \\
&\quad ((\Psi^{s,f}_{f^{-n}(x)} N(f^{-n+1}(x)) \cdots N(x))
\end{aligned} \tag{3.6}$$

Recall that $K(AB) \le K(A)K(B)$ and that if A is diagonal $K(A) = 1$. Since M, N are multiples of the identity, we can estimate (3.6) by

$$K\left((\Psi^{s,g}_{kf^{-n}(x)})^{-1} D_s k(f^{-n}(x)) \Psi^{s,f}_{f^{-n}(x)}\right)$$

which can be clearly estimated by a constant.

We recall that Theorem 2.1 guarantees that h_n converges to a homeomorphism. Even if the pointwise limit of quasi-conformal maps may not be quasi-conformal, when the limit is a homeomorphism it is. (See e.g., Theorem 37.2 and Corollary 37.3 of [V, p.125ff.] or [GP])

■

By standard properties of quasi-conformal mappings (see e.g., [V, §32, p.109ff. and 32.3, p.109ff.], [GP]) we have:

Corollary 3.2. *h is differentiable almost everywhere, it is absolutely continuous when restricted to almost all lines, the partial derivatives are in L^s_{loc} and*

$$\|D_s h(x)\| \le K J(h,x)^{1/d_s}$$

where $J(h,x)$ denotes the Jacobian and d_s is the dimension of the stable manifold.

4. Higher regularity of the conjugacy.

In this section, we want to present two proofs of the following result:

Theorem 4.1. *Assume that f, g are C^r diffeomorphisms $r = 1, 2, \ldots, \infty, \omega$ as before. Assume that h is a homeomorphism such that $h \circ f = g \circ h$ and that h is Lipschitz when restricted to the stable manifold, Then, h is $C_s^{r,f}$.*

We note that the first proof is not a proof of the theorem exactly as stated since it does not cover the case $r = \omega$. This will be obtained with the second proof.

The first part of the proof is common to both approaches, we will show that the conjugacy is C^1.

Remark. We point out that Theorem 6.1 of [L2] shows that, for any high dimensional C^r Anosov map, all the conjugacies to another C^r map that are sufficently smooth are C^r $r = 1, \cdots, \infty$. The condition of this theorem depends on the metric and it is advantageous to choose the conformal metrics constructed in the previous results. In that case, we obtain that the critical regularity is 2. So that, it follows from Theorem 6.1 in [L2] that, under the conditions $i)$,$ii)$ of Theorem 1.1, any C^2 conjugacy h is $C^{r-\epsilon}$.

4.1. The conjugacy is C^1.

First, we will show that the mapping h is C^1 and that the derivatives are Hölder along the manifold. Again, it suffices to show it is in C_s^1 and a similar argument will show that it is in C_u^1. For the case of time one maps of flows, the differentiablility along the flow lines is automatic.

We note that we will be assuming that $r \geq 2$, so that in this step, we will actually able to conclude $C^{1+\alpha}$ for some $\alpha > 0$.

We note that, since we have shown that h restricted to the stable leaf is absolutely continuous, it is differentiable almost everywhere. Taking derivatives with respect to the stable direction of f the conjugacy equation (a) of Theorem 1.1), we have for almost all x:

$$D_s^{(f)} f \circ h(x) D_s^{(f)} h(x) = D_s^{(f)} h \circ g(x) D_s^{(g)} g(x) \tag{4.1}$$

Recall that we had used the notation $D_s^{(f)}, D_s^{(g)}$ to denote the derivative along the stable leaves of f and g. Note that we have used the fact that h maps stable leaves of g into stable leaves of f.

Now, multiplying (4.1) on the right by the trivialization map $\Psi^{s,(f)}_{f\circ h(x)} = \Psi^{s,(f)}_{h\circ g(x)}$ and on the left by $(\Psi^{s,(g)}_x)^{-1}$, we have:

$$\begin{aligned} &\left[\Psi^{s,(f)}_{f\circ h(x)} D_s^{(f)} (\Psi^{s,(f)}_{h(x)})^{-1}\right] \cdot \left[\Psi^{s,(f)}_{h(x)} D_s^{(f)} h(x) (\Psi^{s,(g)}_x)^{-1}\right] \\ &= \left[\Psi^{s,(f)}_{h\circ g(x)} D_s^{(f)} h \circ g(x) (\Psi^{s,(g)}_{g(x)})^{-1}\right] \cdot \left[\Psi^{s,(g)}_{g(x)} D_s^{(g)} g(x) (\Psi^{s,(g)}_x)^{-1}\right] \end{aligned} \tag{4.2}$$

We recall that, by hypothesis, the first factor in the left and the last factor in the right were multiples of the identity.

Hence, introducting the function

$$\Gamma(x) = \Psi^{s,(f)}_{h(x)} D^{(f)}_s h(x) (\Psi^{s,(g)}_x)^{-1} \tag{4.3}$$

, we see that (4.2) can be written as:

$$\gamma^{(f)}_s \circ h(x) \Gamma(x) = \Gamma \circ g(x) \gamma^{(g)}_s(x)$$

Or, using the fact that $\gamma^{(f)}_s, \gamma^{(g)}_s$ are just scalar functions

$$\Gamma(x)\Gamma \circ g(x)^{-1} = \gamma^{(f)}_s(h(x))/\gamma^{(g)}_s(x)\,\mathrm{Id} \tag{4.4}$$

This is an equation of the form considered in Theorem 2.3.

Note also that, since by the structural stability theorem we obtain that h is Hölder when restricted to the stable manifold, the right of (4.4) is Hölder.

More importantly, note that by the hypothesis on periodic orbits, if $g^N(x) = x$

$$\prod_{i=0}^{N-1} \left[\gamma^{(f)}_s(h)/\gamma^{(g)}_s\right] \circ g^i(x) = 1$$

We can apply the scalar Livsic theorem and find a Hölder function σ bounded away from zero such that

$$\sigma(x)/\sigma \circ g(x) = \gamma^{(f)}_s(h(x))/\gamma^{(g)}_s(x)$$

Nothat that $\sigma(x)\,\mathrm{Id}$ is already a solution of (4.4) and it is Hölder. So, our task reduces to proving uniqueness statements for measurable solutions of Livsic equations.

In the generality of the non-commutative case, this results have been obtained recently in [PP],[Wa].

In, our case, however, we can imitate the proof given in the commutative case in [L1].

Using (4.4), we see that

$$[\sigma(x)]^{-1}\Gamma(x) = [\sigma(g(x))]^{-1}\Gamma(g(x))$$

Since g is ergodic with respect to SRB measures, we conclude that $\sigma^{-1}\Gamma$ is constant almost everywhere with respect to the SRB measures.

We also recall that the SRB measures have density on stable leaves. So that we can find a leaf of the stable foliation for which $\sigma(x)\Gamma(x)$ is constant almost everywhere with respect to the Riemann measure on the leaf. In particular, it is constant almost everywhere in an open set.

Since for transitive Anosov maps, these open neighborhoods are dense, using the invariance, we conclude that Γ agrees with a constant multiple of $\sigma\,\mathrm{Id}$ on a dense set of stable leaves. For the case of transitive Anosov flows, we need to observe that if an invariant function is constant in an open neighborhood in a stable manifold, it is also constant in the stable manifold.

Once we have that a continuous function is the derivative in a dense set of leaves, we conclude it is the derivative everywhere because we can justify the finite increment formula

$$h(x) - h(y) = \int_0^1 D_s h \circ \beta(t)\, dt,$$

where β is a path contained in a stable leaf by approximating it by paths contained in stable leafs for which the derivative agrees with our Hölder function.

Of course the same argument can be applied to obtain that the derivative along unstable directions is also Hölder.

■

We note that, as a corollary of the proof, we have obtained that

$$\Gamma^s(x) = \Psi^{s,(f)}_{h(x)} D^{(f)}_s h(x) (\Psi^{s,(g)}_x)^{-1}$$
$$\Gamma^u(x) = \Psi^{u,(f)}_{h(x)} D^{(f)}_u h(x) (\Psi^{u,(g)}_x)^{-1}$$

are multiples of the identity.

Remark. We point out that the only place where we use that r is at least two is in the uniqueness theorem for L^∞ solutions of cohomology equations. The proof we presented depends on the use of ergodic theory – the invariant measures – which is not well controlled for C^1 maps. The arguments in [Wa] establish uniqueness when the flow is C^1 and the solution is almost everywhere with respect to an equilibrium state corresponding to a C^α function. It seems plausible that, for area preserving maps, one can assume only C^1 and push through a similar argument.

4.2. Proof of Theorem 4.1 using the Smooth Livsic theorem.

The method presented in this subsection cannot obtain the result $r = \omega$.

We note that we have shown that the stable deriviative of h exists and that, in the previously introduced notations, Γ,(see (4.3)) the trivialization of the stable derivative of h can be written as $\sigma(x)\,\mathrm{Id}$ and, hence, (4.4) can be written as

$$\sigma(x)/\sigma(g(x)) = \gamma_s^(f))(h(x))/\gamma_s^{(g)}(x).$$

We note that if $h \in C_s^{\ell,(g)}$, the theorem Theorem 2.4 implies that $\sigma \in C_s^{\ell,(g)}$, which in turn implies $h \in C_s^{\ell+1,(g)}$. Since, from previous results we have $h \in C_s^{1,(g)}$, we can start the induction and use that the induction stops only when we cannot obtain more regularity of the RHS of (4.4).

Recall that the RHS of (4.4) is formed out of the derivatives of the maps f and g and the trivializations. If the maps are C^r, obviously the derivatives are C^{r-1} and we showed that the trivializations are in C^{r-1}_s.

Hence, using this argument, we can conclude that $h \in C^r_s$, when $s = 1, 2, \infty$. The reason we cannot decide the case C^ω is because we have a very poor control of the size of derivatives of increasing order. This involves not only substituting, but also obtaining good estimates in Theorem 2.4 depending on the order.

4.3. Proof of Theorem 4.1 using SRB measures.

As it turns out this proof does not obtain the full regularity in Theorem 4.1.

We recall that transitive Anosov systems carry some distinguished invariant measures (which we will denote by $\mu_+(f), \mu_-(f)$) which are equilibrium states corresponding to the unstable or stable Jacobians of the system.

In our context, what makes them particularly useful is that they can be approximated by periodic orbits — so that hypothesis at periodic orbits gives us control over them — and that they have absolutely continuous — in fact with a smooth density — restrictions to stable and unstable foliations. We refer to [Si], [Ne], [Ru] for properties of SRB measures.

Lemma 4.2. *Let f, g be transitive C^r Anosov diffeomorphisms (resp. X_t Y_t transitive Anosov flows). Let h be a homeomorphism. Assume that $h \circ f = g \circ h$ (resp. $h \circ X_t = Y_t \circ h$). If the stable Jacobian at corresponding periodic orbits of f and g (resp. of X and Y) then $h^*\mu_-(f) = \mu_-(g)$ (resp. $h^*\mu_-(X) = \mu_-(Y)$). In particular, denoting by $\omega^{(f}_{-x}(y)$ the density of μ_- on W^s_x, we have*

$$\int_A \omega^{(f}_{-x}(y)\, dy = \int_{h(A)} \omega^{(g}_{-h(x)}(y)\, dy$$

(analogously for flows).

Proof. We will only give the proof for diffeomorphisms. Recall that, by results in [Si], [Ru], [Ne] we have:

$$\mu_-(f) = \lim_{n\to\infty} \frac{1}{Z_n} \sum_{o\in \mathrm{Per}[n]} J^{u,(f)}_o \delta_o$$

where $J^{u,(f)}_o$ is the unstable Jacobian of the return map along the periodic orbit o, δ_0 is the usual Dirac measure at the point o Per$[n]$ denotes the set of periodic orbits of period n and Z_n is a normalization factor to have the measure have mass 1.

Note that, under the hypothesis about the Jacobians at corresponding periodic orbit 1,

$$\begin{aligned} h^* \sum_{o\in \mathrm{Per}_f[n]} (J^{s(f)}_o)^{-1}\delta_o &= \sum_{o\in \mathrm{Per}_f[n]} J^{s(f)^{-1}}_o \delta_{h(o)} = \\ &= \sum_{p\in \mathrm{Per}_g[n]} J^{s(g)^{-1}}_p \delta_{po} \end{aligned},$$

Taking limits, we obtain the first result in the claim. The result about the densities is just a consequence of the definition of the densities.

■

Note that if h is differentiable along the stable directions at a point x this gives us immediately

$$J(D_s h(x)) = \frac{\omega^g_{-h(x)}(h(x))}{\omega^f_{-x}(x)} \tag{4.5}$$

where the Jacobian is understood with respect to the induced Riemann metrics on the stable and unstable manifolds.

We just observe that, in particular, the Jacobian is bounded for any Riemann metric.

The control over the Jacobian complements very nicely the information we have using quasi-conformal theory.

We obtain a formula for the determinant of the function Γ introduced in (4.3) as the expression of $D_s h$ on the trivializations.

Its determinant can also be computed as

$$\det \Gamma(x) = J(\Psi^{s,g}_{h(x)}) J(D_s h(x)) J(\Psi^{s,f}_x)^{-1}$$

where we denote by J the Jacobian of a linear mapping when the metrics in the domain and the range are understood.

We also recall that by (4.5) we can write

$$\det \Gamma(x) = F(x, h(x)) \tag{4.6}$$

where

$$F(x,y) = \frac{\omega^{s,g}_{-y}(y)}{\omega^{s,g}_{-x}(x)}.$$

We emphasize that the function F is C^{r-1} in its arguments. (In this case, r can be ω.)

The reason for that is that the densities $\omega^{s,(f)}(x)$ are in in $C^{r-1,(f)}_s$ because, since they are invariant, they satisfy

$$\omega^{s,(f)}(f(x)) = J_(f,x)\omega^{s,(f)}(x)$$

Since we know that $\omega^{s,(f)}(x)$ is continuous, we can apply Theorem 2.4.

We can write (4.6) as

$$D_s h(x) = G(x, h(x)) \tag{4.7}$$

where

$$G(x,y) = (\Psi^{s,g}_y)^{-1} \Psi^{s,f}_x (F(x,y))^{1/s}$$

Note that the function $G(x,y)$ is C^{r-1} in each of its arguments and, in this case r can be ω.

We consider (4.7) as a first order partial differential equation satisfied by h.

The meaning of this is that if we fix a smooth one-dimensional curve contained in the stable manifold, (4.7) reduces to an ODE. Using standard theory of ODE's we obtain that h is C^r in the parameter of the curve. (Since the parameter is the independent variable, we gain one derivative.)

¿From that, it is standard in ODE theory to show that the function is C^r restricted to the manifold. (The standard dependence on parameters for the ODE gives that the function is C^{r-1}, but then, looking at the differential equation we see that it implies that the partial derivatives are C^{r-1} and, therefore, the function is C^r.

$\blacksquare$

Remark. The differential equation (4.7) was derived in [L4] by a very different method.

5. Proof of Theorem 1.4

The idea of the proof is that we will show that the map h restricted to a leaf of the stable foliation is $C^{1-\epsilon}$ and that the Lipschitz seminorm is bounded uniformly independenly of the leaf. The same argument will establish a similar result for the unstable manifold.

Since the stable and unstable foliations are transversal, the angle between the tangent spaces of the leaves is bounded uniformly from below and the leaves are uniformly C^1 we have that for x, y such that $d(x,y) \le \delta$ (where $\delta > 0$ is a number that only depends on the map)

$$d(x,y) \ge K\big(d_s([x,y],x) + d_u([x,y],y)\big)$$

where $[x,y]$ is the usual construction in the local product structure of hyperbolic theory. Namely, $[x,y] = W_x^{s,\mathrm{loc}} \cap W_y^{u,\mathrm{loc}}$ (that is the only point of the intersection of the local stable manifold of x and the local unstable manifold of y, where we choose the segments of the local stable manifold small enough that for points at a distance δ there is not more than one point in the intersection and big enough so that all points at a distance less that δ indeed have such a point [KH] p. 268 Prop. 6.4.13).

Since we, obviously have

$$d(x,y) \le d_s([x,y],x) + d_u([x,y],y)$$

the metrics $d(x,y)$, $\big(d_s([x,y],x) + d_u([x,y],y)\big)$ are equivalent. It then follows that for an Anosov system and for $0 \le \alpha\,\mathrm{Lip}$, $C_s^\alpha \cap C_u^\alpha = C^\alpha$. (Noting that a function is continuously differentiable if and only if it has continuous partial derivatives, we note that the case for $\alpha = 1$ is also true.)

For flows, the corresponding result is $C_s^\alpha \cap C_u^\alpha \cap C_X^\alpha = C^\alpha$ for $o \le \alpha \le 1$.

Hence, to prove Theorem 1.4 it suffices to establish that $h \in C_s^{1-\epsilon}$. The same argument will then establish that $h \in C_u^{1-\epsilon}$ and this establishes the claimed result for Anosov diffeomorphisms. For Anosov flows we need to supplement this argument with the observation that $X_t \circ h = h \circ Y_t$ implies that h is differentiable along the flow lines.

To prove that $h \in C_s^{1-\epsilon}$ we proceed as in Theorem 2.1. We will consider h as a fixed point of $\mathcal{T}(k) = g \circ k \circ f^{-1}$ acting on a space of mappings k which send the stable foliation of f into the stable foliation of g. Similarly, we can consider h^{-1} as a fixed point of $\mathcal{N}(\ell) = f \circ \ell \circ g^{-1}$ acting on functions that send the stable foliation of g into the stable foliation of f.

Note that, as a consequence of the usual results in structural stability, by assuming that g is C^1 close to f, we have that h, h^{-1} are in an arbitrarily small C^0 neighborhod of the identity.

We observe that, as in the proof of Theorem 2.1 this is a contraction in C_s^0. Hence, if g is close to f, and we take k^0 to be one of the functions produced in Lemma 2.2 smoothing h and ℓ^0 one obtained smoothing h^{-1} – note in special point $d)$ – we have that $\mathcal{T}^n(k^0)$ does not leave a C^0 neighborhood of the identity in $C_s^{0,(f)}$ and $\mathcal{N}^n(\ell^0)$ does not leave a $C_s^{0,(g)}$ neighborhood of the identity. The size of these neighborhoods can be nade as small as desired by assuming that f,g are are sufficiently C^1 close.

Proposition 5.1. *Given $\epsilon > 0$ we can find $\eta > 0$ and fix a C^2 neighborhood of f so that $\mathcal{T}^n(k^0)$ remains bounded in the seminorm*

$$||k||_{\alpha,eta} = \sup_x \sup_{\substack{y,z \in W_x^{s,(g)} \\ 0 < d_s(y,z); d_s(x,y), d_s(x,z) \le \eta}} d(k(y), k(z))/d_s(y,z)^\alpha$$

for $\alpha = 1 - \epsilon$.

Proof. Since this is a local result – notice that we are only considering an η neighborood – we will take coordinates so that we can use the more transparent notation $|y - z|$ rather that $d(x, z)$.

We denote by $k_n = \mathcal{T}^n(k_0)$

Note that $k_{n+1} = g \circ k_n \circ f^{-1}$ and that $||k_n - \mathrm{Id}|| \le \sigma$, where we can make σ as small as we want by assuming f, g are C^1 close.

Note also that, if $\eta' > \eta$, we can bound

$$(5.1)\qquad \begin{aligned} \sup_{\substack{y,z \in B_{\eta'}(x) \\ x \ne y}} |k(y) - k(z)|/|y-z|^\alpha &\le \max\Big(\sup_{\substack{y,z \in B_{\eta'}(x) \\ 0<|y-z|\le\eta}} |k(y) - k(z)|/|y-z|^\alpha, \\ &\qquad \sup_{\substack{y,z \in B_{\eta'}(x) \\ \eta<|y-z|}} |k(y) - k(z)|/|y-z|^\alpha \Big) \\ &\le \max\left(\sup_{\substack{y,z \in B_{\eta'}(x) \\ 0<|y-z|\le\eta}} |k(y) - k(z)|/|y-z|^\alpha, 2 * \eta^{-\alpha} ||k||_{C_0} \right) \end{aligned}$$

If we fix x and take y, z such that $d_s(x,y), d_s(x,z) \le \eta$ we have

$$\begin{aligned}|k_{n+1}(y) - k_{n+1}(z)| &\le \sup_{\xi \in B_{\eta+\sigma}(x)} |g'(\xi)|\, |k_n \circ f^{-1}(y) - k_n \circ f^{-1}(z)| \\ &\le \sup_{\xi \in B_{\eta+\sigma}(x)} |g'(\xi)| \cdot \\ &\qquad \cdot \sup_{a,b \in B_{||Df^{-1}||_{C^0}\eta}(f^{-1}(x))} |k_n(a) - k_n(b)|/|a-b|^\alpha\, |f^{-1}(y) - f^{-1}(z)|^\alpha\end{aligned}$$

If in this inequality we bound the second factor as in (5.1) and also use $|f^{-1}(y) - f^{-1}(z)|^\alpha \le \sup_{\xi \in B_\eta(x)} |Df^{-1}(\xi)|^\alpha |y-z|^\alpha$, we obtain:

$$\begin{aligned}||k_{n+1}||_{\alpha,\eta} \le \max\big(&\sup_{\xi \in B_{\eta+\sigma}(x)} |g'(\xi)| \sup_{\xi \in B_\eta(x)} |Df^{-1}(\xi)|^\alpha ||k_n||_{\alpha,\eta}, \\ &\sup_{\xi \in B_{\eta+\sigma}(x)} |g'(\xi)| \sup_{\xi \in B_\eta(x)} |Df^{-1}(\xi)|^\alpha \eta^{-\alpha} ||k_n||_{C^0}\big)\end{aligned} \tag{5.2}$$

We note that if f is 1-conformal, for any $\epsilon > 0$, if η, σ, are sufficiently small and g is sufficiently C^1 close to f then

$$\sup_{\xi \in B_{\eta+\sigma}(x)} |Dg(\xi)| \sup_{\xi \in B_\eta(x)} |Df^{-1}(\xi)|^{1-\epsilon} \le \kappa < 1 \tag{5.3}$$

where κ is independent of x.

Since $\eta^{-1+\epsilon}$, $||k_n||_{C^0}$ are bounded independently of n, we obtain that (5.2) can be written as

$$||k_{n+1}||_{\alpha,\eta} \le \max \kappa ||k_n||_{\alpha,\eta}, A)$$

where κ and A are independent of n and $\kappa < 1$. It then follows that $||k_n||_{1-\epsilon,\eta}$ remains bounded.

Since k_n is converging uniformly to h, we obtain that h is in $C_s^{1-\epsilon}$.

■

The argument for $\mathcal{N}$ is exactly the same with f and g interchanged, except for a minor modification in the paragraph leading to (5.3).

An identical argument works for the unstable foliation both for $\mathcal{T}$ and $\mathcal{N}$.

For flows, the regularity in the direction of the flow is automatic.

This finishes the proof of Theorem 1.4

■

6. Some examples

In view of the fact that one can have a Livsic theorem for nilpotent groups, one could have hopes that, when the normal forms are just Jordan blocks, one could have rigidity.

This turns out not to be true as shown by the following examples.

Let A be an Anosov linear automorphism of $\mathbb{T}^2$. Consider the automorphism B of $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ defined by:

$$B(x, y) = (Ax + y, Ay)$$

where $x \in \mathbb{T}^2$, $y \in \mathbb{T}^2$. Consider perturbations of the form

$$F(x, y) = (Ax + y + e_s \Phi(y), Ay)$$

where e_s is the stable eigenvalue of A and Φ is a small analytic function.

Denote by λ_s, λ_u the eigenvalues of A. The matrix of the derivative of F in the basis

$$(e_s, 0), (0, e_s), (e_u, 0), (0, e_u)$$

is

$$\begin{pmatrix} \lambda_s & 1 + \alpha(y) & 0 & \beta(y) \\ 0 & \lambda_s & 0 & 0 \\ 0 & 0 & \lambda_u & 1 \\ 0 & 0 & 0 & \lambda_u \end{pmatrix}$$

where α, β are real valued functions that involve only derivatives of Φ. We note that if Φ is C^1 small, then the Jordan normal form of F is exactly B.

Hence, we can find an infinite dimensional manifold around B in which the Jordan form around periodic orbits does not change.

Now, we can compute what is the change of variables satisfying

$$h \circ B = F \circ h \tag{6.1}$$

given by structural stability. Since the structural stability theorem includes uniqueness conclussions we can make an ansatz for the form of the solution and then, provided that we find a solution of this form, it will be the unique solution.

If we use the ansatz

$$h(x, y) = (x + e_u \Gamma(y), y) \tag{6.2}$$

equation (6.1) becomes

$$(Ax + y + e_u \Gamma(Ay), y) = (Ax + Ae_u \Gamma(y) + y + e_u \Phi(y), y)$$

which is equivalent to:

$$\Gamma(Ay) - \lambda_u \Gamma(y) = \Phi(y) \tag{6.3}$$

If Φ is C^1-small, we can get a C^0 small solution of (6.3) by

$$\Gamma(y) = -\sum_{i=1} \lambda_u^{-i} \Phi(A^{i-1}y) \tag{6.4}$$

which converges uniformly.

These series are quite well studied since they are variants of the Riemann-Weierstrass function. We note that in general – even for trigonometric polynomials – the sum will not correspond to a C^1 function.

Note that if we take $\Phi(x) = e^{2\pi i k\cdot x}$ then $\hat{\Gamma}_{(A^t)^{i-1}k} = \lambda_u^{-i}$.

Since $|(A^t)^{i-1}k| \approx \lambda_u^i$ we have that

$$\hat{\Gamma}_{(A^t)^{i-1}k} |(A^t)^{i-1}k|$$

does not tend to zero. We recall that Riemann-Lebesgue theorem says that if φ is C^1, then, $|k|\hat{\varphi}_k$ tends to zero. Hence, the function constructed in (6.4) for this Φ is not even C^1.

There are arguments not based on Fourier series that show that for many choices of Φ, the function given in (6.4) is not C^1.

Of course, trivial modifications of this argument lead to infinite dimensional parameters of familes of examples of real valued functions which preserve the normal form at periodic orbits but for which the conjugacy given by structural stability is not even C^1.

There are many variants to this example, for example, one could do similar constructions using automorphisms of nil-manifolds rather than automorphisms of the torus or construct more complicated Jordan normal forms. For example:

$$B(x_1, \cdots, x_n) = (Ax_1 + x_2, Ax_2 + x_3, \cdots, Ax_n)$$
$$B(x, y, z) = (Ax + y, Ay, Az)$$

which contain longer Jordan blocks or several Jordan blocks, some trivial and some not. Simple variants of the methods given here allow to show that they are not rigid in a very strong sense and that they allow infinite dimensional families of deformations.

We, however, point out that Theorem 6.1 in [L2] shows that given $\Phi \in C^r$ $r = 2, \cdots, \infty$ sufficiently small, any C^2 conjugacy is $C^{r-\epsilon}$.

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